Drinfeld modular forms

IAS Short Talk

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5 October 2023

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- $|f(z)|$ is bounded as $\mathrm{Im}\, z \to +\infty$. *(holomorphic at infinity)*

The second condition can be written as $f(\gamma z) = j(\gamma, z)^k f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, with *factor of automorphy* $j(\gamma, z) = cz + d$.

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Here the group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ from the left by the fractional linear transformation $z \overset{\gamma}{\mapsto} \frac{az+b}{cz+d}$, which is closely related to matrix multiplication on the left:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = \frac{1}{cz+d} \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}.$$

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The left action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ translates into a right action on the functions $f : \mathcal{H} \to \mathbb{C}$:

$$f|_\gamma : \mathcal{H} \to \mathbb{C}, \quad z \mapsto j(\gamma, z)^{-k} f(\gamma z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

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$$f|_\gamma(z) = f(z) \quad \text{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

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The first definition above is that of a modular form for the *full* group $\mathrm{SL}_2(\mathbb{Z})$. More generally, for any $N \in \mathbb{N}$ we can define a modular form for any *congruence subgroup* $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma \supseteq \Gamma(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod{N} \right\} :$$

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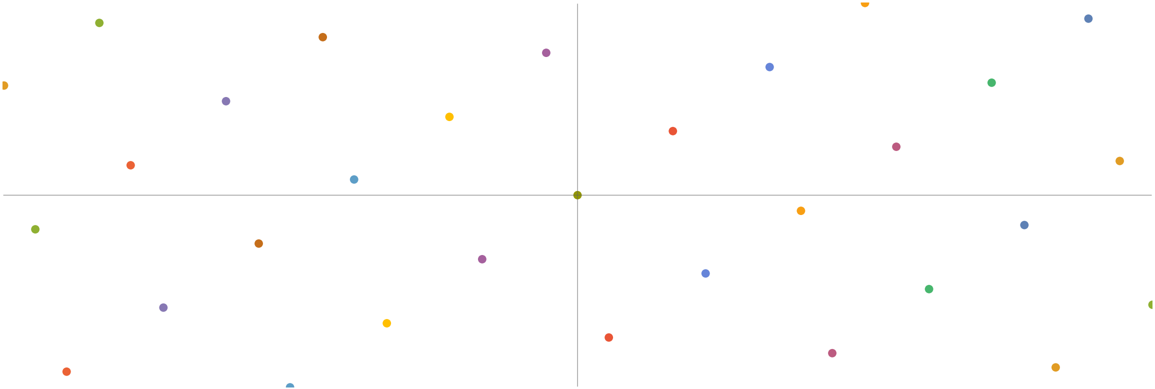

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If f is a modular form in this lattice sense, then $\bar{f}(z) = f(z\mathbb{Z} + \mathbb{Z})$ is a modular form in the complex-variable sense. In fact, the converse is also true, so that these two definitions are equivalent.

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To define a modular form for the congruence subgroup $\Gamma(N)$ as a function of lattices we actually define it as a homogeneous function $f(\Lambda, \alpha)$ of a lattice Λ together with a *level structure* $\alpha : N^{-1}\Lambda/\Lambda \hookrightarrow\!\!\!\!\rightarrow (N^{-1}/\mathbb{Z})^2$.

The Drinfeld setting

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Here, analogously with the classical setting, A is a Dedekind ring. We also consider the positive integer q, which is the cardinality of the field of constants of F, with associated finite field $\mathbb{F}_q$.

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	$\lvert\cdot\rvert$, an absolute value on F with place ∞	the usual absolute value $\lvert\cdot\rvert$ on $\mathbb{C}$
$\mathbb{F}_q[T]$	A, the ring of elements of F regular away from ∞	$\mathbb{Z}$, the integers
$\mathbb{F}_q((\frac{1}{T}))$	$\mathbb{F}_\infty$, the completion of F with respect to $\lvert\cdot\rvert$	$\mathbb{R}$, the real numbers
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- Here $\mathbb{C}_\infty$ has *infinite* dimension as a vector space over $\mathbb{F}_\infty$, whereas $\mathbb{C}$ has dimension 2 as a vector space over $\mathbb{R}$. As a result, whereas lattices in the classical case can have rank at most 2, here lattices can have arbitrarily high rank. This what makes the theory of 'modular forms of higher rank' possible.

My work – higher rank modular forms

Whereas most of the theory of Drinfeld modular forms has focused on modular forms of rank 2, due to the simplicity of dealing with a function of one variable and on analogy with the classical case, recent work by Gekeler (for $F = \mathbb{F}_q(T)$) and Basson, Breuer, and Pink (for general F) have established theories of Drinfeld modular forms of higher rank.

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Their approach has been viewing them as functions of $r - 1$ variables, whereas my PhD thesis establishes a theory viewing them as functions on the space of lattices of higher rank. My current work in this area is on further filling out the theory from this point of view.

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We denote the space $\{(\Lambda, \alpha)\}$ of lattices Λ of rank r with level N structure α by $\mathcal{L}_N^r$, and the space of lattices Λ of rank r without level structure by $\mathcal{L}^r$.

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We denote the space $\{(\Lambda, \alpha)\}$ of lattices Λ of rank r with level N structure α by $\mathcal{L}_N^r$, and the space of lattices Λ of rank r without level structure by $\mathcal{L}^r$.

We prove that these spaces are rigid analytic spaces by identifying them with a double quotient:

$$\mathcal{L}_N^r \simeq \mathrm{GL}_r(F)\Big\backslash\left(\Psi^r \times \mathrm{GL}_r\left(\mathbb{A}_F^{fin}\right)/K(N)\right)$$

and so we can speak of holomorphic functions on these spaces.

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- Defining Hecke operators and proving their recursive properties.

My work – Eisenstein series in rank 2

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Typical examples of modular forms of rank 2 for $\Gamma(N)$ are the (partial) Eisenstein series:

$$E_{r_1,r_2}(z) = \sum_{m,n \in A} \frac{1}{(m+r_1)z + n + r_2} \quad \text{for } r_1, r_2 \in N^{-1}A/A;$$

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I have some partial computational results in this direction: for specific N we can reduce it to linear algebra using the series expansions of these Eisenstein series at the cusps and the known dimension of the space of weight 2 modular forms, but I hope to finish it off analytically.