9.9.5.10

EE24BTECH11010 - Balaji B

Question:
Find the area of the region included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$.
Answer:

Variable	Description
$\mathbf{V}_1, \mathbf{u}_1, f_1$	Parameters of Circle
$\mathbf{V}_2, \mathbf{u}_2, f_2$	Parameters of Parabola
$\mathbf{m}_1, \mathbf{m}_2$	The direction vector of lines
$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$	Points of intersection
$\mathbf{h}_1, \mathbf{h}_2$	Point on lines
A	Area between the conics

TABLE I: Variables used

The conic parameter for the circle and parabola can be expressed as

$$\mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, f_1 = 0 \tag{1}$$

$$\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, f_2 = 0 \tag{2}$$

Substituting the values of $\mathbf{V_i}$, $\mathbf{u_i}$, f_i, $i = 1, 2$, for pair of straight lines

$$\begin{vmatrix} \mathbf{V}_1 + \mu\mathbf{V}_2 & \mathbf{u}_1 + \mu\mathbf{u}_2 \\ (\mathbf{u}_1 + \mu\mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{3}$$

$$\begin{vmatrix} 1 & 0 & -4 - 2\mu \\ 0 & \mu + 1 & 0 \\ -4 - 2\mu & 0 & 0 \end{vmatrix} = 0 \tag{4}$$

$$\mu = -1 \tag{5}$$

The equation of the pair of straight line, is given by

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu\mathbf{V}_2)\mathbf{x} + 2(\mathbf{u}_1 + \mu\mathbf{u}_2)^\top \mathbf{x} + (f_1 + \mu f_2) = 0 \tag{6}$$

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{7}$$

$$\left[\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -4 & 0 \end{pmatrix}\right] \mathbf{x} = 0 \tag{8}$$

$\because$ $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, substituting it we get

$$\left[\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -4 & 0 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} x-4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{10}$$

$$(x-4)(x) = 0 \tag{11}$$

$$\therefore x = 0, x = 4 \tag{12}$$

The equation of the lines are

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{13}$$

$\therefore$The intersection of Circle and Parabola is lines with parameters

$$\mathbf{m}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{h}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{14}$$

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{h}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{15}$$

Substituting the line-1 equation in parabola equation gives the value of κ

$$(\mathbf{h} + \kappa\mathbf{m})^\top \mathbf{V} (\mathbf{h} + \kappa\mathbf{m}) + 2\mathbf{u}^\top (\mathbf{h} + \kappa\mathbf{m}) + f = 0 \tag{16}$$

$$\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \kappa\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \kappa\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + 2\begin{pmatrix} -2 \\ 0 \end{pmatrix}^\top \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \kappa\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + 0 = 0 \tag{17}$$

$$\begin{pmatrix} 0 & \kappa \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \kappa \end{pmatrix} + 2\begin{pmatrix} -2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \kappa \end{pmatrix} = 0 \tag{18}$$

$$\kappa^2 = 0 \tag{19}$$

$$\kappa_1 = 0 \tag{20}$$

The intersection point of line-1 with parabola

$$\mathbf{x}_1 = \mathbf{h}_1 + \kappa_1\mathbf{m} \tag{21}$$

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{22}$$

Substituting the line-2 equation in parabola equation gives the value of k

$$\left(\begin{pmatrix}4\\0\end{pmatrix}+\kappa\begin{pmatrix}0\\1\end{pmatrix}\right)^{\top}\begin{pmatrix}0&0\\0&1\end{pmatrix}\left(\begin{pmatrix}4\\0\end{pmatrix}+\kappa\begin{pmatrix}0\\1\end{pmatrix}\right)+2\begin{pmatrix}-2\\0\end{pmatrix}^{\top}\left(\begin{pmatrix}4\\0\end{pmatrix}+\kappa\begin{pmatrix}0\\1\end{pmatrix}\right)+0=0 \tag{23}$$

$$\begin{pmatrix}4&\kappa\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}4\\\kappa\end{pmatrix}+2\begin{pmatrix}-2&0\end{pmatrix}\begin{pmatrix}4\\\kappa\end{pmatrix}=0 \tag{24}$$

$$\kappa^2-16=0 \tag{25}$$

$$\kappa_1=4 \tag{26}$$

$$\kappa_2=-4 \tag{27}$$

The intersection points of line-2 with parabola

$$\mathbf{x_2}=\begin{pmatrix}4\\4\end{pmatrix} \tag{28}$$

$$\mathbf{x_3}=\begin{pmatrix}4\\-4\end{pmatrix} \tag{29}$$

The desired area is given by

$$A=2\left[\int_0^4\sqrt{4x}dx+\int_4^8\sqrt{8x-x^2}\right] \tag{30}$$

$$A=2\left[\left[\frac{4}{3}x^{\frac{3}{2}}\right]_0^4+\left[\frac{x-4}{2}\sqrt{8x-x^2}+8\sin^{-1}\left(\frac{x-4}{4}\right)\right]_4^8\right] \tag{31}$$

$$A=\frac{64}{3}+8\pi \tag{32}$$

$\therefore$ The Area A is $\frac{64}{3}+8\pi$

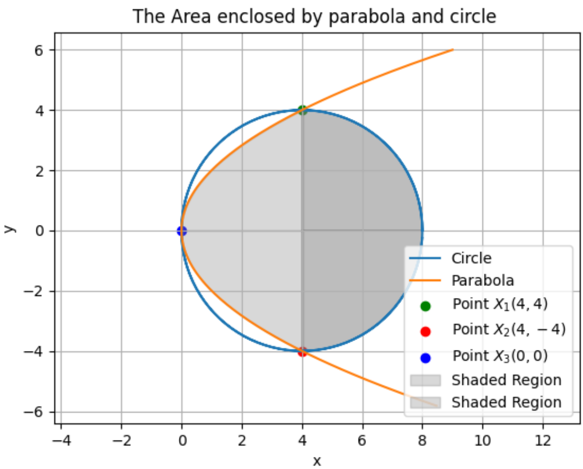

Fig. 1: The Area enclosed by parabola and circle