Unit-3 : First order and First degree O.D.E & its application

Definition - An equation of the form $\frac{dy}{dx} = f(x,y)$ is called a diff eqn of first order and of first degree. In general first order diff eqns can be classified as below :-

i) variable separable type

ii) Homogenous eqns & eqns reducible to homogenous eqns

iii) exact and non-exact diff eqns

iv) linear & bernoulli's type diff eqns

Linear differential eqns of first order :

The eqns of the form $\frac{dy}{dx} + P(x)y = Q(x)$ is called a "linear diff eqn of 1st order in 'y'"

The general solution is given by

$$y \times I.F = \int (I.F)\, Q(x)\, dx + c$$

$$\text{where } \quad I.F = e^{\int P(x) dx}$$

Another form - $\frac{dx}{dy} + P(y)x = Q(y)$. the general soln is $x \times I.F = \int (I.F) Q(y)\, dy + c$ where

$$I.F = e^{\int P(y) dy}$$

Problems :

Q) solve $(1-x^2)\frac{dy}{dx} + xy = ax$

Sol :

$$\frac{dy}{dx} + \frac{x}{1-x^2} y = \frac{ax}{1-x^2} \quad \text{where } P(x) = \frac{x}{1-x^2}, \quad Q(x) = \frac{ax}{1-x^2}$$

The general soln is :

$$y \times (I.F) = \int (I.F)\, Q(x)\, dx + c$$

$$I.F = e^{\int P(x) dx} = e^{\int \frac{x}{1-x^2}} = e^{\frac{-1}{2}\int \frac{-2x}{1-x^2} dx}$$

$$= e^{\frac{-1}{2}\log(1-x^2)} \quad \because \left(\int \frac{f'(x)}{f(x)} dx = \log x \right)$$

$$= (1-x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x^2}}$$

General soln $\Rightarrow$ $y\left(\frac{1}{\sqrt{1-x^2}}\right) = \int \frac{1}{\sqrt{1-x^2}}\left(\frac{ax}{1-x^2}\right) dx + C$

$\Rightarrow \frac{y}{\sqrt{1-x^2}} = \int \frac{ax}{(1-x^2)^{3/2}} dx + C$

Put $1-x^2 = t \Rightarrow -2x\,dx = dt \Rightarrow x\,dx = -dt/2$

$\Rightarrow \frac{y}{\sqrt{1-x^2}} = a\int \frac{-dt}{2 \times t^{3/2}} + C$

$\Rightarrow \frac{y}{\sqrt{1-x^2}} = \frac{-1}{2} a\int t^{-3/2} dt + C$

$\Rightarrow \frac{y}{\sqrt{1-x^2}} = a t^{-1/2} + C$ $\Rightarrow \frac{y}{\sqrt{1-x^2}} = \frac{a}{\sqrt{1-x^2}} + C$

$\Rightarrow \boxed{y = a + C\sqrt{1-x^2}}$

Q) solve $(x+y+1)\frac{dy}{dx} = 1$

Sol : $\frac{dx}{dy} = x+y+1 \Rightarrow \frac{dx}{dy} - x = y+1$

$\Rightarrow \frac{dx}{dy} + P(y)\,x = Q(y)$

where $P(y) = -1$, $Q(y) = y+1$

$I.F = e^{\int P(y)dy} = e^{\int(-1)dy} = e^{-y}$

The general soln is $x(I.F) = \int Q(y)(I.F)\,dy + C$

$\Rightarrow xe^{-y} = \int \underset{u}{(y+1)}\underset{v}{e^{-y}} dy + C$ $\quad \because \left(\int uv = u\int v - \int \frac{du}{dx}\int v\right)dx$

$\Rightarrow xe^{-y} = (y+1)\left(\frac{e^{-y}}{-1}\right) - \int (1)\cdot\frac{e^{-y}}{(-1)} dy + C$

$\Rightarrow xe^{-y} = -e^{-y}(y+1) + \int e^{-y} dy + C$

$\Rightarrow xe^{-y} = -e^{-y}(y+2) + C$

$\Rightarrow \boxed{x = -(y+2) + ce^{y}}$

a) Solve $x\cos x \frac{dy}{dx} + (x\sin x + \cos x)\, y = 1$

Sol: $\frac{dy}{dx} + \left(\frac{x\sin x + \cos x}{x\cos x}\right) y = \frac{1}{x\cos x}$

where $P(x) = \frac{x\sin x + \cos x}{x\cos x}$, $Q(x) = \frac{1}{x\cos x}$

$I \cdot F = e^{\int P(x)\,dx} = e^{\int \frac{x\sin x + \cos x}{x\cos x}\,dx}$

$= e^{\int (\tan x + \frac{1}{x})\,dx} = e^{\log(x\sec x)} = x\sec x$

The general soln is

$y(x\sec x) = \int \frac{x\sec x}{x\cos x}\,dx + c$

$y(x\sec x) = \int \sec^2 x\,dx + c$

$\boxed{y(x\sec x) = \tan x + c}$

9) Solve $(1+y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$

Sol: $(1+y^2)\frac{dx}{dy} + x - e^{\tan^{-1}y} = 0$

$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{e^{\tan^{-1}y}}{1+y^2}$ where $P(y) = \frac{1}{1+y^2}$ & $Q(y) = \frac{e^{\tan^{-1}y}}{1+y^2}$

$I \cdot F = e^{\int P(y)\,dy} = e^{\int \frac{1}{1+y^2}\,dy} = e^{\tan^{-1}y}$

The general soln is

$x \cdot e^{\tan^{-1}y} = \int \frac{e^{2\tan^{-1}y}}{1+y^2}\,dy + c$

Put $\tan^{-1}y = t$

$\frac{1}{1+y^2}\,dy = dt$

$\Rightarrow x \cdot e^{\tan^{-1}y} = \int e^{2t}\,dt + c$

$$\Rightarrow x e^{\tan^{-1} y} = \frac{e^{2t}}{2} + c \Rightarrow x e^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2} + c$$

Q) $x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2$

Sol:

$$\frac{dy}{dx} - \frac{1}{x(x-1)} y = x(x-1)$$

Put $p(x) = \frac{-1}{x(x-1)}$ and $Q(x) = x(x-1)$

$$I.F = e^{\int p(x)\,dx} = e^{\int \frac{-1}{x(x-1)}\,dx}$$

$$= e^{\int \left(\frac{1}{x} - \frac{1}{x-1}\right) dx} = e^{\int \log\left(\frac{x}{x-1}\right) dx} = \frac{x}{x-1}$$

The general solution is

$$y\left(\frac{x}{x-1}\right) = \int x(x-1) \cdot \frac{x\,dx}{x-1} + c$$

$$\frac{xy}{x-1} = \int x^2\,dx + c$$

$$\boxed{\frac{xy}{x-1} = \frac{x^3}{3} + c}$$

a) $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\sin 2x}{\log x}$.

Sol:

$p(x) = \frac{1}{x\log x}$, $Q(x) = \frac{\sin 2x}{\log x}$

$$I.F = e^{\int p(x)\,dx} = e^{\int \frac{1}{x\log x}\,dx} = e^{\log(\log x)} = \log x$$

The general solution is

$$y(\log x) = \int \frac{\sin 2x}{\log x} (\log x)\,dx + c$$

$$y(\log x) = \frac{-\cos 2x}{2} + c$$

$$\boxed{y\log x + \frac{\cos 2x}{2} = c}$$

Q) $(1+y^2)dx = (\tan^{-1} y - x)dy$

(Ans : $x e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$

Q) $\frac{dy}{dx} + (y-1)\cos x = e^{-\sin x}\cos^{-1} x$

(A : $y e^{\sin x} = \frac{x}{2} + \frac{\sin 2x}{4} + e^{\sin x} + c$)

Bernoulli's diff eqn ;

Eqn of the form $\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$ where n is a real constant

Another form - $\frac{dx}{dy} + P(y)\,x = Q(y)\,x^n$

Q) Solve $x\frac{dy}{dx} + y = x^3 y^6$

Sol:

$$\frac{dy}{dx} + \frac{y}{x} = x^2 y^6 \text{ --- } (1)$$

$\Rightarrow$ ~~where~~ $P(x) = \frac{1}{x}$, $Q(x) = x^2$, $n = 6$

multiplying (1) with y^{-6}

$$\therefore (1) \Rightarrow \frac{1}{y^6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2$$

Put $y^{-5} = t$

$$\Rightarrow -5y^{-6}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{y^6}\frac{dy}{dx} = \frac{-1}{5}\cdot\frac{dt}{dx}$$

$$\therefore -\frac{1}{5}\cdot\frac{dt}{dx} + \frac{1}{x}t = x^2$$

$$\Rightarrow \frac{dt}{dx} + \left(\frac{-5}{x}\right)t = -5x^2 \longrightarrow (2)$$ is linear where

$$P(x) = \frac{-5}{x} \ \& \ Q(x) = -5x^2$$

$$I{\cdot}F = e^{\int P(x)\,dx} = e^{-5\int \frac{1}{x}dx} = e^{-5\log x} = x^{-5}$$

The general solution is

$$t\left(\frac{1}{x^5}\right) = \int -5x^2\left(\frac{1}{x^5}\right)dx + c$$

$$\Rightarrow \frac{t}{x^5} = -5\int x^{-3}\,dx + c$$

$$\Rightarrow \frac{t}{x^5} = (-5)\cdot\frac{x^{-2}}{(-2)} + c \Rightarrow \frac{t}{x^5} = \frac{5}{2x^2} + c$$

$$\Rightarrow t = \frac{5}{2}x^3 + cx^5$$

$$\Rightarrow \boxed{y^{-5} = \frac{5x^3}{2} + cx^5}$$

Q) Solve $\frac{dy}{dx}(x^2y^3 + xy) = 1$

Sol: $\frac{dy}{dx} = \frac{1}{x^2y^3 + xy} \Rightarrow \frac{dx}{dy} = x^2y^3 + xy$

$$\Rightarrow \frac{dx}{dy} - xy = x^2y^3 \quad \left(\because \frac{dx}{dy} + P(y)x = Q(y)x^n\right)$$

$$\Rightarrow \frac{1}{x^2}\frac{dx}{dy} - \frac{y}{x} = y^3$$

Put $\frac{1}{x} = t$

$$\Rightarrow \frac{-1}{x^2}\frac{dx}{dy} = \frac{dt}{dy}$$

$$\Rightarrow -\frac{dt}{dy} - yt = y^3$$

$$\Rightarrow \frac{dt}{dy} + yt = -y^3 \text{ is linear where}$$

$P(y) = y$, $Q(y) = -y^3$

$$I.F = e^{\int P dy} = e^{\int y\, dy} = e^{\frac{y^2}{2}}$$

The general soln is

$$t \cdot e^{\frac{y^2}{2}} = -\int e^{\frac{y^2}{2}} y^3\, dy + c$$

$\longrightarrow$ Put $\frac{y^2}{2} = t \Rightarrow -\int e^t (2t)\, dt$

$\Rightarrow \frac{2y}{\cancel{2}} dy = dt$

$-2[te^t - e^t]$

$$t \cdot e^{\frac{y^2}{2}} = -2\left(\frac{y^2}{2} - 1\right)e^{\frac{y^2}{2}} + c$$

$$t \cdot e^{\frac{y^2}{2}} = (2 - y^2) \cdot e^{\frac{y^2}{2}} + c$$

$$\Rightarrow \frac{e^{\frac{y^2}{2}}}{x} = (2 - y^2) \cdot e^{\frac{y^2}{2}} + c$$

$$\Rightarrow e^{\frac{y^2}{2}} = (2 - y^2)e^{\frac{y^2}{2}} \cdot x + cx$$

$$\Rightarrow \boxed{1 = (2 - y^2) \cdot x + cx \cdot e^{-\frac{y^2}{2}}}$$

Q) Solve $\frac{dy}{dx} + y\cos x = y^2 \sin 2x$ [Ans: $\frac{1}{y} = 1 + 2\sin x + ce^{\sin x}$]

5) Solve $\frac{dy}{dx} + \frac{y}{x-1} = xy^{1/3}$ [Ans: $y^{2/3} = (x-1)\left[\frac{2x}{5} - \frac{3}{20}(x-1) + c(x-1)^{-5/3}\right]$]

5) Solve $(1-x^2)\frac{dy}{dx} + xy = y^3\sin^{-1}x$ [Ans: $\frac{1-x^2}{y^2} = -2(x\sin^{-1}x + \sqrt{1-x^2}) + c$]

6) Solve $y' - y\tan x = \frac{\sin x\cos^2 x}{y^2}$ [Ans: $2y^3 = -\cos^3 x + 2c\sec^3 x$]

4) $\frac{dy}{dx} + y\tan x = y^2\sec x$ (Ans: $\cos x / y = -x + c$

Q) $3\frac{dy}{dx} - y\cos x = y^4(\sin 2x - \cos x)$ (Ans: $\frac{e^{\sin x}}{y^3} = e^{\sin x}(3-2\sin x) + c$

Solutions:

1. $\frac{1}{y^3}\cdot\frac{dy}{dx} + \frac{1}{y^2}\cos x = \sin 2x$

$\Rightarrow$ Put $\frac{1}{y^2} = t$

$-2y^{-3}\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dt}{dx}$

$-\frac{1}{2}\cdot\frac{dt}{dx} + t\cos x = \sin 2x$

$\frac{dt}{dx} - 2t\cos x = -2\sin 2x$ is linear in t

$P(x) = -2\cos x$, $Q(x) = -2\sin 2x$

$I.F = e^{\int p\,dx} = e^{-2\int \cos x\,dx} = e^{-2\sin x}$

The general solution is $t(e^{-2\sin x}) = \int (-2\sin 2x)(e^{-2\sin x})\,dx + c$

$\Rightarrow y^{-2}\cdot e^{-2\sin x} = -4\int \sin x\cos x\,(e^{-2\sin x})\,dx + c$

put $\sin x = k$ & $-2k = u$

$\cos x\,dx = dk$ $\Rightarrow -2\,dk = du$

$\Rightarrow y^{-2}\cdot e^{-2\sin x} = -4\int e^{-2k}\cdot k\,dk + c$

$\Rightarrow y^{-2}\cdot e^{-2\sin x} = -4\int e^{u}\cdot\left(\frac{-u}{2}\right)\left(\frac{-du}{2}\right) + c$

$\Rightarrow y^{-2}\cdot e^{-2\sin x} = -\int e^{u}\cdot u + c$

$\Rightarrow y^{-2}\cdot e^{-2\sin x} = -\left[u\cdot e^{u} - \int e^{u}du\right] + c$

$\Rightarrow y^{-2}\cdot e^{-2\sin x} = 2ke^{-2k} + e^{-2k} + c$

$$\Rightarrow y^{-2}\cdot e^{-2\sin x} = 2\sin x\cdot e^{-2\sin x} + e^{-2\sin x} + c$$

$$\Rightarrow \frac{1}{y^2} = 1 + 2\sin x + c e^{2\sin x}$$

2. $$\frac{1}{y^{1/3}}\cdot\frac{dy}{dx} + \frac{y^{2/3}}{x-1} = x$$

put $y^{2/3} = t$

$$\Rightarrow \frac{2}{3} y^{-1/3}\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{1}{y^{1/3}}\cdot\frac{dy}{dx} = \frac{3}{2}\frac{dt}{dx}$$

$$\Rightarrow \frac{3}{2}\cdot\frac{dt}{dx} + \frac{t}{x-1} = x$$

$$\Rightarrow \frac{dt}{dx} + \frac{2}{3}\cdot\frac{t}{x-1} = \frac{2}{3}x \quad \text{where}$$

$$P(x) = \frac{2}{3(x-1)} \quad \text{and} \quad Q(x) = \frac{2x}{3}$$

$$I.F = e^{\int P dx} = e^{\int \frac{2}{3(x-1)}dx} = e^{\frac{2}{3}\ln(x-1)} = (x-1)^{2/3}$$

The general soln is $$t(x-1)^{2/3} = \int \frac{2x}{3}\times(x-1)^{2/3}dx + c$$

$$y^{2/3}(x-1)^{2/3} = \frac{2}{3}\int \underset{\mathrm{I}}{x}\cdot \underset{\mathrm{II}}{(x-1)^{2/3}}dx + c$$

$$y^{2/3}(x-1)^{2/3} = \frac{2}{3}\left[x\cdot\frac{(x-1)^{5/3}}{5/3} - \frac{3}{5}\int(x-1)^{5/3}dx\right] + c$$

$$y^{2/3}(x-1)^{2/3} = \frac{2}{5}x(x-1)^{5/3} - \frac{3}{20}(x-1)^{8/3} + c$$

$$y^{2/3} = \frac{2}{5}x\cdot(x-1) - \frac{3}{20}(x-1)^2 + c(x-1)^{-2/3}$$

$$y^{2/3} = (x-1)\left[\frac{2x}{5} - \frac{3}{20}(x-1) + c(x-1)^{-5/3}\right]$$

3. $\frac{dy}{dx} + \frac{x}{1-x^2} y = \frac{y^3}{1-x^2} \sin^{-1}x$

$\frac{1}{y^3} \cdot \frac{dy}{dx} + \frac{1}{y^2} \cdot \frac{x}{1-x^2} = \frac{\sin^{-1}x}{1-x^2}$

Put $\frac{1}{y^2} = t$

$-\frac{2}{y^3} \cdot \frac{dy}{dx} = \frac{dt}{dx}$

$-\frac{1}{2} \cdot \frac{dt}{dx} + \frac{x}{1-x^2} t = \frac{\sin^{-1}x}{1-x^2}$

$\frac{dt}{dx} - \frac{2x}{1-x^2} t = \frac{-2\sin^{-1}x}{1-x^2}$ where

$P(x) = \frac{-2x}{1-x^2}$, $Q(x) = \frac{-2\sin^{-1}x}{1-x^2}$

$I.F = e^{\int P(x)dx} = e^{\log(1-x^2)} = 1-x^2$

The general solution is

$t(1-x^2) = \int \frac{-2\sin^{-1}x}{1-x^2}(1-x^2) + c$

$y^{-2}(1-x^2) = -2\int \underset{u}{\sin^{-1}x} \underset{v}{(1)} + c$

$y^{-2}(1-x^2) = -2\left(x\sin^{-1}x + \sqrt{1-x^2}\right) + c$

4. $\frac{dy}{dx} - \tan x \cdot y = \frac{\sin x \cdot \cos^2 x}{y^2}$

$y^2 \frac{dy}{dx} - y^3 \tan x = \sin x \cdot \cos^2 x$

Put $y^3 = t$

$3y^2 \frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \frac{1}{3} \frac{dt}{dx} - t \cdot \tan x = \sin x \cdot \cos^2 x$

$\Rightarrow \frac{dt}{dx} - (3\tan x) \cdot t = 3\sin x \cdot \cos^2 x$ where

$P(x) = -3\tan x$, $Q(x) = 3\sin x \cdot \cos^2 x$

$I.F = e^{\int P(x)\,dx} = e^{-3\int \tan x\,dx} = e^{-3\log|\sec x|} = (\cos x)^{3} = \cos^3 x$ $\cos^3 x$

The general soln is $t(\cos^3 x) = \int (3\sin x \cos^2 x) \cdot \cos^3 x \, dx + c$

$$y^3 \cdot \cos^3 x = 3\int \sin x \cos^5 x \, dx + C$$

Put $\cos x = u$

$-\sin x \, dx = du$

$$y^3 \cdot \cos^3 x = -3\int u^5 \cdot du + c$$

$$y^3 \cdot \cos^3 x = -3\left(\frac{\cos^6 x}{6}\right) + c$$

$$y^3 = -\frac{\cos^3 x}{2} + \frac{c}{\cos^3 x}$$

$$\Rightarrow 2y^3 = -\cos^3 x + 2c\sec^3 x$$

Exact differential eqns

Definition - Let $M dx + N dy = 0$ be a first order and first degree diff eqn, where M & N are real valued fns for some x, y. Then the eqn $M dx + N dy = 0$ is said to be exact D.E if $\exists$ a fn f such that

$$\frac{\partial f}{\partial x} = M \quad \& \quad \frac{\partial f}{\partial y} = N$$

Condn for exactness If M & N are 2 real valued fns which have continuous partial derivatives then a necessary & suff condn for the D.E $M dx + N dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Procedure to solve exact D.E

Step-1: Check the condn for exactness i.e $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Step-2: If $M dx + N dy = 0$ is exact then the general soln of exact diff eqn is given by

$$\underbrace{\int M dx}_{\text{Put } y \text{ as constant}} + \underbrace{\int N dy}_{\text{terms without } x \text{ should be taken}} = c$$

Problems:

1. Solve $(y^2-2xy)\,dx = (x^2-2xy)\,dy$

Sol: $M = y^2-2xy$ & $N = -(x^2-2xy)$

$$\frac{\partial M}{\partial y} = 2y-2x \;,\quad \frac{\partial N}{\partial x} = -2x+2y \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ i.e given diff eqn is exact.

The general soln of the given D.E is

$$\int M\,dx + \int N\,dy = c$$

$$\int (y^2-2xy)\,dx + \int 0\,dy = c$$

$$xy^2 - 2y\left(\frac{x^2}{2}\right) = c \Rightarrow \boxed{c = xy^2 - x^2y}$$

2. Solve $(1+e^{x/y})\,dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$

Sol: $M = 1+e^{\frac{x}{y}}$, $N = e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$

$$\frac{\partial M}{\partial y} = 0 + e^{\frac{x}{y}}\cdot\left(\frac{-x}{y^2}\right) = \frac{-x}{y^2}e^{\frac{x}{y}}$$

$$\frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{1}{y}\right) - \left[\frac{x}{y}e^{\frac{x}{y}}\left(\frac{1}{y}\right) + \left(\frac{1}{y}\right)e^{\frac{x}{y}}\right] = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right)$$. The given diff eqn is exact.

The general soln of the given D.E is

$$\int M\,dx + \int N\,dy = c$$

$$\int 1+e^{\frac{x}{y}}\,dx + \int 0\,dy = c$$

$$x + e^{\frac{x}{y}}\Big/\left(\frac{1}{y}\right) = c \Rightarrow c = \frac{e^{\frac{x}{y}}}{1/y} + x \Rightarrow \boxed{c = y\cdot e^{\frac{x}{y}} + x}$$

3. Solve $(xy\cos xy + \sin xy)\,dx + (x^2\cos(xy))\,dy = 0$

Sol: $M = xy\cos(xy) + \sin xy$, $N = x^2\cos xy$

$$\frac{\partial M}{\partial y} = x\left[y(-\sin xy)(x) + \cos xy\right] + x\cos(xy) = 2x\cos xy - x^2y\sin xy$$

$$\frac{\partial N}{\partial x} = x^2(-\sin xy)(y) + 2x\cos xy = 2x\cos xy - x^2 y\sin xy$$

$$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \text{ — exact}$$

The general soln is $\int M\,dx + \int N\,dy = C$

$$\int (xy\cos xy + \sin xy)\,dx + \int 0\,dy = C$$

$$\int d(x\sin xy)\,dx = C$$

$$\therefore \boxed{C = x\sin xy}$$

4. Solve $(xe^{xy} + 2y)\frac{dy}{dx} + ye^{xy} = 0$ (Ans: $e^{xy} + y^2 = c$)

5. Solve $\left[y\left(1+\frac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\sin y)\,dy = 0$ (Ans: $y(x + \log x) + x\cos y = c$)

6. Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$ (Ans: $y\sin x + x\sin y + xy = c$)

7. Solve $(hx + by + f)\,dy + (ax + hy + g)\,dx = 0$ (Ans: y sin

8. Solve $(r + \sin\theta - \cos\theta)\,dr + r(\sin\theta + \cos\theta)\,d\theta = 0$

SOLUTIONS

4. $M = y\cdot e^{xy}$, $N = xe^{xy} + 2y$

$$\frac{\partial M}{\partial y} = e^{xy} + xy\cdot e^{xy}, \quad \frac{\partial N}{\partial x} = [e^{xy} + xye^{xy}] + 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ - exact}$$

The general soln is

$$\int y\cdot e^{xy}\,dx + \int 2y\,dy = C$$

$$y\cdot\frac{e^{xy}}{y} + 2\left(\frac{y^2}{2}\right) = C$$

$$C = e^{xy} + y^2$$

5. $M = y + \frac{y}{x} + \cos y \quad , \quad N = x + \log x - x\sin y$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{- exact}$$

The general soln is

$$\int \left(y + \frac{y}{x} + \cos y\right) dx + \int c \, dy = c$$

$$\therefore \ yx + y\log x + x\cos y = c$$

6. $$\frac{dy}{dx} = -\left(\frac{y\cos x + \sin y + y}{\sin x + x\cos y + x}\right)$$

$$(\sin x + x\cos y + x)\,dy + (y\cos x + \sin y + y)\,dx = 0$$

$$M = y\cos x + \sin y + y \quad , \quad N = \sin x + x\cos y + x$$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1 \quad , \quad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{- exact}$$

The general soln is

$$\int (y\cos x + \sin y + y)\,dx + \int 0\,dy = c$$

$$y\sin x + (y + \sin y)\,x = c$$

7. $M = ax + hy + g \quad , \quad N = hx + by + f$

$$\frac{\partial M}{\partial y} = 0 + h + 0 = h \quad , \quad \frac{\partial N}{\partial x} = h + 0 + 0 = h \quad \text{- exact}$$

The general soln is

$$\int(ax+hy+g)\,dx + \int(\cancel{a}+by+f)\,dy = c$$

$$a\left(\frac{x^2}{2}\right) + hxy + gx + b\left(\frac{y^2}{2}\right) + fy = c$$

$$\boxed{ax^2+by^2+2hx+2gx+2fy-2c=0}$$

8. $\boxed{M\,d\theta + N\,dr = 0}$

$M = r(\sin\theta+\cos\theta)$, $N = r+\sin\theta-\cos\theta$

$\frac{\partial M}{\partial r} = \sin\theta+\cos\theta$, $\frac{\partial N}{\partial \theta} = 0+\cos\theta-(-\sin\theta) = \sin\theta+\cos\theta$

$\boxed{\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}}$ — exact

The general soln is : $\int M\,d\theta + \int N\,dr = c$

(r→constant) (don't contain θ)

$\Rightarrow r\int(\sin\theta+\cos\theta)\,d\theta + \int r\,dr = c$

$\Rightarrow r(\cos\theta+\sin\theta) + \frac{r^2}{2} = c$

$\Rightarrow \boxed{2r(\sin\theta-\cos\theta)+r^2 = 2c}$ is the required soln

Non-exact Diff eqn -

Defn - If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then $M\,dx + N\,dy = 0$ is called a non-exact diff eqn

Integrating factor - It is a fn which is used to made a non-exact diff eqn to an exact diff eqn.

Let $M\,dx + N\,dy = 0$ be a non-exact diff eqn. It can be made exact by multiplying with suitable I.F i.e $u(x,y) \neq 0$

NOTE - An I.F of a differencial eqn isn't unique ~~but~~ if it exists.

Methods to find I F:

Method-I : If the D.E $Mdx + Ndy = 0$ is a homogenous diff eqn and $Mx + Ny \neq 0$

then I.F $\frac{1}{Mx + Ny}$

Method-II : If the D.E $Mdx + Ndy = 0$ is of the form $y \cdot f(xy)\,dx + x\,g(xy)\,dy = 0$

and $Mx - Ny \neq 0$ then $I \cdot F = \frac{1}{Mx - Ny}$

Problems:

1. Solve $x^2y\,dx - (x^3 + y^3)\,dy = 0$

Sol: $M = x^2y$, $N = -(x^3 + y^3)$

$\frac{\partial M}{\partial y} = x^2$, $\frac{\partial N}{\partial x} = -3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ — non-exact D.E

The given D.E is homogenous $\Rightarrow$ $IF = \frac{1}{Mx + Ny} = \frac{1}{(x^2y)x + (-x^3 - y^3)y}$

$\Rightarrow$ $\boxed{I \cdot F = \frac{-1}{y^4}}$

Multiplying given d.e with $I \cdot F = \frac{-1}{y^4}$, we get

$\Rightarrow$ $-\frac{x^2y}{y^4}\,dx + \left(\frac{x^3 + y^3}{y^4}\right)dy = 0 \longrightarrow (1)$

$\Rightarrow$ $-\frac{x^2}{y^3}\,dx + \left[\frac{x^3}{y^4} + \frac{1}{y}\right]dy = 0$

$M_1 = -\frac{x^2}{y^3}$, $N_1 = \frac{x^3}{y^4} + \frac{1}{y}$

$\Rightarrow$ $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ is exact

The general soln is $\int M_1\,dx + \int N_1\,dy = c$

$\Rightarrow$ $\int -\frac{x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = c$

(y→constant)

$$\Rightarrow \frac{-1}{y^3}\left(\frac{x^3}{3}\right) + \log y = c \Rightarrow \boxed{-x^3 + 3y^3 \log y = 3cy^3}$$

2. Solve $y(1+xy)dx + x(1-xy)dy = 0$

Sol: $M = y + xy^2$, $N = x - x^2y$

$$\frac{\partial M}{\partial y} = 1 + 2xy, \quad \frac{\partial N}{\partial x} = 1 - 2xy$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad - \text{non-exact}$$

The given diff eqn is in the form $y f(xy)dx + x g(xy)dy = 0$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{xy(1+xy) - xy(1-xy)} = \frac{1}{2x^2y^2}$$

∴ multiplying the given d.e with the I.F we get

$$\Rightarrow \frac{y + xy^2}{2x^2y^2} dx + \frac{x - x^2y}{2x^2y^2} dy = 0$$

$$\Rightarrow \left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0 \quad \text{—① is exact}$$

$$M_1 = \frac{1}{2x^2y} + \frac{1}{2x}, \quad N_1 = \frac{1}{2x^2y} - \frac{1}{2y}$$

The general soln is

$$\int_{y \to \text{constant}} \left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \int \left(\frac{-1}{2y}\right)dy = c$$

$$\frac{1}{2y}\left(\frac{-1}{x}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$\log x - \log y - \frac{1}{xy} = 2c$$

$$\log\left(\frac{x}{y}\right) - \frac{1}{xy} = 2c$$

$$xy \log\left(\frac{x}{y}\right) - 1 = 2cxy$$

$$\Rightarrow \boxed{xy\left(\log\left(\frac{x}{y}\right) - 2c\right) = 1}$$

3. $xy\,dx - (x^2+2y^2)\,dy = 0$ $\left(\text{Ans}: \frac{-x^2}{4y^2} + \log y = c\right)$

4. $(xy\sin xy + \cos xy)\,y\,dx + (xy\sin xy - \cos xy)\,x\,dy = 0$ $\left(\text{Ans}: \frac{x\sec xy}{y} = c\right)$

5. $(x^2y^2+xy+1)\,y\,dx + (x^2y^2-xy+1)\,x\,dy = 0$ $\left(\text{Ans}: xy - \frac{1}{xy} + \log\left(\frac{x}{y}\right) = c\right)$

6. $(2xy+1)\,y\,dx + (1+2xy-x^3y^3)\,x\,dy = 0$ $\left(\text{Ans}: \log y + \frac{1}{x^2y^2} + \frac{1}{3x^3y^3} = c\right)$

✓ 7. $y - xy' = x + yy'$ $\left(\text{Ans}: \log\sqrt{x^2+y^2} + \tan^{-1}\left(\frac{x}{y}\right) = c\right)$

✓ 8. $y(y^2-2x^2)\,dx + x(2y^2-x^2)\,dy = 0$ $\left(A: \text{I.F.} = \frac{1}{3xy(y^2-x^2)},\ \log(x^2y^2(y^2-x^2)) = c_1\right.$ (or) $x^2y^2(y^2-x^2) = c_1^*$

✓ 9. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

$\left(A: \frac{x}{y} + \log\left(\frac{y^3}{x^2}\right) = C\right.$

SOLUTIONS

3. $$\text{I.F} = \frac{1}{Mx+Ny} = \frac{1}{x^2y - x^2y - 2y^3} = \frac{-1}{2y^3}$$

$$\left(\frac{-xy}{2y^3}\right)dx + \left(\frac{x^2+2y^2}{2y^3}\right)dy = 0$$

$$\Rightarrow \left(\frac{-x}{2y^2}\right)dx + \left(\frac{x^2}{2y^3} + \frac{1}{y}\right)dy = 0$$

The general soln is $$\int_{y\to\text{constant}} \left(\frac{-x}{2y^2}\right)dx + \int \frac{1}{y}\,dy = c$$

$$\frac{-1}{2y^2}\left(\frac{x^2}{2}\right) + \log y = c$$

$$\frac{-x^2}{4y^2} + \log y = c$$

4. $$\text{I.F} = \frac{1}{Mx - Ny} = \frac{1}{xy(xy\sin xy + \cos xy) - xy(xy\sin xy - \cos xy)} = \frac{1}{2xy\cos xy}$$

$$\left(\frac{xy^2\sin xy + y\cos xy}{2xy\cos xy}\right)dx + \left(\frac{x^2y\sin xy - x\cos xy}{2xy\cos xy}\right)dy = 0$$

$$\left(\frac{y\tan xy}{2} + \frac{1}{2x}\right)dx + \left(\frac{x\tan xy}{2} - \frac{1}{2y}\right)dy = 0 \text{ is exact}$$

The general soln is $$\int_{y\to\text{constant}} \left(\frac{y\tan xy}{2} + \frac{1}{2x}\right)dx + \int\left(\frac{-1}{2y}\right)dy = c$$

$$\frac{\frac{y}{2}\cdot\log|\sec(xy)|}{y}+\frac{1}{2}\log x-\frac{1}{2}\log y=c$$

$$\log|\sec xy|+\log x-\log y=2c$$

$$\log\left|\frac{x\sec xy}{y}\right|=2c$$

$$\therefore\ \frac{x\sec xy}{y}=e^{2c}=k$$

5. $$\text{I.F}=\frac{1}{Mx-Ny}=\frac{1}{xy(x^2y^2+xy+1)-xy(x^2y^2-xy+1)}=\frac{1}{2x^2y^2}$$

$$\left(\frac{x^2y^3+xy^2+y}{2x^2y^2}\right)dx+\left(\frac{x^3y^2-x^2y+x}{2x^2y^2}\right)dy=0$$

$$\left(\frac{y}{2}+\frac{1}{2x}+\frac{1}{2x^2y}\right)dx+\left(\frac{x}{2}-\frac{1}{2y}+\frac{1}{2xy^2}\right)dy=0 \text{ is exact diff eqn}$$

The general soln is

$$\int_{y\to\text{constant}}\left(\frac{y}{2}+\frac{1}{2x}+\frac{1}{2x^2y}\right)dx+\int\left(\frac{-1}{2y}\right)dy=c$$

$$\frac{xy}{2}+\frac{1}{2}\log x+\frac{1}{2y}\left(\frac{-1}{x}\right)-\frac{1}{2}\log y=c$$

$$\frac{xy}{2}-\frac{1}{2xy}+\frac{1}{2}\log\left(\frac{x}{y}\right)=c$$

$$xy-\frac{1}{xy}+\log\left(\frac{x}{y}\right)=2c=k$$

6. $$\text{I.F}=\frac{1}{Mx-Ny}=\frac{1}{xy(2xy+1)-xy(2xy+1-x^3y^3)}=\frac{1}{x^4y^4}$$

$$\left(\frac{2xy^2+y}{x^4y^4}\right)dx+\left(\frac{2x^2y+x-x^4y^3}{x^4y^4}\right)dy=0$$

$$\Rightarrow\left(\frac{2}{x^3y^2}+\frac{1}{x^4y^3}\right)dx+\left(\frac{2}{x^2y^3}+\frac{1}{x^3y^4}-\frac{1}{y}\right)dy=0 \text{ is exact}$$

The general soln is

$$\int_{y\to constant} \left(\frac{2}{x^3y^2}+\frac{1}{x^4y^3}\right)dx + \int\left(\frac{-1}{y}\right)dy = c$$

$$\frac{2}{y^2}\left(\frac{-1}{2x^2}\right)+\frac{1}{y^3}\left(\frac{-1}{3x^3}\right)-\log y = c$$

$$\log y + \frac{1}{x^2y^2}+\frac{1}{3x^3y^3} = -c = k$$

7. $y - x\frac{dy}{dx} = x + y\cdot\frac{dy}{dx}$

$$(x+y)\frac{dy}{dx} = (y-x)$$

$$(x+y)dy = -(x-y)dx \Rightarrow (x+y)dy + (x-y)dx = 0$$

Homogenous eqn

$$y = Vx$$

$$\Rightarrow \frac{dy}{dx} = x\cdot\frac{dv}{dx}+V$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x-y)}{x+y}$$

$$\Rightarrow V + x\cdot\frac{dv}{dx} = -\left(\frac{x-Vx}{x+Vx}\right)$$

$$\Rightarrow V + x\cdot\frac{dv}{dx} = \frac{V-1}{V+1} \Rightarrow x\cdot\frac{dv}{dx} = \frac{V-1}{V+1} - V$$

$$\Rightarrow x\cdot\frac{dv}{dx} = -\left(\frac{1+V^2}{1+V}\right)$$

$$\Rightarrow \int\left(\frac{1+V}{1+V^2}\right)dV = \int\frac{-dx}{x}$$

$$\Rightarrow \int\frac{1}{V^2+1}dv + \int\frac{V}{V^2+1}dv = -\log|x| + c$$

$$\Rightarrow \tan^{-1}v + \frac{1}{2}\int\frac{2V}{V^2+1}dv = -\log|x|+c$$

$$\Rightarrow \tan^{-1}v + \frac{1}{2}\log(V^2+1) = -\log|x|+c$$

$\Rightarrow$ Put $v = \frac{y}{x}$

$$\Rightarrow \tan^{-1}\frac{y}{x} + \frac{1}{2}\log\left|\left(\frac{y}{x}\right)^2+1\right| = -\log|x| + c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} + \frac{1}{2}\log\left|\left(\frac{y}{x}\right)^2+1\right| + \frac{2}{2}\log|x| = c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} + \frac{1}{2}\left(\log\left(\frac{y^2+x^2}{x^2}\right) + \log x^2\right) = c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} + \frac{1}{2}\log(x^2+y^2) = c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} + \log\sqrt{x^2+y^2} = c$$

method-III- If $\exists$ a continuous single variable fn $f(x)$ such that $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$

then $\boxed{I.F = e^{\int f(x)\,dx}}$ where $Mdx + Ndy = 0$ is non-exact D.E.

method-IV- If $\exists$ a continuous single variable fn $g(y)$ such that $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = g(y)$

then $\boxed{I.F = e^{\int g(y)\,dy}}$ when $Mdx + Ndy = 0$ is non-exact D.E

Problems:

1. Solve $2xy\,dy - (x^2+y^2+1)\,dx = 0$

Sol: $M = -(x^2+y^2+1)$, $N = 2xy$

$$\frac{\partial M}{\partial y} = -2y, \quad \frac{\partial N}{\partial x} = 2y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ non-exact}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}(-2y-2y) = \frac{-4y}{2xy} = \frac{-2}{x} = f(x)$$

$$\therefore I.F = e^{\int f(x)\,dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$$

multiplying given d.e with I.F

$$\frac{2xy}{x^2}\,dy - \left(\frac{x^2+y^2+1}{x^2}\right)dx = 0$$

$$-\left[1+\frac{y^2+1}{x^2}\right]dx + \frac{2y}{x}\,dy = 0$$

$$M_1 = -\left[1+\frac{y^2+1}{x^2}\right], \quad N_1 = \frac{2y}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \quad \text{- exact}$$

General soln is $\displaystyle\int_{y\to\text{constant}} M_1\,dx + \int_{x\text{ x-terms}} N_1\,dy = C$

$$\Rightarrow \int\left(-1-\frac{y^2+1}{x^2}\right)dx + \int 0\,dy = C$$

$$\Rightarrow -x - (y^2+1)\times\left(\frac{-1}{x}\right) = C$$

$$\Rightarrow x^2-(y^2+1) = -Cx \Rightarrow \boxed{x^2-y^2+Cx = 1}$$

2. Solve $(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0$

Sol:

$$M = y^4+2y, \quad N = xy^3+2y^4-4x$$

$$\frac{\partial M}{\partial y} = 4y^3+2, \quad \frac{\partial N}{\partial x} = y^3-4 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{non-exact}$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y^4+2y}(y^3-4-4y^3-2) = \frac{1}{y(y^3+2)}(-3y^3-6) = -\frac{3}{y}$$

$$I.F = e^{\int g(y)\,dy} = e^{-3\int\frac{1}{y}dy} = e^{-3\log y} = \frac{1}{y^3}$$

multipling d.e with I.F, we get

$$\left(\frac{y^4+2y}{y^3}\right)dx + \left(\frac{xy^3+2y^4-4x}{y^3}\right)dy = 0$$

$$\underbrace{\left[y+\frac{2}{y^2}\right]}_{M_1}dx + \underbrace{\left[x+2y-\frac{4x}{y^3}\right]}_{N_1}dy = 0$$

The general soln is $\displaystyle\int_{y\to\text{constant}}\left(y+\frac{2}{y^2}\right)dx + \int 2y\,dy = C$

$$xy + \frac{2x}{y^2} + 2\left(\frac{y^2}{2}\right) = c \Rightarrow \boxed{xy + \frac{2x}{y^2} + y^2 = c}$$

3. Solve $y(2x^2 - xy + 1)\,dx + (x-y)\,dy = 0$

Sol: $\frac{\partial M}{\partial y} = (2x^2 - xy + 1) + y(-x) = 2x^2 - 2xy + 1$

$\frac{\partial N}{\partial x} = 1$ Not exact

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x-y}(2x^2 - 2xy + 1 - 1) = \frac{2x(x-y)}{x-y} = 2x = f(x)$$

$$I.F = e^{\int 2x\,dx} = e^{2\left(\frac{x^2}{2}\right)} = e^{x^2}$$

multiply d.e with I.F

$$\underbrace{y \cdot e^{x^2}(2x^2 - xy + 1)}_{M_1}\,dx + \underbrace{e^{x^2}(x-y)}_{N_1}\,dy = 0 \quad \text{is exact}$$

The general soln is

$$\underset{y \to \text{constant}}{\int M_1\,dx} + \underset{\text{don't contain } x}{\int N_1\,dy} = c$$

$$\int y \cdot e^{x^2}(2x^2 - xy + 1)\,dx + \int 0\,dy = c$$

$$y\int\left(e^{x^2}(2x^2)\,dx + e^{x^2}\,dx\right) - y^2\int x \cdot e^{x^2}\,dx = c$$

$$y\int d(e^{x^2} \cdot x)\,dx - y^2\left(\frac{e^{x^2}}{2}\right) = c \qquad (\because d(uv) = uv' + u'v)$$

$$xy \cdot e^{x^2} - y^2\left(\frac{e^{x^2}}{2}\right) = c$$

$$\boxed{ye^{x^2}\left(x - \frac{y}{2}\right) = c}$$

[OR]

The general soln is

$$\underset{\text{don't contain } y}{\int M_1\,dx} + \underset{x \to \text{constant}}{\int N_1\,dy} = c$$

$$\int 0\, dx + \int e^{x^2}(x-y)\, dy = c$$

$$yx\cdot e^{x^2} - e^{x^2}\left(\frac{y^2}{2}\right) = c \Rightarrow \boxed{y\cdot e^{x^2}\left(x-\frac{y}{2}\right) = c}$$

Method - V - method of inspection

1. Solve $(1+xy)x\, dy + (1-xy)y\, dx = 0$

Sol :

$$x\,dy + y\,dx + xy\,(x\,dy - y\,dx) = 0$$

dividing with x^2y^2 we get

$$\frac{x\,dy + y\,dx}{x^2y^2} + \frac{x\,dy - y\,dx}{xy} = 0$$

$$\Rightarrow \frac{d(xy)}{(xy)^2} + \frac{1}{y}dy - \frac{1}{x}dx = 0$$

Integrating this,

$$\int \frac{d(xy)}{(xy)^2} + \int \frac{1}{y}dy - \int \frac{1}{x}dx = c$$

$$\frac{-1}{(xy)} + \log y - \log x = \log c$$

2. Solve $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$

Sol :

$$\frac{y\,dx - x\,dy}{y^2} + 3x^2 e^{x^3}. dx = 0$$

$$d\left(\frac{x}{y}\right) + d\left(e^{x^3}\right) = 0 \quad \text{Integrating},$$

$$\Rightarrow \frac{x}{y} + e^{x^3} = c$$

Solve

(H.W) $\rightarrow y(x^3e^{xy} - y)dx + x(xy + x^3e^{xy})dy = 0$ $\quad$ (A: $e^{xy} + \left(\frac{y}{x}\right)^2\frac{1}{2} = c$)

Questions :

1. Solve $y(x+y+1)\,dx + x(x+3y+2)\,dy = 0$ $\quad$ (Ans: $xy^2(x+2y+2) = c$) & IF $= y$

2. Solve $(x^2+y^2+x)\,dx + xy\,dy = 0$ $\quad$ (Ans; $3x^4 + 6x^2y^2 + 4x^3 = c$) & IF $= x$

3. Solve $y(2xy+e^x)\,dx - e^x\,dy = 0$ $\quad$ (Ans; $x^2 + \frac{e^x}{y} = c$)

4. Solve $(y+y^2)\,dx + xy\,dy = 0$ $\quad$ (Ans: $x + xy = c$) & IF $= y$ exact

5. Solve $(3xy-2ay^2)\,dx + (x^2-2axy)\,dy = 0$ $\quad I.F = x$ & $x^3y - ax^2y^2 = C$

SOLUTIONS

3. $2xy^2\,dx + y\cdot e^x\,dx - e^x\,dy = 0$

$$2xy^2\,dx = e^x\,dy - y\cdot e^x\,dx$$

$$\int 2x\,dx = \int \frac{e^x\,dy - y\cdot e^x\,dx}{y^2}$$

$$x^2 = -\frac{e^x}{y} + c \Rightarrow x^2 + \frac{e^x}{y} = c$$

Applications of 1st order D.E

Newton's law of cooling -

The rate of change of temperature of an object is proportional to the difference of the temp of an object & that of the surrounding medium (usually air)

θ - temp of an obj at time 't'

θ_0 - temp of surrounding medium.

$$\frac{d\theta}{dt} \propto \theta - \theta_0$$

$$\Rightarrow \frac{d\theta}{dt} = -k(\theta-\theta_0)$$

$$\Rightarrow \frac{1}{\theta-\theta_0}\,d\theta = -k\,dt \qquad \text{(variable seperable)}$$

$$\Rightarrow \int \frac{1}{\theta-\theta_0}\,d\theta = -k\int 1\,dt$$

$$\Rightarrow \log(\theta-\theta_0) = -kt + \log c$$

$$\Rightarrow \log\left(\frac{\theta-\theta_0}{c}\right) = -kt$$

$$\Rightarrow \frac{\theta-\theta_0}{c} = e^{-kt} \Rightarrow \boxed{\theta-\theta_0 = c\cdot e^{-kt}}$$

Problems:

1) A body is originally at 80°C & cools down to 60°C in 20 minutes, if the temp of air is 40°C

Find the temp of the body after 40 minutes.

Sol: According to newton's law of cooling, we have

$$\frac{d\theta}{dt} \propto \theta-\theta_0 \Rightarrow \frac{d\theta}{dt} = -k(\theta-\theta_0)$$

$$\therefore \theta-\theta_0 = c\cdot e^{-kt} \longrightarrow (1)$$

Initially i.e $t=0 \Rightarrow \theta=80°C$, the temp of air is 40°C i.e $\theta_0 = 40°C$

$$(1) \Rightarrow 80-40 = c\cdot e^{-k(0)}$$

$$\Rightarrow 40 = c\,(1) \Rightarrow \boxed{c=40}$$

after 20 minutes, $\theta = 60°C$.

$$(1) \Rightarrow 60-40 = 40\, e^{-k(20)}.$$

$$\Rightarrow 20 = 40\cdot e^{-20k}.$$

$$\Rightarrow e^{20k} = 2 \Rightarrow 20k = \log 2$$

$$\Rightarrow \boxed{k = \frac{1}{20}\log 2}$$

after 40 minutes

$$(1) \Rightarrow \theta-40 = 40\, e^{-\frac{\log 2}{20}(40)}$$

$$\Rightarrow \theta-40 = 40\, e^{-\log 4}$$

$$\Rightarrow \theta-40 = \frac{40}{4} = 10 \Rightarrow \boxed{\theta = 50}$$

2) If the temp of body is changing from 100°C to 70°C in 15 min find when the temp will be 40°C, if $\theta_0 = 30°C$.

Sol: at $t=0$, $\theta = 100°C$ $\quad (1) \Rightarrow 100-30 = c\cdot e^{-k(0)} \Rightarrow \boxed{c=70}$

at $t=15$, $\theta=70^\circ C$ $\quad$ ① $\Rightarrow 70-30 = 70\cdot e^{-15k}$

$$\Rightarrow e^{-15k} = \frac{4}{7}$$

$$\Rightarrow e^{15k} = 7/4$$

$$\Rightarrow 15k = \log(7/4) \Rightarrow k = \frac{1}{15}\log\left(\frac{7}{4}\right)$$

$t=?$ ① $\Rightarrow 40-30 = 70\cdot e^{-\frac{1}{15}\log\left(\frac{7}{4}\right)t}$

$$\Rightarrow \frac{1}{7} = e^{-\frac{1}{15}\log\left(\frac{7}{4}\right)t}$$

$$\Rightarrow 7 = e^{\frac{1}{15}\log\left(\frac{7}{4}\right)t}$$

$$\Rightarrow \frac{1}{15}\log\left(\frac{7}{4}\right)t = \log 7$$

$$\Rightarrow t = \frac{15\cdot\log 7}{\log\left(\frac{7}{4}\right)} \simeq 52.16$$

3. The temp of a cup of coffee is $92^\circ C$, when freshly poured the room temp being $24^\circ C$. In one minute it has cooled to $80^\circ C$. How long must elapse, before the temp of the cup becomes $65^\circ C$? $\left(\text{Ans}: t = \frac{\log\left(\frac{41}{68}\right)}{\log\left(\frac{56}{68}\right)}\right) \simeq 2.61$

4. The temp of the body drops from $100^\circ c$ to $75^\circ C$ in 10 min. and $\theta_0 = 20^\circ C$. what will be it's temp after 30 min? when will the temp be $25^\circ C$? $\left(\text{Ans}: \theta \approx 46^\circ C \ \& \ t \simeq \frac{10\times\log\left(\frac{5}{80}\right)}{\log\left(\frac{11}{16}\right)} \simeq 74.86 \text{ min}\right)$

5. A object is heated to $300^\circ F$ & allowed to cool in a room whose air temp is $80^\circ F$, if after 10 min, The temp of the object is $250^\circ F$. What will be its temp after 20 min? $(\text{Ans}; \theta \simeq 283^\circ F)$

6. An obj cools from $120^\circ F$ to $95^\circ F$ in 30 min when surrounded by air whose temp is $70^\circ F$. Find its temp at the end of another half an hour. $(\text{Ans}: \theta \simeq 82.5^\circ F)$

7. A murder victim is discoved & a lieutenant from the forensic labaratory is summoned to estimate the time of death. The body is located in a room that is kept at a constant temp of $68^\circ F$. The lieutenant arrived at 9:40 pm & measured the body temp as $94.4^\circ F$. Another

measurement of body temp at 11 pm is 89.2°F. Find the estimated time of death.

Sol: According to newton's law of cooling, $\theta - \theta_0 = c \cdot e^{-kt}$

Given $\theta_0 = 68°F$

at 9:40, $t=0$, $\theta = 94.4°F$ (1) $\Rightarrow 94.4 - 68 = c \cdot e^{-0} \Rightarrow c = 26.4$

at 11:00, $t = 80$, $\theta = 89.2°F$ (1) $\Rightarrow 89.2 - 68 = 26.4 e^{-80k}$

$\Rightarrow 21.2 = 26.4 e^{-80k}$

$\Rightarrow -80k = \log\left(\frac{21.2}{26.4}\right)$

$\Rightarrow k = \frac{-1}{80} \log\left(\frac{21.2}{26.4}\right)$

$t = ?$, $\theta = 98.6$ (1) $\Rightarrow 98.6 - 68 = 26.4\, e^{\frac{t}{80} \log\left(\frac{21.2}{26.4}\right)}$

$\Rightarrow \frac{30.6}{26.4} = \left(e^{\log\left(\frac{21.2}{26.4}\right)}\right)^{t/80}$

$\Rightarrow \log\left(\frac{30.6}{26.4}\right) = \log\left(\frac{21.2}{26.4}\right) \times \frac{t}{80}$

$\Rightarrow t = 80 \times \frac{\log\left(\frac{30.6}{26.4}\right)}{\log\left(\frac{21.2}{26.4}\right)} \simeq 53.8$ min

$\therefore$ Death occures approx 54 min before first measurement i.e 9:40 pm. So, the approx time of death is 8:46 pm.

Orthogonal trajectories:

Definition: A curve which cuts every member of a given family of curves at right angles is called "orthogonal trajectory" of the given family of curves.

Procedure to find orthogonal trajectory:

1. Let $f(x, y, c) = 0$ be the eqn of given family of curves.
   ↳ (1)

2. Eliminate 'c' from (1) & obtain the d.e $F(x, y, \frac{dy}{dx} = 0) \longrightarrow$ (2)

3. Replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$ & obtain the d.e $G\left(x, y, \frac{-dx}{dy}\right) = 0 \longrightarrow$ (3)

4. Solve d.e (3) to get the eqn of family of orthogonal trajectories of eqn i.e

$g(x, y, k) = 0 \longrightarrow$ (4) is the O.T.

Problems:

Q) Find the O.T of semicubical parabolas $ay^2 = x^3$ (a is constant)

Sol: Let $ay^2 = x^3 \longrightarrow$ (1)

diff (1) w.r.t x

$$a(2y)\frac{dy}{dx} = 3x^2 \longrightarrow (2)$$

(1) $\Rightarrow a = x^3/y^2$, sub a in (2)

$$\frac{x^3}{y^2}(2y)\frac{dy}{dx} = 3x^2$$

$$\Rightarrow 2x\frac{dy}{dx} = 3y \longrightarrow (3)$$

Replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$ in (3)

$$(3) \Rightarrow 2x\left(\frac{-dx}{dy}\right) = 3y$$

$$\Rightarrow 2x\,dx + 3y\,dy = 0 \quad \text{—} (4)$$

on Integrating,

$$\Rightarrow 2\int x\,dx + 3\int y\,dy = c$$

$$\Rightarrow x^2 + \frac{3y^2}{2} = c$$

$$\Rightarrow \frac{x^2}{c} + \frac{y^2}{(2c/3)} = 1 \text{ is ellipse (O.T)}$$

Q) Find O.T of the following curves

(i) $x^{2/3} + y^{2/3} = a^{2/3}$ (Ans: $x^{4/3} - y^{4/3} = c$)

(ii) $x^2 + y^2 = r^2$ (Ans: $y = xk$)

Q) Find the O.T of the circles $x^2+(y-c)^2=c^2$

Sol: Let $x^2+(y-c)^2=c^2 \rightarrow (1)$

Diff (1) w.r.t x, we get

$$2x+2(y-c)\frac{dy}{dx}=0$$

$$\Rightarrow y-c=\frac{-x}{\frac{dy}{dx}} \Rightarrow c=y+\frac{x}{\frac{dy}{dx}}$$

$$\therefore (1) \Rightarrow x^2+\frac{x^2}{\left(\frac{dy}{dx}\right)^2}=\left(y+x\frac{dy}{dx}\right)^2$$

$$\Rightarrow x^2+x^2\left(\frac{dx}{dy}\right)^2=y^2+x^2\left(\frac{dx}{dy}\right)^2+2xy\frac{dx}{dy}$$

$$\Rightarrow x^2-y^2=2xy\frac{dx}{dy}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2xy}{x^2-y^2} \longrightarrow (2)$$

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

$$\Rightarrow -\frac{dx}{dy}=\frac{2xy}{x^2-y^2}$$

$$\Rightarrow (x^2-y^2)dx+2xy\,dy=0 \longrightarrow (3)$$

$$Mdx+Ndy=0$$

$$\frac{\partial M}{\partial y}=-2y,\quad \frac{\partial N}{\partial x}=2y \Rightarrow \text{non-exact}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2xy}(-4y)=\frac{-2}{x}=f(x)$$

$$IF=e^{\int f(x)dx}=e^{-2\int\frac{1}{x}dx}=e^{-2\log x}=x^{-2}=\frac{1}{x^2}$$

mul (3) with $I.F=\frac{1}{x^2}$

$$\left(\frac{x^2-y^2}{x^2}\right)dx + \frac{2xy}{x^2}\,dy = 0$$

$$\left(1-\frac{y^2}{x^2}\right)dx + \frac{2y}{x}\,dy = 0 \longrightarrow (4) \text{ is exact}$$

The general soln

$$\int_{y\to \text{constant}} \left(1-\frac{y^2}{x^2}\right)dx + \int 0\,dy = C$$

$$x - y^2\left(\frac{-1}{x}\right) = C$$

$$\Rightarrow x^2+y^2 = Cx \quad \text{is the O.T}$$

Q) P.T the system of parabolas $y^2 = 4a(x+a)$ is self orthogonal.

Sol : Let $y^2 = 4a(x+a) \longrightarrow (1)$

Diff (1) w.r.t x

$$2y\frac{dy}{dx} = 4a \Rightarrow a = \frac{y}{2}\cdot\frac{dy}{dx}$$

$$\therefore (1) \Rightarrow y^2 = 2y\frac{dy}{dx}\left(x+\frac{y}{2}\cdot\frac{dy}{dx}\right)$$

$$\Rightarrow y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2 \longrightarrow (2)$$

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

$$\Rightarrow y^2 = -2xy\cdot\frac{dx}{dy} + y^2\left(\frac{dx}{dy}\right)^2$$

$$\Rightarrow y^2 = \frac{-2xy}{\left(\frac{dy}{dx}\right)} + \frac{y^2}{\left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow y^2 \times \left(\frac{dy}{dx}\right)^2 = -2xy\frac{dy}{dx} + y^2$$

$$\Rightarrow y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2 \longrightarrow (3)$$

The d.e 2 & 3 are same, hence the eqn $y^2 = 4a(x+a)$ is self orthogonal.

Q) S.T the system of rectangular parabolas $x^2-y^2=c^2$ & $xy=c^2$ are mutually orthogonal.

Sol :

let $x^2-y^2=c^2 \longrightarrow (1)$

Diff (1) w.r.t x

$$2x-2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{x}{y} \longrightarrow (2)$$

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

$$\therefore (2) \Rightarrow -\frac{dx}{dy}=\frac{x}{y} \Rightarrow \frac{1}{x}dx+\frac{1}{y}dy=0$$

On integrating, $\int\frac{1}{x}dx+\int\frac{1}{y}dy=c$

$$\Rightarrow \log x+\log y=\log c$$

$$\Rightarrow xy=c=k^2$$

Q) S.T the system of confocal conics $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ is self orthogonal.

Sol :

Let $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \longrightarrow (1)$

Diff (1) w.r.t 'x', we get

$$\frac{2x}{a^2+\lambda}+\frac{2yy_1}{b^2+\lambda}=0$$

$$\frac{2x(b^2+\lambda)+2y(a^2+\lambda)y'}{(a^2+\lambda)(b^2+\lambda)}=0$$

$$x(b^2+\lambda)+(a^2+\lambda)yy_1=0$$

$$\lambda(x+yy_1)=-(xb^2+a^2yy_1)$$

$$\therefore \lambda=\frac{-(b^2x+a^2yy_1)}{x+yy_1}$$

$$\therefore a^2+\lambda=a^2-\frac{b^2x+a^2yy_1}{x+yy_1}=\frac{x(a^2-b^2)}{x+yy_1}$$

$$\therefore b^2+\lambda=b^2-\frac{b^2x+a^2yy_1}{x+yy_1}=\frac{yy_1(b^2-a^2)}{x+yy_1}$$

use $a^2+\lambda$ & $b^2+\lambda$ in ①

we get

$$\frac{x^2}{\left[\frac{x(a^2-b^2)}{x+yy_1}\right]} + \frac{y^2}{\left[\frac{yy_1(b^2-a^2)}{x+yy_1}\right]} = 1$$

$$\Rightarrow \left(\frac{x+yy_1}{a^2-b^2}\right)\left[x-\frac{y}{y'}\right] = 1$$

$$\Rightarrow (x+yy_1)\left(x-\frac{y}{y_1}\right) = a^2-b^2 \longrightarrow ②$$

Replace y_1 with $-\frac{1}{y_1}$

$$\therefore ② \Rightarrow \left(x-\frac{y}{y_1}\right)\left(x+yy_1\right) = a^2-b^2 \longrightarrow ③$$

$\therefore$ Eqns ② & ③ are same. Hence, the given eqn is self orthogonal

Q) Find the O.T of family of circles $x^2+y^2+2gx+c=0$ where 'g' is the parameter.

Sol:

$$x^2+y^2+2gx+c=0 \longrightarrow ①$$

Diff ① wrt 'x', we get

$$2x+2yy_1+2g+0=0$$

$$2g = -\left(2x+2y\frac{dy}{dx}\right)$$

$$\therefore ① \Rightarrow x^2+y^2-\left(2x+2y\frac{dy}{dx}\right)x+c=0$$

$$x^2+y^2-2x^2-2xy\frac{dy}{dx}+c=0$$

$$y^2-x^2-2xy\frac{dy}{dx}+c=0 \longrightarrow ②$$

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

$$\therefore ② \Rightarrow y^2-x^2+2xy\frac{dx}{dy}+c=0$$

$\Rightarrow 2xy\frac{dx}{dy} - x^2 = -(c+y^2)$

$\Rightarrow 2x\frac{dx}{dy} - \frac{x^2}{y} = -\left(\frac{c+y^2}{y}\right) \longrightarrow (3)$

sub $x^2 = t$ in (3)

diff w.r.t y

$\Rightarrow 2x\frac{dx}{dy} = \frac{dt}{dy}$

$\therefore (3) \Rightarrow \frac{dt}{dy} - \frac{t}{y} = -\left(\frac{c+y^2}{y}\right) \longrightarrow (4)$

$\frac{dt}{dy} + P(y)t = Q(y)$ is linear 't'

$P(y) = \frac{-1}{y}$, $Q(y) = -\left(\frac{c+y^2}{y}\right)$

$I.F = e^{\int P(y)dy} = e^{-\int \frac{1}{y}dx} = e^{-\log y} = y^{-1} = \frac{1}{y}$

$\therefore$ The general soln of (4) is

$t(I.F) = \int Q(y).\, IF\, dy + c_1$

$\Rightarrow x^2\left(\frac{1}{y}\right) = \int \frac{1}{y} \times \left(\frac{-c-y^2}{y}\right) dy + c_1$

$\Rightarrow \frac{x^2}{y} = -\int \frac{c+y^2}{y^2} dy + c_1$

$\Rightarrow \frac{x^2}{y} = -\int \left(\frac{c}{y^2}+1\right) dy + c_1$

$\Rightarrow \frac{x^2}{y} = -c\left(\frac{-1}{y}\right) - y + c_1$

$\Rightarrow \frac{x^2}{y} = \frac{c - y^2 + c_1 y}{y}$

$\Rightarrow \boxed{x^2 + y^2 - c_1 y = c}$ is the O.T

Q) Find particular member (or) O.T of $x^2 + cy^2 = 1$ passing through the pt $(2,1)$.

Sol:

Let $x^2 + cy^2 = 1 \longrightarrow (1)$ $\Rightarrow c = \frac{1-x^2}{y^2}$

Diff ① wrt x, we get

$$2x + 2cy\frac{dy}{dx} = 0 \Rightarrow x + cy\frac{dy}{dx} = 0 \rightarrow ②$$

Sub c' in ② $\Rightarrow$ $x + \left(\frac{1-x^2}{y^2}\right) \cdot y \frac{dy}{dx} = 0$

$$\Rightarrow x + \left(\frac{1-x^2}{y}\right)\frac{dy}{dx} = 0 \rightarrow ③$$

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ in ③

$$\therefore ③ \Rightarrow x + \left(\frac{1-x^2}{y}\right)\left(-\frac{dx}{dy}\right) = 0$$

$$\Rightarrow yx\,dy + (x^2-1)\,dx = 0$$

$$\Rightarrow y\,dy + \left(\frac{x^2-1}{x}\right)dx = 0 \rightarrow ④$$

Integrating on both sides.

$$\Rightarrow \int y\,dy + \int x\,dx - \int \frac{1}{x}\,dx = C_1$$

$\Rightarrow$ $\boxed{\frac{y^2}{2} + \frac{x^2}{2} - \log x = C_1}$ is the ~~req~~ O.T

(or)

$$\Rightarrow y^2 + x^2 = 2\log x + 2C_1$$

$$\Rightarrow y^2 + x^2 = \log x^2 + 2C_1$$

$\Rightarrow$ $\boxed{x^2 = C_2 e^{x^2+y^2}}$ is the ~~req~~ O.T

We need the particular member passing through $(2,1)$

$$\Rightarrow 4 = C_2 \cdot e^5 \Rightarrow C_2 = 4e^{-5}$$

$\therefore$ The req O.T is $x^2 = 4e^{x^2+y^2-5}$

6) Find the O.T of the family of parabolas through the origin & whose foci on y-axis (Hint: $x^2 = 4ay$) (Ans: $x^2+2y^2 = c$ is an ellipse)

Q) Find O.T of $x^2+y^2+2fy+1=0$ ($f \to$ parameter) (Ans: $x^2+y^2-cx=1$)

Q) Find the O.T of circles on x-axis passing through the origin (Hint: $x^2+y^2-2gx=0$)

(Ans: $x^2+y^2-cy=0$)

Q) Find the O.T of $y=ax^n$ (Ans: $x^2+ny^2=c^2$)

Law of Natural growth / Decay:

Law of chemical conversion:

Let $x(t)$ be the amt of a substance at time "t" & let the substance be getting converted chemically such that rate of change of amt of substance $x(t)$ is proportional to the amt of substance available.

i.e $\frac{dx}{dt} \propto x$

As $t\uparrow$, $x\uparrow$, we can take $\frac{dx}{dt} = kx$ $(K>0)$

$\Rightarrow \boxed{x(t) = c \cdot e^{kt}}$ is law of natural growth

As $t\uparrow$, $x\downarrow$, we can take $\frac{dx}{dt} = -kx$ $(K>0)$

$\Rightarrow \boxed{x(t) = c \cdot e^{-kt}}$ is law of natural decay

Problems:

Q) The number 'N' of bacteria in a culture grew at a rate proportional to N. The value of 'N' was initially 100 & increases to 332 in one hour, what would be the value of 'N' after $1\frac{1}{2}$ hrs.

Sol: Given $\frac{dN}{dt} \propto N \Rightarrow \frac{dN}{dt} = KN$

$\Rightarrow N = ce^{kt} \longrightarrow (1)$

at $t=0$, $N=100$, (1) $\Rightarrow 100 = c \cdot e^0 \Rightarrow \boxed{c=100}$

at $t=1$, $N=332$, (1) $\Rightarrow 332 = 100e^k \Rightarrow \boxed{K = \log\left(\frac{332}{100}\right)}$

at $t=\frac{3}{2}$, $N=?$ (1) $\Rightarrow N = 100e^{\frac{3}{2}\log\left(\frac{332}{100}\right)} = 100\left(\frac{332}{100}\right)^{3/2} \simeq 605$

Q) If the radioactive C_{14} has a half life of 5750 yrs. What will remain of 1 gm after 3000 yrs?

Sol:

$$\frac{dm}{dt} = -km \Rightarrow m(t) = c\cdot e^{-kt} \longrightarrow (1)$$

at $t=0$, $m=1 \Rightarrow 1 = c\cdot e^{0} \Rightarrow \boxed{c=1}$

at 5750, $m=\frac{1}{2} \Rightarrow \frac{1}{2} = e^{-k(5750)} \Rightarrow \boxed{k = \frac{1}{5750}\log 2}$

at 3000, $m=? \Rightarrow m = 1\cdot e^{-\frac{\log 2}{5750}(3000)}$

$$\Rightarrow m = (2)^{-\frac{3000}{5750}} \simeq 0.7 \text{ gms}$$

Q) Uranium disintegrates at a rate proportional to amt present at any instant. If m_1 & m_2 grms of uranium that is present at time T_1 & T_2 respectively, find the half life of uranium.

Sol:

$$\frac{dm}{dt} \propto m(t) \Rightarrow m = c\cdot e^{-kt} \longrightarrow (1)$$

let at $t=0$, $m=m_0$ $(1) \Rightarrow m_0 = c\cdot e^{-k(0)} \Rightarrow c = m_0$

at $t=T_1$, $m=m_1$ $(1) \Rightarrow m_1 = m_0\cdot e^{-k(T_1)} \longrightarrow (2)$

at $t=T_2$, $m=m_2$ $(1) \Rightarrow m_2 = m_0\cdot e^{-k(T_2)} \longrightarrow (3)$

from (2) & (3),

$$\frac{(2)}{(3)} \Rightarrow \frac{m_1}{m_2} = e^{-k(T_1-T_2)} \Rightarrow \log\left(\frac{m_1}{m_2}\right) = -k(T_1-T_2)$$

$$\Rightarrow k = \frac{1}{T_2-T_1}\log\left(\frac{m_1}{m_2}\right)$$

at $t=?$, $m=\frac{m_0}{2}$

$$(1) \Rightarrow \frac{m_0}{2} = m_0\cdot e^{-\frac{\log\left(\frac{m_1}{m_2}\right)}{T_2-T_1}t}$$

$$\Rightarrow t = \frac{\log 2\,(T_2-T_1)}{\log\left(\frac{m_1}{m_2}\right)}$$