Homework 3.2 for DLDC

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Homework 3.2: Rational Quadratic Kernel

Use Rational Quadratic Kernel:

$$K(x) = \left(1 + \frac{x^2}{2\alpha k^2}\right)^{-\alpha}$$

and see how the regression changes with different hyper parameters (α, k).

Draw regression plots with different hyper parameters and discuss their effect on the regression.

Answer

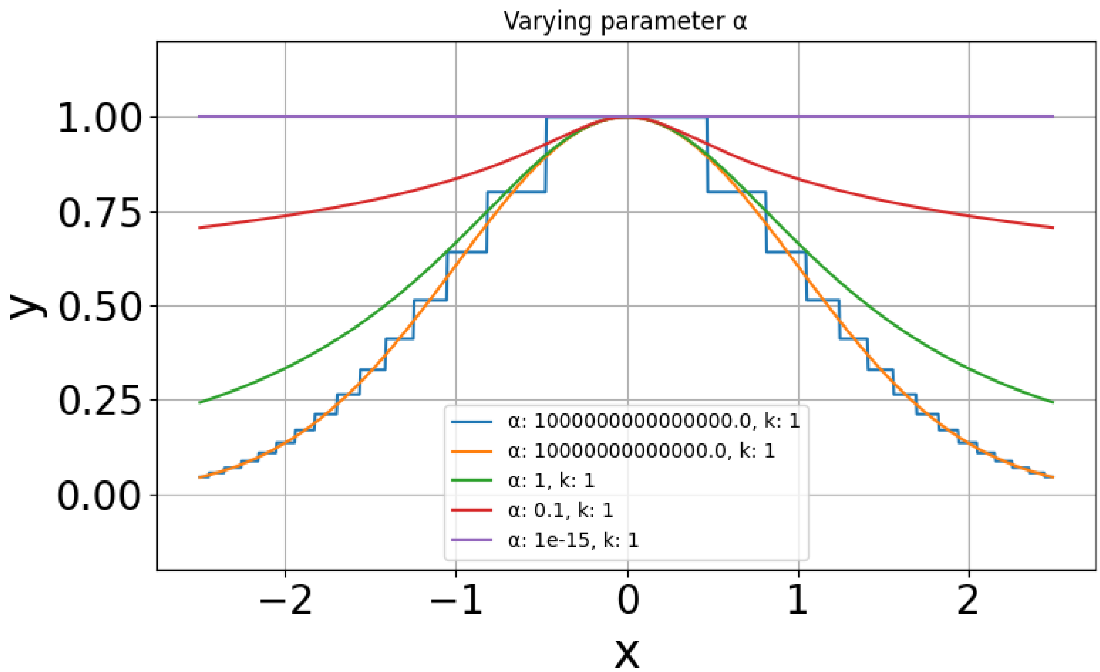
We plot the Rational Quadratic Kernel using the plotting function from the previous homework with different hyperparameters below and see the following effects:

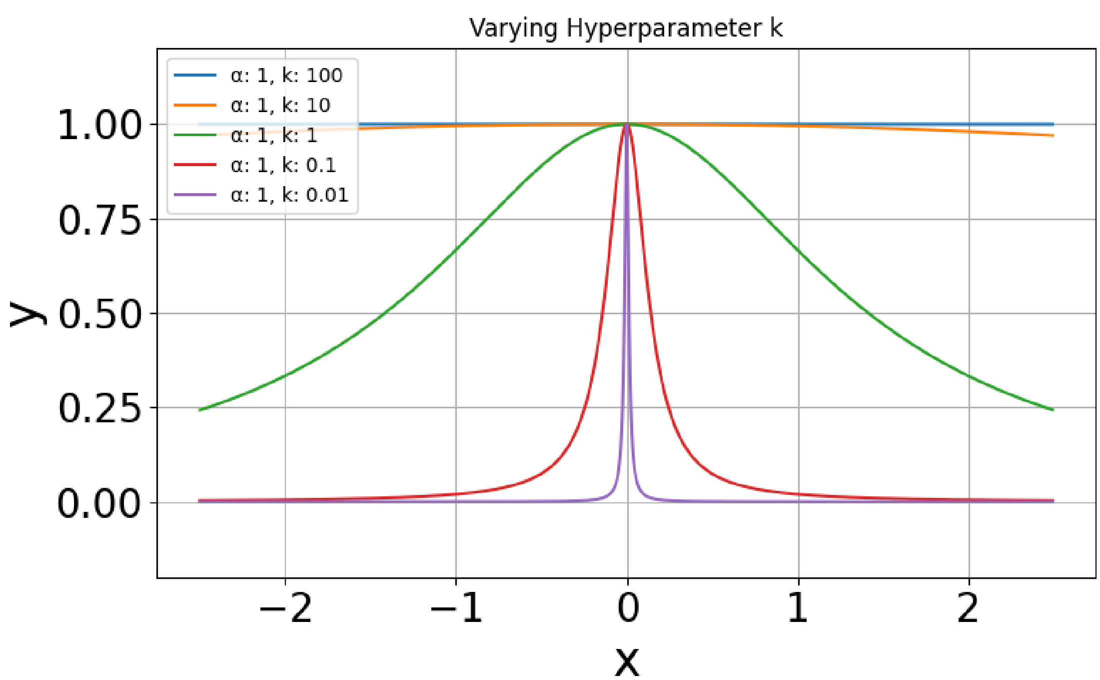
Varying the parameter α controls the smoothness of the Kernel function: The smaller α is, the smoother (and flatter) the curve.

Varying the parameter k also varies the smoothness of the Kernel function, now being completely flat for large k and getting "pointier" the smaller k gets.

[1]:
```python
import numpy as np
import matplotlib.pyplot as plt

def RationalQuadratic(x, dx, alpha, k):
  x_n = x # / dx normalize x
  kernel = (1 + (x_n ** 2) / (2 * alpha * (k ** 2))) ** (-alpha)
  return kernel

def PlotKernel(kernel_fn, dx, alpha, k):
  # plot Kernel
  x_space = np.arange(-2.5, 2.5, 0.005)
  y_space = np.zeros_like(x_space)

  for idx, xii in enumerate(x_space):
    y_space[idx] = kernel_fn(xii, dx, alpha, k)
  label = " : {}, k: {}".format(alpha,k)
  line, = ax.plot(x_space, y_space,  markersize=5, label=label)

dx = 0.5
```

```
fig, ax = plt.subplots(figsize = (8,5))
plt.title("Varying parameter  ")
PlotKernel(RationalQuadratic, dx, alpha=1E15, k=1)
PlotKernel(RationalQuadratic, dx, alpha=1E13, k=1)
PlotKernel(RationalQuadratic, dx, alpha=1, k=1)
PlotKernel(RationalQuadratic, dx, alpha=1E-1, k=1)
PlotKernel(RationalQuadratic, dx, alpha=1E-15, k=1)
ax.legend()
ax.axes.tick_params(labelsize=20)
ax.set_xlabel("x", fontsize=24)
ax.set_ylabel("y", fontsize=24)
ax.set_ylim([-0.2,1.2])
ax.grid()
fig.tight_layout()

dx = 0.5
fig, ax = plt.subplots(figsize = (8,5))
plt.title("Varying Hyperparameter k")
PlotKernel(RationalQuadratic, dx, alpha=1, k=100)
PlotKernel(RationalQuadratic, dx, alpha=1, k=10)
PlotKernel(RationalQuadratic, dx, alpha=1, k=1)
PlotKernel(RationalQuadratic, dx, alpha=1, k=0.1)
PlotKernel(RationalQuadratic, dx, alpha=1, k=0.01)
ax.legend(loc=2)
ax.axes.tick_params(labelsize=20)
ax.set_xlabel("x", fontsize=24)
ax.set_ylabel("y", fontsize=24)
ax.set_ylim([-0.2,1.2])
ax.grid()
fig.tight_layout()
```





Since both parameters have a similar effect on the Kernel (they affect the bandwidth, though that particular term is usually used only for the Gaussian Kernel), we perform Kernel interpolation using the given diamond-carat dataset with varying α and k and note, that as seen in the lecture, a smoother/wider Kernel (large k, small α) will emphasize global trends, while a narrower/pointier Kernel (small k, large α) will emphasize local trends.

On the extremes, a completely flat Kernel function results in the regression function just yielding the mean of all known points, while a completely narrow Kernel function yields an almost step-like function (i.e. nearest neighbour interpolation) .

[2]:
```
def RationalQuadratic(x, alpha, k):
  x_n = x # / dx normalize x
  kernel = (1 + (x_n ** 2) / (2 * alpha * (k ** 2))) ** (-alpha)
  return kernel

def Regression(x, x_sample, y_sample, alpha, k):
  y = np.zeros_like(x)
  x_sample_norm = (x_sample - x_sample.mean()) / x_sample.std()
  x_norm = (x - x_sample.mean()) / x_sample.std()

  for idx, y_i in enumerate(y):
      kernel_sum = 0
      for x_sample_i, y_sample_i in zip(x_sample_norm, y_sample):
        y_i += y_sample_i*RationalQuadratic(x_norm[idx]-x_sample_i, alpha, k)
        kernel_sum += RationalQuadratic(x_norm[idx]-x_sample_i, alpha, k)
      y[idx] = y_i / kernel_sum
  return y

x_sample = np.array([0.3, 0.5, 0.58, 0.64, 0.7, 0.8, 0.9, 1.0])
y_sample = np.array([630, 1098, 1641, 1733, 2000, 2885, 3508, 4140])
x = np.linspace(x_sample.min(), x_sample.max(), num=1000)

fig, ax = plt.subplots(figsize = (8,5))
plt.title("Varying Hyperparameter   (constant k=0.1)")

for alpha in [1E-15,1E-1,0.5,1,10]:
  k = 0.1
  y = Regression(x,x_sample,y_sample, alpha, k)
  label = "  = {}".format(alpha)
  ax.plot(x, y, label = label)
ax.legend()
plt.plot(x_sample, y_sample, 'o')
plt.show()

fig, ax = plt.subplots(figsize = (8,5))
plt.title("Varying Hyperparameter k (constant  =2)")
for k in [.01,.1,.3,1,10]:
  alpha = 2
  y = Regression(x,x_sample,y_sample, alpha, k)
  label = "k = {}".format(k)
  ax.plot(x, y, label = label)
```

```python
ax.legend()
plt.plot(x_sample, y_sample, 'o')
plt.show()
```