# First-order Logic

## First-order Language: The Language of Predicate Logic

### Definition 1:

The language of first-order logic is a

$$L^{(1)} = \langle LC, Var, \text{Con}, \text{ Term}, \text{ Form}\rangle$$

ordered 5-tuple, where

1. $LC = \{\neg, \supset, \wedge, \vee, \equiv, =, \forall, \exists, ()\}$ :, (the set of logical constants).
2. Var $(= \{x_n : n = 0, 1, 2, \ldots\})$ : countable infinite set of variables
3. Con $= \bigcup_{n=0}^{\infty}(\mathcal{F}(n) \cup \mathcal{P}(n))$ the set of non-logical constants (at best countable infinite), where
   (a) $\mathcal{F}(0)$ : the set of name parameters,
   (b) $\mathcal{F}(n)$ : the set of $n$ argument function parameters,
   (c) $\mathcal{P}(0)$ : the set of proposition parameters,
   (d) $\mathcal{P}(n)$ : the set of predicate parameters.
4. The sets $LC$, Var, $\mathcal{F}(n), \mathcal{P}(n)$ are pairwise disjoint $(n = 0, 1, 2, \ldots)$.
5. The set of terms, i.e. the set Term is given by the following inductive definition:
   (a) Var $\cup \mathcal{F}(0) \subseteq$ Term
   (b) If $f \in \mathcal{F}(n), (n = 1, 2, \ldots)$, and $t_1, t_2, \ldots, t_n \in$ Term, then $f(t_1, t_2, \ldots, t_n) \in$ Term.
6. The set of formulas, i.e. the set Form is given by the following inductive definition:
   (a) $\mathcal{P}(0) \subseteq$ Form
   (b) If $t_1, t_2 \in$ Term, then $(t_1 = t_2) \in$ Form
   (c) If $P \in \mathcal{P}(n), (n \geq 1)$, and $t_1, t_2, \ldots, t_n \in$ Term, then $P(t_1, t_2, \ldots, t_n) \in$ Form.
   (d) If $A \in$ Form, then $\neg A \in$ Form.
   (e) If $A, B \in$ Form, then $(A \supset B), (A \wedge B), (A \vee B), (A \equiv B) \in$ Form.

(f) If $x \in$ Var, $A \in$ Form, then $\forall x A, \exists x A \in$ Form.

**Definition 2:**

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form} \rangle$ be a language of first-order logic Then the set of atomic formulas of $L^{(1)}$ (in notation AtForm) is the following:

1. $\mathcal{P}(0) \subseteq$ AtForm
2. If $t_1, t_2 \in$ Term, then $(t_1 = t_2) \in$ AtForm
3. If $P \in \mathcal{P}(n), (n \geq 1)$, and $t_1, t_2, \ldots, t_n \in$ Term, then $P(t_1, t_2, \ldots, t_n) \in$ AtForm.

## Syntactical Properties of Variables: Free and Bound Variables

**Definition:**

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form} \rangle$ be a first order language and $A \in$ Form be a formula. The set of free variables of the formula $A$ (in notation: FreeVar($A$)) is given by the following inductive definition:

1. If $A$ is an atomic formula (i.e. $A \in$ AtForm), then the members of the set FreeVar($A$) are the variables occuring in $A$.
2. If the formula $A$ is $\neg B$, then FreeVar($A$) = FreeVar($B$).
3. If the formula $A$ is $(B \supset C), (B \wedge C), (B \vee C)$ or $(B \equiv C)$, then FreeVar($A$) = FreeVar($B$) $\bigcup$ FreeVar($C$).
4. If the formula $A$ is $\forall x B$ or $\exists x B$, then FreeVar($A$) = FreeVar($B$)$\setminus\{x\}$.

**Definition:**

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form} \rangle$ be a first order language and $A \in$ Form be a formula. The set of bound variables of the formula $A$ (in notation: BoundVar($A$)) is given by the following inductive definition:

1. If $A$ is an atomic formula (i.e. $A \in At$ Form), then BoundVar($A$) = $\emptyset$.
2. If the formula $A$ is $\neg B$, then BoundVar($A$) = FreeVar($B$).
3. If the formula $A$ is $(B \supset C), (B \wedge C), (B \vee C)$ or $(B \equiv C)$, then BoundVar($A$) = BoundVar($B$) $\cup$ BoundVar($C$).
4. If the formula $A$ is $\forall x B$ or $\exists x B$, then BoundVar($A$) = BoundVar($B$) $\cup \{x\}$.

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## Syntactical Properties of Variables: Free and Bound Occurrences

**Definition:**

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form} \rangle$ be a first order language, $A \in$ Form be a formula, and $x \in Var$ be a variable.

1. A fixed occurrence of the variable $x$ in the formula $A$ is free if it is not in the subformulae $\forall x B$ or $\exists x B$ of the formula $A$.
2. A fixed occurrence of the variable $x$ in the formula $A$ is bound if it is not free.

**Remarks:**

1. If $x$ is a free variable of the formula $A$ (i.e. $x \in \text{FreeVar}(A)$ ), then it has at least one free occurrence in $A$.
2. If $x$ is a bound variable of the formula $A$ (i.e. $x \in \text{BoundVar}(A)$ ), then it has at least one bound occurrence in $A$.
3. A fixed occurrence of a variable $x$ in the formula $A$ is free if
   - it does not follow a universal or an existential quantifier, or
   - it is not in a scope of a $\forall x$ or a $\exists x$ quantification.
4. A variable $x$ may be a free and a bound variable of the formula $A:\ (P(x) \wedge \exists x R(x))$

**Definition:**

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form}\rangle$ be a first order language and $A \in \text{Form}$ be a formula.

1. If $\text{FreeVar}(A) \neq \emptyset$, then the formula $A$ is an open formula.
2. If $\text{FreeVar}(A) = \emptyset$, then the formula $A$ is a closed formula.

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## Interpretation and Assignment in First-order Logic

The concept of interpretation is a crucial component of the semantics of any logical system. It shows the possibilities how to gives 'meanings' (semantic values) to parameters (nonlogical constants). In first-order logic

- name parameters (members of $\mathcal{F}(0)$) represent proper names;
- function parameters (members of $\mathcal{F}(n)$) represent operations;
- propositional parameters (members of $\mathcal{P}(0)$ ) represent propositions;
- one-argument predicate parameters (members of $\mathcal{P}(1)$) represent properties;
- $n$-argument predicate parameters (members of $\mathcal{P}(n), n \geq 1$) represent $n$-argument relations.

**Definition:** (Interpretation of first-order logic)

The ordered pair $\langle U, \rho\rangle$ is an interpretation of the language $L^{(1)}$ if

1. $U \neq \emptyset$ (i.e. $U$ is a nonempty set);
2. $\text{Dom}(\rho) = \text{Con}$;
   (a) If $a \in \mathcal{F}(0)$, then $\rho(a) \in U$;
   (b) If $f \in \mathcal{F}(n) \quad (n \neq 0)$, then $\rho(f)$ is a function from $U^{(n)}$ to $U$;
   (c) If $p \in \mathcal{P}(0)$, then $\rho(p) \in \{0, 1\}$;
   (d) If $P \in \mathcal{P}(n) \quad (n \neq 0)$, then $\rho(P) \subseteq U^{(n)}$.

**Definition:** (Assignment in a given interpretation)

The function $v$ is an assignment relying on the interpretation $\langle U, \rho\rangle$ if the followings hold:

1. $\mathrm{Dom}(v) = \mathrm{Var}$;
2. If $x \in Var$, then $v(x) \in U$.

**Definition:** (Modified assignment)

Let $v$ be an assignment relying on the interpretation $\langle U, \rho \rangle, x \in Var$ and $u \in U$.

$$v[x:u](y) = \begin{cases} u, & \text{if } y = x; \\ v(y), & \text{otherwise.} \end{cases}$$

for all $y \in Var$.

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**Definition:** (Semantic rules)

Let $\langle U, \rho \rangle$ be a given interpretation and $v$ be an assignment relying on $\langle U, \rho \rangle$.

1. If $a \in \mathcal{F}(0)$, then $|a|_v^{\langle U, \rho \rangle} = \rho(a)$.
2. If $x \in Var$, then $|x|_v^{\langle U, \rho \rangle} = v(x)$.
3. If $f \in \mathcal{F}(n), (n = 1, 2, \ldots)$, and $t_1, t_2, \ldots, t_n \in$ Term, then $|f\,(t_1)\,(t_2) \ldots (t_n)|_v^{\langle U, \rho \rangle} = \rho(f)\left(\left\langle |t_1|_v^{\langle U, \rho \rangle}, |t_2|_v^{\langle U, \rho \rangle}, \ldots, |t_n|^{\langle U, \rho \rangle_v} \right\rangle\right)$
4. If $p \in \mathcal{P}(0)$, then $|p|_v^{\langle U, \rho \rangle} = \rho(p)$
5. If $t_1, t_2 \in$ Term, then $|(t_1 = t_2)|_v^{\langle U, \rho \rangle} = \begin{cases} 1, & \text{if } |t_1|_v^{\langle U, \rho \rangle} = |t_2|_v^{\langle U, \rho \rangle} \\ 0, & \text{otherwise.} \end{cases}$
6. If $P \in \mathcal{P}(n)(n \neq 0), t_1, \ldots, t_n \in$ Term,

   then $|P\,(t_1) \ldots (t_n)|_v^{\langle U, \rho \rangle} = \begin{cases} 1, & \text{if } \left\langle |t_1|\, v_v^{\langle U, \rho \rangle}, \ldots, |t_n|_v^{\langle U, \rho \rangle} \right\rangle \in \rho(P); \\ 0, & \text{otherwise.} \end{cases}$
7. If $A \in$ Form, then $|\neg A|_v^{\langle U, \rho \rangle} = 1 - |A|_v^{\langle U, \rho \rangle}$.
8. If $A, B \in$ Form, then

$$|(A \supset B)|_v^{\langle U, \rho \rangle} = \begin{cases} 0 & \text{if } |A|_v^{\langle U, \rho \rangle} = 1, \text{ and } |B|_v^{\langle U, \rho \rangle} = 0; \\ 1, & \text{otherwise.} \end{cases}$$

$$|(A \wedge B)|_v^{\langle U, \rho \rangle} = \begin{cases} 1 & \text{if } |A|_v^{\langle U, \rho \rangle} = 1, \text{ and } |B|_v^{\langle U, \rho \rangle} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$|(A \vee B)|_v^{\langle U, \rho \rangle} = \begin{cases} 0 & \text{if } |A|_v^{\langle U, \rho \rangle} = 0, \text{ and } |B|_v^{\langle U, \rho \rangle} = 0; \\ 1, & \text{otherwise.} \end{cases}$$

$$|(A \equiv B)|_v^{\langle U, \rho \rangle} = \begin{cases} 1 & \text{if } |A|_v^{\langle U, \rho \rangle} = |B|_v^{\langle U, \rho \rangle} = 0; \\ 0, & \text{otherwise.} \end{cases}$$

9. If $A \in \text{Form}$, $x \in Var$, then

$$|\forall x A|_v^{\langle U,\rho\rangle} = \begin{cases} 0, & \text{if there is an } u \in U \text{ such that } |A|_{v[x:u]}^{\langle U,\rho\rangle} = 0; \\ 1, & \text{otherwise.} \end{cases}$$

$$|\exists x A|_v^{\langle U,\rho\rangle} = \begin{cases} 1, & \text{if there is an } u \in U \text{ such that } |A|_{v[x:u]}^{\langle U,\rho\rangle} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The semantic value of an expression belonging to the set Term$\cup$Form depends on the given interpretation and assignment, therefore the precise notation is the following: $|\langle\text{expression}\rangle|_v^{\langle U,\rho\rangle}$.

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## Central Logical Concepts of Classical First-order Logic

**Definition:** (Model of a set of formulae)

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form}\rangle$ be a first order language and $\Gamma \subseteq \text{Form}$ be a set of formulae. An ordered triple $\langle U, \rho, v\rangle$ is a model of the set $\Gamma$, if

1. $\langle U, \rho\rangle$ is an interpretation of $L^{(1)}$;
2. $v$ is an assignment relying on $\langle U, \rho\rangle$;
3. $|A|_v^{\langle U,\rho\rangle} = 1$ for all $A \in \Gamma$.

**Definition:** (Model of a formula)

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form}\rangle$ be a first order language and $A \in \text{Form}$ be a formula. A model of a formula $A$ is the model of the singleton $\{A\}$.

**Definition:** (Satisfiability of a set of formulae)

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form}\rangle$ be a first order language and $\Gamma \subseteq \text{Form}$ be a set of formulae. $\Gamma \subseteq \text{Form}$ is satisfiable if it has a model. (If there is an interpretation and an assignment in which all members of the set $\Gamma$ are true.)

**Definition:** (Satisfiability of a formula)

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form}\rangle$ be a first order language and $A \in \text{Form}$ be a formula. The formula $A$ is satisfiable, if the singleton $\{A\}$ is satisfiable. (If there is an interpretation and an assignment in which the formula $A$ is true.)

**Remarks:**

- A satisfiable set of formulas does not involve a logical contradiction; its formulae may be true together.
- A satisfiable formula may be true.
- If a set of formulas is satisfiable, then its members are satisfiable.
- But: all members of the set $\{P(a), \neg P(a)\}$ are satisfiable, and the set is not satisfiable.

**Definition:** (Unsatisfiability of a set of formulae)

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form}\rangle$ be a first order language and $\Gamma \subseteq \text{Form}$ be a set of formulae. The set $\Gamma \subseteq \text{Form}$ is unsatisfiable if it is not satisfiable.

**Remark:**

An unsatisfiable set of formulae involves a logical contradiction. (Its members cannot be true together.)

**Definition:** (Unsatisfiability of a formula)

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form}\rangle$ be a first order language and $A \in \text{Form}$ be a formula. The formula $A$ is unsatisfiable if the singleton $\{A\}$ is unsatisfiable.

**Remark:**

An unsatisfiable formula involves a logical contradiction. (It cannot be true, i.e. it is false with respect to all interpretations and assignment.)

**Definition:** (Logical consequence of a set of formulae)

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form}\rangle$ be a first order language, $\Gamma, \subseteq \text{Form}$ be a set of formulae and $A \in \text{Form}$ be a formula. The formula $A$ is the logical consequence of the set of formulae $\Gamma$ if the set $\Gamma \cup \{\neg A\}$ is unsatifiable. (Notation: $\Gamma \models A$ )

**Definition:** (Logical consequence of a formula)

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form}\rangle$ be a first order language, and $A, B \in \text{Form}$ be formulae. The formula $B$ is the logical consequence of the formula $A$ if $\{A\} \vDash B$. (Notation: $A \models B$ )

**Definition:** (Validity of a formula)

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form}\rangle$ be a first order language, and $A \in \text{Form}$ be a formula. The formula $A$ is valid if $\emptyset \models A$. (Notation: $\models A$ )

**Definition:** (Logical equivalence)

Let $L^{(1)} = \langle LC, Var, \text{Con, Term, Form}\rangle$ be a first order language, and $A, B \in \text{Form}$ be formulae. The formulae $A$ and $B$ are logically equivalent if $A \vDash B$ and $B \vDash A$. (Notation: $A \Leftrightarrow B$ )

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## Normal Forms

- A base is a set of truth functors whose members can express all truth functors.
  - For example: $\{\neg, \supset), \{\neg, \wedge\}, \{\neg, \vee\}$
    1. $(p \wedge q) \Leftrightarrow \neg(p \supset \neg q)$
    2. $(p \vee q) \Leftrightarrow (\neg p \supset q)$
  - Truth functor Sheffer: $(p \mid q) \Leftrightarrow_{\text{def}} \neg(p \wedge q)$
  - Truth functor neither-nor: $(p \| q) \Leftrightarrow_{\text{def}} (\neg p \wedge \neg q)$
  - Remark: Singleton bases: $(p \mid q), (p \| q)$

**Definition:** Let $L^{(0)} = \langle LC, \text{Con}, \text{Form} \rangle$ be a language of propositional logic and $p \in \text{Con}$ a propositional parameter. Then the formulae $p, \neg p$ are literals (where $p$ is the base of the literals).

**Definition:** If the formula $A$ is a literal or a conjunction of literals, then $A$ is an elementary conjunction.

**Definition:** If the formula $A$ is a literal or a disjunction of literals, the $A$ is an elementary disjunction.

**Remark:** If the literals of an elementary conjunction/disjunction have different bases, then the elementary conjunction/disjunction represents an interpretation (or a family of interpretations).

**Definition:** A disjunction of elementary conjunctions is a disjunctive normal form.

**Definition:** A conjunction of elementary disjunctions is a conjunctive normal form.

**Definition:**

Let $L^{(1)} = \langle LC, Var, \text{Con}, \text{Term,Form} \rangle$ be a first order language and $A \in \text{Form}$ be a formula.

The formula $A$ is prenex if

1. there is no quantifier in $A$ or
2. the formula $A$ is in the form $Q_1 x_1 Q_2 x_2 \ldots Q_n x_n B (n = 1, 2, \ldots)$, where
   (a) there is no quantifier in the formula $B \in \text{Form}$;
   (b) $x_1, x_2 \ldots x_n \in \text{Var}$ are diffrent variables;
   (c) $Q_1, Q_2, \ldots, Q_n \in \{\forall, \exists\}$ are quantifiers.

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## Sequent calculus

Truth tables can be used to determine valid formulas, but if we have too many non-logical constants, it is hard to construct these tables, even for computers. Let's consider a method based on syntax.

**Definition:**

If $\Gamma$ and $\Delta$ are two-possibly empty—set of formulae, then $\Gamma \vdash \Delta$ is a sequent.

The axioms of the sequent calculus are $\Gamma \cup \{A\} \vdash \Delta \cup \{A\}$, where $A$ is an atomic formula, $\Gamma$ and $\Delta$ are set of formulae.

Let $S$ be the sequent $\Gamma \vdash \Delta$, where $\Gamma = \{A_1, \ldots, A_n\}$ and $\Delta = \{B_1, \ldots, B_m\}$;

The sequent $S$ is valid, if for every interpretation $\varrho$ where $|A_1|_\varrho = \cdots = |A_n|_\varrho = 1$ then $|B_i|_\varrho = 1$ for some $i$.

**Remark:** If a sequent is not valid-i.e. falsifiable-then there exists an interpretation $\varrho$ for which $|A_1|_\varrho = \dots = |A_n|_\varrho = 1$, but $|B_1|_\varrho \dots = |B_m|_\varrho = 0$.

**Inference rules**

For the sake of simplicity we write „$\Gamma, A$" in the following instead of $\Gamma \cup \{A\}$. In the following rules the upper sequent(s) and the lower sequent are called the premise(s) and the conclusion of the rule, respectively.

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \qquad \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}$$

$$\frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B} \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \supset B \vdash \Delta}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta}$$