9.9.3.9

EE24BTECH11010 - Balaji B

Question:
If the area of the region bounded by the curve $y^2 = 4ax$ and the line $x = 4a$ is $\frac{256}{3}$ sq. units, then using integration, find the value of a, where $a > 0$.

Solution:

Variable	Description
$\mathbf{x_1}$	First intersection point
$\mathbf{x_2}$	Second intersection point
$\mathbf{h}$	Point on the given line
$\mathbf{m}$	Direction vector of given line
A	Area of the region

TABLE I: Variables Used

The equation of a parabola in Matrix form is

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

The equation of a line in vector form is

$$\mathbf{x} = \mathbf{h} + \kappa \mathbf{m} \tag{2}$$

For the given parabola $y^2 = 4ax$, The values of $\mathbf{V}$,$\mathbf{u}$,f are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

$$\mathbf{u} = \begin{pmatrix} -2a \\ 0 \end{pmatrix} \tag{4}$$

$$f = 0 \tag{5}$$

For the given line $x = 4a$, The values of $\mathbf{h}$, $\mathbf{m}$ are

$$\mathbf{h} = \begin{pmatrix} 4a \\ 0 \end{pmatrix} \tag{6}$$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7}$$

Substituting the line equation in parabola equation gives the values of κ

$$(\mathbf{h}+\kappa\mathbf{m})^\top \mathbf{V}(\mathbf{h}+\kappa\mathbf{m})+2\mathbf{u}^\top(\mathbf{h}+\kappa\mathbf{m})+f=0 \tag{8}$$

$$\left(\begin{pmatrix}4a\\0\end{pmatrix}+\kappa\begin{pmatrix}0\\1\end{pmatrix}\right)^\top\begin{pmatrix}0&0\\0&1\end{pmatrix}\left(\begin{pmatrix}4a\\0\end{pmatrix}+\kappa\begin{pmatrix}0\\1\end{pmatrix}\right)+2\begin{pmatrix}-2a\\0\end{pmatrix}^\top\left(\begin{pmatrix}4a\\0\end{pmatrix}+\kappa\begin{pmatrix}0\\1\end{pmatrix}\right)+0=0 \tag{9}$$

$$\begin{pmatrix}4a&\kappa\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}4a\\\kappa\end{pmatrix}+2\begin{pmatrix}-2a&0\end{pmatrix}\begin{pmatrix}4a\\\kappa\end{pmatrix}=0 \tag{10}$$

$$\begin{pmatrix}4a&\kappa\end{pmatrix}\begin{pmatrix}0\\\kappa\end{pmatrix}+2\left(-8a^2\right)=0 \tag{11}$$

$$\kappa^2-16a^2=0 \tag{12}$$

$$\kappa_1=4a \tag{13}$$

$$\kappa_2=-4a \tag{14}$$

$$\tag{15}$$

The intersection points are

$$\mathbf{x_1}=\mathbf{h}+\kappa_1\mathbf{m} \tag{16}$$

$$\mathbf{x_1}=\begin{pmatrix}4a\\4a\end{pmatrix} \tag{17}$$

$$\mathbf{x_2}=\mathbf{h}+\kappa_2\mathbf{m} \tag{18}$$

$$\mathbf{x_2}=\begin{pmatrix}4a\\-4a\end{pmatrix} \tag{19}$$

The Area under the curve is given by

$$A=\int_{-4a}^{4a}\left(\frac{y^2}{4a}\right)dy \tag{20}$$

$$A=\left(\frac{1}{4a}\right)\left(\frac{(4a)^3-\left((-4a)^3\right)}{3}\right) \tag{21}$$

$$A=\left(\frac{1}{12a}\right)\left(128a^3\right) \tag{22}$$

$$A=\frac{32a^2}{3} \tag{23}$$

The area of region bounded by the line $x=4a$ and the parabola $y^2=4ax$ is given as $\frac{256}{3}$

Comparing the above area with the given area we get, the value of a as

$$\frac{32a^2}{3}=\frac{256}{3} \tag{24}$$

$$a=2\sqrt{2} \tag{25}$$

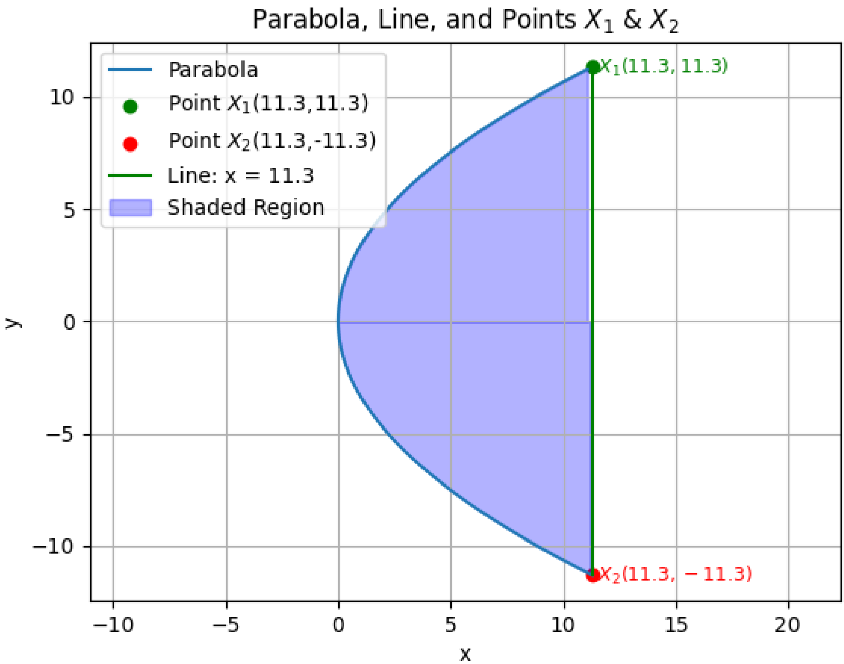

Fig. 1: The parabola along with the line