12.6.5.3.4

EE24BTECH11010 - Balaji B

Question:
Find the roots of equation $2x^2 - 7x + 3 = 0$

Theoretical Solution:
Applying the Quadratic formula, we get

$$x_1 = \frac{7 + \sqrt{49 - 24}}{4} \tag{0.1}$$

$$x_1 = 3 \tag{0.2}$$

$$x_2 = \frac{7 - \sqrt{49 - 24}}{4} \tag{0.3}$$

$$x_2 = \frac{1}{2} \tag{0.4}$$

$\therefore$ The roots of the equation $2x^2 - 7x + 3 = 0$ are $x_1 = 3$ and $x_2 = \frac{1}{2}$

Computational Solution:

Newton-Raphson Method

1) Update Equation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1.1}$$

2) Steps:
   1. Start with an initial guess x_0.
   2. Define the function $f(x)$ and its derivative $f'(x)$.
   3. Iterate using:

   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.1}$$

   until convergence, i.e.,

   $$|x_{n+1} - x_n| < \text{tolerance} \tag{2.2}$$

   4. Stop if $f'(x_n)$ is close to zero to avoid division by zero.
3) Convergence Criteria: The method converges quadratically if the initial guess is sufficiently close to the root and $f'(x) \neq 0$.

   For our question $f(x) = 4x - 7$ and $f'(x) = 4x - 7$, on substituting we get

   $$x_{n+1} = x_n - \frac{4x - 7}{4x - 7} \tag{3.1}$$

Secant Method:

a) Update Formula:

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \tag{3.2}$$

b) Steps:
1. Start with two initial guesses x_0 and x_1.
2. Define the function $f(x)$.
3. Iterate using:

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \tag{3.3}$$

until convergence, i.e.,

$$|x_{n+1} - x_n| < \text{tolerance}. \tag{3.4}$$

4. Stop if $f(x_n) - f(x_{n-1})$ is close to zero to avoid division by zero.

c) Convergence Criteria: The method converges superlinearly and does not require the derivative $f'(x)$.

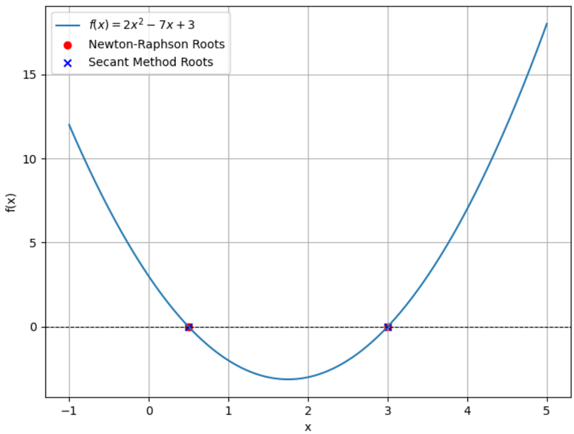

Fig. 3.1: Roots of the quadratic equation $2x^2 - 7x + 3 = 0$

Finding Eigen-Value:

A general quadratic equation $ax^2 + bx + c$ is written in matrix form as

$$\text{Matrix} = \begin{pmatrix} 0 & -\frac{c}{a} \\ 1 & -\frac{b}{a} \end{pmatrix} \tag{3.5}$$

For our question $a = 2$, $b = -7$ and $c = 3$, on substituting

$$\text{Matrix} = \begin{pmatrix} 0 & -\frac{3}{2} \\ 1 & \frac{7}{2} \end{pmatrix} \tag{3.6}$$

QR-DECOMPOSITION:-GRAM-SCHMIDT METHOD

1) QR decomposition

$$A = QR \tag{1.1}$$

   a) Q is an $m \times n$ orthogonal matrix
   b) R is an $n \times n$ upper triangular matrix.

   Given a matrix $A = [a_1, a_2, \ldots, a_n]$, where each a_i is a column vector of size $m \times 1$.
2) Normalize the first column of A:

$$q_1 = \frac{a_1}{\|a_1\|} \tag{2.1}$$

3) For each subsequent column a_i, subtract the projections of the previously obtained orthonormal vectors from a_i :

$$a_i' = a_i - \sum_{k=1}^{i-1} \langle a_i, q_k \rangle q_k \tag{3.1}$$

   Normalize the result to obtain the next column of Q:

$$q_i = \frac{a_i'}{\|a_i'\|} \tag{3.2}$$

   Repeat this process for all columns of A.
4) Finding R:-
   After constructing the ortho-normal columns $q_1, q_2, \ldots, q_n$ of Q, we can compute the elements of R by taking the dot product of the original columns of A with the columns of Q:

$$r_{ij} = \langle a_j, q_i \rangle \text{ , for } i \leq j \tag{4.1}$$

QR-Algorithm

1) Initialization
   Let $A_0 = A$, where A is the given matrix.
2) QR Decomposition
   For each iteration $k = 0, 1, 2, \ldots$:
   a) Compute the QR decomposition of A_k, such that:

$$A_k = Q_k R_k \tag{2.1}$$

      where:
      i) Q_k is an orthogonal matrix ($Q_k^\top Q_k = I$).

ii) R_k is an upper triangular matrix.

The decomposition ensures $A_k = Q_k R_k$.

b) Form the next matrix A_{k+1} as:

$$A_{k+1} = R_k Q_k \tag{2.2}$$

3) Convergence
   Repeat Step 2 until A_k converges to an upper triangular matrix T. The diagonal entries of T are the eigenvalues of A.

4) The eigenvalues of matrix will be the roots of the equation.

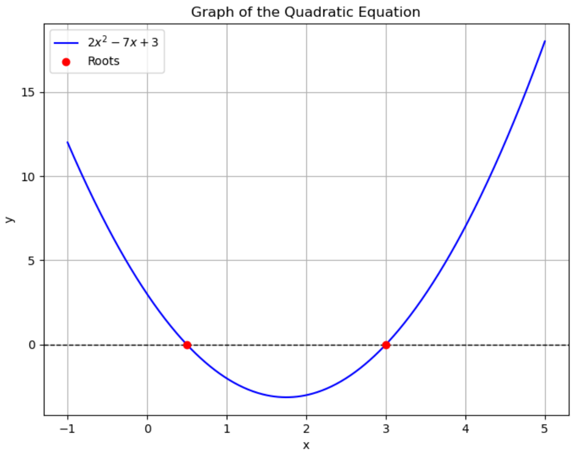

Fig. 4.1: Roots of the quadratic equation $2x^2 - 7x + 3 = 0$