IMO Stream Test 3

June Camp 2017

Time: $4\frac{1}{2}$ hours

1. For any positive integer k, denote the sum of the digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for all positive integers $n \geq 2016$, the integer $P(n)$ is positive and
$$S(P(n)) = P(S(n)).$$

2. Find the smallest real constant C such that for any positive real numbers a_1, a_2, a_3, a_4 and a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k and l such that
$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leq C.$$

3. Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty bounded subset S of the plane is said to be *nice* if

   (i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and

   (ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

   Prove that there exist two distinct nice subsets S and S' of the quadrant $\{(x, y) : x \geq 0, y \geq 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.