

# 9.9.5.10

EE24BTECH11010 - Balaji B

## Question:

Find the area of the region included between the circle  $x^2 + y^2 = 8x$  and inside of the parabola  $y^2 = 4x$ .

## Answer:

Variable	Description
$\mathbf{V}_1, \mathbf{u}_1, f_1$	Parameters of Circle
$\mathbf{V}_2, \mathbf{u}_2, f_2$	Parameters of Parabola
$\mathbf{m}_1, \mathbf{m}_2$	The direction vector of lines
$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$	Points of intersection
$\mathbf{h}_1, \mathbf{h}_2$	Point on lines
$A$	Area between the conics

TABLE I: Variables used

The conic parameter for the circle and parabola can be expressed as

$$\mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, f_1 = 0 \quad (1)$$

$$\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, f_2 = 0 \quad (2)$$

Substituting the values of  $\mathbf{V}_i, \mathbf{u}_i, f_i, i = 1, 2$ , for pair of straight lines

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \quad (3)$$

$$\begin{vmatrix} 1 & 0 & -4 - 2\mu \\ 0 & \mu + 1 & 0 \\ -4 - 2\mu & 0 & 0 \end{vmatrix} = 0 \quad (4)$$

$$\mu = -1 \quad (5)$$

The equation of the pair of straight line, is given by

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu \mathbf{V}_2) \mathbf{x} + 2(\mathbf{u}_1 + \mu \mathbf{u}_2)^\top \mathbf{x} + (f_1 + \mu f_2) = 0 \quad (6)$$

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (7)$$

$$\left[ \mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -4 & 0 \end{pmatrix} \right] \mathbf{x} = 0 \quad (8)$$

$$\left[ \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -4 & 0 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix} \quad (9)$$

$$\begin{pmatrix} x-4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad (10)$$

$$(x-4)(x) = 0 \quad (11)$$

$$\therefore x = 0, x = 4 \quad (12)$$

The equation of the lines are

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (13)$$

∴ The intersection of Circle and Parabola is lines with parameters

$$\mathbf{m}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{h}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (14)$$

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{h}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (15)$$

Substituting the line-1 equation in parabola equation gives the value of  $\kappa$

$$(\mathbf{h} + \kappa \mathbf{m})^\top \mathbf{V} (\mathbf{h} + \kappa \mathbf{m}) + 2\mathbf{u}^\top (\mathbf{h} + \kappa \mathbf{m}) + f = 0 \quad (16)$$

$$\left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) + 2 \begin{pmatrix} -2 \\ 0 \end{pmatrix}^\top \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) + 0 = 0 \quad (17)$$

$$\begin{pmatrix} 0 & \kappa \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \kappa \end{pmatrix} + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \kappa \end{pmatrix} = 0 \quad (18)$$

$$\kappa^2 = 0 \quad (19)$$

$$\kappa_1 = 0 \quad (20)$$

The intersection point of line-1 with parabola

$$\mathbf{x}_1 = \mathbf{h}_1 + \kappa_1 \mathbf{m} \quad (21)$$

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (22)$$

Substituting the line-2 equation in parabola equation gives the value of  $\kappa$

$$\left( \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left( \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) + 2 \begin{pmatrix} -2 \\ 0 \end{pmatrix}^\top \left( \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) + 0 = 0 \quad (23)$$

$$\begin{pmatrix} 4 & \kappa \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ \kappa \end{pmatrix} + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ \kappa \end{pmatrix} = 0 \quad (24)$$

$$\kappa^2 - 16 = 0 \quad (25)$$

$$\kappa_1 = 4 \quad (26)$$

$$\kappa_2 = -4 \quad (27)$$

The intersection points of line-2 with parabola

$$\mathbf{x}_2 = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (28)$$

$$\mathbf{x}_3 = \begin{pmatrix} 4 \\ -4 \end{pmatrix} \quad (29)$$

The Area enclosed by parabola is

$$A_1 = 2 \int_0^4 \sqrt{4x} dx \quad (30)$$

$$A_1 = 2 \left[ \frac{4}{3} x^{\frac{3}{2}} \right]_0^4 \quad (31)$$

$$A_1 = \frac{64}{3} \quad (32)$$

The Area enclosed by circle is

$$A_2 = \int_4^8 \sqrt{8x - x^2} dx \quad (33)$$

$$A_2 = \left[ \frac{x-4}{2} \sqrt{8x-x^2} + 8 \sin^{-1} \left( \frac{x-4}{4} \right) \right]_4^8 \quad (34)$$

$$A_2 = 8\pi \quad (35)$$

The Total area enclosed is

$$A = A_1 + A_2 \quad (36)$$

$$A = \frac{64}{3} + 8\pi \quad (37)$$

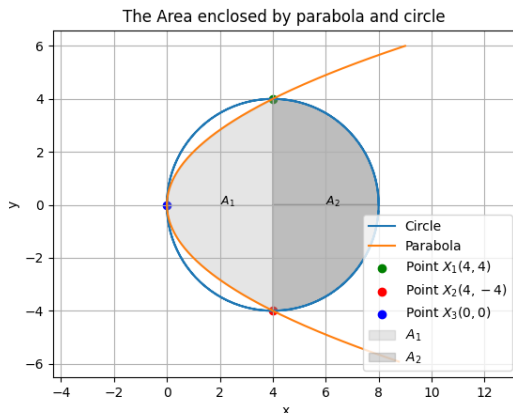


Fig. 1: The Area enclosed by parabola and circle