Department of Mathematics, Indian Institute of Technology, Kanpur

MTH101A: Quiz 1 B- 24-8-2012

Maximum Marks- 15 **5:15-5:35 p.m.**

Name: Roll No: Section:

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(1) Find the supremum of the set $X = \{\pi - 1, \pi - \frac{1}{2}, \pi - \frac{1}{3}, \dots\}$. [5]

Solution: Clearly, π is an upper bound for the set X. Let $\epsilon > 0$.
By the Archimedean property, there exists a natural number n_ϵ such that $\frac{1}{n_\epsilon} < \epsilon$. [2]
$\pi - \epsilon < \pi + \dfrac{1}{n_\epsilon} < \pi$. [2]
Hence, Supremum of the set X is π. [1]

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} x^2 sin\frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Compute $f^{'}(x)$, for all $x \in \mathbb{R}^n$. [5]

Solution: $f^{'}(x) = 2x \sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}$, at each $x \neq 0$. [1]
$f^{'}(0) = \lim\limits_{h \to 0} \frac{f(h) - f(0)}{h}$, if the limit exists.
$\lim\limits_{h \to 0} \frac{f(h) - f(0)}{h} = \lim\limits_{h \to 0} h \sin\frac{1}{h^2}$ [2]
$-h \leq h \sin\frac{1}{h^2} \leq h$ [1]
Taking limits as $h \to 0$, we get $f^{'}(0) = 0$. [1]

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} 0, & x \text{ is rational} \\ -1, & x \text{ is irrational} \end{cases}$$

Show that f is not continuous at any point in $\mathbb{R}$. What can you say about the continuity of $|f|$? Justify. [5]

Solution: Let $x \in \mathbb{Q}$ and $\{x_n\}$ be an irrational sequence converging to x. [1]
$\{f(x_n)\} = \{-1\} \to -1 \neq f(x) = 0$. [1]
Hence f is not continuous at any rational point. [1]
By a similar argument f is not continuous at any irrational point. [1]

$$|f|(x) = \begin{cases} 0, & x \text{ is rational} \\ 1, & x \text{ is irrational} \end{cases}$$

By the same argument as the one above, we see that $|f|$ is discontinuous at each point of $\mathbb{R}$. [1]