AIS - COHOMOLOGY OF COMMUTATIVE ALGEBRAS

LECTURE MK1

Lecture 1: Koszul Complexes, Gorenstein Rings and Poincaré Algebras

Convention 1. R is always assumed to be a noetherian commutative ring with unity, unless mentioned otherwise.

Let M be a finitely generated R-module and $f \in M^* := \mathrm{Hom}_R(M, R)$. We want to define the Koszul complex $K_\bullet(M, f)$ in a coordinate free way. Recall the wedge product

$$\bigwedge M = \frac{\bigoplus_{n \geqslant 0} M^{\otimes n}}{\text{“A certain graded ideal”}} = \bigoplus_{i \geqslant 0} \bigwedge^i M$$

$\bigwedge M$ is a skew commutative associative graded algebra.

Definition 2. We can view $\bigwedge M$ as a complex with the boundary maps given by

$$d : \bigwedge^i M \to \bigwedge^{i-1} M$$
$$x_1 \wedge \ldots \wedge x_i \mapsto \sum_{j=1}^{i} (-1)^j f(x_j)(x_1 \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_i)$$

It is easily checked that $d^2 = 0$ (this is the same computation as in the case of composition of boundary maps of the singular chain complex in topology). We call this complex the *Koszul complex* (with respect to M and $f \in M^*$) and denote it by $K_\bullet(M, f)$.

Observation 3. For homogeneous $x, y \in K_\bullet(M, f)$, $d(x \wedge y) = dx \wedge y + (-1)^{\deg(x)} x \wedge dy$. If $x \in K_1(M, f) \cong M$, then $dx = -f(x)$.

Example 4. The form in which one has probably seen the Koszul complex is something like

$$0 \to R \xrightarrow{\binom{-b}{a}} R \oplus R \xrightarrow{(a \quad b)} R$$

This is in fact a special case of the above definition. Put $M = R \oplus R$, with $(e_1 := (1,0), e_2 := (0,1))$ as a basis and let $f : M \to R$ be given by $e_1 \mapsto a$ and $e_2 \mapsto b$. A similar construction can be carried out for $R^{\oplus n}$ and this reduces to the (probably) more familiar version. For $x_1, \ldots, x_n \in R$, we will write $K_\bullet(x_1, \ldots, x_m, R)$ for the Koszul complex $K_\bullet(R^{\oplus n}, f)$, where $f : R^{\oplus n} \to R$, $e_i \mapsto x_i$.

Convention 5. From now on, we will assume that $(R, \mathfrak{m}, k)$ is noetherian local, with maximal ideal $\mathfrak{m}$, residue field k and with minimal generating set $\{x_1, \ldots, x_n\}$ for $\mathfrak{m}$. We will write either of $K_\bullet^R$, $K_\bullet$ or $K_\bullet(x_1, \ldots, x_n)$ to mean the same as $K_\bullet(x_1, \ldots, x_n, R)$

Remark 6. Recall that by NAK, any generating set of $\mathfrak{m}/\mathfrak{m}^2$ as a k-vector space can be lifted to a generating set of $\mathfrak{m}$. So, any two minimal generating sets of the maximal ideal of a noetherian local ring have the same finite number of elements.

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Remark 7. If $\{y_1, \ldots, y_n\}$ is another minimal generating set for $\mathfrak{m}$, then using NAK, it can be shown that there exists an isomorphism $\phi : R^n \to R^n$ that makes the following diagram commute.

$$\begin{tikzcd} R^n \arrow[r, "e_i \mapsto x_i"] \arrow[d, "\phi"'] & R \arrow[d, equal] \\ R^n \arrow[r, "e_i \mapsto y_i"'] & R \end{tikzcd}$$

By functoriality, $K_\bullet(x_1, \ldots, x_n, R) \cong K_\bullet(y_1, \ldots, y_n, R)$. This justifies the notation adopted in Convention 5.

Definition 8. Define $Z_i := \ker(K_i \to K_{i-1})$ (called the i-cycles of $K_\bullet$), $B_i := \operatorname{Im}(K_{i+1} \to K_i)$ (called the i-boundaries of $K_\bullet$), $Z := \bigoplus Z_i$ and $B := \bigoplus B_i$. Observe that Z is canonically a subalgebra of $K_\bullet$ and that B is an ideal of Z. Define the *Koszul homology* of R as $H(R) := Z/B$. $H(R)$ is canonically an R-algebra.

Example 9 (Characterisation of regular local rings)**.** A noetherian local ring $(R, \mathfrak{m}, k)$ is regular if and only if $H(R) = k$ (in degree 0).

Example 10. Let $(S, \mathfrak{n}, k)$ be a regular local ring and $R = S/I$. Then,

$$H_\bullet(R) = \operatorname{Tor}^S_\bullet(R, k)$$

The results holds in general but to simplify the argument, we will assume that $I \subseteq \mathfrak{n}^2$. This is to ensure $\operatorname{embdim}(R) = \operatorname{embdim}(S)$ as

$$\operatorname{embdim}(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim_k(\mathfrak{n} + I/\mathfrak{n}^2 + I) = \dim_k(\mathfrak{n}/\mathfrak{n}^2) = \operatorname{embdim}(S)$$

Let $x_1, \ldots, x_n$ be a minimal generating set of $\mathfrak{n}$. Then, $\overline{x}_1, \ldots, \overline{x}_n$ is a minimal generating set of $\mathfrak{m}$. Let $K^R_\bullet = K_\bullet(\overline{x}_1, \ldots, \overline{x}_n)$ and $K^S_\bullet = K_\bullet(x_1, \ldots, x_n)$. Then, $K^R_\bullet = K^S_\bullet \otimes_S R$. Hence,

$$H_i(R) = H_i(K^R_\bullet) = H_i(K^S_\bullet \otimes_S R) = \operatorname{Tor}^S_i(R, k)$$

If $F_\bullet \to R$ is a minimal free resolution of R as an S-module, then $\dim_k(H_i(R)) = \operatorname{rk}_S(F_i)$.
Remark 11 (Depth Sensitivity of the Koszul Complex).
Let $s := \operatorname{embdim}(R) - \operatorname{depth}(R)$. Then, $H_i(R) = 0$ for all $i > s$ and $H_s(R) \neq 0$.

Proposition 12. *Let $y \in \mathfrak{m}$ be a non-zerodivisor in R. Then,*

(1) *For all i, there is an exact sequence*

$$0 \to H_i(R) \to H_i(R/(y)) \to H_{i-1}(R) \to 0$$

(2) *Let $s = \operatorname{embdim}(R) - \operatorname{depth}(R)$. Then, $H_{i+1}(R/(y)) \cong H_i(R)$ for all $i \geqslant s$.*

Proof. See [Mat89, Theorem 16.4] for (1). (2) follows from (1). □

Corollary 13. *Let $t = depth(R)$, $y_1, \ldots, y_t$ be a maximal regular sequence, $J = (y_1, \ldots, y_t)$, $n = embdim(R)$ and $s = n - t$. Then,*

$$\begin{aligned} H_s(R) = H_n(R/J) &= \ker((R/J) \xrightarrow{(\pm x_1\ \pm x_2\ \ldots\ \pm x_n)^T} (R/J)^n) \\ &= 0 :_{R/J} \mathfrak{m} \\ &= \operatorname{Hom}_R(k, R/J) \end{aligned}$$

Definition 14. R is said to be *Gorenstein* if it is Cohen-Macaulay and $H_s(R) \cong k$, where $s := \operatorname{embdim}(R) - \operatorname{depth}(R)$. (Recall Convention 5)

The following are well-known equivalent reformulations, possibly under further assumptions.

Proposition 15.

(1) *Let $(R, \mathfrak{m}, k)$ be an artinian local ring. Then, the following are equivalent*
  (a) *R is Gorenstein*
  (b) $(0 :_R \mathfrak{m}) \cong k$
  (c) *R is an injective R-module*

(2) *R is Gorenstein if and only if $R/(y)$ is Gorenstein for all non-zerodivisors $y \in \mathfrak{m}$.*

(3) *R is Gorenstein if and only if the injective dimension of R as a module over itself is finite.*

Definition 16. A finite dimensional associative (and not necessarily commutative) graded k-algebra $A = \bigoplus_{i=0}^{g} A_i$, with (assuming for safety) $A_0 = k$, is called a *Poincaré algebra* if

$$\begin{aligned} A_i &\to \mathrm{Hom}_k(A_{g-i}, A_g) \\ a &\mapsto [b \mapsto ab] \end{aligned}$$

is an isomorphism of A_0-modules for all i.

Example 17. Let $R = R_0 \oplus R_1 \oplus \ldots \oplus R_g$ (with $g > 0$ and $R_g \neq 0$) be a standard graded artinian k-algebra with $R_0 = k$. So, R is local with maximal ideal $\mathfrak{m} := \bigoplus_{1 \leqslant i \leqslant g} R_i$. Assume that R is Gorenstein. Since $R_g\mathfrak{m} = 0$, $R_g \subseteq \mathrm{soc}(R)$. Since R is Gorenstein, local and artininan, dimension of $\dim_k(\mathrm{soc}(R)) = 1$ and since $R_g\mathfrak{m} = 0$, $\mathrm{soc}(R) = R_g$. Let $a \in R_i$ be non-zero. Then, since R is artinian (in which case the socle is an essential submodule), $(a) \cap \mathrm{soc}(R) \neq (0)$. Since R is Gorenstein, $\mathrm{soc}(R) \subseteq (a)$. Hence, there exists $b \in R_{g-i}$ such that ab generates $\mathrm{soc}(R)$. This means that the map $R_i \to \mathrm{Hom}_k(R_{g-i}, R_g)$ induced by multiplication is injective for all i. If it is further assumed that R is a finite dimensional over k, then by a dimension argument, we conclude that $R_i \to \mathrm{Hom}_k(R_{g-i}, R_g)$ is an isomorphism for all i and that R is a Poincaré algebra.

Example 18. Let V be a finite dimensional k-vector space with an ordered basis $(v_1, \ldots, v_n)$. We will show that $\bigwedge V$ is a Poincaré algebra. Clearly, the multiplication map $k \cong \bigwedge^0 V \to \mathrm{Hom}_k(\bigwedge^n V, \bigwedge^n V) \cong \mathrm{Hom}_k(k,k)$ is an isomorphism. Fix $i \geqslant 1$ and let $\mu := \sum_{1 \leqslant j_1 < j_2 < \ldots < j_i \leqslant n} \alpha_{j_1, \ldots, j_n}(v_{j_1} \wedge \ldots \wedge v_{j_i}) \in \bigwedge^i V$ be non-zero with $\alpha_{k_1,\ldots,k_n} \neq 0$ for some $1 \leqslant k_1 < \ldots < k_n \leqslant n$. Then, $\mu \wedge (v_1 \wedge \ldots \wedge \hat{v}_{j_1} \ldots \wedge \hat{v}_{j_n} \wedge \ldots \wedge v_n) \neq 0$. Thus, the map $\bigwedge^i V \to \mathrm{Hom}_k(\bigwedge^{n-i} V, \bigwedge^n V)$ is injective. Comparing dimensions as k-vector spaces, it is further an isomorphism.

References

[Mat89] Matsumura, Hideyuki. *Commutative ring theory.* No. 8. Cambridge university press, 1989.