Data-driven Mathematical Optimization with Python – Exercises and solutions

Krzysztof Postek Alessandro Zocca Joaquim Gromicho Jeff Kantor

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Chapter 1

Mathematical optimization

No exercises are provided for this chapter.

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Chapter 2

Linear optimization

Exercises

*Exercises are labeled as **C** (calculating), **M** (modeling), **T** (theoretical) or combination of these, reflecting on which of these three aspects they focus. The most difficult/lengthy exercises are labeled with a $\star$.*

Ex. 2.1 (M) Consider a company which is working on a project. The project consists of two activities on which the company can work simultaneously. The project terminates when both the two activities are completed. The aim of the company is to complete the project as soon as possible.

According to the current estimations, activity 1 will be completed in 200 days while activity 2 requires 100 days. To speed up the process, the company is willing to deploy additional resources. A total budget of €500.000 that can be used to reduce the completing time of the project is made available. The company estimates that additional resources will impact on the two activities as follows:

- for each €10.000 spent on activity 1, the completion time of the activity is reduced by 3 days;
- for each €10.000 spent on activity 2, the completion time of the activity is reduced by 2 days;
- for each activity, investing less than €10.000 will not decrease the completion time (e.g., investing €9.999 does not reduce the completion time, investing €19.999 reduces the time by 3 days for activity 1 and 2 days for activity 2).

What is the allocation of the budget that allows to complete the project as soon as possible? Formulate this problem as a mathematical optimization problem.

Sol. 2.1

Observe that we are minimizing a max function here. To model such a function, we will need to introduce an auxiliary variable $z = \max\{200 - 3x_1, 100 - 2x_2\}$. The model can then be formulated as follows, where $x_i \times 10.000$ euros denotes the amount spent on activity i:

$$
\begin{array}{ll}
\min & z \\
\text{s.t.} & 200 - 3x_1 \leq z \\
& 100 - 2x_2 \leq z \\
& x_1 + x_2 \leq 50 \\
& z \geq 0 \\
& x_1, x_2 \in \mathbb{Z}^+
\end{array}
$$

Ex. 2.2 (C+T) Consider the following problem

$$\begin{aligned} \min \quad & |x_1| + |x_2| \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 3 \\ & x_1 \geq -1 \\ & x_2 \geq 2. \end{aligned}$$

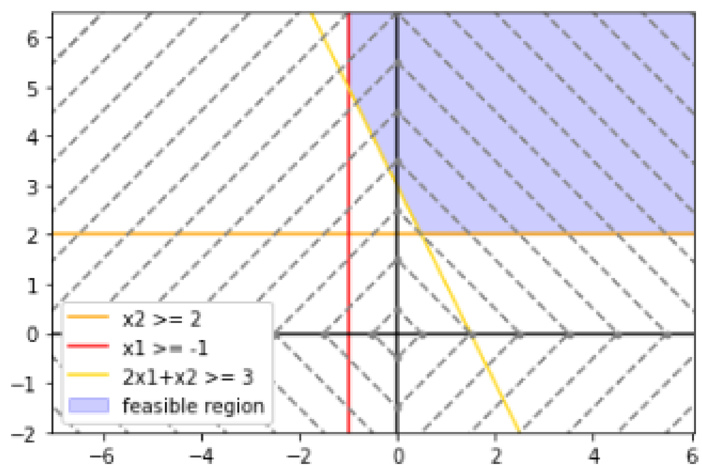
(a) Represent graphically and solve it.

(b) Reformulate the problem as a linear problem

(⋆ c) Prove that the technique used in (b) correctly models the minimization of a sum of absolute values.
Note: Your proof should hold for any optimization problem in which we minimize the sum of absolute values, not just for the specific instance of this question.

Sol. 2.2

(a) The feasible region (in blue) and the isolines (or countour lines) of the objective function $|x_1| + |x_2|$ (in gray) look as follow:



It is easy to graphically deduce that the optimal solution is $x_1 = 0.5$, $x_2 = 2$.

(b) Replace $|x_i|$ by $x_i^+ + x_i^-$ and define $x_i := x_i^+ - x_i^-$. The problem rewrites as

$$\begin{aligned} \min \ & x_1^- + x_1^+ + x_2^- + x_2^+ \\ \text{s.t.} \quad & 2x_1^+ - 2x_1^- + x_2^+ - x_2^- \geq 3, \\ & x_1^+ - x_1^- \geq -1, \\ & x_2^+ - x_2^- \geq 2, \\ & x_1^+, x_1^-, x_2^+, x_2^- \geq 0. \end{aligned}$$

and admits the following solution: $x_1^+ = 0.5$, $x_2^+ = 2$. Another way to reformulate the problem is to apply the same procedure for x_1 and to "forget" about the absolute value of x_2 (i.e., you can substitute $|x_2|$ with x_2 without introducing variables x_2^+ and x_2^-) since x_2 can take only positive values in the feasible region.

(c) It is easy to check by hand that, if either x_i^+ or x_i^- is zero for every i, the problems with and without absolute values are equivalent.

Let us now prove that if x_i is (a component of the) optimal solution then either x_i^+ or x_i^- is zero for every $i = 1, \ldots, n$.

By contradiction, assume that there is an optimal solution for which there exists an index i such that both x_i^+ and x_i^- are larger than zero. Let $\delta = \min\{x_i^+, x_i^-\}$ and subtract δ from both x_i^+ and x_i^-, obtaining two new values, namely $y_i^+ = x_i^+ - \delta$ and $y_i^- = x_i^- - \delta$. Consider the candidate solution equal to x in all the components but the ones corresponding to the index i, where we insert the new values y_i^+ and y_i^-. The constraints are still all satisfied, because $y_i = y_i^+ - y_i^- = x_i^+ - \delta - (x_i^- - \delta) = x_i^+ - x_i^- = x_i$. However, for this new candidate solution the objective function has been decreased by 2δ with respect to the original solution x, contradicting its optimality. Thus, there cannot be an optimal solution with both x_i^+ and x_i^- strictly positive.

Note that this argument only holds for minimization.

Ex. 2.3 (C) Consider the following optimization problem.

$$\begin{aligned} \max \quad & \frac{5x_1 + 6x_2}{2x_2 + 7} \\ \text{s.t.} \quad & 2x_1 + 3x_2 \leq 6 \\ & 2x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Formulate the problem as a linear problem and solve it.

Sol. 2.3

Letting $t := (2x_2 + 7)^{-1}$, we can equivalently rewrite the objective function as $5x_1t + 6x_2t$. Define y_j as x_jt for $j = 1, 2$. Using this notation we can reformulate the LP as:

$$\begin{aligned} \max \quad & 5y_1 + 6y_2 \\ \text{s.t.} \quad & 2y_1 + 3y_2 \leq 6t, \\ & 2y_1 + y_2 \leq 3t, \\ & 2y_2 + 7t = 1, \\ & t > 0, \\ & y_1, y_2 \geq 0. \end{aligned}$$

Solving this problem via Pyomo gives the optimal solution: $y_1 = 0.075$, $y_2 = 0.15$ and $t = 0.1$, which result in the maximum value of 1.275.

Translating these values back to the original x-variables gives us: $x_1 = \frac{0.075}{0.1} = 0.75$ and $x_2 = \frac{0.15}{0.1} = 1.5$.

Ex. 2.4 (M) A truck driver has to deliver packages to houses 1, 2 and 3, in this specific order. When driving at normal speed it takes him a_1 to get from its current position to house 1, a_2 from house 1 to house 2 and a_3 to drive from house 2 to house 3. The driver has a deadline d_i by which the corresponding package should be delivered at the house i, for $i = 1, 2, 3$.

The driver has the option to slow down or speed up his normal speed, more specifically it takes the driver a_ix_i to arrive at house i from the previous stop for $x_i \in \mathbb{R}^+$. More specifically, if $x_i > 1$, then the drivers is driving slower than the nominal speed and the trip to house i takes longer and the reverse scenario when $0 < x_i < 1$. However, the driver does not want to drive too fast for a long time, thus we impose the constraints $x_1 + x_2 + x_3 \geq 2.5$ and $x_1, x_2, x_3 \geq 0.5$.

The goal is to minimize the <u>total</u> delivery delay. The delivery delay to each house is equal to the difference between the arrival time of the driver and the scheduled delivery, but only when he/she is late.
Note: You may assume that once the driver arrives at a customer, he/she immediately drives to the next customer.

(a) Formulate the problem above as a LP.

(b) In the aforementioned model it may occur that the driver arrives before d_1 at the first house, in order to be on time for the second delivery. In some situations it might be undesirable that a driver arrives before its deadline (e.g., when the driver has to deliver a parcel in a specific time window). Adjust the previous problem such that the driver is penalized not only when he/she is too late, but also (proportionally) when he/she is too early.

Sol. 2.4

a) $a_i x_i$ is the time it takes driver to drive part i of the trip. Let z_i be the time describe how late the driver arrives at house i. We can formulate the following LP:

$$\begin{aligned}\min \quad & z_1 + z_2 + z_3 \\ \text{s.t.} \quad & a_1x_1 - d_1 \le z_1, \\ & a_1x_1 + a_2x_2 - d_2 \le z_2, \\ & a_1x_1 + a_2x_2 + a_3x_3 - d_3 \le z_3, \\ & x_1 + x_2 + x_3 \ge 2.5, \\ & x_1, x_2, x_3 \ge 0.5, \\ & z_1, z_2, z_3 \ge 0.\end{aligned}$$

Note that defining z_i in this way is equivalent to setting $z_i := \max\{0, \sum_{j=1}^{i}(a_j x_j) - d_i\}$ for every $i = 1, 2, 3$.

b) Let y_i be the time describing how early the driver arrives at house i. We extend the LP in (a) as follows:

$$\begin{aligned}\min \quad & z_1 + z_2 + z_3 + y_1 + y_2 + y_3 \\ \text{s.t.} \quad & a_1x_1 - d_1 \le z_1, \\ & a_1x_1 + a_2x_2 - d_2 \le z_2, \\ & a_1x_1 + a_2x_2 + a_3x_3 - d_3 \le z_3, \\ & d_1 - a_1x_1 \le y_1, \\ & d_2 - a_1x_1 - a_2x_2 \le y_2, \\ & d_3 - a_1x_1 - a_2x_2 - a_3x_3 \le y_3, \\ & x_1 + x_2 + x_3 \ge 2.5, \\ & x_1, x_2, x_3 \ge 0.5, \\ & y_1, y_2, y_3 \ge 0, \\ & z_1, z_2, z_3 \ge 0.\end{aligned}$$

Verify that y_i and z_i are never simultaneously strictly positive at each optimal solution.

Another equivalent reformulation that does not require variables y_1, y_2, y_3 can be obtained by defining $z_i := \max\{d_i - \sum_{j=1}^{i}(a_j x_j), \sum_{j=1}^{i}(a_j x_j) - d_i\}$ for every $i = 1, 2, 3$. The corresponding reformulation states as follows:

$$\begin{aligned}\min \quad & z_1 + z_2 + z_3 \\ \text{s.t.} \quad & a_1x_1 - d_1 \le z_1, \\ & a_1x_1 + a_2x_2 - d_2 \le z_2, \\ & a_1x_1 + a_2x_2 + a_3x_3 - d_3 \le z_3, \\ & d_1 - a_1x_1 \le z_1, \\ & d_2 - a_1x_1 - a_2x_2 \le z_2, \\ & d_3 - a_1x_1 - a_2x_2 - a_3x_3 \le z_3, \\ & x_1 + x_2 + x_3 \ge 2.5, \\ & x_1, x_2, x_3 \ge 0.5, \\ & z_1, z_2, z_3 \ge 0.\end{aligned}$$

Ex. 2.5 (C) Consider the dual problem (2.1) derived in Chapter 2, treat it as if it was a primal problem and derives its dual, and check that the original primal problem (2.2) is re-obtained.

$$\begin{aligned}
\max \quad & 3\lambda_1 - 5\lambda_2 + 7\lambda_3 + 11\lambda_4 \\
\text{s.t.} \quad & \lambda_1 \in \mathbb{R} \\
& \lambda_2 \geq 0 \\
& \lambda_3 \leq 0 \\
& \lambda_4 \leq 0 \\
& -\lambda_1 + 2\lambda_2 + \lambda_4 \leq 1 \\
& 3\lambda_1 - \lambda_2 + \lambda_4 \geq -2 \\
& 3\lambda_2 + \lambda_3 - \lambda_4 = 4.
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
\min \quad & x_1 - 2x_2 + 4x_3 \\
\text{s.t.} \quad & -x_1 + 3x_2 = 3 \\
& 2x_1 - x_2 + 3x_3 \geq -5 \\
& x_3 \leq 7 \\
& x_1 + x_2 - x_3 \leq 11 \\
& x_1 \geq 0 \\
& x_2 \leq 0 \\
& x_3 \in \mathbb{R}.
\end{aligned} \tag{2.2}$$

Ex. 2.6 (M) In a smartphone shop, in view of the arrival of new models, a salesman wants to sell off quickly its stock composed of eight phones, four hands-free kits and nineteen prepaid cards. Thanks to a market study, she knows that she can propose an offer with a phone and two prepaid cards and that this offer will bring in a profit of seven euros. Similarly, she can prepare a box with a phone, a hands-free kit and three prepaid cards, yielding a profit of nine euros. She is assured to be able to sell any quantity of these two offers within the availability of the stock.

(a) Give a LP that determines which quantity of each offer should the salesman prepare to maximize its net profit.

(b) A sales representative of a supermarket chain proposes to buy its stock (the products, not the offers). The salesman wants to know the minimum unit prices she should negotiate for each product (phone, hands-free kits, and prepaid cards) in order to obtain at least the same profit as in (a). Formulate a LP to decides these prices.

Sol. 2.6

(a) The LP is:

$$\begin{aligned}
\max \quad & 7x_1 + 9x_2 \\
\text{s.t.} \quad & x_1 + x_2 \leq 8, \\
& x_2 \leq 4, \\
& 2x_1 + 3x_2 \leq 19, \\
& \boldsymbol{x} \geq \boldsymbol{0},
\end{aligned}$$

which can be written in the canonical form by taking

$$\boldsymbol{c} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 8 \\ 4 \\ 19 \end{bmatrix}.$$

(b) The desired LP problem is the dual of the LP stated in (a). It is formulated as a minimization problem with $\boldsymbol{b}^\top \boldsymbol{\lambda}$ as objective function and both $\boldsymbol{\lambda}^\top A \geq \boldsymbol{c}$ and $\boldsymbol{\lambda} \geq 0$ as constraints, namely

$$\begin{aligned}
\min \quad & 8\lambda_1 + 4\lambda_2 + 19\lambda_3 \\
\text{s.t.} \quad & \lambda_1 + 2\lambda_3 \geq 7, \\
& \lambda_1 + \lambda_2 + 3\lambda_3 \geq 9, \\
& \boldsymbol{\lambda} \geq \boldsymbol{0}.
\end{aligned}$$

In view of their association with the constraints in the original problem, the dual variables λ_1, λ_2 and λ_3 indicate the prices the salesman needs to pay to get extra phones, hands-free kits and prepaid cards, respectively.

Ex. 2.7 (C) Consider the following LP and use its dual to decide if $\boldsymbol{x} = (x_1, x_2) = (1, 4)$ is the optimal solution or not.

$$\begin{aligned} \max \quad & x_1 - x_2 \\ \text{s.t.} \quad & -2x_1 + x_2 \leq 2, \\ & x_1 - 2x_2 \leq 2, \\ & x_1 + x_2 \leq 5, \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

Sol. 2.7

Firstly, it is easy to check that $\boldsymbol{x} = (1, 4)$ is a feasible solution by showing that it satisfies all the constraints. Using the standard theory for LP duals (cf. Lecture notes), we can formulate the dual problem as:

$$\begin{aligned} \min \quad & 2\lambda_1 + 2\lambda_2 + 5\lambda_3 \\ \text{s.t.} \quad & -2\lambda_1 + \lambda_2 + \lambda_3 \geq 1, \\ & \lambda_1 - 2\lambda_2 + \lambda_3 \geq -1, \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{aligned}$$

Assume that the solution $\boldsymbol{x} = (1, 4)$ is the optimal solution. Then, the KKT conditions should hold for the pair $(\boldsymbol{x}, \boldsymbol{\lambda})$, for some dual variables $\boldsymbol{\lambda} \in \mathbb{R}^3$. Plugging in the values $\boldsymbol{x} = (1, 4)$, we note that, by the complementary slackness, the two nontrivial inequality constraints of the dual must be attained and thus hold as equations. Moreover, $\lambda_2 = 0$, because the second constraint of the primal is not binding if $\boldsymbol{x} = (1, 4)$. These considerations result into the following two system of equations for the dual variables

$$\begin{cases} -2\lambda_1 + \lambda_3 = 1 \\ \lambda_1 + \lambda_3 = -1, \end{cases}$$

whose solution is $\lambda_1 = -\frac{2}{3}$ and $\lambda_3 = -\frac{1}{3}$. If $\boldsymbol{x} = (1, 4)$ was the optimal solution for the primal, then $\boldsymbol{\lambda} = (-\frac{2}{3}, 0, -\frac{1}{3})$ would be the optimal solution for the dual. However, this solution is infeasible, since not all the entries of y are non-negative. Thus, $\boldsymbol{x} = (1, 4)$ is *not* the optimal solution.

Ex. 2.8 (C) Use duality to show that the following problem is infeasible:

$$\begin{aligned} \min \quad & 2x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 1, \\ & -x_1 - x_2 \geq 1, \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

Sol. 2.8

The dual of the problem is:

$$\begin{aligned} \max \quad & \lambda_1 + \lambda_2 \\ \text{s.t.} \quad & \lambda_1 - \lambda_2 \leq 2, \\ & \lambda_1 - \lambda_2 \leq -1, \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{aligned}$$

Note that the variable λ_1 can take any arbitrary large value, say K, as long as we make sure that $\lambda_2 \geq K + 1$ (and we can do so without violating any of the other constraints). Therefore, the dual is unbounded, which implies that the primal problem does not have a feasible solution.

Ex. 2.9 (T) Consider the following linear optimization problem and assume that it has a feasible and bounded solution.

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \le \boldsymbol{0}, \\ & \boldsymbol{x} \ge \boldsymbol{0}. \end{aligned}$$

(a) Write down the dual of the above problem.

(b) Find the optimal solution of the above problem.

(c) What can you say about the constraint set for the above problem?
(Hint: if it helps, think first at the situation with only two x-variables).

Sol.Sol. 2.92.9

(a) Since the vector $\boldsymbol{b} = \boldsymbol{0}$ in the original problem, the objective function of the dual is $\boldsymbol{\lambda}^\top \boldsymbol{b} = 0$ and thus constantly equal to zero. Hence, the dual is

$$\begin{aligned} \min \quad & 0 \\ \text{s.t.} \quad & A^\top \boldsymbol{\lambda} \ge \boldsymbol{c}, \\ & \boldsymbol{\lambda} \ge \boldsymbol{0}. \end{aligned}$$

(b) It is easy to see that the value of the objective function of the dual is always 0 and thus is the optimal value of the primal problem. Hence, the primal solution $\boldsymbol{x} = \boldsymbol{0}$ attains the same value and is feasible, hence is the optimal solution.

(c) If there is a constraint with only positive coefficients, it follows directly that $\boldsymbol{x} = \boldsymbol{0}$ is the only solution. However, even if there are negative coefficients, the solution $\boldsymbol{x} = \boldsymbol{0}$ is always feasible. In a two-dimensional case, one can draw all the constraints and if there is a non-empty feasible set, the only corner point of that feasible set is the origin. Hence, not matter how one chooses the objective function, the optimal solution will always be the origin.

Exercise 5 (M)

A textile firm is capable of producing three products and let x_1, x_2 and x_3 denote the quantities of these products. The production plan for the next month must satisfy the constraints:

$$\begin{aligned} & x_1 + 2x_2 + 2x_3 \le 12, \\ & 2x_1 + 4x_2 + x_3 \le c, \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

The first constraint is determined by equipment availability and is fixed. The second constraint is determined by the availability of cotton c. The net profits of the products are 2, 3 and 3 respectively, minus the cost of cotton and fixed costs. Find the value of the dual variable corresponding to the cotton constraint as a function of c. Moreover, also determine the profit minus the cost of cotton as a function of c.

Sol. 2.9

The dual of the problem is:

$$\begin{aligned} \min \quad & 12\lambda_1 + c\lambda_2 \\ \text{s.t.} \quad & \lambda_1 + 2\lambda_2 \ge 2, \\ & 2\lambda_1 + 4\lambda_2 \ge 3, \\ & 2\lambda_1 + \lambda_2 \ge 3, \\ & \boldsymbol{\lambda} \ge \boldsymbol{0}. \end{aligned}$$

Note that the second constraint is redundant, because from the first constraint follows already that $2\lambda_1 + 4\lambda_2 \ge 4$. You could also conclude from the original problem that you will never produce any item of type 2.

If c is close to 0, the optimal solution of the dual is $\lambda_1 = 0$ and $\lambda_2 = 3$. This fact can be deduced either from solving the dual or by realizing that if c is sufficiently small, the fixed capacity constraint will never be binding. Similarly if c is extremely large, the optimal solution of the dual is $\lambda_1 = 2$ and $\lambda_2 = 0$ and the cotton constraint is not active. For intermediate values of c, both constraints are active and $\lambda_1 = \frac{4}{3}$ and $\lambda_2 = \frac{1}{3}$.

Using these three scenarios and the active constraints in each one of them, we can calculate the value of the objective function of the dual as a function of c. The three values, listed for increasing values of c, are $3c$, $16 + \frac{1}{3}c$ and 24. We can find the value of c for which the values of λ_1 and λ_2 change by calculating for which two threshold values of c holds first that $3c = 16 + \frac{1}{3}c$ and then that $16 + \frac{1}{3}c = 24$. Therefore, the variables λ_1 and λ_2 are defined as function of c in the following way:

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2) = \begin{cases} (0,3) & \text{if } c \leq 6, \\ (\frac{4}{3}, \frac{1}{3}) & \text{if } 6 \leq c \leq 24, \\ (2,0) & \text{if } c \geq 24. \end{cases}$$

Thanks to strong duality, the profit as a function of c is equal to the objective function of the dual.

Ex. 2.10 (T, $\star$) Given positive constants C, p_i, and w_i for $i = 1, \dots, n$, consider the following LP:

$$\begin{aligned} \min \quad & Cy + \sum_{i=1}^{n} z_i \\ \text{s.t.} \quad & w_i y + z_i \geq p_i, && i = 1, \dots, n, \\ & z_i \geq 0, && i = 1, \dots, n, \\ & y \geq 0. \end{aligned}$$

(a) Intuitively, what is the largest number of n for which you can solve *by hand* (i.e., without a computer) this problem for any values of C, p and w? How would you do that?

(b) Formulate the dual of the problem.

(c) Intuitively, what is the largest number of n for which you can solve *by hand* (i.e., without a computer) the dual? How would you do that? *(Hint: See Exercise 9 of the Exercise Sheet 1.1 on ILP)*

(d) Could you use the solution from (c) to get the optimal values for y and z_i's in the original problem?

Sol. 2.10

(a) For $n = 1$, it is possible to find the solution graphically. For larger n it not trivial to find solutions for general values of the parameters.

(b) Using the fact that it is a linear problem, it is easy to derive the dual problem using the standard method seen at lecture. Let us first rewrite the primal problem in canonical form, by introducing the vector of variables $\boldsymbol{x} = (y, z_1, \dots, z_n) \in \mathbb{R}^{n+1}$, the vector $c = (C, 1, \dots, 1) \in \mathbb{R}^{n+1}$, the vector $b = (p_1, \dots, p_n)$ and the matrix $A = [w \mid I] \in \mathbb{R}^{n \times n+1}$, with I being the $n \times n$ identity matrix and w the column vector $w = (w_1, \dots, w_n)$. Using this notation, the primal rewrites as

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0. \end{aligned}$$

whose dual we now from the theory is equal to

$$\begin{aligned} \max \quad & \boldsymbol{b}^\top \boldsymbol{\lambda} \\ \text{s.t.} \quad & \boldsymbol{\lambda}^\top A \leq \boldsymbol{c}^\top \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{aligned}$$

We can now express the dual explicitly in the original variables y, z and parameters C, w, and p, namely

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} p_i \lambda_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} w_i \lambda_i \leq C, \\ & \lambda_i \leq 1, \qquad i = 1, \dots, n \\ & \boldsymbol{\lambda} \geq \mathbf{0}. \end{aligned}$$

(c) The given problem should be recognized as the *bounded knapsack problem (BKP)*, i.e., the one where it is possible to pick a fraction $\lambda_i \in [0, 1]$ of item i. Note that the BKP is the LP relaxation of the 0-1 knapsack problem (cf. Exercise 9 of the Exercise Sheet 1.1). The optimal solution of the BKP is obtained by adding the items in non-increasing order of their efficiency $\frac{p_i}{w_i}$, until the capacity C is reached (cf. Exercise 8(a) in Exercise Sheet 1.1). Thus, using this policy, the problem can easily be solved for any number n.

(d) Dealing with a linear problem, we know that strong duality holds and thus the optimal values of the primal and the dual problem must be equal. Furthermore, the complementary slackness conditions $\lambda_i(p_i - w_i y - z_i) = 0$ for $i = 1, \dots, n$, tell us that for every item i that is assigned to the knapsack (i.e., $\lambda_i > 0$), it must hold that $w_i y + z_i = p_i$. Moreover, using our intuition about the bounded knapsack problem in (b), the variable z_i can be seen as the dual variables corresponding to the constraints $\lambda_i \leq 1$. Therefore, again by complementarity slackness, every item that is not fully assigned, i.e., $\lambda_i < 1$, we must have $z_i = 0$. Using all these ingredients, one can find the optimal values for y and z's.

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Chapter 3

Mixed-integer linear programming

Exercises

*Exercises are labeled as **C** (calculating), **M** (modeling), **T** (theoretical) or combination of these, reflecting on which of these three aspects they focus. The most difficult/lengthy exercises are labeled with a ⋆.*

Ex. 3.1 (M+T ⋆) Assume there are three constraints of the type $\sum_j^J a_{ij}x_j \leq b_i$, $i = 1, 2, 3$, where a_j is a generic real number and x_j is a decision variable for each $j = 1, \dots, J$. We want to construct an objective function that reflects the way the constraints are satisfied. More specifically:

- For each satisfied constraint, the objective function increases by 1;
- If at least one constraint is satisfied, then the objective function increases by 1;
- If constraints 1 and 2 are satisfied simultaneously, the objective function gets an extra 1.

Examples:

- *Satisfying all constraints yields* $3 \times 1 + 1 + 1 = 5$.
- *Satisfying only constraint 3 yields* $1 + 1 = 2$.
- *Satisfying constraints 1 and 2 yields* $2 \times 1 + 1 + 1 = 4$.
- *Satisfying no constraint yields* 0.

Formulate a mixed ILP for which the objective function is maximized.

Sol. 3.1

Let y_i be a variable indicating if constraint i is satisfied for $i = 1, 2, 3$. If we allow for a product to appear in the objective function, we get the following problem:

$$\begin{aligned}
\max \quad & y_1 + y_2 + y_3 + (1 - (1 - y_1)(1 - y_2)(1 - y_3)) + y_1 y_2 \\
\text{s.t.} \quad & \sum_j a_{1j}x_j \leq b_1 + M(1 - y_1), \\
& \sum_j a_{2j}x_j \leq b_2 + M(1 - y_2), \\
& \sum_j a_{3j}x_j \leq b_3 + M(1 - y_3), \\
& x_j \in \mathbb{R} \quad \forall j, \\
& y_1, y_2, y_3 \in \{0, 1\}.
\end{aligned}$$

Having a product in the objective function means it is not a MILP yet. We replace product $(1-y_1)(1-y_2)(1-y_3)$ by a binary variable z_1 using the following constraints:

$$\begin{aligned}
z_1 &\leq 1 - y_1 \\
z_1 &\leq 1 - y_2 \\
z_1 &\leq 1 - y_3 \\
z_1 &\geq (1-y_1) + (1-y_2) + (1-y_3) - 2 \implies z_1 \geq 1 - y_1 - y_2 - y_3
\end{aligned}$$

The first three constraints ensure that z_1 cannot be 1 if one of the three y_i-variables is 1 and the last constraint forces z_1 to be 1 if all three y_i-variables are 0. Similarly, we replace $y_1 y_2$ by a binary variable z_2:

$$\begin{aligned}
z_2 &\leq y_1 \\
z_2 &\leq y_2 \\
z_2 &\geq y_1 + y_2 - 1
\end{aligned}$$

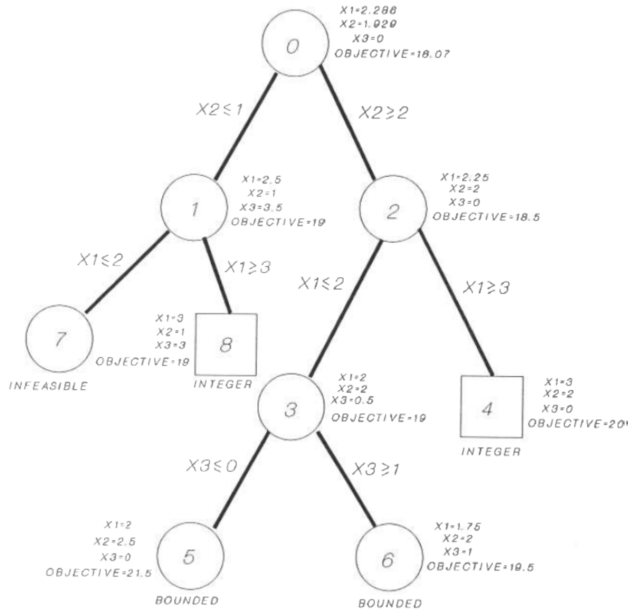
The complete MILP is:

$$\begin{aligned}
\max \quad & y_1 + y_2 + y_3 + (1 - z_1) + z_2 \\
\text{s.t.} \quad & \sum_j a_{1j} x_{ij} \leq b_1 + M(1 - y_1), \\
& \sum_j a_{2j} x_{ij} \leq b_2 + M(1 - y_2), \\
& \sum_j a_{3j} x_{ij} \leq b_3 + M(1 - y_3), \\
& z_1 \leq 1 - y_1, \\
& z_1 \leq 1 - y_2, \\
& z_1 \leq 1 - y_3, \\
& z_1 \geq 1 - y_1 - y_2 - y_3, \\
& z_2 \leq y_1, \\
& z_2 \leq y_2, \\
& z_2 \geq y_1 + y_2 - 1, \\
& x_{ij} \in \mathbb{R} \quad \forall i, j, \\
& y_1, y_2, y_3 \in \{0, 1\}, \\
& z_1, z_2 \in \{0, 1\}.
\end{aligned}$$

Ex. 3.2 (C) Consider the following ILP:

$$\begin{aligned}
\min \quad & 2x_1 + 7x_2 + 2x_3 \\
\text{s.t.} \quad & x_1 + 4x_2 + x_3 \geq 10, \\
& 4x_1 + 2x_2 + 2x_3 \geq 13, \\
& x_1 + x_2 - x_1 \geq 10, \\
& x_1, x_2, x_3 \geq 0 \text{ and integer.}
\end{aligned}$$

Which branching sequence in the tree below results in the fewest number of subproblems to be solved?



Sol. 3.2

Order $1, 7, 8, 2, 3, 4$ results in the fewest number of subproblems to be solved. Multiple sequences are equivalent, but it is important that node 8 is visited before node 3, because one can stop at node 3. The lower bound in node 3 is 19, which is also the feasible solution from node 8.

Ex. 3.3 (M, $\star$) A company produces and sells m different products. For each product $j = 1, \ldots, m$ the forecasted demand is equal to d_j. To produce each product, the company uses n types of ingredients. In particular, the amount of ingredient i that must be used to produce product j is denoted with a_{ij}. The company purchases ingredients from a supplier that applies the following pricing policy: buying one unit of ingredient i costs c_i, but, if more than q_i units of ingredient i are bought, the company gets volume discount and the cost per unit of ingredient i goes down to c'_i for the entire order. The supply chain department must decide on the amount of ingredients to purchase in order to satisfy the demand of each product while minimizing the purchasing costs. Formulate an ILP that can be used to help the supply-chain department.

Note: Depending on the values of the parameters, it could be cheaper buying more ingredients than necessary. This happens when the cost of ordering $q_i - 1$ units of ingredient i, i.e., $(q_i - 1)c_i$, is higher than $q_i c'_i$.

Sol. 3.3

Let u_i be the ordered amount of ingredient i at cost c_i and v_i be the ordered amount of ingredient i at cost c'_i. It follows that, if more than q_i units are ordered, then u_i is zero or, otherwise, v_i is zero. Thus, for every $i = 1, \ldots, n$, the two types variables have the following range:

$$0 \le u_i \le q_i, \quad \text{and} \quad v_i \ge q_i,$$

Let y_i be a binary variable indicating that more than q_i of ingredient i is ordered. If $y_i = 1$, then $u_i = 0$ and $v_i \ge q_i$. Otherwise, if $y_i = 0$, then $0 \le u_i \le q_i$ and $v_i = 0$. This is equivalent to the following set of constraints:

$$\begin{aligned} v_i &\ge y_i q_i, \\ v_i &\le y_i M, \\ u_i &\le (1 - y_i) q_i. \end{aligned}$$

The complete MILP is given as:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n}(c_i u_i + c_i' v_i) & \\
\text{s.t.} \quad & (u_i + v_i) \geq \sum_{j=1}^{m} a_{ij} d_j, & i = 1, \dots, n, \\
& v_i \geq y_i q_i, & i = 1, \dots, n, \\
& v_i \leq y_i M, & i = 1, \dots, n, \\
& u_i \leq (1 - y_i) q_i, & = 1, \dots, n, \\
& u_i \geq 0, & i = 1, \dots, n, \\
& v_i \geq 0, & i = 1, \dots, n, \\
& y_i \in \{0, 1\}, & i = 1, \dots, n.
\end{aligned}$$

Ex. 3.4 (T+C) The knapsack problem is a well-studied problem in Operations Research. In its general form, there are n items and each item $i = 1, \dots, n$ yields a profit p_i and has weight w_i. The goal of the problem is to decide which items to put in your knapsack to maximize your total profit under the constraint that the total weight of the packed items does not exceed C, the capacity of your knapsack . Assume the total capacity is $C = 11$ and consider the items as listed in the following table:

i	1	2	3	4	5	6	7
p_i	60	60	40	15	20	10	3
w_i	3	5	4	2	3	3	1

In the *fractional knapsack problem*, it is assumed that fractions of items can be chosen (the items are divisible). For such a problem the following simple greedy algorithm produces an optimal solution. At each step we select the item with maximum value of p_i/w_i and we add as much as possible of that item to our knapsack (taking into account the capacity constraint). We update the used capacity and we repeat this process with the next item until the knapsack is full or until all the items have been processed.

(a) Prove (in full generality) that the greedy solution indeed produces an optimal solution.

(b) In practice, it is often not possible to split an item. The situation in which items are indivisible is harder to solve than the divisible case and give rise to the so-called *0-1 knapsack problem.* Formulate this new problem as an ILP. Verify that the optimal solution of the *linear relaxation* of the 0-1 knapsack problem for the items described in the table above is to assign items 1 and 2 completely to the knapsack and item 3 for 0.75, resulting in a total profit of 150.

(c) Solve the ILP formulated in (b) by hand using the branch-and-bound method. Use as initial upper bound of the solution the optimal fractional solution of (a) and as lower bound we consider only assigning items 1 and 2, which yields a profit of 120. In each node you can use the greedy heuristic of the fractional knapsack problem to find the solution of that node.

(d) Solve the ILP formulated in (b) by hand using (again) the branch-and-bound method. This time, use the following strategy: branch first on the nodes with the highest upper bound (i.e., the nodes with highest optimal solution value).

Sol. 3.4

(a) Without loss of generality, assumed that the items are numbered such that $\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \cdots \geq \frac{p_n}{w_n}$. Let G be the solution obtained by the greedy algorithm. Assume by contradiction that G is not the optimal solution, i.e., that G yields less profit than O.

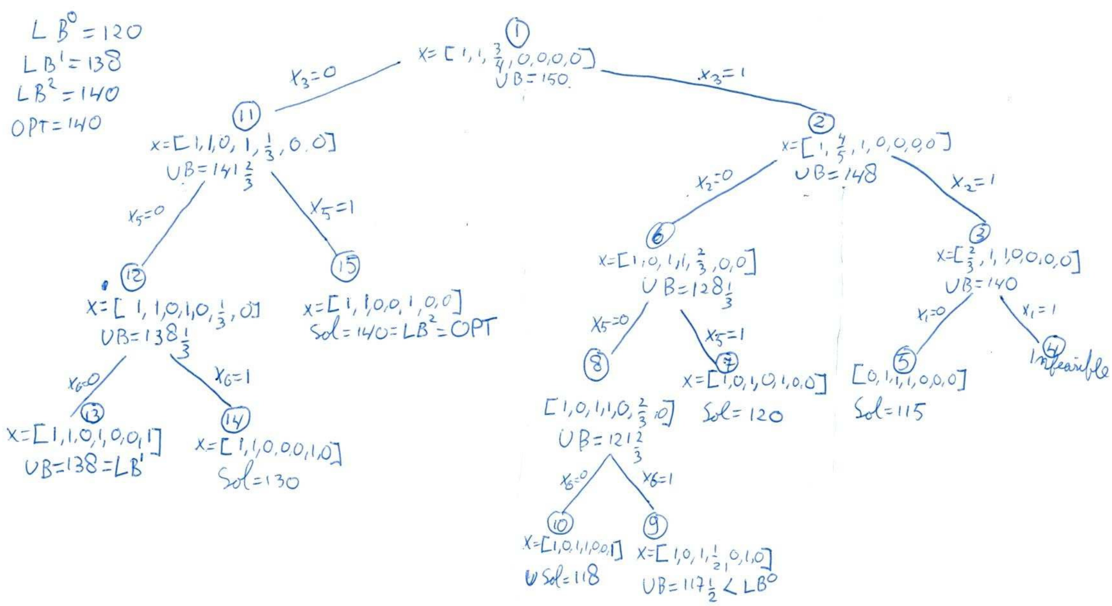
Let us denote f_i^G and f_i^O denote the fraction of item i that is assigned to the knapsack in the greedy and optimal solution respectively. The first item that is assigned differently by the two solution is item $k := \arg\min_{i=1,\dots,n}\{f_i^G \neq f_i^O\}$. Note that it is impossible that $f_k^G < f_k^O$, because in the greedy algorithm all but one f^G are equal to 1 and the only item for which $0 < f^G < 1$ is the last one assigned by the greedy algorithm (and in particular assigning a larger fraction of this last item would violated the capacity constraint).

Hence, we must have that $f_k^O < f_k^G$, meaning that there should be another item l for which $f_l^O > f_l^G$. If such an item l does not exist, the optimal solution O would not use all capacity, contradicting its optimality. Let u be the minimum between the weight of item l in the optimal solution and the extra weight of item k that is used in the greedy solution with respect solution O, i.e., $u = \min\{f_l^O w_l, (f_k^G - f_k^O)w_k\}$. We can modify solution O and generate a new solution O' by moving the quantity u from item l to item k. Since $\frac{p_k}{w_k} > \frac{p_l}{w_l}$, the profit per weight is larger for item k is larger than for item l. Thus, the total profit in O' is larger than in O, contradicting the optimality of O.

(b) Let x_i be the binary variable indicating that item i is assigned to the knapsack. The general formulation of the $0-1$ knapsack problem then reads:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} p_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} w_i x_i \le C, \\ & x_i \in \{0,1\} \quad i = 1, \ldots, n. \end{aligned}$$

(c) The actual branch-and-bound tree depends on the chosen branching strategy. A possible tree is given in the figure below. The nodes are numbered in the order in which they are explored. The strategy prescribes to explore the branch $x_i = 1$ first if $x_i \ge \frac{1}{2}$ and otherwise the branch $x_i = 0$. Moreover, a depth-first strategy is performed, going down in the tree as much as possible before exploring other branches in the tree. Independently of the chosen strategy, the optimal solution is $x = [1, 1, 0, 0, 1, 0, 0]$ resulting in a profit of 140.



Ex. 3.5 (M+C, $\star$) Formulate as a mixed integer optimization problem and solve the following problem:

$$\frac{abc}{def} = \frac{1}{5},$$

where a, b, c, d, e and f can take <u>distinct</u> integer values in the set $\{1, 2, \ldots, 6\}$.

Hint 1: Each number/variable appears either in the numerator or in the denominator.
Hint 2: $\log(ab) = \log(a) + \log(b)$.

Sol. 3.5

First we use the logarithm to get rid of the product and the division in the constraint:

$$\frac{abc}{def} = \frac{1}{5} \iff \log\left(\frac{abc}{def}\right) = \log\left(\frac{1}{5}\right) \iff \log(a) + \log(b) + \log(c) - \log(d) - \log(e) - \log(f) = -\log(5).$$

Let x_i be a binary variable that equals 1 if number i is in the numerator and 0 if i is in the denominator. Note that we are not optimizing, as we are only looking for a feasible solution. Therefore, one may choose any objective function, including a trivial one constantly equal to 0. The resulting MILP formulation is:

$$\begin{aligned} \max \quad & 0 \\ \text{s.t.} \quad & \sum_{i=1}^{6} \log(i)(2x_i - 1) = -\log(5), \\ & \sum_{i=1}^{6} x_i = 3, \\ & x_i \in \{0, 1\}, \quad i = 1, \dots, 6. \end{aligned}$$

The unique solution is $x = (1, 1, 0, 0, 0, 1)$, corresponding to $a = 1$, $b = 2$, $c = 6$, $d = 3$, $e = 4$, and $f = 5$.

Ex. 3.6 (originally appeared as Exercise 3.4-8 of the book by Hillier and Lieberman)
Larry Edison is the director of the Computer Center for Buckly College. He now needs to schedule the staffing of the center. It is open from 8 A.M. until midnight. Larry has monitored the usage of the center at various times of the day, and determined that the following number of computer consultants are required:

Time of day	**Minimum number of consultants required to be on duty**
8 am – noon	4
Noon – 4 pm	8
4 pm. – 8 pm	10
8 pm. – midnight	6

Two types of computer consultants can be hired: full-time and part-time. The full-time consultants work for 8 consecutive hours in any of the following shifts: morning (8 A.M.–4 P.M.), afternoon (noon–8 P.M.), and evening (4 P.M.–midnight). Full-time consultants are paid \$40 per hour. Part-time consultants can be hired to work any of the four shifts listed in the above table. Part-time consultants are paid \$30 per hour. An additional requirement is that during every time period, there must be at least two full-time consultants on duty for every part-time consultant on duty. Larry would like to determine how many full-time and how many part-time workers should work each shift to meet the above requirements at the minimum possible cost. Formulate a LP for this problem.

Sol. 3.6

First we define our decision variables as follows:

f_1 = number of full-time consultants working the morning shift (8 A.M.-4 P.M.)
f_2 = number of full-time consultants working the afternoon shift (noon-8 P.M.)
f_3 = number of full-time consultants working the evening shift (4 P.M.-midnight)
p_1 = number of part-time consultants working the first shift (8 A.M.-noon)
p_2 = number of part-time consultants working the second shift (Noon-4 P.M.)
p_3 = number of part-time consultants working the third shift (4 P.M.-8 P.M.)
p_4 = number of part-time consultants working the fourth shift (8 P.M.-midnight).

Using these, we can write the LP problem as follows:

$$\begin{array}{rl} \min & 320(f_1+f_2+f_3)+120(p_1+p_2+p_3+p_4) \\ \text{s.t.} & f_1+p_1 \geq 4 \\ & f_1+f_2+p_2 \geq 8 \\ & f_2+f_3+p_3 \geq 10 \\ & f_3+p_4 \geq 6 \\ & f_1 \geq 2p_1 \\ & f_1+f_2 \geq 2p_2 \\ & f_2+f_3 \geq 2p_3 \\ & f_3 \geq 2p_4 \\ & f_1,f_2,f_3,p_1,p_2,p_3,p_4 \geq 0. \end{array}$$

Ex. 3.7 (originally appeared as Exercise 3.4-14 of the book by Hillier and Lieberman)
Oxbridge University maintains a powerful mainframe computer for research use by its faculty, Ph.D. students, and research associates. During all working hours, an operator must be available to operate and maintain the computer, as well as to perform some programming services. Beryl Ingram, the director of the computer facility, oversees the operation.

It is now the beginning of the fall semester, and Beryl is confronted with the problem of assigning different working hours to her operators. Because all the operators are currently enrolled in the university, they are available to work only a limited number of hours each day, as shown in the following table.

		Maximum Hours of Availability				
Operators	**Wage Rate**	**Mon.**	**Tue.**	**Wed.**	**Thurs.**	**Fri.**
K. C.	\$25/hour	6	0	6	0	6
D. H.	\$26/hour	0	6	0	6	0
H. B.	\$24/hour	4	8	4	0	4
S. C.	\$23/hour	5	5	5	0	5
K. S.	\$28/hour	3	0	3	8	0
N. K.	\$30/hour	0	0	0	6	2

There are six operators (four undergraduate students and two graduate students). They all have different wage rates because of differences in their experience with computers and in their programming ability. The above table shows their wage rates, along with the maximum number of hours that each can work each day. Each operator is guaranteed a certain minimum number of hours per week that will maintain an adequate knowledge of the operation. This level is set arbitrarily at 8 hours per week for the undergraduate students (K. C., D. H., H. B., and S. C.) and 7 hours per week for the graduate students (K. S. and N. K.). The computer facility is to be open for operation from 8 A.M. to 10 P.M. Monday through Friday with exactly one operator on duty during these hours. On Saturdays and Sundays, the computer is to be operated by other staff. Because of a tight budget, Beryl has to minimize cost. She wishes to determine the number of hours she should assign to each operator on each day. Formulate a LP for this problem.

Sol. 3.7

Let x_{ij} be the number of hours operator i is assigned to work on day j for $i = KC, DH, HB, SC, KS, NK$ and $j = M, Tu, W, Th, F$. Then we can write our objective function as the minimization of the total sum of wages paid (hours worked $\times$ wage rate per operator).

$$\begin{array}{rl} \min & 25(x_{KC,M}+x_{KC,W}+x_{KC,F}) \\ & 26(x_{DH,Tu}+x_{DH,Th}) \\ & 24(x_{HB,M}+x_{HB,Tu}+x_{HB,W}+x_{HB,F}) \\ & 23(x_{SC,M}+x_{SC,Tu}+x_{SC,W}+x_{SC,F}) \\ & 28(x_{KS,M}+x_{KS,W}+x_{KS,Th}) \\ & 30(x_{NK,Th}+x_{NK,F}) \end{array}$$

Then, we have our maximum hours of availability constraints. This restriction varies per operator and day.

$$
\begin{aligned}
\text{s.t.} \quad & x_{KC,M} \le 6, x_{KC,W} \le 6, x_{KC,F} \le 6, \\
& x_{DH,Tu} \le 6, x_{DH,Th} \le 6, \\
& x_{HB,M} \le 4, x_{HB,Tu} \le 8, x_{HB,W} \le 4, x_{HB,F} \le 4, \\
& x_{SC,M} \le 5, x_{SC,Tu} \le 5, x_{SC,W} \le 5, x_{SC,F} \le 5, \\
& x_{KS,M} \le 3, x_{KS,W} \le 3, x_{KS,Th} \le 8, \\
& x_{NK,Th} \le 6, x_{NK,F} \le 2
\end{aligned}
$$

There is also the requirement that each operator is guaranteed a minimum number of hours per week. This differs for the undergraduate and graduate students and can be represented as follows.

$$
\begin{array}{ccccccccccc}
x_{KC,M} & & & + & x_{KC,W} & & & + & x_{KC,F} & \ge 8 \\
 & & x_{DH,Tu} & & & + & x_{DH,Th} & & & \ge 8 \\
x_{HB,M} & + & x_{HB,Tu} & + & x_{HB,W} & & & + & x_{HB,F} & \ge 8 \\
x_{SC,M} & + & x_{SC,Tu} & + & x_{SC,W} & & & + & x_{SC,F} & \ge 8 \\
x_{KS,M} & + & & + & x_{KS,W} & + & x_{KS,Th} & & & \ge 7 \\
 & & & & & & x_{NK,Th} & + & x_{NK,F} & \ge 7
\end{array}
$$

Additionally, it is specified that there must be exactly one operator on duty during the week, from 8 A.M. to 10 P.M. (14 hours), which can be modelled using the following constraints.

$$
\begin{array}{cccccccc}
x_{KC,M} & + & x_{HB,M} & + & x_{SC,M} & + & x_{KS,M} & = 14 \\
x_{DH,Tu} & + & x_{HB,Tu} & + & x_{SC,Tu} & & & = 14 \\
x_{KC,W} & + & x_{HB,W} & + & x_{SC,W} & + & x_{KS,W} & = 14 \\
x_{DH,Th} & + & x_{HB,Th} & + & x_{NK,Th} & & & = 14 \\
x_{KC,F} & + & x_{HB,F} & + & x_{SC,F} & + & x_{NK,F} & = 14
\end{array}
$$

Finally, as the number of hours assigned to an operator cannot be negative, we add non-negativity constraints to our variables.

$$x_{ij} \ge 0 \text{ for all } i, j$$

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