Drinfeld modular forms

IAS Short Talk

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Classical modular forms – connections

Modular forms are analytic functions on the complex upper half plane which have important number theoretical properties.

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- $|f(z)|$ is bounded as $\operatorname{Im} z \to +\infty$. *(holomorphic at infinity)*

The second condition can be written as $f(\gamma z) = j(\gamma, z)^{-k} f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, with *factor of automorphy* $j(\gamma, z) = cz + d$.

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Here the group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ from the left by the fractional linear transformation $z \overset{\gamma}{\mapsto} \frac{az+b}{cz+d}$, which is closely related to matrix multiplication on the left:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = \frac{1}{cz+d} \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}.$$

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The left action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ translates into a right action on the functions $f : \mathcal{H} \to \mathbb{C}$:

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$$f|_\gamma(z) = f(z) \quad \text{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

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The first definition above is that of a modular form for the *full* group $\mathrm{SL}_2(\mathbb{Z})$. More generally, for any $N \in \mathbb{N}$ we can define a modular form for any *congruence subgroup* $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma \supseteq \Gamma(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod{N} \right\} :$$

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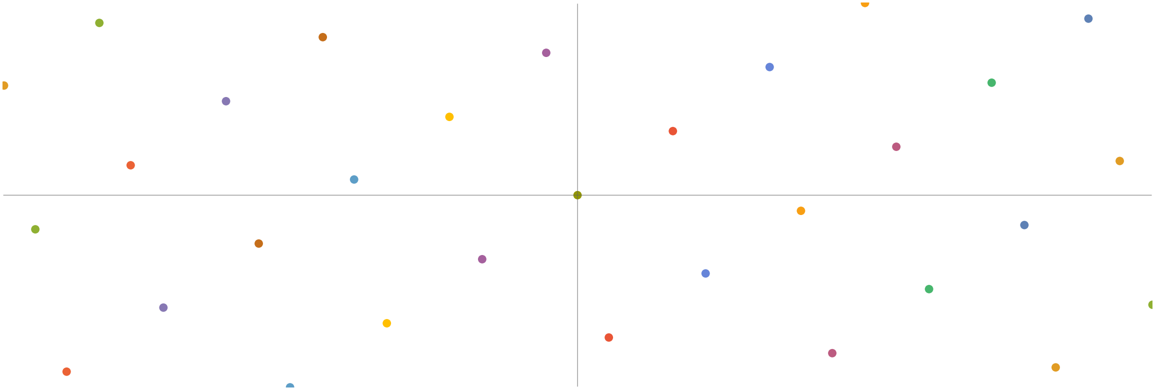

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To define a modular form for the congruence subgroup $\Gamma(N)$ as a function of lattices we actually define it as a homogeneous function $f(\Lambda, \alpha)$ of a lattice Λ together with a *level structure* $\alpha : N^{-1}\Lambda/\Lambda \hookrightarrow\!\!\!\!\rightarrow (N^{-1}/\mathbb{Z})^2$.

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Here, analogously with the classical setting, A is a Dedekind ring. We also consider the positive integer q, which is the cardinality of the field of constants of F, with associated finite field $\mathbb{F}_q$.

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$\mathbb{F}_q[T]$	A, the ring of elements of F regular away from ∞	$\mathbb{Z}$, the integers
$\mathbb{F}_q((\frac{1}{T}))$	$\mathbb{F}_\infty$, the completion of F with respect to $\lvert\cdot\rvert$	$\mathbb{R}$, the real numbers
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- Here $\mathbb{C}_\infty$ has *infinite* dimension as a vector space over $\mathbb{F}_\infty$, whereas $\mathbb{C}$ has dimension 2 as a vector space over $\mathbb{R}$. As a result, whereas lattices in the classical case can have rank at most 2, here lattices can have arbitrarily high rank. This what makes the theory of 'modular forms of higher rank' possible.

My work – higher rank modular forms

Whereas most of the theory of Drinfeld modular forms has focused on modular forms of rank 2, due to the simplicity of dealing with a function of one variable and on analogy with the classical case, recent work by Gekeler (for $F = \mathbb{F}_q(T)$) and Basson, Breuer, and Pink (for general F) have established theories of Drinfeld modular forms of higher rank.

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Their approach has been viewing them as functions of $r - 1$ variables, whereas my PhD thesis establishes a theory viewing them as functions on the space of lattices of higher rank. My current work in this area is on further filling out the theory from this point of view.

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We prove that these spaces are rigid analytic spaces by identifying them with a double quotient:

$$\mathcal{L}_N^r \simeq \mathrm{GL}_r(F)\Big\backslash\left(\Psi^r \times \mathrm{GL}_r\left(\mathbb{A}_F^{fin}\right)/K(N)\right)$$

and so we can speak of holomorphic functions on these spaces.

Higher rank modular forms

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We define metrics $\mathrm{d}_{\mathcal{L}}$ and $\mathrm{d}_{\mathcal{L}_N^r}$ on the rigid analytic spaces $\mathcal{L}^r$ and $\mathcal{L}_N^r$, leading to their completions $\mathcal{L}^{\leq r}$ and $\overleftarrow{\mathcal{L}_N^r}$, where the boundaries consist of spaces of lattices of lower rank.

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- Defining Hecke operators and proving their recursive properties.

My work – Eisenstein series in rank 2

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Typical examples of modular forms of rank 2 for $\Gamma(N)$ are the (partial) Eisenstein series:

$$E_{r_1,r_2}(z) = \sum_{m,n\in A} \frac{1}{(m+r_1)z+n+r_2} \quad \text{for } r_1, r_2 \in N^{-1}A/A;$$

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I have some partial computational results in this direction: for specific N we can reduce it to linear algebra using the series expansions of these Eisenstein series at the cusps and the known dimension of the space of weight 2 modular forms, but I hope to finish it off analytically.