12.8.2.6

Balaji Balamurugan
Dept. of Electrical Engineering
IIT Hyderabad

January 9, 2025

Question

Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

Parameters

The parameters for the problem are given as follows:

Variable	**Description**	**values**
$\mathbf{V}$	Quadratic form of the matrix	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathbf{u}$	Linear coefficient vector	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
f	constant term	-4
$\mathbf{m}$	The direction vector of line	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
$\mathbf{h}$	Point on line	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Table: Parameter Used

Theoretical Solution

The point of intersection of the line with the circle is $x_i = h + k_i m$, where, k_i is a constant and is calculated as follows:-

$$k_i = \frac{1}{m^\top V m}\left(-m^\top (Vh + u) \pm \sqrt{\left[m^\top (Vh + u)\right]^2 - g(h)\left(m^\top V m\right)}\right)$$

Substituting the input parameters into k_i,

$$k_i = \frac{1}{\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}}\left(-\begin{pmatrix}1 & -1\end{pmatrix}\left(\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}\right)\right) \pm$$
$$\sqrt{\left[\begin{pmatrix}1 & -1\end{pmatrix}\left(\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}\right)\right]^2 - g(h)\left(\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}\right)} \tag{3.1}$$

We get,
$k_i = 0, -2$
Substituting k_i into $x_i = h + k_i m$ we get
For the given line $x = 4a$, The values of **h**, **m** are

$$x_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + (0) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.2}$$

$$\implies x_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{3.3}$$

$$x_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + (-2) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.4}$$

$$\implies x_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{3.5}$$

$$\implies x_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{3.6}$$

The area of the smaller region bounded by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

$$= \int_0^2 \sqrt{4 - x^2} dx - \int_0^2 (2 - x)\, dx \tag{3.7}$$

$$= \left(\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} - 2x + \frac{x^2}{2} \right)_0^2 \tag{3.8}$$

$$= (\pi - 2) \tag{3.9}$$

Computational Solution

Taking trapezoid-shaped strips of small area and adding them all up. Say we have to find the area of y_x from $x = x_0$ to $x = x_n$, discretize the points on the x axis $x_0, x_1, x_2, \ldots, x_n$ such that they are equally spaced with the step size h.
Sum of all trapezoidal areas is given by,

$$A = \frac{1}{2}h\left(y\left(x_1\right) + y\left(x_0\right)\right) + \frac{1}{2}h\left(y\left(x_2\right) + y\left(x_1\right)\right) + \cdots + \frac{1}{2}h\left(y\left(x_n\right) + y\left(x_{n-1}\right)\right) \tag{3.10}$$

$$= h\left[\frac{1}{2}\left(y\left(x_0\right) + y\left(x_n\right)\right) + y\left(x_1\right) + \cdots + y\left(x_{n-1}\right)\right] \tag{3.11}$$

Let $A\left(x_n\right)$ be the area enclosed by the curve $y\left(x\right)$ from $x = x_0$ to $x = x_n$, $\left(x_0, x_1, \ldots x_n\right)$ be equidistant points with step-size h.

$$A\left(x_n + h\right) = A\left(x_n\right) + \frac{1}{2}h\left(y\left(x_n + h\right) + y\left(x_n\right)\right) \tag{3.12}$$

We can repeat this till we get required area.
Discretizing the steps, making $A(x_n) = A_n, y(x_n) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2}h(y_{n+1} + y_n) \tag{3.13}$$

We can write y_{n+1} in terms of y_n using first principle of derivative.
$y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left(\left(y_n + hy'_n\right) + y_n\right) \tag{3.14}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left(2y_n + hy'_n\right) \tag{3.15}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{3.16}$$

$$x_{n+1} = x_n + h \tag{3.17}$$

In the given question, $y_n = \sqrt{4 - x_n^2} + x_n - 2$ and $y_n' = \frac{-x_n}{\left(\sqrt{4 - x_n^2}\right)} + 1$

The general difference equation will be given by

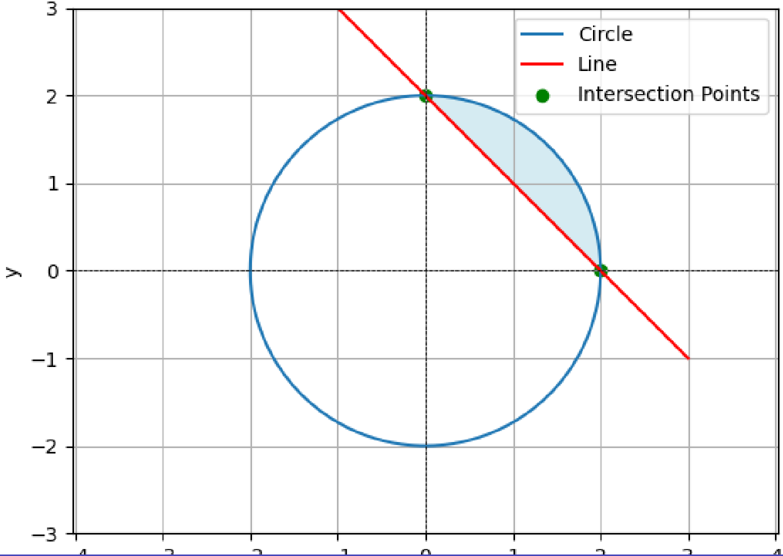
$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y_n' \tag{3.18}$$

$$A_{n+1} = A_n + h\left(\sqrt{4 - x_n^2} + x_n - 2\right) + \frac{1}{2}h^2\left(\frac{-x_n}{\left(\sqrt{4 - x_n^2}\right)} + 1\right) \tag{3.19}$$

$$x_{n+1} = x_n + h \tag{3.20}$$

Iterating till we reach $x_n = 2$ will return required area.
Area obtained computationally: 1.41555 sq. units
Area obtained theoretically: $(\pi - 2)$ sq. units = 1.14 sq.unis

Plot



Codes

Code:
https://github.com/Balaji29-code/EE1003/tree/main/Problem-2/codes