Probability Theory

Random experiment: the outcome of the experiment is not determined before performing the experiment.

Outcome: An outcome of an experiment is any possible observation of that experiment.

Sample Space: The sample space of an experiment is the finest-grain, mutually exclusive, collectively exhaustive set of all possible outcomes. (Ω)

Event: An event is a set of outcomes of an experiment. Subset of Ω.

Event Space: An event space is a collectively exhaustive, mutually exclusive set of events.

Axioms of Probability

Axiom 1: For any event A, $P[A] \geq 0$

Axiom 2: $P[S] = 1$

Axiom 3: For any countable collection $A_1, A_2, \ldots$ of mutually exclusive events

$$P[A_1 \cup A_2 \cup \ldots] = P[A_1] + P[A_2] + \ldots$$

Theorem

For an experiment with sample Space $S = \{s_1, \ldots, s_n\}$ in which each outcome s_i is equaly likely

$$P[s_i] = 1/n \quad 1 \leq i \leq n$$

Proof using axiom 2.

Theorem

The probability measure $P[\cdot]$ satisfies

(a) $P[\phi] = 0$

(b) $P[A^c] = 1 - P[A]$

(c) For any A and B (not necessarily disjoint),

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

(d) If $A \subset B$,then $P[A] \leq P[B]$

Conditional Probability

The conditional probability of the event A given the occurrence of the event B is

$$P[A|B] = \frac{P[AB]}{P[B]}$$

Law of Total Probability

For an event space $\{B_1, B_2, \ldots, B_m\}$ with $P[B_i] > 0$ for all i,

$$P[A] = \sum_{i=1}^{m} P[A|B_i]\ P[B_i]$$

Bayes' Theorem

In many situations, we have advance information about ***P[A/B]*** and need to calculate ***P[B/A]***. To do so we have the following formula: *proved using Conditional Probability.*

$$P[B|A] = \frac{P[A|B]\ P[B]}{P[A]}$$

Independence

Events A and B are independent if and only if

$$P[AB] = P[A]\ P[B]$$

When events A and B have nonzero probabilities, the following formulas are equivalent to the definition of independent events:

$$P[A|B] = P[A], \quad P[B|A] = P[B]$$

3 Independent Events

A_1, A_2, and A_3 are ***independent*** if and only if

(a) A_1 and A_2 are independent,

(b) A_2 and A_3 are independent,

(c) A_1 and A_3 are independent,

(d) $P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$.

More than Two Independent Events

If $n \geq 3$, the sets $A_1, A_2, \ldots, A_n$ are independent if and only if

(a) every set on $n-1$ sets taken from $A_1, A_2, \ldots, A_n$ is independent,

(b) $P[A_1 \cap A_2 \cap \cdots \cap A_3] = P[A_1]P[A_2]\ldots P[A_n]$

Fundamental Principle of Counting

If subexperiment A has n possible outcomes, and subexperiment B has k possible outcomes, then there are nk possible outcomes when you perform both subexperiments.

Theorem

The number of k-permutations of n distinguishable objects is

$$(n)_k = \frac{n!}{(n-k)!}$$

Sample without Replacement

Theorem

The number of ways to choose **k** objects out of **n** distinguishable objects is

$$\binom{n}{k} = \binom{n}{n-k}$$

n choose k

For an integer ≥ 0, we define

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & k = 0, 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

Sample with Replacement

Theorem

Given m distinguishable objects, there are m^n ways to choose with replacement an ordered sample of n objects.

Permutations:

Definition: An ordered sequence of n distinguishable objects is called an n-permutation.
Theorem: The number of n-permutations is n-factorial, that is $n! = n \cdot (n-1) \cdots 2 \cdot 1$ $0! = 1$

Identical Elements

Let h red, k blue and m white elements *($h + k + m = n$)*. Denote by X the number of permutations of these n elements. Then

$$Xh!k!m! = n!$$

So,

$$X = \frac{n!}{h!k!m!}$$

Ordered Selections

Without Replacement

Theorem: The number of ordered selections without replacement is

$$n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}$$

With Replacement

Theorem: The number of ordered selections with replacement is

$$n \cdot n \cdots n \cdot n = n^k$$

Combinations

Order is not important. **Theorem:** The number of ways to choose k objects out of n distinguishable objects is

$$\binom{n}{k}, \text{ where } \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

With Replacement

Theorem: The number of ways to choose k objects out of n distinguishable objects when we replace the chosen elements is

$$\binom{n+k-1}{k}$$

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \cdots + \binom{n}{n} a^0 b^n.$$

The binomial coefficients are contained in the Pascal triangle. **Remark:** The rule of the Pascal triangle is

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

Number of Subsets

Theorem: The number of subsets of an n-element set is 2^n.