Milnor's Definition of a Manifold

The aim of this document is to compare the definitions of an n-manifold as in [MS74] and [Lee00]. For convenience, we shall refer to smooth n-manifolds according to [MS74] "Milnor n-manifolds". What we call a "chart n-manifold" is precisely a smooth n-manifolds according to, for example, [Lee00] *except* that we do not assume the underlying topological space to be second countable.

Convention 1. $\mathbb{R}^n$ will be viewed both as a chart n-manifold and a Milnor n-manifold with the singleton chart and the singleton local parametrisation respectively.

Notation 2. For topological spaces A and B, "$A \subseteq_{\text{o}} B$" means "A is an open subspace of B".

Notation 3. Given a chart n-manifold M, let F_M denote the set of smooth (in the "chart sense") real valued functions on M. Let $i : M \to \mathbb{R}^{F_M}$ denote the evaluation map.

Proposition 4. Let M be a chart n-manifold. Then, $i(M)$ is canonically a Milnor manifold.

Proof. Fix $x \in M$. Let $\alpha : \mathbb{R}^n \to U \subseteq_{\text{o}} M$ be a chart around x in the maximal atlas of M. We claim that the composition

$$\beta : \mathbb{R}^n \xrightarrow{\alpha} U \xrightarrow{i} i(U) \subseteq \mathbb{R}^{F_M}$$

is a local parametrisation around $i(x)$. In order to prove this, it is sufficient to show the following:

(1) β is smooth.
(2) For all $y \in \mathbb{R}^n$, $D\beta|_y$ has rank n.
(3) $\mathrm{Im}(\beta)$ is open and β is a homeomorphism onto its image.

For $f \in F_M$, let $\pi_f : \mathbb{R}^{F_M} \to \mathbb{R}$ denote the projection to the f^{th} coordinate. Then, for any $f \in F_M$, $\pi_f \circ \beta = f|_U \circ \alpha$, which is smooth. Hence, β is smooth.

For $i = 1, \ldots, n$, let $\psi_i : U \to \mathbb{R}$ such that $\psi_i \circ \alpha = \pi_i$. Clearly, ψ_i is smooth on U. Fix $y \in \mathbb{R}^n$. Let $g_i \in F_M$ such that g_i agrees with ψ_i in an open neighbourhood of $\alpha(y)$. Consider the composition

$$\mathbb{R}^n \xrightarrow{\beta} i(M) \subseteq \mathbb{R}^{F_M} \xrightarrow{\pi_{g_1}, \ldots, \pi_{g_n}} \mathbb{R}^n$$

whose components are clearly $g_i \circ \alpha$. Thus,

$$\frac{\partial}{\partial u_j}(g_i \circ \alpha)\bigg|_y = \frac{\partial}{\partial u_j}(\psi_i \circ \alpha)\bigg|_y = \frac{\partial u_i}{\partial u_j}\bigg|_y = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

This proves (2). Lastly, suppose that $V \subseteq_{\text{o}} U$. For all $v \in V$, let $g_v \in F_M$ such that $g_v(v) > 0$ and $\mathrm{supp}(g_v) \subseteq V$. Let $\pi_{g_v} : \mathbb{R}^{F_M} \to \mathbb{R}$ denote the projection to the g_v^{th} coordinate. Then,

- $\pi_{g_v}^{-1}(\mathbb{R} \setminus \{0\}) \subseteq_{\text{o}} \mathbb{R}^{F_M}$.
- $i(v) \in \pi_{g_v}^{-1}(\mathbb{R} \setminus \{0\}) \cap i(M) \subseteq_{\text{o}} i(M)$.
- $i(m) \notin \pi_{g_v}^{-1}(\mathbb{R} \setminus \{0\}) \cap i(M)$ for all $m \in M \setminus V$.

Hence,

$$i(V) = \bigcup_{v \in V} \pi_{g_v}^{-1}(\mathbb{R} \setminus \{0\}) \cap i(M) \subseteq_{\text{o}} i(M)$$

This proves (3). $\square$

Remark 5. In the proof of Proposition 4, we might as well have relaxed the assumption that the domain of α is $\mathbb{R}^n$ and the same argument could have been used to construct a local parametrisation. The constructed Milnor manifold structure is said to be canonical as any chart translates to a local parametrisation by pre-composition.

Remark 6. Proposition 4 essentially says that a chart n-manifold admits a smooth embedding into $\mathbb{R}^A$ for some set A.

Remark 7. Conversely, given a Milnor n-manifold N, the local parametrisations on the underlying topological space provide an atlas for N, making it a chart n-manifold. It is also easy to see, under this correspondence, that the definitions of smooth functions (see [MS74, P.17] and [Lee00, P.34]) coincide.

References

[MS74] John Milnor and James Stasheff. *Characteristic Classes.*
[Lee00] John Lee. *Introduction to Smooth Manifolds.*