Function Analysis

Monotonicity of Real Functions

Definition: Let $I \subseteq \mathbb{R}$ be a nonempty open interval.

A real function $f : I \to \mathbb{R}$ is decreasing or increasing if, for all $x, y \in I$ with $x \leq y$, we have

$$f(x) \geq f(y) \quad \text{or} \quad f(x) \leq f(y),$$

respectively. If we have a strict inequality whenever x and y are different, we are speaking about a strictly decreasing or strictly increasing function, respectively.

The point of the domain where the function changes its monotonicity is called a critical point. More precisely, we have the following definition.

Definition: Let $f : I \to \mathbb{R}$ and $p \in I$.

- The point p is called a place of a local minimum of the function f if there exists $r > 0$ such that $f(x) \geq f(p)$ if $x \in]p - r, p + r[\cap I$.
- The point p is called a place of a local maximum of the function f if there exists $r > 0$ such that $f(x) \leq f(p)$ if $x \in]p - r, p + r[\cap I$.

The places of local minimum and maximum of the functions are called the critical points of the function.

Remark: It is easy to observe that, having a differentiable function, the derivative at p is zero provided that p is a place of a local minimum or a maximum. In general, the converse is not true.

Theorem: (Location of Absolute Extrema)

Let f be a continuous function over a closed, bounded interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in I.

Monotonicity of Differentiable Functions

Theorem (Necessary condition for critical points):

Let $f : l \to \mathbb{R}$ be a differentiable function and $p \in I$ be a critical point. Then $f'(p) = 0$.

Theorem (First order sufficient condition for critical points):

Let $f : l \to \mathbb{R}$ be a differentiable function and $p \in I$ such that $f'(p) = 0$.

- If there exists $r > 0$ such that $f'(x) < 0$ if $x \in]\, p - r, p\, [\cap /$ and $f'(x) > 0$ if $x \in]p, p + r[\cap I$, then p is a place of a local minimum of f.
- If there exists $r > 0$ such that $f'(x) > 0$ if $x \in]\, p - r, p\, [\cap /$ and $f'(x) < 0$ if $x \in]p, p + r[\cap I$, then p is a place of a local maximum of f.

Theorem (Second order sufficient condition for critical points):

Let the function $f : l \to \mathbb{R}$ be twice differentiable and $p \in I$ such that $f'(p) = 0$. If $f''(p) \neq 0$, then p is a critical point of f and

- p is a place of local minimum if $f''(p) > 0$ and
- p is a place of local maximum if $f''(p) < 0$.

Convexity of Real Functions

Definition. We say that the function $f : I \to \mathbb{R}$ is convex or concave on its domain if, for all $t \in [0, 1]$ and for all $x, y \in I$, we have

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

or

$$f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y),$$

respectively.

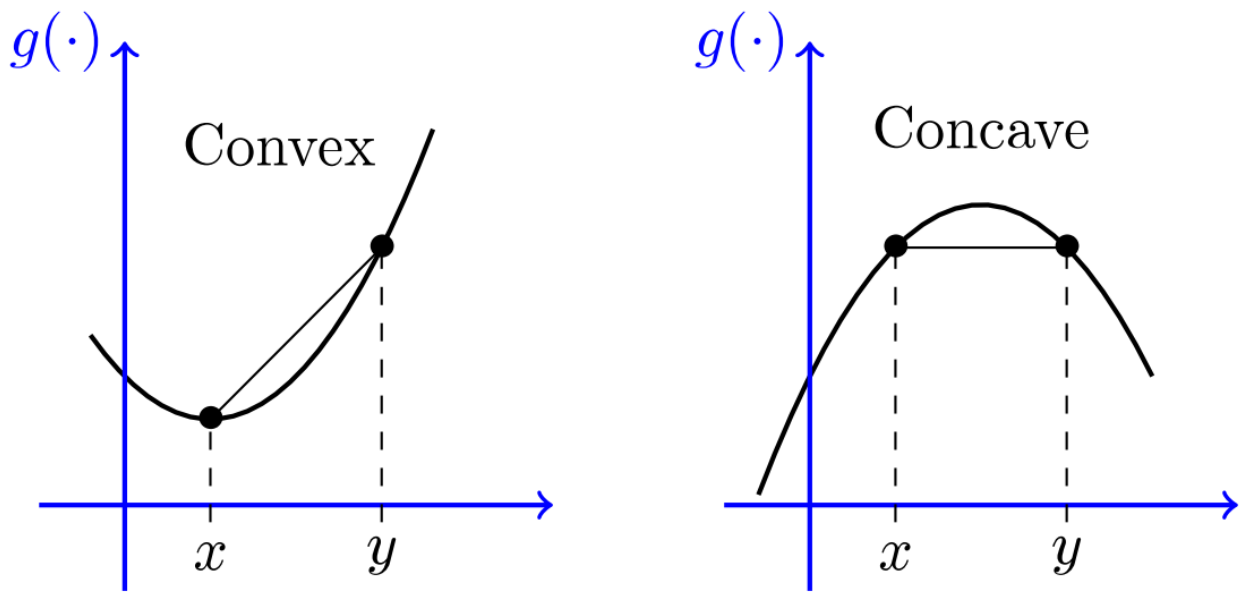

Figure 1: **Geometrical meaning:** If f is convex (concave) on its domain then any chord connecting two different points of its graph lies above (below) the corresponding arc of the graph.

The point of the domain where the function changes its convexity property is called a point of inflection.

Definition: Let $f : I \to \mathbb{R}$ and $p \in I$. The point p is called a point of inflection if there exists $r > 0$ such that

- f is convex on the interval $]p - r, p[\cap I$ and concave on the interval $]p, p + r[\cap I$ or
- f is concave on the interval $]p - r, p[\cap I$ and convex on the interval $]p, p + r[\cap I$

Theorem (Necessary condition for point of inflection):

Let $f : I \to \mathbb{R}$ be a twice-differentiable function and $p \in I$ be a point of inflection. Then $f''(p) = 0$.

Theorem (Second order sufficient condition for point of inflection):

Let $f : I \to \mathbb{R}$ be a twice differentiable function and $p \in I$ such that $f''(p) = 0$.

- If there exists $r > 0$ such that $f''(x) < 0$ if $x \in]\, p - r, p\, [\cap I$ and $f''(x) > 0$ if $x \in]p, p + r[\cap I$, then f changes from concave to convex at p.
- If there exists $r > 0$ such that $f''(x) > 0$ if $x \in]\, p - r, p\, [\cap I$ and $f''(x) < 0$ if $x \in]p, p + r[\cap I$, then f changes from convex to concave at p.

Theorem (Third order sufficient condition for point of inflection):

Let the function $f : I \to \mathbb{R}$ be three-times differentiable and $p \in I$ such that $f''(p) = 0$. If $f'''(p) \neq 0$, then p is a point of inflection and

- f changes from concave to convex if $f'''(p) > 0$ and
- f changes from convex to concave if $f'''(p) < 0$.

Least Squares Method

- Least square method is the process of finding a regression line or best-fitted line for any data set that is described by an equation.
- This method requires reducing the sum of the squares of the residual parts of the points from the curve or line and the trend of outcomes is found quantitatively.

$$m = \frac{(n \sum xy - \sum y \sum x)}{\left[n \sum x^2 - (\sum x)^2\right]}$$
$$b = \left(\sum y - m \sum x\right) / n$$

Advantages:

- Easy to apply and understand
- Highlights relationship between two variables
- Can be used to make predictions about future performance

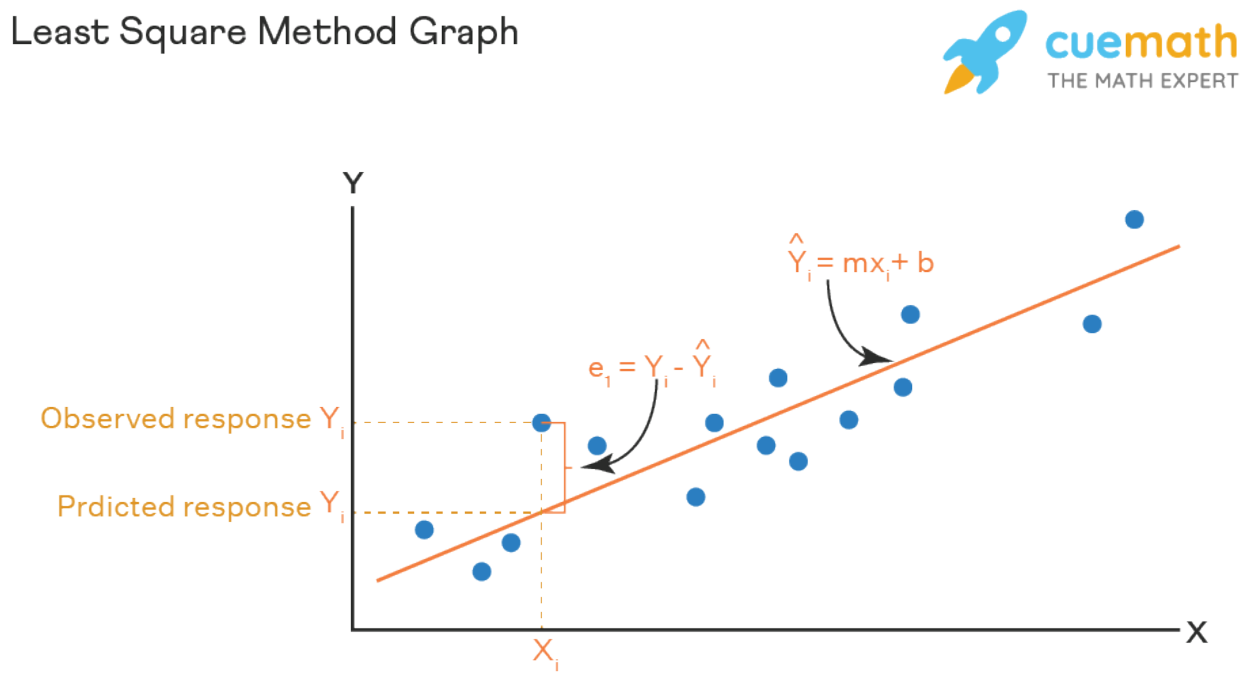

Figure 2: Least Square Method

Disadvantages:

- Only highlights relationship between two variables
- Doesn't account for outliers
- May be skewed if data isn't evenly distributed