12.9.2.2

EE24BTECH11010 - Balaji B

Question:
Solve the differential equation $y' - 2x - 2 = 0$ with initial conditions $y(0) = 0$
Theoretical Solution:
We apply the Laplace transform to each term in the equation. The Laplace transforms for the derivatives of $y(x)$ are:

$$\mathcal{L}\{y'(x)\} = sY(s) - y(0) \tag{0.1}$$

$$\mathcal{L}\{2x\} = 2 \cdot \frac{1!}{s^2} = \frac{2}{s^2}, \tag{0.2}$$

$$\mathcal{L}\{2\} = \frac{2}{s}. \tag{0.3}$$

Now, applying the Laplace transform to the entire differential equation:

$$\mathcal{L}\{y' - 2x - 2\} = 0 \tag{0.4}$$

$$\mathcal{L}\{y'(x)\} - \mathcal{L}\{2x\} - \mathcal{L}\{2\} = 0 \tag{0.5}$$

$$sY(s) - y(0) - \frac{2}{s^2} - \frac{2}{s} = 0 \tag{0.6}$$

Substitute the initial conditions $y(0) = 0$, we get

$$sY(s) - 0 - \frac{2}{s^2} - \frac{2}{s} = 0 \tag{0.7}$$

Simplify the equation:

$$sY(s) = \frac{2}{s^2} + \frac{2}{s} \tag{0.8}$$

$$Y(s) = \frac{2}{s^3} + \frac{2}{s^2} \tag{0.9}$$

Now, take the inverse Laplace transform

$$y(x) = \mathcal{L}^{-1}\{Y(s)\} \tag{0.10}$$

$$y(x) = \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s^2}\right\} \tag{0.11}$$

$$y(x) = 2x + x^2 \tag{0.12}$$

Region of Convergence:
The denominator indicates a pole at $s = 0$. To ensure convergence of the Laplace transform integral, the real part of s must satisfy:

$$Re(s) > 0 \tag{0.13}$$

So the Laplace transform converges for values of s with real part greater than 0

Computational Solution:

The **bilinear transform** is used to approximate the continuous derivative y' and reformulate the differential equation into a discrete time difference equation.
Applying the Laplace transform to both sides of the differential equation, we get

$$sY(s) - 0 - \frac{2}{s^2} - \frac{2}{s} = 0 \tag{0.14}$$

$$sY(s) = \frac{2}{s^2} + \frac{2}{s} \tag{0.15}$$

$$Y(s) = \frac{2}{s^3} + \frac{2}{s^2} \tag{0.16}$$

Apply Bilinear Transform with $T = h$

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{0.17}$$

Substituting the above in our Laplace equation we get,

$$Y(z) = \frac{2}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^3} + \frac{2}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2} \tag{0.18}$$

$$Y(z) = \frac{2}{\left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^3} + \frac{2}{\left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^2} \tag{0.19}$$

On rearranging and neglecting the powers greater than 2 of h we get,

$$(1 - z^{-1})^3 Y(z) = \frac{h^2\left(1 - z^{-2}\right)\left(1 + z^{-1}\right)}{2} \tag{0.20}$$

$$\left(1 - 3z^{-1} + 3z^{-2} - z^{-3}\right) Y(z) = \frac{h^2\left(1 + z^{-1} - z^{-2} - z^{-3}\right)}{2} \tag{0.21}$$

Applying Z transform we get

$$y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3} = \frac{h^2\left(\delta[n] + \delta[n-1] - \delta[n-2] - \delta[n-3]\right)}{2} \tag{0.22}$$

The above equation is the required difference equation.

Taking the value of $h = 0.1$, we get the following plot

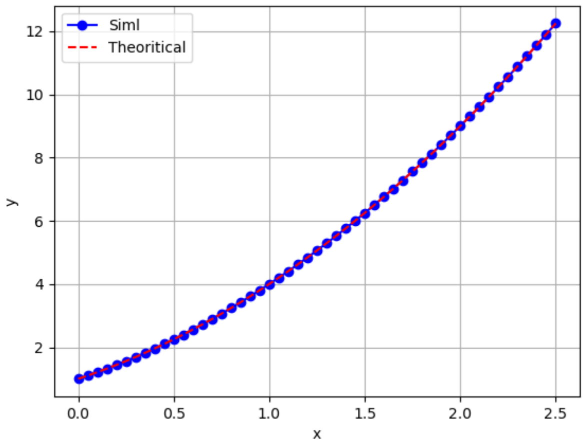

Fig. 0.1: Approximate solution of the DE