SOLUTIONS TO EXERCISES IN HOCHSTER'S NOTES ON LOCAL COHOMOLOGY

This is a set of solutions to exercises in [Hoc] along with some random comments

(1.4) Pick a non-zero n in $\mathrm{Soc}(N)$. Then, $\mathrm{Ann}_R(n) = \mathfrak{m}$ and the submodule Rn of N is simple. Since S intersects this submodule non-trivially, it contains all of Rn. Thus, $\mathrm{Soc}(N) \subseteq S$. $\square$

(1.5) Let K be a non-zero submodule of $N_1 \oplus N_2$. Let $0 \neq (k_1, k_2) \in K$. If $k_1 \in M_1$ and $k_2 \in M_2$, then K intersects $M_1 \oplus M_2$ non-trivially. So, WLOG, assume that $k_1 \notin M_1$. Then, there exists $r \in R$ such that $0 \neq rk_1 \in M_1$. If $rk_2 \in M_2$, then we are done. Otherwise, there exists $s \in R$ such that $0 \neq srk_2 \in M_2$. This implies $0 \neq (srk_1, srk_2) \in M_1 \oplus M_2$. $\square$

(1.7) By (1.5), $E(M_1) \oplus E(M_2)$ is an essential extension of $M_1 \oplus M_2$ and also injective, being a finite direct sum of injective modules. As injective modules do not admit proper essential extensions, $E(M_1 \oplus M_2) \cong E(M_1) \oplus E(M_2)$. This immediately generalises to finite direct sums. The case of infinite direct sums reduces to the finite one as follows – every element of say $\bigoplus N_i$ is contained in $N_{i_1} \oplus \ldots \oplus N_{i_k}$ for some $i_1, \ldots, i_k$. Now use the fact that $M_{i_1} \oplus \ldots \oplus M_{i_k}$ is an essential submodule of $N_{i_1} \oplus \ldots \oplus N_{i_k}$ $\square$

(2.5) Obviously, $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(N)$. Suppose $\mathfrak{p} \in \mathrm{Ass}(N)$. Then, there is an element $n \in N$ with $\mathrm{Ann}_R(n) = \mathfrak{p}$. Since $M \subseteq N$ is essential, there exists $r \in R$ such that $0 \neq rn \in M$. Then, $r \notin \mathfrak{p}$ and since $\mathfrak{p}$ is prime, $\mathrm{Ann}_R(rn) = \mathfrak{p}$. $\square$

(8.2) This is an immediate consequence of [Har77].

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Comments

(1) Partial Converse to Corollary 0.2

If R is a domain and M is a divisible and torsion free R-module, then M is injective. Consequently, if R is a domain, then $E(R) = \mathrm{Frac}(R)$.

Proof. [1] Let I be an ideal of R and $f : I \to M$ be an R-module map. If $I = 0$, then f obviously extends to R. Assuming otherwise, let $0 \neq i \in I$. Since M is divisible, there exists m' such that $i.m' = f(i)$. We claim that $\tilde{f} : R \to M$, $1 \mapsto m'$ is the required map. If $0 \neq j \in I$, then $if(j) = f(ij) = jf(i) = jim = i\tilde{f}(j)$. Since M is torsion-free, this implies that $f(j) = \tilde{f}(j)$. The second statement follows easily as $\mathrm{Frac}(R)$ is torsion-free, divisible and an essential extension of R. □

(2) Observation preceding Theorem 0.4

The C-bilinar map $M \otimes_R N \to C$ that he is referring to is obtained by considering the concrete construction and sending $r_{m,n} \mapsto B(rm, n) = B(m, rn)$. This turns out to well defined and C-bilinear because B is.

(3) Possible error in Example 1.4

Let $R = k[x_i]_{i\in\mathbb{N}}$ for a field k. Let $\mathfrak{m} = (x_i)_{i\in\mathbb{N}}$ and $R_\mathfrak{m}$ be the localisation of R at the maximal ideal $\mathfrak{m}$. Then, $(R, \mathfrak{m}, k)$ is a local ring and $R_\mathfrak{m}/(\mathfrak{m}R_\mathfrak{m})^2$ is a finitely generated $R_\mathfrak{m}$-module. But

$$x_1 R_\mathfrak{m} \subsetneq (x_1, x_2)R_\mathfrak{m} \subsetneq \dots$$

is a strictly increasing chain of submodules in $R_\mathfrak{m}/(\mathfrak{m}R_\mathfrak{m})^2$. This means that $R_\mathfrak{m}/(\mathfrak{m}R_\mathfrak{m})^2$ is not noetherian and hence not of finite length.

However, if we assume that $(R, \mathfrak{m}, k)$ is a noetherian local ring, then the statement is true. For a finitely generated submodule N' of N, some power of $\mathfrak{m}$, say $\mathfrak{m}^n$ kills all of N'. Hence, $\mathrm{Ann}_R(N')$ contains $\mathfrak{m}^n$ and consequently, the only prime ideal that contains $\mathrm{Ann}_R(N')$ is $\mathfrak{m}$. This implies that $R/\mathrm{Ann}_R(N')$ has dimension 0. Thus, $R/\mathrm{Ann}_R(N')$ is artinian. Being a finitely generated module over the artinian ring $R/\mathrm{Ann}_R(N')$, N' is artinian and hence of finite length.

(4) Direct summands of injective modules are injective

Useful little fact. For a quick proof, if $E \oplus M \cong E'$ and E' is injective, then take the pushout of a given $E \hookrightarrow K$ along the inclusion $E \hookrightarrow E'$. Compose a left inverse of the base change with the projection to the first coordinate.

(5) Essential extensions are not closed under taking pushouts

$$\begin{tikzcd}
\mathbb{Z}/p\mathbb{Z} \arrow[r, hook, "\cdot p"] \arrow[d, hook, "\cdot p"'] \arrow[dr, phantom, "\ulcorner"] & \mathbb{Z}/p^2\mathbb{Z} \arrow[d, hook] \\
\mathbb{Z}/p^2\mathbb{Z} \arrow[r, hook] & \dfrac{\frac{\mathbb{Z}}{p^2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^2\mathbb{Z}}}{(p,p)}
\end{tikzcd}$$

(6) Remark preceding Proposition 2.3

That R is noetherian is important although we don't need $0 \neq M$ to be finitely generated (for $\mathrm{Ass}(M) \neq \emptyset$). Consider the collection of ideals $\mathrm{Ann}_R(x)$ indexed by $x \in M$. Since R is notherian, this collection contains a maximal element with respect to inclusion, say $\mathfrak{p} = \mathrm{Ann}_R(n)$. We claim that $\mathfrak{p}$ is prime. Suppose for the sake of contradiction that there exist $r, s \in R$ such that $rs \in \mathfrak{p}$, $r \notin \mathfrak{p}$ and $s \notin \mathfrak{p}$. Then, $rn \neq 0$ and $s \in \mathrm{Ann}_R(rn) \supset \mathfrak{p}$ contradicting maximality. So, $\mathfrak{p} \in \mathrm{Ass}(M)$.

[1] Proof sourced from this post

(7) Some results towards a clearer proof of Theorem 2.4 (a)

(a) If $C \to R$ is a ring map and E is an injective C-module, then $\mathrm{Hom}_C(R, E)$ is an injective R-module.

Proof. Since E is an injective C-module, $\mathrm{Hom}_C(\_, E)$ is exact. As endofunctors on R-Mod,

$$\mathrm{Hom}_R(\_, \mathrm{Hom}_C(R, E)) \cong \mathrm{Hom}_C(\_ \otimes_R R, E) \cong \mathrm{Hom}_C(\_, E)$$

The statement immediately follows. It is tempting to extend this to an if and only if statement but then remember that $\mathrm{Hom}_C(\_, E)$ needs to be exact for all C-modules (not just R-modules) for E to be an injective C-module. $\square$

Quick remark: Extension of scalars takes projective S-modules to projective R-modules.

(b) If I is an ideal of C, $R = C/I$ and M is a C-module, We think of $\mathrm{Hom}_C(R, M)$ as the C-submodule of M consisting of elements that are killed by I. This also has an R-module structure.

(c) Using the above facts, if $\mathfrak{p}$ is a prime ideal in a noetherian ring as in [Hoc, Theorem 2.4], it follows that $\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{p}))$ is an injective $R/\mathfrak{p}$-module. As an R-module, we may canonically identify $\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{p}))$ as an R-submodule of $E(R/\mathfrak{p})$ containing $R/\mathfrak{p}$. Hence, $\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{p}))$ is an essential extension (since R-submodules of $\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{p}))$ coincide with $R/\mathfrak{p}$-submodules) of $R/\mathfrak{p}$. Thus, $\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{p})) = E_{R/\mathfrak{p}}(R/\mathfrak{p})$.

(d) By Remark (1), $E_{R/\mathfrak{p}}(R/\mathfrak{p}) = \mathrm{Frac}(R/\mathfrak{p}) \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$

Questions

1. Can the construction of the injective hull be made functorial?
2. Is the homomorphic image of an injective module injective? Probably not true.

References

[Hoc] Hochster, Mel. *Local Cohomology* (https://dept.math.lsa.umich.edu/~hochster/615W11/loc.pdf)

[Har77] Hartshorne, Robin. *Algebraic Geometry*