Punto 3

$$U_{ij}^{l+1} = V^2\left[U'_{i+j} - 2U'_{ij} + U'_{i-j} + \frac{\Delta\rho}{\rho[i]}\cdot\left(U'_{ji} - U'_{i-1j}\right) + \left(\frac{\lambda^2}{\rho[i]}\right)\left(U'_{ij+1} - 2U'_{ij}\right.\right.$$

$$\left.\left. + U'_{ij-1}\right)\right] + 2U'_{ji} - U_{ij}^{l-1}$$

★ Teniendo

$$\lambda = \frac{\Delta\rho}{\Delta\theta} \qquad V = \frac{\alpha\Delta t}{\Delta\rho}$$

en coordenadas polares $x = \rho\cos\theta$

$\rho = \sqrt{x^2+y^2}$ $\quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$ $\quad y = \rho\,\mathrm{sen}\,\theta$

$$\star\ \frac{\partial\rho}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{\rho\cos\theta}{\rho} = \cos\theta$$

$$\star\ \frac{\partial\rho}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{\rho\,\mathrm{sen}\,\theta}{\rho} = \mathrm{sen}\,\theta$$

$$\Delta\ \frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\rho\cos\theta}{\rho^2} = \frac{\cos\theta}{\rho}$$

$\Delta \; \frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} = \frac{\rho \operatorname{sen}\theta}{\rho^2} = -\frac{\operatorname{sen}\theta}{\rho}$

considerando $\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$

Donde $\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)$ • $\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x}$

$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x}\right]$ expandimos y reemplazamos

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial \rho^2}\cos^2\theta + \frac{\partial^2 U}{\partial \theta^2}\frac{\operatorname{sen}^2\theta}{\rho^2} - \frac{\partial^2 U}{\partial \rho \partial \theta}\frac{\operatorname{sen} 2\theta}{\rho} + \frac{\partial U}{\partial \theta}\frac{\operatorname{sen} 2\theta}{\rho^2} + \frac{\partial U}{\partial \rho}\frac{\operatorname{sen}^2\theta}{\rho}$$

y donde $\frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right)$ • $\frac{\partial U}{\partial y} = \frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial y}$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial y}\right]$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 U}{\partial \rho^2}\operatorname{sen}^2\theta + \frac{\partial^2 U}{\partial \theta^2}\frac{\cos^2\theta}{\rho^2} + \frac{\partial U^2}{\partial \rho \partial \theta}\frac{\operatorname{sen}(2\theta)}{\rho} + \frac{\partial U}{\partial \rho}\frac{\cos^2\theta}{\rho} - \frac{\partial U}{\partial \theta}\frac{\operatorname{sen} 2\theta}{\rho^2}$$

Como

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$$

hay términos que se cancelan

$$\nabla^2 U = \frac{\partial^2 U}{\partial \rho^2}\left[\cos^2\theta + \operatorname{sen}^2\theta\right] + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \theta^2}\left[\cos^2\theta + \operatorname{sen}^2\theta\right]$$

$$+ \frac{1}{\rho}\frac{\partial U}{\partial \rho}\left[\cos^2\theta + \operatorname{sen}^2\theta\right]$$

$$\nabla^2 U = \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho}$$

- consideramos que la ecuación de onda puede describirse de la forma

$$\frac{\partial^2 U}{\partial t^2} = \alpha^2 \nabla^2 U$$

discretizamos la ecuación

$$\frac{1}{\Delta t^2}\left(U_{ij}^{l+1} - 2U_{ji}^{l} + U_{ij}^{l-1}\right) = \frac{\alpha^2}{\Delta \rho^2}\left(U_{i+1j}^{l} - 2U_{ji}^{l} + U_{i-1j}^{l}\right)$$

$$\frac{\alpha^2}{\rho[i^2]\Delta\theta^2}\left(U_{ij+1}^{l} - 2U_{ji}^{l} + U_{ij-1}^{l}\right) + \frac{\alpha^2}{\rho[i]\Delta\rho}\left(U_{ij}^{l} - U_{i-1j}^{l}\right)$$

Juntamos términos iguales y despejamos U_{ij}^{l+1}

$$U_{ij}^{l+1} = \left(\frac{\alpha \Delta t}{\Delta \rho}\right)^2 \left[U_{i+1j}^{l} - 2U_{ij}^{l} + U_{i-1j}^{l} + \left(\frac{\Delta \rho}{\rho[i]}\right)\left(U_{ij}^{l} - U_{ij-1}^{l}\right) + \left(\frac{\Delta \rho^2}{\Delta \theta^2 \rho[i]}\right)\left(U_{ij+1}^{l} - 2U_{ij}^{l} + U_{ij-1}^{l}\right) \right] + 2U_{ij}^{l} - U_{ij}^{l-1}$$

Y como $\lambda = \dfrac{\Delta \rho}{\Delta \theta}$ $\quad V = \dfrac{\alpha \Delta t}{\Delta \rho}$

$$U_{ij}^{l+1} = V^2 \left[U_{i+1j}^{l} - 2U_{ij}^{l} + U_{i-1j} + \left(\frac{\Delta \rho}{\rho[i]}\right)\left(U_{ij}^{l} - U_{ij-1}^{l}\right) + \left(\frac{\lambda^2}{\rho[i]}\right)\left(U_{ij+1}^{l} - 2U_{ij}^{l} + U_{ij-1}^{l}\right) \right] + 2U_{ij}^{l} - U_{ij}^{l-1}$$