The Generation of Rank 2 Drinfeld Modular Forms by Eisenstein Series: A Computational Approach

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Abstract

A computational method is presented for determining whether or not the space of Drinfeld modular forms for a principal congruence subgroup $\Gamma(N)$ is generated by Eisenstein series, which is currently an open question. This method boils down to expressing the dimension of the weight 2 space generated by Eisenstein series as the rank of a matrix of coefficients of expansions of products of Eisenstein series at the cusps. As a result, the question is answered in the case of $q = 2$ and N quadratic or cubic and in the case of $q = 3$ and N quadratic.

1 Introduction and Outline

1 Outline of the Paper

In Section 1, we give an outline of the paper, a review of the work published so far on this specific topic, and a description of the main objects in our playing field. Then in Section 2 we describe the algorithm which is the main subject of this paper, together with proof of its validity and bounds on its computational complexity. In the subsequent Section 3 we present some refinements to the algorithm which speed up its execution; this section also includes some previously unseen relations between Eisenstein series which may even be of interest outside of the context of this algorithm. Finally in Section 4 we present the results of the algorithm applied to small values of N and present some concluding remarks.

2 Review of the Relevant Literature

Drinfeld modules are analogous to elliptic curves in the function field setting, and were first defined by ¡Drinfeld¿, who called them *elliptic modules* and used them to prove ¡Langlands¿. Later, ¡Goss¿ defined what he called *Drinfeld modular forms* of rank 2 analogously with classical modular forms (in this paper, we will call Drinfeld modular forms simply modular forms, and explicitly mention when their classical counterparts are named). These functions form a graded algebra graded by their *weight*. Goss also imposed a condition of 'boundedness at infinity' which allowed him to show the finite-dimensionality of the spaces of D-modular forms of any given weight. ¡Gekeler¿ defined Eisenstein series as cases of modular forms, and showed that Eisenstein series of higher weight are generated by Eisenstein series of weight 1.

The question then naturally arose of whether or not the space of all modular forms is generated as an algebra by the Eisenstein series of weight 1; this was answered in the affirmative by ¡Cornellissen¿

for principal congruence subgroups $\Gamma(N)$ where N is linear, but the question for general N is still open. The most significant progress thus far has been by ¡someone¿, who showed that the space of all modular forms is generated by the Eisenstein series of weight 1, possibly together with some cusp forms of weight 2 (these are modular forms which are zero at all the cusps). It thus suffices to establish whether or not the Eisenstein series of weight 1 generates the space of modular forms of weight 2, and in this paper we give a method for doing so for any specific characteristic q and principal congruence subgroup $\Gamma(N)$, as well as an implementation of this method in SageMath.

3 The Mathematical Setup

Let q be a prime power, and let $\mathbb{F}_q$ denote the finite field of cardinality q. Let $A = \mathbb{F}_q[T]$ denote the ring of polynomials with coefficients in $\mathbb{F}_q$, with $F = \mathbb{F}_q(T)$ its field of fractions; these can be thought of in analogy with the integers $\mathbb{Z}$ and rational numbers $\mathbb{Q}$ respectively. A can be equipped with an absolute value $|a| = q^{\deg_T a}$ which is extended to F in the natural way; when F is completed with respect to this absolute value we obtain the field $F_\infty = \mathbb{F}_q((T))$ of Laurent series in T with finitely many positive powers of T, which is analogous to the real numbers $\mathbb{R}$. Finally, the completion of an algebraic closure of F_∞ is denoted by $\mathbb{C}_\infty$; this is analogous to the complex numbers $\mathbb{C}$, and is where most of the action will happen.

$\mathbb{C}_\infty$ has rigid analytic structure, as does $\Omega = \mathbb{C}_\infty - F_\infty$, and as such we can speak of rigid analytic functions on them. Moreover, the group $\mathrm{GL}_2(A)$ acts on Ω on the left by fractional linear transformations $\gamma z = \frac{az+b}{cz+d}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A)$ and $z \in \mathbb{C}_\infty$. That this is a group action can be seen in that it arises from the action of $\mathrm{GL}_2(A)$ as matrices multiplying on the left of $\mathbb{P}^2(\mathbb{C}_\infty)$ with rational lines removed, seen as column vectors: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} az+b \\ cz+d \end{smallmatrix}\right) = \left(\begin{smallmatrix} (az+b)/(cz+d) \\ 1 \end{smallmatrix}\right)$.

As special subgroups of $\mathrm{GL}_2(A)$ we consider the full congruence subgroups

$$\Gamma(N) = \ker\left(\mathrm{GL}_2(A) \twoheadrightarrow \mathrm{GL}_2(A/N)\right) = \left\{\gamma \in \mathrm{GL}_2(A) \mid \gamma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod N\right\}$$

where N is an ideal of A, or equivalently a monic polynomial since $A = \mathbb{F}_q[T]$ is a principal ideal domain.

The quotient of $\Gamma(N)\backslash\Omega$ is compactified by adding finitely many points $\Gamma(N)\backslash F$; these points are called the *cusps* of the compactified space, which is denoted by $\overline{M}_{\Gamma(N)}$.

For a nonnegative integer k,

2 The Algorithm

3 Refinements To The Algorithm

4 Conclusion

Due to computational constraints we are only able to run the code for some small nonlinear values of N, but we hope that this progress will inspire others to improve on our method or given the evidence above to revisit this interesting question with fresh eyes.