Chapter 9: $\mathbb{R}^n$, Take 2!

The big take aways:

- Subspaces of $\mathbb{R}^n$ (special subsets that are "closed" under linear combinations)
- "Range" of a matrix and the "null space" of a matrix: two subspaces that tell you everything about the solutions of linear equations
- basis vectors: finite "generators of subspaces
- orthogonal vectors:  "right angles in $\mathbb{R}^n$!
- a super cool matrix factorization

  QR $\quad Q^T Q = I_m$, R = Upper Triangular

The only "better" matrix factorization is called SVD Singular value decomposition

(Appendix of our book)

Review

$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \,\middle|\, x_i \in \mathbb{R} \right\}$ collection of all column vectors with n-components.

(1) $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$ vector addition

(2) $\alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$ scalar times vector multiplication

TFAE

(a) (1) and (2) hold

(b) For all scalars $\alpha, \beta \in \mathbb{R}$, vectors $x, y \in \mathbb{R}^n$

$$\alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \vdots \\ \alpha x_n + \beta y_n \end{bmatrix}$$ linear combinations

Subset notation: $V \subset W$, V is a subset of W, $V \subset W \Leftrightarrow v \in V \Rightarrow v \in W$.

We allow $V = W$. In other words, any set is a subset of itself.

Def. Let $V \subset \mathbb{R}^n$ be a subset.
V is a subspace of $\mathbb{R}^n$ if V is closed under linear combinations.
That is, for all $\alpha_1, \alpha_2 \in \mathbb{R}$, and vectors $v_1, v_2 \in V$, the linear combination

$$\alpha_1 v_1 + \alpha_2 v_2 \in V.$$

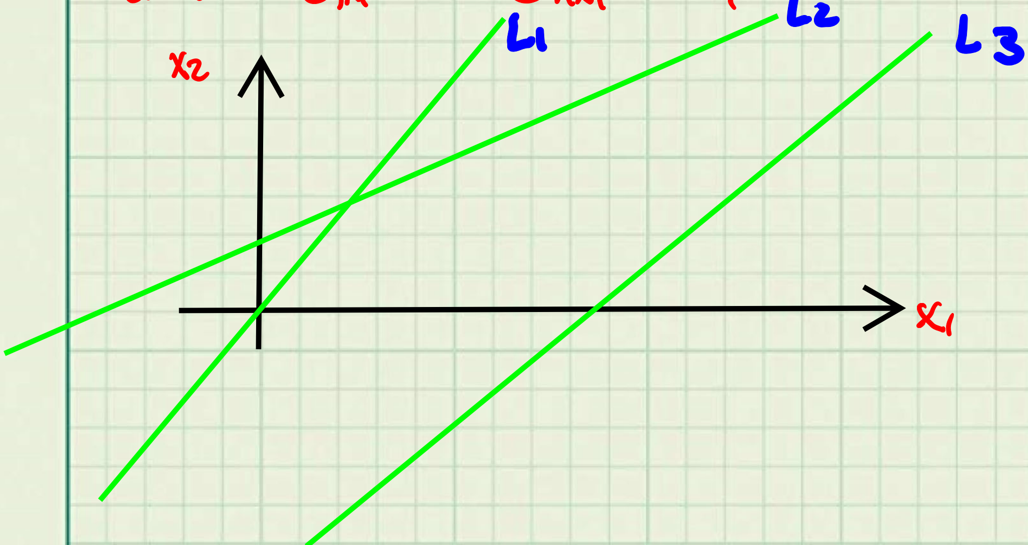
Fact: TFAE

(a) V is closed under linear combinations

(b) V is closed under vector addition and scalar times vector multiplication.

Elementary Screening Test to Always Apply First: If $V \subset \mathbb{R}^n$ is a subspace, then $0_{n\times 1} \in V$.

Why? Suppose $v \in V$. Then $0_{1\times 1} \cdot v \in V$ and $0_{1\times 1} \cdot v = 0_{n\times 1}$!!!



Which of the lines L_1, L_2, L_3 is potentially a subspace?

Only L_1 is potentially a subspace!

We can view $n \times m$ matrices as functions!

$A = n \times m$

$f: \mathbb{R}^m \to \mathbb{R}^n$ by $f(x) := Ax \in \mathbb{R}^n$

The following subsets are naturally motivated by the function view of a matrix.

Def.

(a) $\mathrm{null}(A) := \{x \in \mathbb{R}^m \mid A \cdot x = 0_{n \times 1}\}$
called the null space of A.

(b) $R(A) := \{y \in \mathbb{R}^n \mid y = Ax \text{ for some } x \in \mathbb{R}^m\}$
is called the range of A.

Relation to $Ax = b$, system of linear equations.

Suppose that $\bar{x}$ is a solution of $Ax = b$. $\therefore A\bar{x} = b$.

Let $\tilde{x} \in \text{null}(A)$. Then,

$$A(\bar{x}+\tilde{x}) = \cancelto{b}{A\bar{x}} + \cancelto{0_{n\times 1}}{A\tilde{x}} = A\bar{x} = b$$

Once you know one solution to $Ax=b$, you can generate all solutions to $Ax=b$ once you have computed null(A)!!

One solution: $\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{3\times 1} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{2\times 1}$

What about the range, R(A)?

$$R(A) := \{ y \in \mathbb{R}^n \mid y = Ax \text{ for some } x \in \mathbb{R}^m \}$$
$$= \{ b \in \mathbb{R}^n \mid b = Ax \text{ for some } x \in \mathbb{R}^m \}$$

(I can call my vectors in $\mathbb{R}^n$ anything I want)

$R(A) = \{ b \in \mathbb{R}^n \mid Ax = b, \text{ has a solution} \}$

$\therefore$ $Ax = b$ has a solution $\Leftrightarrow b \in R(A)$.

<u>Claim</u>: null(A) is a subspace of $\mathbb{R}^m$ and $R(A)$ is a subspace of $\mathbb{R}^n$.

<u>Why?</u> Will give explanation for null(A) and leave you to ponder $R(A)$.

Seek to show null(A) is closed under linear combinations. Let $x, \bar{x} \in$ null(A) and $\alpha, \bar{\alpha} \in \mathbb{R}$.

$$
\begin{aligned}
A(\alpha x + \bar{\alpha}\bar{x}) &= A(\alpha x) + A(\bar{\alpha}\bar{x}) \\
&= \alpha A x + \bar{\alpha} A \bar{x} \\
&= \alpha \, 0_{n\times 1} + \bar{\alpha} \, 0_{n\times 1} \\
&= 0_{n\times 1}
\end{aligned}
$$

because $Ax = 0 \Leftrightarrow x \in$ null(A) and

$A\bar{x} = 0 \Leftrightarrow \bar{x} \in$ null(A)

$\therefore$ $\alpha x + \bar{\alpha}\bar{x} \in$ null(A). $\blacksquare$

Does null(A) pass the "smell test" $0 \in null(A)$?

Yes $f(x) = Ax$, $f(0_{m\times 1}) = A \cdot 0_{m\times 1} = 0_{n\times 1}$

$0_{m\times 1} \in null(A)$!