

9.9.3.9

EE24BTECH11010 - Balaji B

Question:

If the area of the region bounded by the curve $y^2 = 4ax$ and the line $x = 4a$ is $\frac{256}{3}$ sq. units, then using integration, find the value of a , where $a > 0$.

Solution:

Variable	Description
\mathbf{x}_1	First intersection point
\mathbf{x}_2	Second intersection point
\mathbf{h}	Point on the given line
\mathbf{m}	Direction vector of given line
A	Area of the region

TABLE I: Variables Used

The equation of a parabola in Matrix form is

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1)$$

The equation of a line in vector form is

$$\mathbf{x} = \mathbf{h} + \kappa \mathbf{m} \quad (2)$$

For the given parabola $y^2 = 4ax$, The values of $\mathbf{V}, \mathbf{u}, f$ are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3)$$

$$\mathbf{u} = \begin{pmatrix} -2a \\ 0 \end{pmatrix} \quad (4)$$

$$f = 0 \quad (5)$$

For the given line $x = 4a$, The values of \mathbf{h}, \mathbf{m} are

$$\mathbf{h} = \begin{pmatrix} 4a \\ 0 \end{pmatrix} \quad (6)$$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (7)$$

Substituting the line equation in parabola equation gives the values of κ

$$(\mathbf{h} + \kappa \mathbf{m})^\top \mathbf{V} (\mathbf{h} + \kappa \mathbf{m}) + 2\mathbf{u}^\top (\mathbf{h} + \kappa \mathbf{m}) + f = 0 \quad (8)$$

$$\left(\begin{pmatrix} 4a \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 4a \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) + 2 \begin{pmatrix} -2a \\ 0 \end{pmatrix}^\top \left(\begin{pmatrix} 4a \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) + 0 = 0 \quad (9)$$

$$\begin{pmatrix} 4a & \kappa \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4a \\ \kappa \end{pmatrix} + 2 \begin{pmatrix} -2a & 0 \end{pmatrix} \begin{pmatrix} 4a \\ \kappa \end{pmatrix} = 0 \quad (10)$$

$$\begin{pmatrix} 4a & \kappa \end{pmatrix} \begin{pmatrix} 0 \\ \kappa \end{pmatrix} + 2 \begin{pmatrix} -8a^2 \end{pmatrix} = 0 \quad (11)$$

$$\kappa^2 - 16a^2 = 0 \quad (12)$$

$$\kappa_1 = 4a \quad (13)$$

$$\kappa_2 = -4a \quad (14)$$

$$(15)$$

The intersection points are

$$\mathbf{x}_1 = \mathbf{h} + \kappa_1 \mathbf{m} \quad (16)$$

$$\mathbf{x}_1 = \begin{pmatrix} 4a \\ 4a \end{pmatrix} \quad (17)$$

$$\mathbf{x}_2 = \mathbf{h} + \kappa_2 \mathbf{m} \quad (18)$$

$$\mathbf{x}_2 = \begin{pmatrix} 4a \\ -4a \end{pmatrix} \quad (19)$$

The Area under the curve is given by

$$A = \int_{-4a}^{4a} \left(\frac{y^2}{4a} \right) dy \quad (20)$$

$$A = \left(\frac{1}{4a} \right) \left(\frac{(4a)^3 - (-4a)^3}{3} \right) \quad (21)$$

$$A = \left(\frac{1}{12a} \right) (128a^3) \quad (22)$$

$$A = \frac{32a^2}{3} \quad (23)$$

The area of region bounded by the line $x = 4a$ and the parabola $y^2 = 4ax$ is given as $\frac{256}{3}$

Comparing the above area with the given area we get, the value of a as

$$\frac{32a^2}{3} = \frac{256}{3} \quad (24)$$

$$a = 2\sqrt{2} \quad (25)$$

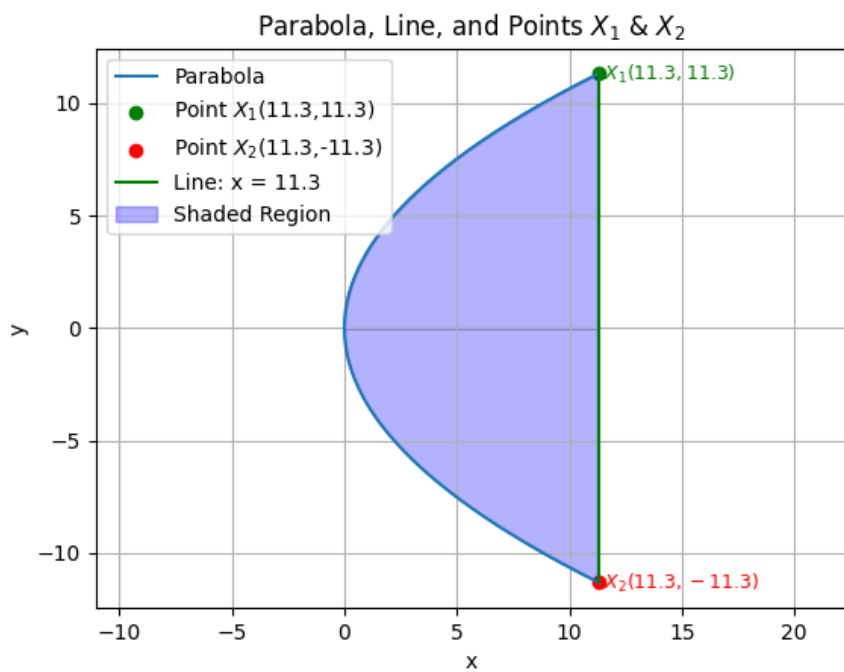


Fig. 1: The parabola along with the line