NSMB
Fall 2017

Simple Pendulum: Not So Simple

Mallory Tollefson[1], Bakir Hajdarevic[1], Hernan Bernabe[1]

Abstract
In the absence of air resistance and other frictional forces, an ideal, simple pendulum will oscillate continuously at a constant magnitude. In such a case, damping—a decrease in the amplitude of oscillation as a result of energy loss—will not occur. The ideal case of a pendulum lacks damping and is quite simple to model using mathematical equations, however, such an ideal case is not often encountered in the reality of day-to-day life where air resistance and sliding friction can affect a pendulum. In this study, data was collected from a simple pendulum system that experiences frictional forces. The data shows that a simple pendulum with a concentrated mass of 0.3648 kg at the end of two 11-inch strings released from a 60 degree angle is an underdamped system. Two different exponential equations (1) Y = a * exp(b*t) and (2) Y = a * exp(b*t) + c * exp(d*t) were then used to model the underdamped system. The damping coefficients were calculated using constrained optimization and are 0.0169 and 0.018, respectively. Equation 1 provided the best fit to model this pendulum with its corresponding damping coefficient.

[1]*Department of Biomedical Engineering, The University of Iowa, Iowa City, IA*

Introduction

Simple pendulums are present in a variety of commonplace tools such as a swing used by a child on a playground or a metronome used by a musician to keep beat. Under ideal conditions, a simple pendulum is represented by a point mass at the end of a weightless and frictionless string/rod. A diagram and the motion of an ideal, simple pendulum is shown in Figure 1 and described in equations 1, 2 and 3.

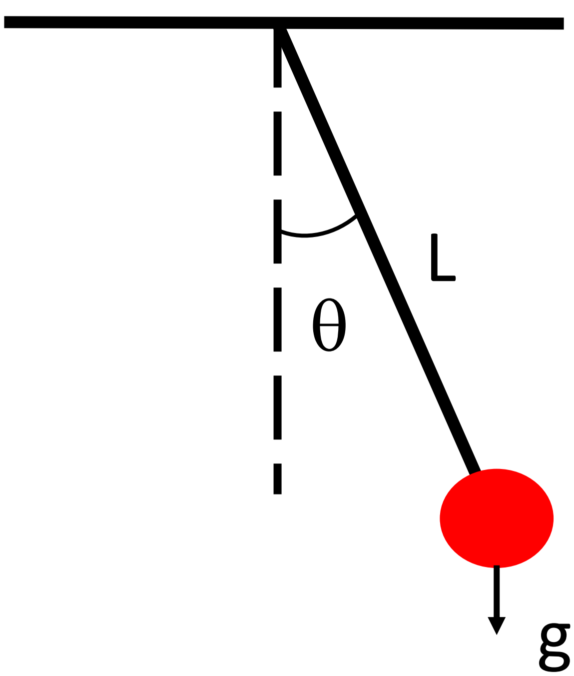

Figure 1. Simple pendulum with motion represented by equations 1, 2 and 3.

Equation 1 represents the angle of the pendulum, θ, (from a theoretical line that is orthogonal to the cross bar from which the pendulum rod/string hangs). The pendulum angle is calculated by the product of the initial angle, θ_0, and the cosine of radial frequency, ω, multiplied by time.

$$\theta = \theta_0 * \cos(\omega t) \qquad (1)$$

Equation 2 represents the radial frequency of the pendulum, which is calculated as the square root of length of the pendulum string/rod, L (figure 1), and acceleration due to gravity, g (figure 1).

$$\omega = \sqrt{L/g} \qquad (2)$$

Equation 3 represents the period of the pendulum oscillations, T, which is equal to the quotient of $2 * \pi$ and the radial frequency.

$$T = 2\pi/\omega \qquad (3)$$

In the absence of air resistance and other frictional forces, an ideal, simple pendulum will oscillate continuously at a constant magnitude. However, in the case of a child being pushed on a swing, air resistance and the sliding friction on the swing components will gradually damp the system, which decreases the amplitude of oscillations on the swing until it stops moving entirely. Modeling an ideal, undamped, simple pendulum using mathematical equations is quite simple, however, modeling damped oscillations that account for frictional forces presents a more challenging problem. Different damping characteristics are displayed in Figure 2. A typical children's playground swing will experience underdamping. The equations used to model an underdamped pendulum are available in equations 4, 5 and 6.

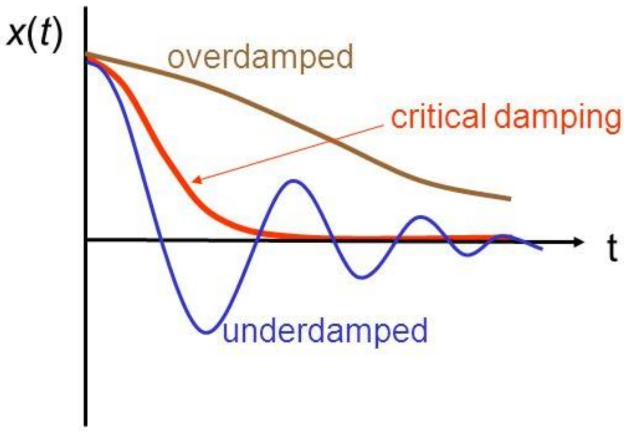

Figure 2. Damping diagram. A playground swing-style pendulum experiences underdamping.

Equation 4 represents the angle of the pendulum, θ, (from a theoretical line that is orthogonal to the cross bar from which the pendulum rod/string hangs), where A is the initial amplitude, b is the damping coefficient, t is time, m is the concentrated mass, ω is the radial frequency, and ϕ is the phase shift.

$$\theta(t) = Ae^{-bt/_{2m}} * \cos(\omega t + \phi) \qquad (4)$$

Equation 5 describes the damping coefficient as an inequality that is less than the product of two times the mass and the initial radial frequency.

$$b < 2m\omega_0 \qquad (5)$$

Equation 6 shows the radial frequency in terms of the damping coefficient, initial radial frequency and mass.

$$\omega = \omega_0^2 - (^b/_{2m})^2 \qquad (6)$$

Accurately fitting the data collected from an underdamped pendulum and calculating the damping coefficient requires the use of optimization and numerical techniques. The goal of our project was to collect data from a small-scale version of an underdamped children's swing, optimize/fit an equation to model the underdamped system, calculate the damping coefficient, and evaluate how often a child needs to be pushed on the swing to stay within 25 percent of the maximum amplitude.

Methods

Data was obtained by connecting the mobile application MATLAB Mobile to MATLAB R2017b (MATLAB v9.0, The Mathworks Inc.). For our experiment, we were only interested in acquiring the orientation (in degrees) and the approximate time (in seconds). When orientation data is logged from the mobile, three orientation measurements are obtained as Azimuth, Pitch, and Roll, where each measurement is assessed from 180° to -180° and reflect a rotation about the z, x, and y axis respectively.

The initial experimental setup is illustrated in Figure 3. In this setup, the mobile was suspended at 90 degrees while attached to one string. While this setup seems simple enough, the group neglected that with one string, the mobile will experience some rotation about all axes. We found that this effect skews the orientation data and made it impossible to accurately assess the periodicity of the pendulum's oscillations. In addition, by suspending the mobile at 90 degrees, the mobile's orientation sensor seemed to confuse the correct polarity of orientation.

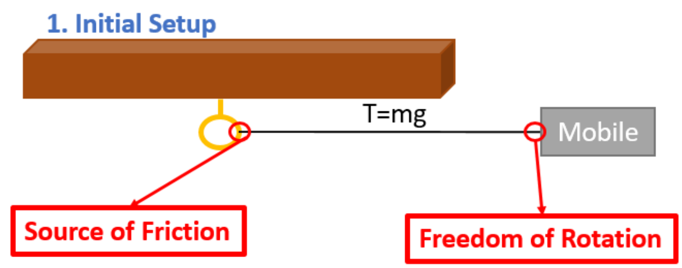

Figure 3. The initial experimental setup of a mobile pendulum where the mobile is initially suspended at 90 degrees with respect to the vertical axis. This setup failed as the group neglected that with one string, the mobile will experience some rotation about all axes. In addition, the initial angle, 90 degrees, also skewed the orientation data. Friction was neglected.

To limit the rotation of the mobile about all axes, the team devised a new setup which is illustrated in Figure 4. Within this new setup, an additional string was tied to the mobile on a separate hook and then suspended at 30 degrees with respect to the horizontal. By suspending the mobile with two strings, rotation about all axes was minimized. However, in the theoretical angular displacement calculation of the observed pendulum, mass must be halved to compensate for how tension is now divided between two strings.

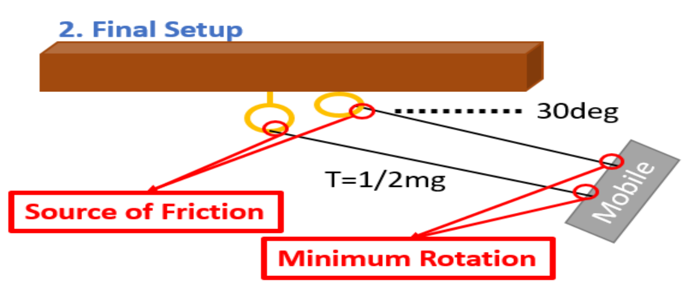

Figure 4. The final experimental setup where the mobile is suspended by two strings and at an angle of 30 degrees with respect to the horizontal. By suspending the mobile with two strings, rotation about all axes was minimized.

The mass of the phone was halved in the calculation as the tension of the mobile was split between the two strings.

Once the orientation and time data was successfully logged, we processed it using MATLAB (MATLAB v9.0, The Mathworks Inc.). Only a single orientation was needed to model the observed periodicity of the mobile which in our case was the Roll. To compare the observed data to the theoretical model, we first had to offset the observed to center around 0 degrees. Next, the positive peak amplitudes were found and then fit with an exponential regression. This function was defined as:

$$\text{Function fit} = P_{\text{obs}}(b, t) \qquad (7)$$

Where b is the damping term and t is time (in seconds). A single and bi-exponential regressions were fit to the observed data. These function fits were used in conjugation with a theoretical model of the pendulum's exponential regression:

$$P_{\text{mod}}(b, t) = \theta(t) = Ae^{-bt/2m} \qquad (8)$$

where all terms are known a prior except for b, the damping term.

MATLAB's nonlinear function solver fmincon (MATLAB v9.0, The Mathworks Inc.) was used as part of a constrained optimization approach to derive the damping term, b. An objective function was defined:

$$SSE = \underbrace{\left(\sum_{t=0}^{T}(P_{\text{obs}}(t) - P_{\text{mod}}(b,t))^2\right)}_{\text{Objective Function}} \qquad (9)$$

where two P_{obs} were used, one for each separate exponential regression of the observed data. The constraint of the optimization is defined by:

$$0 \leq b \leq 2 * m * \omega_0 \qquad (10)$$

where the inequality reflects (6).

Results

Raw data collected from the small-scale swing was plotted using MATLAB and is shown in Figure 5. Two adjustments were made to the data prior to analysis optimization/fitting and analysis. Noise at the beginning of the signal (prior to release of the pendulum) was removed from the data (Figure 6). Second, the roll data oscillations were centered at zero degrees to make reading the data easier (Figure 7).

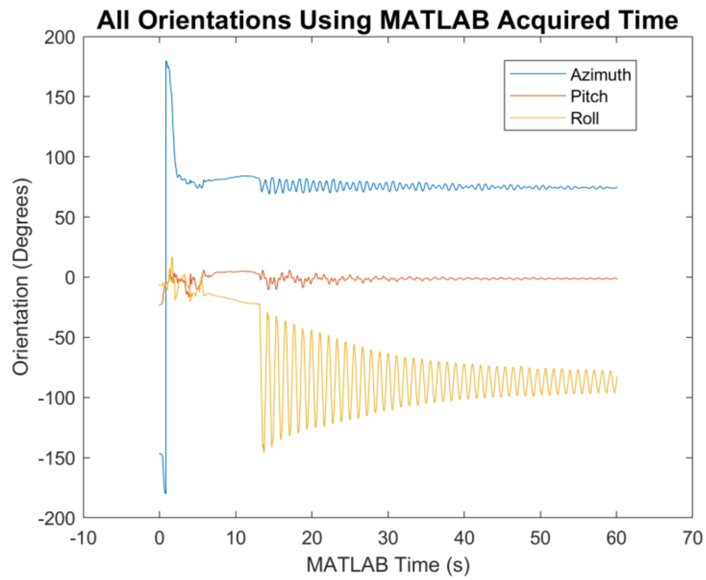

Figure 5. Raw data collected from the experimental swing set-up. Noise at the beginning of the run time data collection occurred before the pendulum was released from its initial position. The azimuth and pitch each show a small amount of oscillating noise that would not be present in an ideal situation, but the noise is small enough to be considered negligible. The roll shows underdamped oscillations as expected.

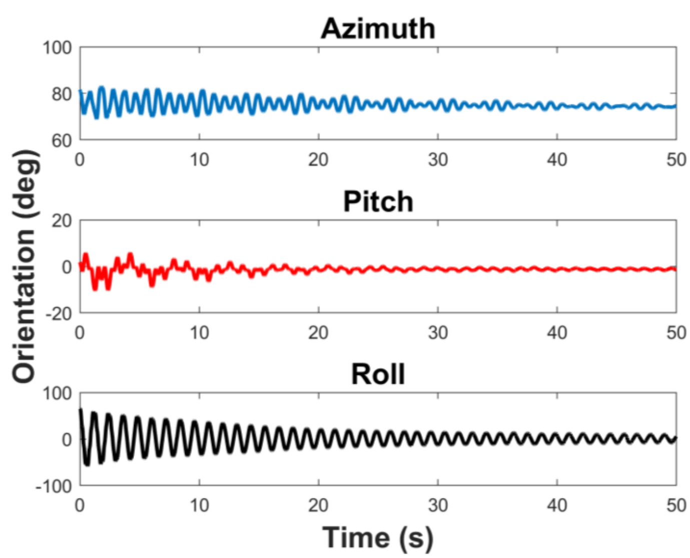

Figure 6. Signal after pre-release noise was removed.

All positive roll oscillation peaks were tagged using MATLAB's peak-to-peak function, and an exponential equation was fit to the data (Figure 8). Using a constrained optimization, the damping coefficient was calculated at 0.0169 with a cost function of $2.360 * 10^{-7}$. The maximum pendulum roll angle is 60 degrees for this experiment. Even with a small damping coefficient, air resistance and sliding friction cause the system to lower to a pendulum roll angle of 45 degrees in under 10 seconds. According to our model, in order to keep a child swinging with a maximum decrease in pendulum angle of 15 degrees, the child needs to receive a push approximately every 8 seconds.

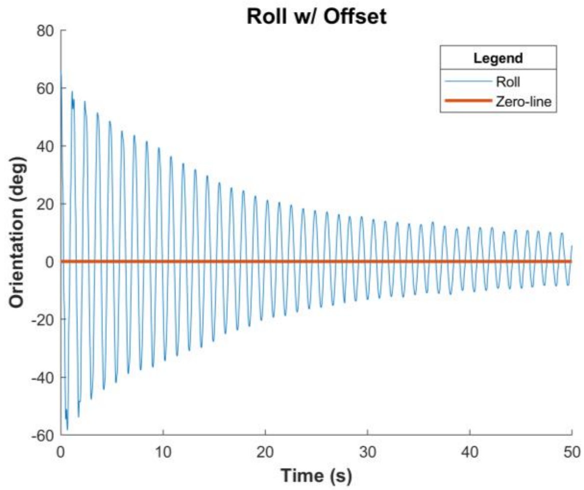

Figure 7. Roll oscillations centered at zero degrees.

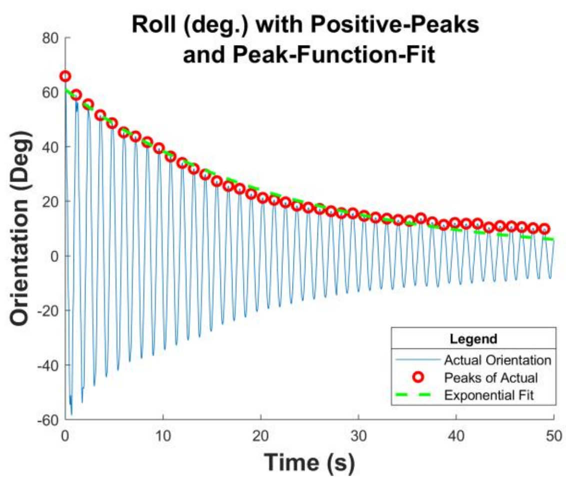

Figure 8. Positive peaks are tagged with a red circle, and the fitted exponential equation is shown in green.

Finally, the damping of a theoretical pendulum with a coefficient of 0.0169 was plotted against the exponential fitted function to demonstrate the accuracy of our method (Figure 9).

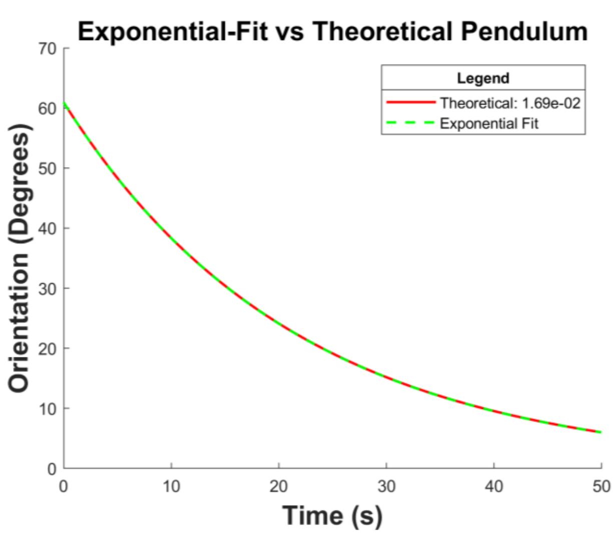

Figure 9. Theoretical damping with coefficient of 0.0169 plotted agains exponential fit function.

Discussion

Attempts at modeling the system with more advanced equations (such as two exponentials) were unsuccessful. However, we accurately modeled the swing-style pendulum using only one exponential.

Possible sources of error during our experiment include an estimating the mass of the mobile device used to collect data and neglecting friction. Additionally, the weight proportions of the materials and amount of friction experienced by our small-scale experimental set-up are very different from an actual child swinging on a playground. While our modeling techniques could be applied to model data collected from a swinging child, our damping coefficient and final result (8 seconds until a 25 percent drop occurs in pendulum angle) are likely different from real-world scenarios.

Future experimentation includes collecting data from real-world scenarios of children in swings (with variation in the children's masses and wind creating air resistance). We also hope to more accurately account for frictional forces in future experiments.

Conclusion

Constrained optimization resulted in a reasonably accurate calculation of the damping coefficient for this data set. Non-linear properties made this system challenging to model, so much future experimentation remains possible. In summary, our model provides an estimate that a child on a swing will have an underdamped oscillation (coefficient of 0.0169). In order to have the best swinging experience, a maximum pendulum angle needs to be maintained; our calculations estimate that pushing a swinging child every eight seconds will prevent a drop in pendulum angle that is greater than 25 percent.

References

[1] "MATLAB Answers." *MathWorks*, 30 Nov. 2017, https://www.mathworks.com/matlabcentral/answers/335305-matlab-mobile-orientation-sensor-direction-of-the-z-axis-and-pitch-angle-ambiguity.

[2] "Free Vibration of a Damped, Single Degree of Freedom, Linear Spring Mass System." *Introduction to Dynamics and Vibrations*, 30 Nov. 2017, https://www.brown.edu/Departments/Engineering/Courses/En4/Notes/vibrations_free_damped/vibrations_free_damped.htm.

[3] "Damped Oscillations." *SlidePlayer*, 30 Nov. 2017, http://slideplayer.com/slide/7344417/.

Appendix

The final code and raw data used for this project are located in the attached/submitted files called FinalCodePendulum.m and RawData.m.