Introduction to Finite Element Method

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Overview

1. Introduction
   - History
   - Idea
   - Terminologies

2. Boundary Value Problems
   - Second Order Ordinary Differential Equations
   - Time Dependent Problems

Boris Galerkin (March 4, 1871 - July 12, 1945)

- Born in Polotsk, Vitebsk Governorate, Russian Empire.
- Jewish Soivet Mathematician.
- Huge contributions to finite element method. Some of them include:
  - *Galerkin Method*
  - *Petrov Galerkin Method*
  - *Discontinuous Galerkin Method*

What is Finite Element Method?

- Numerical method to find out solutions to PDEs and ODEs
- Idea:
  - Divide the whole domain into many simpler (sub)domains.
  - Construct Approximation functions over these (sub)domain.
  - Assemble the elements to obtain solutions at the nodes.
- Applications:
  - Computational Structural Mechanics (CSM).
  - Computational Fluid Dynamics (CFD).
  - And much more ...

Terminologies

Subdomains/Finite Elements

In Finite Element Method, a domain is generally divided into simpler, regular parts known as *finite elements, elements or subdomains*

Nodes

Each subdomain/element is connected to other elements at points known as *Nodes.*

Approximation Functions

The solution curve in each element is *approximated* by a simple curve. Eg.Polynomial.

Interpolation Functions

Functions that are used to derive the *Approximation Functions*.

Basic Steps of finite element method

- Deriving weak form
- Discretization
- Deriving element equations
- Assembling the element equations
- Deriving the solution
- Postprocessing

Consider the general form of the second order differential equation

$$p(x)\frac{d^2u}{dx^2} + q(x)\frac{du}{dx} + r(x)u = f(x) \qquad a < x < b \tag{1}$$

subject to the following boundary conditions

$$u(a) = u_1 \quad u(b) = u_2 \tag{2}$$

Step 1 : Deriving weak form

- The steps to write the weak form is stated below:
  1. Write down the ODE such that it reads $D[u] = 0$
  2. Multiply the LHS with a function w. This function is known as the *weight function*.
  3. Apply integration by parts on the first term $\left(w\frac{d^2u}{dx^2}\right)$.
- The weak form corresponding to the problem (1) is given by

$$\int_a^b \left(-\frac{dw}{dx}\frac{du}{dx} + w\frac{q(x)}{p(x)}\frac{du}{dx} + w\frac{q(x)}{p(x)}u - w\frac{f(x)}{p(x)}\right) dx + \left[w\frac{du}{dx}\right]_a^b = 0$$

- The word "weak" refers to the weakened continuity of the function u in the problem.

Step 2: Discretization of the domain

- The next step in finite element method is to discretize the domain.
- For 1D problems,
  - Divide into N (not necessarily) equal subintervals.
    $a = x_0 < x_1 < \cdots < x_N = b$
  - For simplicity, consider equal subintervals. Then,
    Step size of each *element* $h_e = \frac{b-a}{N}$
    Number of *nodes* $= N + 1$

Step 3: Deriving element equations

- **Galerkin Finite Element Model**
  Substitute $w = \psi_i^e$ and $u = \sum_{i=1}^{n} u_j^e \psi_j^e$ in the weak form.

  where,
  n is the *local degrees of freedom* i.e, number of nodes in each element.
  ψ_i^e are the interpolating functions corresponding to the element e.
- The element equations corresponding to the *Galerkin Finite Element model* are thus given by,

Element Equations (Cont'd)

$$
\begin{aligned}
\sum_{j=1}^{n} K_{ij}^e u_j^e &= f_i^e + Q_i^e \qquad i = 1, 2 \ldots n \\
K_{ij}^e &= \int_a^b \left(-\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} + \frac{p(x)}{q(x)}\psi_i^e \frac{d\psi_j^e}{dx} + \frac{r(x)}{p(x)}\psi_i^e \psi_j^e \right) dx \\
f_i^e &= \int_a^b \psi_i^e \frac{f(x)}{p(x)}\, dx
\end{aligned}
$$

and Q_i^e denotes the secondary variable matrix.

Step 4 : Assembly

- **Connectivity matrix**
  - $B = [b_{ij}]$
    b_{ij} : Global Node number corresponding to the j^{th} node of element i
  - Example: If

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ \vdots & \vdots \end{bmatrix} \tag{3}$$

  Then $K_{11} = K^1_{11}, \quad K_{12} = K^1_{12}, \quad K_{22} = K^1_{22} + K^2_{11}$ and so on

The final assembled matrices for the model using linear interpolation functions is given by

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & \dots & 0 \\ 0 & K_{22}^1 + K_{11}^2 & K_{12}^2 & \dots & 0 \\ 0 & K_{21}^2 & K_{22}^2 + K_{11}^3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & K_{22}^N \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_N \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ \vdots \\ f_2^N \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ \vdots \\ Q_2^N \end{Bmatrix}$$

Step 5 : Deriving solution

- If any of the $U_i = 0$, then remove the corresponding row and column i.
- If any of the U_i is known but non-zero, then set $K_{ii} = 1$ and $RHS\{i\} = U_i$
- If the derivative of the function is specified at the end points, set the value in $\{Q\}$ appropriately.

Step 6 : Postprocessing

- The last and final procedure in FEM.
- Involves finding out the secondary variables using the nodal values.
- In engineering applications, common post processing involves finding out forces, pressure etc.

Time Dependent Problems

Introduction

- Some important one dimensional time dependent problems includes,
  - One dimensional heat conduction in a rod.
  - Propogation of wave in a string.
- Coupled system vs Decoupled system:
  - In coupled system we seek an approximate solution to the problem of the form, (time as another coordinate)

$$u(x,t) \approx u_h(x,t) = \sum_{j=1}^{n} u_j^e \psi_j^e(x,t)$$

  - In a decoupled system, we seek an approximate solution to the problem of the form,

$$u(x,t) \approx u_h(x,t) = \sum_{j=1}^{n} u_j^e(t) \psi_j^e(x)$$

Idea

The idea is to use the finite element method approach to discretize the *space coordinate* x and use methods like Crank Nicolson Method to develop an iterative scheme to find solutions at different time t on a particular node.

The Problem

Consider the following boundary value problem,

$$-\frac{\partial^2 u}{\partial x^2} + c^2 \frac{\partial u}{\partial t} = f(x, t)$$

subject to boundary conditions say,

$$u(0, t) = 0, \quad u(1, t) = 0 \tag{4}$$

and initial condtion say,

$$u(x, 0) = 1 \tag{5}$$

Semi-discrete form

Using the decoupled form of approximation and using the Galerkin finite element model, we have

$$0 = \sum_{j=0}^{n} \int_0^1 \left(c^2 \psi_i \psi_j \frac{du_j(t)}{dt} + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} u_j(t) - f\ \psi_i^e \right) dx - Q_i^e \tag{6}$$

This is known as the **semi-discrete form** as the problem is discretized only w.r.t x.

Fully Discrete Form

- The α *family of approximation* is a method of approximating derivatives that are obtained by taking a weighted mean of time derivatives at two consecutive time steps.

$$(1-\alpha)\dot{u}_s + (\alpha)\dot{u}_{s+1} = \frac{u_{s+1} - u_s}{\Delta t_{s+1}} \tag{7}$$

- For matrices we have,

$$\Delta t\,[(1-\alpha)\{\dot{u}\}_s + \alpha\{\dot{u}\}_{s+1}] = \{u\}_{s+1} - \{u\}_s \tag{8}$$

where $\{u\}_s$ is the value of $\{u\}$ at the nodes(local), at time $t = t_s$

Fully Discrete form (Cont'd)

Writing the semi-discrete form at time $t = t_s$,

$$[M^e]\{\dot{u}\}_s + [K^e]\{u\}_s = \{F\} \tag{9}$$

$$M^e = \int_0^1 \left(c^2\psi_i\psi_j\right) dx$$

$$K^e = \int_0^1 \left(\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\right) dx$$

$$F^e = \int_0^1 \psi_i(x) f \quad dx$$

With (8) and (9) at $t = t_s$ and $t = t_{s+1}$, we obtain the fully discrete form for one element,

Fully Discrete form (Cont'd)

$$\hat{K}_{s+1}\{u\}_{s+1} = \bar{K}_s\{u\}_s + \bar{F}_{s,s+1} \tag{10}$$

where,

$$\begin{aligned}
\hat{K_{s+1}} &= M + \alpha\Delta t K_{s+1} \\
\bar{K}_s &= M - (1-\alpha)\,\Delta t K_s \\
\bar{F}_{s,s+1} &= \Delta t\,[\alpha F_{s+1} + (1-\alpha) F_s]
\end{aligned}$$

Single variable 2D Problems

Introduction

- Analysis more complicated than 1D Problems.
- Two types of errors:
  - Approximation error
  - Discretization error
- Basic steps remain the same as that of 1D problems
- Poisson equation, Laplace Equation etc.

The problem

Consider the following 2D problem,

$$-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f \quad in\,\Omega$$

where Ω a square region given by

$$\Omega=\{(x,y)/-A<x<A,\ -A<y<A\}$$

and a boundary condition,

$$u=0 \text{ in the boundary } \Gamma$$

Discretization of the domain

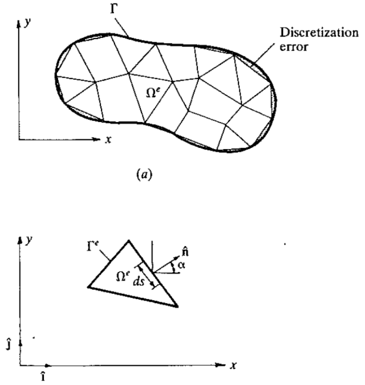


Figure: Discretization of a 2D Domain

Weak form

- Integrating the equation over the element Ω_e
- The weak form,

$$\int_{\Omega_e} \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} - wf \right) dxdy - \int_{\Gamma_e} \left(w \left[n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y} \right] \right) ds = 0$$

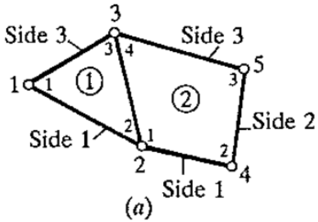
$$q_n \equiv \left[n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y} \right]$$

Element Equations

- Using the *Galerkin Finite element model*, $w = \psi_j^e$ and $u = \sum_{j=1}^{n} u_j^e \psi_j^e$
- The element equations,

$$0 = \sum_{j=1}^{N} \left[\int_{\Omega_e} \left(\frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} + \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} - \psi_i^e f \right) dxdy \right] - \int_{\Gamma_e} (\psi_i^e q_n ds) \tag{11}$$

Assembly



Global →	Local
K_{11}	K_{11}^{1}
K_{12}	K_{12}^{1}
K_{22}	$K_{22}^{1} + K_{11}^{2}$
K_{14}	0
K_{15}	0
K_{23}	$K_{23}^{1} + K_{14}^{2}$

Figure: Assembling the elements

Further Study

- NonLinear Finite Element Method.
- Uniqueness of the Finite element solution.
- Methods other than the Galerkin Method.

THANK YOU