Drinfeld modular forms

IAS Short Talk

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5 October 2023

Classical modular forms – connections

Modular forms are analytic functions on the complex upper half plane which have important number theoretical properties.

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- $|f(z)|$ is bounded as $\mathrm{Im}\, z \to +\infty$. *(holomorphic at infinity)*

The second condition can be written as $f(\gamma z) = j(\gamma, z)^{-k} f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, with *factor of automorphy* $j(\gamma, z) = cz + d$.

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Here the group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ from the left by the fractional linear transformation $z \overset{\gamma}{\mapsto} \frac{az+b}{cz+d}$, which is closely related to matrix multiplication on the left:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = \frac{1}{cz+d} \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}.$$

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The left action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ translates into a right action on the functions $f : \mathcal{H} \to \mathbb{C}$:

$$f|_\gamma : \mathcal{H} \to \mathbb{C}, \quad z \mapsto j(\gamma, z)^k f(\gamma z) = (cz+d)^k f\left(\frac{az+b}{cz+d}\right).$$

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The transformation condition in the previous definition can then be restated more simply:

$$f|_\gamma(z) = f(z) \quad \text{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

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The first definition above is that of a modular form for the *full* modular group $\mathrm{SL}_2(\mathbb{Z})$. More generally, for any $N \in \mathbb{N}$ we can define a modular form for any *congruence subgroup* $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma \supseteq \Gamma(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod{N} \right\} :$$

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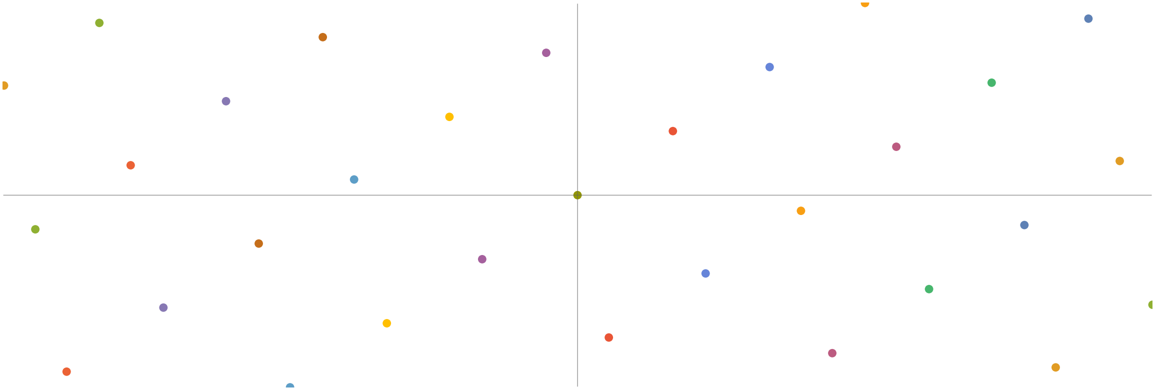

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To define a modular form for the congruence subgroup $\Gamma(N)$ as a function of lattices we actually define it as a homogeneous function $f(\Lambda, \alpha)$ of a lattice Λ together with a *level structure* $\alpha : N^{-1}\Lambda/\Lambda \hookrightarrow\!\!\!\!\rightarrow (N^{-1}/\mathbb{Z})^2$.

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Here, analogously with the classical setting, A is a Dedekind ring. We also consider the positive integer q, which is the cardinality of the field of constants of F, with associated finite field $\mathbb{F}_q$.

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- Here $\mathbb{C}_\infty$ has infinite dimension as a vector space over $\mathbb{F}_\infty$, whereas $\mathbb{C}$ has dimension 2 as a vector space over $\mathbb{R}$. As a result, whereas lattices in the classical case can have rank at most 2, here lattices can have arbitrarily high rank. This what makes the theory of 'modular forms of higher rank' possible.

My work – higher rank

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Every lattice has $\#\mathrm{GL}_r(A/N)$ different level N structures, since A is a Dedekind ring.

We denote the space of lattices Λ of rank r with level N structure α by $\mathcal{L}_N^r$, and the space of lattices Λ of rank r without level structure by $\mathcal{L}^r$.

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We prove that these spaces are rigid analytic spaces by identifying them with a double quotient:

$$\mathcal{L}_N^r \simeq \mathrm{GL}_r(F)\Big\backslash\left(\Psi^r \times \mathrm{GL}_r\left(\mathbb{A}_F^{fin}\right)/K(N)\right)$$

and so we can speak of holomorphic functions on these spaces.

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We define metrics $\mathrm{d}_{\mathcal{L}}$ and $\mathrm{d}_{\mathcal{L}_N^r}$ on the rigid analytic spaces $\mathcal{L}^r$ and $\mathcal{L}_N^r$, leading to their completions $\mathcal{L}^{\leq r}$ and $\overleftarrow{\mathcal{L}_N^r}$, where the boundaries consist of spaces of lattices of lower rank.

Definition

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- Defining Hecke operators and proving their recursive properties

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$$E_{r_1,r_2}(z) = \sum_{m,n\in A} \frac{1}{(m+r_1)z+n+r_2} \quad \text{for } r_1, r_2 \in N^{-1}A/A;$$

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I have some partial computational results in this direction: for specific N we can reduce it to linear algebra using the series expansions of these Eisenstein series at the cusps and the known dimension of the space of weight 2 modular forms, but I hope to finish it off analytically.