PAMO Stream Test 1

June Camp 2017

Time: $4\frac{1}{2}$ hours

1. Let $2\mathbb{N}$ denote the set of all positive even integers. Find all increasing functions $f : 2\mathbb{N} \to \mathbb{Z}$ such that for all $x, y \in 2\mathbb{N}$,
$$f(x) + f(y) = 2f(xy).$$

2. Let ABC be a triangle, and let I_A be the centre of a circle which is tangent to the extensions of AB, BC and CA such that A and I_A are on opposite sides of BC. Let I_B and I_C be defined similarly. Prove that the altitude from I_A onto BC, the altitude from I_B onto CA and the altitude from I_C onto AB are concurrent.

3. Let A be a set of positive integers such that
   (a) if $a \in A$, the all the positive divisors of a are also in A;
   (b) if $a, b \in A$, with $1 < a < b$, then $1 + ab \in A$.

   Prove that if A has at least 3 elements, then A is the set of all positive integers.