# 2D Simpson Integration

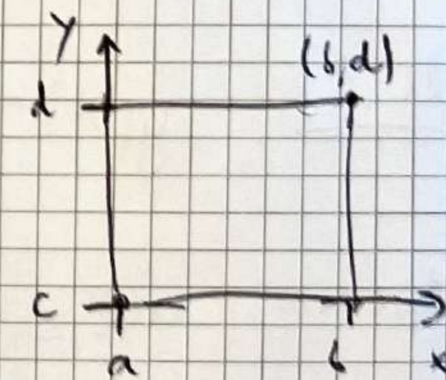
Recall Simpson's Integration: in 1D

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left( f(a) + 4 f\left(\frac{a+b}{2}\right) + f(b)\right)$$

The filter is, in 1D: $\boxed{\frac{h}{3}}\boxed{\frac{4h}{3}}\boxed{\frac{h}{3}} * \boxed{f(x_{i-1})}\boxed{f(x_i)}\boxed{f(x_{i+1})}$

where $(*)$ denotes the convolution operation.

In 2D, we want to integrate:




$$I = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d g(y)\,dy$$

with $$g(y) = \int_a^b f(x,y)\,dx$$

$$\approx \frac{b-a}{6}\left(f(a,y) + 4 f\left(\frac{a+b}{2}, y\right) + f(b,y)\right)$$

$$I = \int_c^d \left(\frac{b-a}{6}\right)\left(f(a,y) + 4 f\left(\frac{a+b}{2}, y\right) + f(b,y)\right) dy$$

$$= \left(\frac{d-c}{6}\right)\left(\frac{b-a}{6}\right)\Big[ f(a,c) + 4 f\left(a, \frac{c+d}{2}\right) + f(a,d) + \ldots$$

$$4\left( f\left(\frac{a+b}{2}, c\right) + 4 f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, d\right)\right) + \ldots$$

$$f(b,c) + 4 f\left(b, \frac{c+d}{2}\right) + f(b,d) \Big]$$

Now, transform into a general grid with:

$$x_{i-1} = a, \quad x_i = \frac{a+b}{2}, \quad x_{i+1} = b, \quad 2h_x = b-a$$

$$y_{j-1} = c, \quad y_j = \frac{c+d}{2}, \quad y_{j+1} = d, \quad 2h_y = d-c$$

The integral becomes:

$$I = \frac{h_x \cdot h_y}{9} \cdot \underbrace{\begin{bmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{bmatrix}}_{\text{2D simpson filter.}} * \begin{bmatrix} f(x_{i-1}, y_{j+1}) & f(x_i, y_{j+1}) & f(x_{i+1}, y_{j+1}) \\ f(x_{i-1}, y_j) & f(x_i, y_j) & f(x_{i+1}, y_j) \\ f(x_{i-1}, y_{j-1}) & f(x_i, y_{j-1}) & f(x_{i+1}, y_{j-1}) \end{bmatrix}$$