Drinfeld modular forms

IAS Short Talk

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5 October 2023

Classical modular forms – connections

Modular forms are analytic functions on the complex upper half plane which have important number theoretical properties.

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- $|f(z)|$ is bounded as $\mathrm{Im}\, z \to +\infty$. *(holomorphic at infinity)*

The second condition can be written as $f(\gamma z) = j(\gamma, z)^{-k} f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, with *factor of automorphy* $j(\gamma, z) = cz + d$.

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Here the group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ from the left by the fractional linear transformation $z \overset{\gamma}{\mapsto} \frac{az+b}{cz+d}$, which is closely related to matrix multiplication on the left:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = \frac{1}{cz+d} \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}.$$

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The left action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ translates into a right action on the functions $f : \mathcal{H} \to \mathbb{C}$:

$$f|_\gamma : \mathcal{H} \to \mathbb{C}, \quad z \mapsto j(\gamma, z)^k f(\gamma z) = (cz+d)^k f\left(\frac{az+b}{cz+d}\right).$$

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The transformation condition in the previous definition can then be restated more simply:

$$f|_\gamma(z) = f(z) \quad \text{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

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The first definition above is that of a modular form for the *full* modular group $\mathrm{SL}_2(\mathbb{Z})$. More generally, for any $N \in \mathbb{N}$ we can define a modular form for any *congruence subgroup* $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma \supseteq \Gamma(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod{N} \right\} :$$

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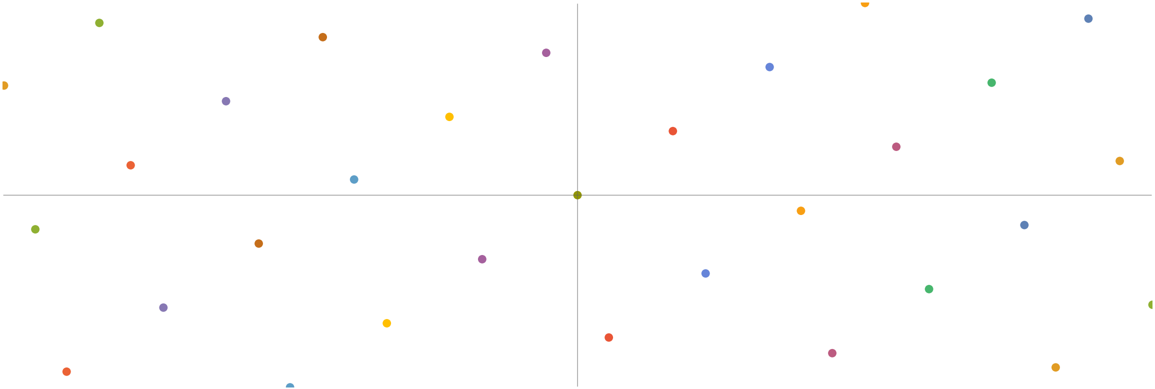

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Defining a modular form for the congruence subgroup $\Gamma(N)$ as a function of lattices is not as simple, but more on this later!

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Here, analogously with the classical setting, A is a Dedekind ring. We also consider the positive integer q, which is the cardinality of the field of constants of F, with associated finite field $\mathbb{F}_q$.

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- Here $\mathbb{C}_\infty$ has infinite dimension as a vector space over $\mathbb{F}_\infty$, whereas $\mathbb{C}$ has dimension 2 as a vector space over $\mathbb{R}$. As a result, whereas lattices in the classical case can have rank at most 2, here lattices can have arbitrarily high rank. This what makes the theory of 'modular forms of higher rank' possible.

My work

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Their approach has been mainly as viewing them as functions of $r-1$ variables, whereas my PhD thesis established a theory viewing them as functions on the space of lattices of higher rank. My current work in this area is on a theory of Hecke operators from this point of view.

Additionally, in the rank 2 case there is the question of generators for the (graded) algebra of modular forms for the principal congruence subgroup $\Gamma(N)$, where I have some partial computational results.

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Every lattice has $\#\mathrm{GL}_r(A/N)$ different level N structures, since A is a Dedekind ring.

We denote the space of lattices Λ of rank r with level N structure α by $\mathcal{L}_N^r$, and the space of lattices Λ of rank r without level structure by $\mathcal{L}^r$.

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- holomorphic on the interior $\mathcal{L}^r_N$ of $\overleftarrow{\mathcal{L}^r_N}$,
- homogeneous of degree $-k$

Higher rank modular forms

We define metrics $\mathrm{d}_{\mathcal{L}}$ and $\mathrm{d}_{\mathcal{L}_N^r}$ on the rigid analytic spaces $\mathcal{L}^r$ and $\mathcal{L}_N^r$, leading to their completions $\mathcal{L}^{\leq r}$ and $\overleftarrow{\mathcal{L}_N^r}$, where the boundaries consist of spaces of lattices of lower rank.

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My PhD thesis was on the general theory of modular forms of higher rank as functions of lattices, and I am currently working on translating the theory of Hecke operators to this viewpoint.

Eisenstein series

Typical examples of modular forms of rank 2 for $\Gamma(N)$ are the (partial) Eisenstein series:

$$E_{r_1,r_2}(z) = \sum_{m,n \in A} \frac{1}{(m+r_1)z+n+r_2} \quad \text{for } r_1, r_2 \in N^{-1}A/A;$$

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Cornelissen proved that the graded algebra of Drinfeld modular forms of rank 2 for $\Gamma(N)$ are generated by these Eisenstein series and possibly some cusp forms of weight 2, but it is not known whether or not these cusp forms are necessary. I have some partial computational results in this direction, but hope to finish it off analytically.