

# System of Linear Equations

**Example** A manufacturer produces two kinds of toys: wooden cars and wooden trains. The amounts of two raw materials needed to product 1 piece of toys are given in the table.

	car	train
wood	2	3
paint	5	4

Determine the number of cars and trains produced on a given day, when we know that for the production there were used 540 units of wood and 1070 units of paint.

## Systems of linear equations (SLE)

### Definition

The system of equations

$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
 &\vdots \\
 a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
 \end{aligned}$$

where the numbers  $a_{ij}$  ( $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ ) and  $b_k$  ( $k \in \{1, \dots, m\}$ ) are known, the variables  $x_1, \dots, x_n$  are unknown, is called a system of linear equations.

- $a_{ij}$  : the coefficients of the system of linear equations
- $b_k$  : the constant terms

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The coefficient matrix and the augmented matrix:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad A|b = \left( \begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array} \right).$$

The right-hand side vector and the solution vector

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

The corresponding matrix-vector equation:  $Ax = b$ .

## Solvability of systems of linear equations

### Definition

The system of linear equations is

- solvable if there exists at least one solution, that is, an  $x$  such that  $Ax = b$  holds; determined if there is exactly 1 solution; undetermined if there are more than 1 solutions;
- inconsistent if it doesn't have a solution.

### Remark

The SLE can be solved if and only if the vector  $b$  can be expressed as a linear combination of the column vectors of  $A$ . This means, that  $b$  is in the subspace spanned by the column vectors of  $A$ .

If  $b$  can be expressed uniquely (i.e. the columns vectors of  $A$  are linearly independent), then there exists a unique solution.

If the column vectors of  $A$  are linearly dependent, and  $b$  is in the subspace spanned by the column vectors of  $A$ , then there are infinitely many solutions.

### Definition

The rank of a matrix is the rank of the system of column vectors of the matrix. Notation:  $\text{rank}(A)$ .

### Condition on solvability

- A system of lin. eq.s is solvable if, and only if  $\text{rank}(A) = \text{rank}(A | b)$ .
- If it is solvable and  $\text{rank}(A) = n$  (where  $n$  is the number of unknown parameters), then the system is determined, if  $\text{rank}(A) < n$ , then undetermined.

## Solutions of a system of linear equations

### Definition

A system of linear equations is homogeneous if  $b = 0$ , thus then the matrix equation has the form  $Ax = 0$ . Otherwise it's called nonhomogeneous.

**Remark:** 0 is a solution of any homogenous system of linear equations (it is the trivial solution).

### Theorem

A homogeneous system of linear equations has a nontrivial solution if and only if the column vectors of  $A$  are linearly dependent.

### Solutions of a homogeneous system of linear equations

The solutions of a real homogeneous system of linear equations form a vector subspace of  $\mathbb{R}^n$  with dimension  $n - \text{rank}(A)$ .

### Solutions of a nonhomogeneous system of linear equations

The solutions of a (solvable) nonhomogeneous system of linear equations  $Ax = b$  are of the form  $x^* + y$ , where

- $x^*$  is a particular solution of the system of linear equations;
- $y$  is an arbitrary solution of the corresponding homogeneous system of linear equation, that is  $Ax = 0$ .

### Solving a system of linear equations with Gaussian elimination

The set of solutions of a system of linear equations does not change, if we

- multiply an equation by a nonzero constant;
- add a scalar multiple of an equation to another equation;
- interchange two equations;
- discard an equation which is a scalar multiple of another equation.

We eliminate the numbers under the main diagonal with the modifications above. The resulting system is easier to solve.

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- If during the process we obtain a row like  $(0 \dots 0 \mid \neq 0)$ , then the system of linear equations has no solution.
  - If at the end of the process there are  $n$  number of not identically 0 rows, then the system is determined (there is a unique solution), if fewer number of rows remains, then undetermined (infinitely many solutions). (Here  $n$  is the number of the unknown parameters.)

### Example

Solve the system  $Ax = b$  using Gaussian elimination. .

$$A = \begin{pmatrix} 2 & 2 & 3 \\ -4 & -1 & -5 \\ -2 & 4 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} -4 \\ 12 \\ 10 \end{pmatrix}$$

**Solution:**

$$\begin{aligned} \left( \begin{array}{ccc|c} 2 & 2 & 3 & -4 \\ -4 & -1 & -5 & 12 \\ -2 & 4 & 0 & 10 \end{array} \right) &\longrightarrow \left( \begin{array}{ccc|c} 2 & 2 & 3 & -4 \\ 0 & 3 & 1 & 4 \\ 0 & 6 & 3 & 6 \end{array} \right) \\ &\longrightarrow \left( \begin{array}{ccc|c} 2 & 2 & 3 & -4 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 1 & -2 \end{array} \right) \\ &\qquad \qquad \qquad \left( \begin{array}{ccc|c} 2 & 2 & 3 & -4 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 1 & -2 \end{array} \right) \end{aligned}$$

Backward substitution:

$$\begin{aligned} x_3 &= -2 \\ 3x_2 + x_3 &= 4 \quad \rightarrow x_2 = 2 \\ 2x_1 + 2x_2 + 3x_3 &= -4 \quad \rightarrow x_1 = -1 \end{aligned}$$

## Rank, Determinant

The following operations do not change the rank of a matrix  $A$  :

- Interchanging 2 rows of  $A$ .
- Multiplying a row of  $A$  by a scalar  $\lambda \neq 0$ .
- Adding a scalar multiple of a row to another row.
- The determinant doesn't change if we add a scalar multiple of a row to another row.
- If we interchange 2 rows of  $A$ , then the sign of the determinant changes.
- The determinant of a triangular matrix is the product of the elements of the main diagonal.

$\implies$  Gaussian elimination can be used for computation of  $\text{rank}(A)$  and  $\det(A)$ .

### Example

Calculate  $\text{rank}(A)$  and  $\det(A)$ .

$$A = \begin{pmatrix} 3 & 5 & -6 \\ -1 & -2 & 1 \\ 2 & 6 & 5 \end{pmatrix}$$

**Solution:**

$$\begin{aligned} A = \begin{pmatrix} 3 & 5 & -6 \\ -1 & -2 & 1 \\ 2 & 6 & 5 \end{pmatrix} &\longrightarrow \begin{pmatrix} -1 & -2 & 1 \\ 3 & 5 & -6 \\ 2 & 6 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & -2 & 1 \\ 0 & -1 & -3 \\ 0 & 2 & 7 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} -1 & -2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \implies \text{rank}(A) = 3 \end{aligned}$$

The determinant of the last matrix is  $(-1)(-1) \cdot 1 = 1$ . During the calculations we interchanged the rows of  $A$  only once, so  $\det(A) = -1$ .