UMPU Interval Computation

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Abstract

This document explains how to compute UMPU rejection regions for a truncated Gaussian distribution, and invert them to obtain UMAU confidence intervals for μ.

Let $X \sim \mathcal{L}(N(\mu, \sigma^2) \,|\, X \in S)$, for $S = \bigcup_{k=1}^K (a_k, b_k)$. Without loss of generality, assume $\sigma^2 = 1$.

Write

$$p_k(\mu) = \Phi(b_k - \mu) - \Phi(a_k - \mu), \quad p(\mu) = \sum_{k=1}^K p_k(\mu) \tag{1}$$

The density function of X is

$$f_\mu(x) = p(\mu)^{-1}\phi(x-\mu)\mathbf{1}\{x \in S\}, \tag{2}$$

which is a one-parameter exponential family with sufficient statistic x and natural parameter μ. Then, according to Lehmann & Romano, the UMPU acceptance region for $H_0 : \mu_0$ is an interval (c_1, c_2), satisfying

$$\mathbb{P}_{\mu_0}(c_1 \le X \le c_2) = 1 - \alpha \tag{3}$$

$$\mathbb{E}_{\mu_0}(X; c_1 \le X \le c_2) = (1-\alpha)\mathbb{E}_{\mu_0} X \tag{4}$$

which can be written as

$$\int_{(c_1,c_2)\cap S} \phi(x-\mu)\,dx = (1-\alpha)\int_S \phi(x-\mu)\,dx = (1-\alpha)p, \tag{5}$$

$$\int_{(c_1,c_2)\cap S} x\phi(x-\mu)\,dx = (1-\alpha)\int_S x\phi(x-\mu)\,dx = m(\mu) \tag{6}$$

The first thing we do is to write c_2 implicitly as a function of c_1 (for $F_\mu(c_1) \le \alpha$):

$$c_2(c_1) = F_\mu^{-1}(F_\mu(c_1) + 1 - \alpha) \tag{7}$$

Note that $c_2'(c_1) = \frac{\phi(c_1-\mu)}{\phi(c_2(c_1)-\mu)}$ for $c_1, c_2 \in S$.

Now, we need to solve the following:

$$0 = g_\mu(c_1) = \int_{(c_1,c_2(c_1))\cap S} x\phi(x-\mu)\,dx - m(\mu) \tag{8}$$

Next, exploiting the identity

$$\delta_\mu(a,b) \triangleq \int_a^b x\phi(x-\mu)\,dx \tag{9}$$

$$= -\phi(b-\mu) + \phi(a-\mu) + \mu\left(\Phi(b-\mu) - \Phi(a-\mu)\right), \tag{10}$$

we can evaluate $g_\mu(c_1)$ with relative ease in terms of sums of $\delta_\mu(a,b)$.

We can also easily evaluate first derivatives of g_μ for $c_1, c_2(c_1) \in S$:

$$g'_\mu(c_1) = c_2(c_1)\phi(c_2(c_1)-\mu)c'_2(c_1) - c_1\phi(c_1-\mu) \tag{11}$$

$$= (c_2(c_1) - c_1)\phi(c_1-\mu) > 0 \tag{12}$$

That is, we are just finding a root of a continuous function which is monotone-increasing for $c_1 \in S$.

The easiest way to get confidence intervals is to use bisection, noting that

$$g_\mu(x) > 0 \Leftrightarrow x > c_1(\mu) \Leftrightarrow \mu_{\text{hi}} > \mu \tag{13}$$