System of Linear Equations

Example A manufacturer produces two kinds of toys: wooden cars and wooden trains. The amounts of two raw materials needed to product 1 piece of toys are given in the table.

	car	train
wood	2	3
paint	5	4

Determine the number of cars and trains produced on a given day, when we know that for the production there were used 540 units of wood and 1070 units of paint.

Systems of linear equations (SLE)

Definition

The system of equations

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1\\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2\\
&\vdots\\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{aligned}$$

where the numbers $a_{ij} (i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\})$ and b_k $(k \in \{1, \ldots, m\})$ are known, the variables $x_1, \ldots, x_n$ are unknown, is called a system of linear equations.

- a_{ij} : the coefficients of the system of linear equations
- b_k : the constant terms

The coefficient matrix and the augmented matrix:

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \quad \text{and} \quad A|b = \left(\begin{array}{ccc|c} a_{11} & \ldots & a_{1n} & b_1 \\ a_{21} & \ldots & a_{2n} & b_2 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \ldots & a_{mn} & b_m \end{array}\right).$$

The right-hand side vector and the solution vector

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

The corresponding matrix-vector equation: $Ax = b$.

Solvability of systems of linear equations

Definition

The system of linear equations is

- solvable if there exists at least one solution, that is, an x such that $Ax = b$ holds; determined if there is exactly 1 solution; undetermined if there are more than 1 solutions;
- inconsistent if it doesn't have a solution.

Remark

The SLE can be solved if and only if the vector b can be expressed as a linear combination of the column vectors of A. This means, that b is in the subspace spanned by the column vectors of A.

If b can be expressed uniquely (i.e. the columns vectors of A are linearly independent), then there exists a unique solution.

If the column vectors of A are linearly dependent, and b is in the subspace spanned by the column vectors of A, then there are infinitely many solutions.

Definition

The rank of a matrix is the rank of the system of column vectors of the matrix. Notation: $\text{rank}(A)$.

Condition on solvability

- A system of lin. eq.s is solvable if, and only if $\text{rank}(A) = \text{rank}(A \mid b)$.
- If it is solvable and $\text{rank}(A) = n$ (where n is the number of unknown parameters), then the system is determined, if $\text{rank}(A) < n$, then undetermined.

Solutions of a system of linear equations

Definition

A system of linear equations is homogeneous if $b = 0$, thus then the matrix equation has the form $Ax = 0$. Otherwise it's called nonhomogeneous.

Remark: 0 is a solution of any homogenous system of linear equations (it is the trivial solution).

Theorem

A homogeneous system of linear equations has a nontrivial solution if and only if the column vectors of A are linearly dependent.

Solutions of a homogeneous system of linear equations

The solutions of a real homogeneous system of linear equations form a vector subspace of $\mathbb{R}^n$ with dimension $n - \operatorname{rank}(A)$.

Solutions of a nonhomogeneous system of linear equations

The solutions of a (solvable) nonhomogeneous system of linear equations $Ax = b$ are of the form $x^* + y$, where

- x^* is a particular solution of the system of linear equations;
- y is an arbitrary solution of the corresponding homogeneous system of linear equation, that is $Ax = 0$.

Solving a system of linear equations with Gaussian elimination

The set of solutions of a system of linear equations does not change, if we

- multiply an equation by a nonzero constant;
- add a scalar multiple of an equation to another equation;
- interchange two equations;
- discard an equation which is a scalar multiple of another equation.

We eliminate the numbers under the main diagonal with the modifications above. The resulting system is easier to solve.

- If during the process we obtain a row like $(0 \dots 0 \mid \neq 0)$, then the system of linear equations has no solution.
- If at the end of the process there are n number of not identically 0 rows, then the system is determined (there is a unique solution), if fewer number of rows remains, then undetermined (infinitely many solutions). (Here n is the number of the unknown parameters.)

Example

Solve the system $Ax = b$ using Gaussian elimination. .

$$A = \begin{pmatrix} 2 & 2 & 3 \\ -4 & -1 & -5 \\ -2 & 4 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} -4 \\ 12 \\ 10 \end{pmatrix}$$

Solution:

$$\left(\begin{array}{ccc|c} 2 & 2 & 3 & -4 \\ -4 & -1 & -5 & 12 \\ -2 & 4 & 0 & 10 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 2 & 3 & -4 \\ 0 & 3 & 1 & 4 \\ 0 & 6 & 3 & 6 \end{array}\right)$$
$$\longrightarrow \left(\begin{array}{ccc|c} 2 & 2 & 3 & -4 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 3 & -4 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

Backward substitution:

$$\begin{aligned} &x_3 = -2 \\ &3x_2 + x_3 = 4 \quad \rightarrow x_2 = 2 \\ &2x_1 + 2x_2 + 3x_3 = -4 \quad \rightarrow x_1 = -1 \end{aligned}$$

Rank, Determinant

The following operations do not change the rank of a matrix A :

- Interchanging 2 rows of A.
- Multiplying a row of A by a scalar $\lambda \neq 0$.
- Adding a scalar multiple of a row to another row.
- The determinant doesn't change if we add a scalar multiple of a row to another row.
- If we interchange 2 rows of A, then the sign of the determinant changes.
- The determinant of a triangular matrix is the product of the elements of the main diagonal.

$\Longrightarrow$ Gaussian elimination can be used for computation of $\operatorname{rank}(A)$ and $\det(A)$.

Example

Calculate $\operatorname{rank}(A)$ and $\det(A)$.

$$A = \begin{pmatrix} 3 & 5 & -6 \\ -1 & -2 & 1 \\ 2 & 6 & 5 \end{pmatrix}$$

Solution:

$$A = \begin{pmatrix} 3 & 5 & -6 \\ -1 & -2 & 1 \\ 2 & 6 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & -2 & 1 \\ 3 & 5 & -6 \\ 2 & 6 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & -2 & 1 \\ 0 & -1 & -3 \\ 0 & 2 & 7 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} -1 & -2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad \mathrm{rank}(A) = 3$$

The determinant of the last matrix is $(-1)(-1)\cdot 1 = 1$. During the calculations we interchanged the rows of A only once, so $\det(A) = -1$.