Stochastic Processes (SE-410)

Assignment

Application of Brownian Motion

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BESE Section A

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Introduction to Brownian Motion

Brownian Motion, also known as **Wiener Process**, is one of the fundamental building blocks of modern stochastic processes. It was first observed in 1827 by botanist Robert Brown, who noticed that pollen particles suspended in water moved randomly, even in still water. This motion arises due to the particle being bombarded by molecules of the fluid, causing a jittery, unpredictable path.

Mathematical Properties:

Let B(t) represent Brownian motion at time ttt. It satisfies the following properties:

1. **Initial Condition**: $B(0)= 0$
2. **Independent Increments**: For all $0 \leq s \leq t$, the increment $B(t) - B(s)$ is independent of the past.
3. **Normally Distributed Increments**: $B(t)-B(s) \sim N(0, t-s)$
4. **Continuous Paths**: The function $t \mapsto B(t)$ is continuous, though nowhere differentiable.

Brownian motion is a key example of a **stochastic process**, which is a collection of random variables indexed by time.

Application of Brownian Motion in Finance

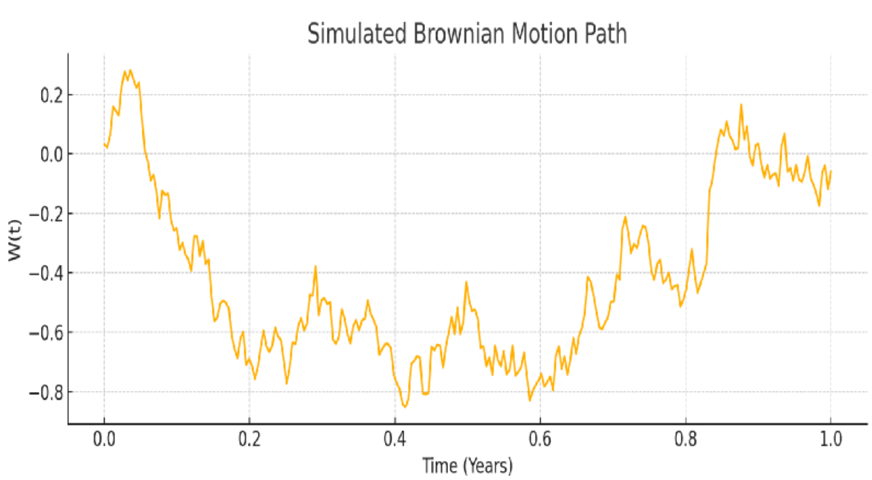
In finance, Brownian motion is widely used to model the unpredictable movement of asset prices over time. It plays a crucial role in the formulation of the Black-Scholes model for option pricing, as well as in the modeling of stock prices through Geometric Brownian Motion (GBM). GBM assumes that the logarithm of asset prices follows a Brownian motion with drift, capturing both the trend (expected return) and the random shocks (volatility).

The stochastic differential equation for a stock price S(t) in GBM is:

dS(t) = mu * S(t) * dt + sigma * S(t) * dW(t)

where:

- mu is the expected return (drift),
- sigma is the volatility,
- dW(t) is the increment of a standard Brownian motion.



Geometric Brownian Motion (GBM)

Geometric Brownian Motion is a type of stochastic process where the logarithm of the underlying variable follows a Brownian motion. It is used to describe the evolution of stock prices in time.

Black-Scholes Model

Developed in 1973 by **Fischer Black**, **Myron Scholes**, and **Robert Merton**, the Black-Scholes model uses GBM to price **European options**.

Assumptions of the Black-Scholes Model:

- No arbitrage opportunities.
- Markets are frictionless (no taxes or transaction costs).
- Constant risk-free interest rate.
- Asset prices follow GBM.
- The option can only be exercised at expiration.

How This Works in the Real World

Scenario: Stock Price Simulation

Financial analysts use GBM to **simulate future stock prices** to estimate risk, expected return, and option value. A single simulation of GBM might look like this:

- Time Horizon: 1 year
- Initial Price: \$100
- Drift (μ): 10%
- Volatility (σ): 20%

The simulated stock price will jitter over time due to Brownian motion, but will trend upward due to drift.

Scenario: Risk Management

By simulating thousands of possible price paths using GBM, financial institutions can estimate **Value at Risk (VaR)**, a measure of potential loss in a portfolio.

Advantages

- **Mathematical tractability**: Easy to work with and simulate.
- **Flexibility**: Can be extended to include jumps, mean-reversion, etc.
- **Widely accepted**: Used by investment banks, hedge funds, and regulators.