SOLUTIONS TO EXERCISES IN HOCHSTER'S NOTES ON LOCAL COHOMOLOGY

This is a set of solutions to exercises in [Hoc1] along with some random comments

(1.4) Pick a non-zero n in $\mathrm{Soc}(N)$. Then, $\mathrm{Ann}_R(n) = \mathfrak{m}$ and the submodule Rn of N is simple. Since S intersects this submodule non-trivially, it contains all of Rn. Thus, $\mathrm{Soc}(N) \subseteq S$. □

(1.5) Let K be a non-zero submodule of $N_1 \oplus N_2$. Let $0 \neq (k_1, k_2) \in K$. If $k_1 \in M_1$ and $k_2 \in M_2$, then K intersects $M_1 \oplus M_2$ non-trivially. So, WLOG, assume that $k_1 \notin M_1$. Then, there exists $r \in R$ such that $0 \neq rk_1 \in M_1$. If $rk_2 \in M_2$, then we are done. Otherwise, there exists $s \in R$ such that $0 \neq srk_2 \in M_2$. This implies $0 \neq (srk_1, srk_2) \in M_1 \oplus M_2$. □

(1.7) By (1.5), $E(M_1) \oplus E(M_2)$ is an essential extension of $M_1 \oplus M_2$ and also injective, being a finite direct sum of injective modules. As injective modules do not admit proper essential extensions, $E(M_1 \oplus M_2) \cong E(M_1) \oplus E(M_2)$. This immediately generalises to finite direct sums. The case of infinite direct sums reduces to the finite one as follows – every element of say $\bigoplus N_i$ is contained in $N_{i_1} \oplus \ldots \oplus N_{i_k}$ for some $i_1, \ldots, i_k$. Now use the fact that $M_{i_1} \oplus \ldots \oplus M_{i_k}$ is an essential submodule of $N_{i_1} \oplus \ldots \oplus N_{i_k}$ □

(2.5) Obviously, $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(N)$. Suppose $\mathfrak{p} \in \mathrm{Ass}(N)$. Then, there is an element $n \in N$ with $\mathrm{Ann}_R(n) = \mathfrak{p}$. Since $M \subseteq N$ is essential, there exists $r \in R$ such that $0 \neq rn \in M$. Then, $r \notin \mathfrak{p}$ and since $\mathfrak{p}$ is prime, $\mathrm{Ann}_R(rn) = \mathfrak{p}$. □

(8.2) This is an immediate consequence of [Har77].

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Comments

(1) Partial Converse to Corollary 0.2

If R is a domain and M is a divisible and torsion free R-module, then M is injective. Consequently, if R is a domain, then $E(R) = \mathrm{Frac}(R)$.

Proof. [1] Let I be an ideal of R and $f : I \to M$ be an R-module map. If $I = 0$, then f obviously extends to R. Assuming otherwise, let $0 \neq i \in I$. Since M is divisible, there exists m' such that $i.m' = f(i)$. We claim that $\tilde{f} : R \to M$, $1 \mapsto m'$ is the required map. If $0 \neq j \in I$, then $if(j) = f(ij) = jf(i) = jim = i\tilde{f}(j)$. Since M is torsion-free, this implies that $f(j) = \tilde{f}(j)$. The second statement follows easily as $\mathrm{Frac}(R)$ is torsion-free, divisible and an essential extension of R. ◻

(2) Observation preceding Theorem 0.4

The C-bilinar map $M \otimes_R N \to C$ that he is referring to is obtained by considering the concrete construction and sending $r_{m,n} \mapsto B(rm, n) = B(m, rn)$. This turns out to well defined and C-bilinear because B is.

(3) Possible error in Example 1.4

Let $R = k[x_i]_{i \in \mathbb{N}}$ for a field k. Let $\mathfrak{m} = (x_i)_{i \in \mathbb{N}}$ and $R_\mathfrak{m}$ be the localisation of R at the maximal ideal $\mathfrak{m}$. Then, $(R, \mathfrak{m}, k)$ is a local ring and $R_\mathfrak{m}/(\mathfrak{m}R_\mathfrak{m})^2$ is a finitely generated $R_\mathfrak{m}$-module. But

$$x_1 R_\mathfrak{m} \subsetneq (x_1, x_2) R_\mathfrak{m} \subsetneq \dots$$

is a strictly increasing chain of submodules in $R_\mathfrak{m}/(\mathfrak{m}R_\mathfrak{m})^2$. This means that $R_\mathfrak{m}/(\mathfrak{m}R_\mathfrak{m})^2$ is not noetherian and hence not of finite length.

However, if we assume that $(R, \mathfrak{m}, k)$ is a noetherian local ring, then the statement is true. For a finitely generated submodule N' of N, some power of $\mathfrak{m}$, say $\mathfrak{m}^n$ kills all of N'. Hence, $\mathrm{Ann}_R(N')$ contains $\mathfrak{m}^n$ and consequently, the only prime ideal that contains $\mathrm{Ann}_R(N')$ is $\mathfrak{m}$. This implies that $R/\mathrm{Ann}_R(N')$ has dimension 0. Thus, $R/\mathrm{Ann}_R(N')$ is artinian. Being a finitely generated module over the artinian ring $R/\mathrm{Ann}_R(N')$, N' is artinian and hence of finite length.

(4) Direct summands of injective modules are injective

Useful little fact. For a quick proof, if $E \oplus M \cong E'$ and E' is injective, then take the pushout of a given $E \hookrightarrow K$ along the inclusion $E \hookrightarrow E'$. Compose a left inverse of the base change with the projection to the first coordinate.

(5) Essential extensions are not closed under taking pushouts

$$\begin{tikzcd}
\mathbb{Z}/p\mathbb{Z} \arrow[r, hook, "\cdot p"] \arrow[d, hook, "\cdot p"'] & \mathbb{Z}/p^2\mathbb{Z} \arrow[d, hook] \\
\mathbb{Z}/p^2\mathbb{Z} \arrow[r, hook] & \frac{\frac{\mathbb{Z}}{p^2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^2\mathbb{Z}}}{(p,p)} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

(6) Remark preceding Proposition 2.3

That R is noetherian is important although we don't need $0 \neq M$ to be finitely generated (for $\mathrm{Ass}(M) \neq \emptyset$). Consider the collection of ideals $\mathrm{Ann}_R(x)$ indexed by $x \in M$. Since R is notherian, this collection contains a maximal element with respect to inclusion, say $\mathfrak{p} = \mathrm{Ann}_R(n)$. We claim that $\mathfrak{p}$ is prime. Suppose for the sake of contradiction that there exist $r, s \in R$ such that $rs \in \mathfrak{p}$, $r \notin \mathfrak{p}$ and $s \notin \mathfrak{p}$. Then, $rn \neq 0$ and $s \in \mathrm{Ann}_R(rn) \supset \mathfrak{p}$ contradicting maximality. So, $\mathfrak{p} \in \mathrm{Ass}(M)$.

[1]Proof sourced from this post

(7) PROOF OF A LARGE PART OF THEOREM 2.4

We record the proof of [Hoc1, Theorem 2.4] in a slightly more convenient form. The substance of the theorem is summarised in the following diagram. Throughout, this discussion, we assume that R is noetherian.

$$\begin{array}{ccccc} & & E_{R/\mathfrak{p}}(R/\mathfrak{p}) & & \\ & & \| & & \\ R/\mathfrak{p} & \subseteq & \mathrm{Frac}(R/\mathfrak{p}) = \kappa_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} & \subseteq & E_R(R/\mathfrak{p}) = E_{R_{\mathfrak{p}}}(\kappa_{\mathfrak{p}}) \\ & & \| & & \\ & & \mathrm{Hom}_R(R/\mathfrak{p}, E_R(R/\mathfrak{p})) & & \end{array}$$

(a) Recall that we've already shown that a direct sum of injective modules over R is injective (because R is noetherian) and that every injective module over R is a direct sum of modules of the form $E(R/\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of R. So, the study of injective modules over R is reduced to the study of $E(R/\mathfrak{p})$'s.

(b) Fix $\mathfrak{p} \in \mathrm{Spec}(R)$. We make a first set of interesting observations.
   (i) The elements of $E(R/\mathfrak{p})$ that are annihilated by $\mathfrak{p}$ constitute the submodule $\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{p}))$ of $E(R/\mathfrak{p})$.
   (ii) By [Hoc1, Corollary 0.5], $\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{p}))$ is an injective module.
   (iii) Further, $R/\mathfrak{p} \subseteq \mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{p}))$ is essential. So, $\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{p})) \cong E_{R/\mathfrak{p}}(R/\mathfrak{p})$.
   (iv) $R/\mathfrak{p}$ is a domain. So, by Comment (1), $E_{R/\mathfrak{p}}(R/\mathfrak{p}) \cong \kappa_{\mathfrak{p}}$.

(c) Next, we claim that $R \setminus \mathfrak{p}$ acts via automorphisms on $E(R/\mathfrak{p})$. This will make $E(R/\mathfrak{p})$ an $R_{\mathfrak{p}}$-module.
   (i) Fix $x \in R \setminus \mathfrak{p}$. Then, we know that x acts via automorphisms on $\kappa_{\mathfrak{p}}$, the submodule of $E(R/\mathfrak{p})$ killed by $\mathfrak{p}$.
   (ii) Assume for the sake of contradiction that $xe = 0$ for some non-zero $e \in E(R/\mathfrak{p})$. Since $R/\mathfrak{p} \subseteq E(R/\mathfrak{p})$ is essential, there exists $r \in R$ such that $0 \neq re \in R/\mathfrak{p}$. But by the above observation, $x(re) \neq 0$ contradicting the above assumption. This shows that any element of $R \setminus \mathfrak{p}$ acts injectively on $E_R(R/\mathfrak{p})$.
   (iii) Observe that $xE \subseteq E$, $R/\mathfrak{p} \subseteq xE$ and xE, being isomorphic to E as R-modules is injective. Since injective modules do not have proper essential extensions, $xE = E$.

(d) Now that $E(R/\mathfrak{p})$ is an $R_{\mathfrak{p}}$-module, we claim that $E(R/\mathfrak{p}) = E_{R_{\mathfrak{p}}}(\kappa_{\mathfrak{p}})$. This further reduces the task of understanding injective modules over noetherian rings to understanding the injective hulls of residue fields of noetherian local rings.
   (i) $\kappa_{\mathfrak{p}} \subseteq E_R(R/\mathfrak{p})$ is a maximal essential extension as R-modules and hence, also as $R_{\mathfrak{p}}$-modules. So, $E(R/\mathfrak{p}) \cong E_{R_{\mathfrak{p}}}(\kappa_{\mathfrak{p}})$.

(e) Lastly, we show that $\mathfrak{p}$ is the only associated prime of $E(R/\mathfrak{p})$. Clearly, $\mathfrak{p} \in \mathrm{Ass}(E(R/\mathfrak{p}))$. Suppose that $\mathfrak{q} \neq \mathfrak{p}$. If $\mathfrak{q}$ contains an element not contained in $\mathfrak{p}$, then $\mathfrak{q} \notin \mathrm{Ass}(E(R/\mathfrak{p}))$ as elements of $R \setminus \mathfrak{p}$ act via automorphisms on $E(R/\mathfrak{p})$. Now, if $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q} \in \mathrm{Ass}(E(R/\mathfrak{p}))$, then the copy of $R/\mathfrak{q}$ in $\mathrm{Ass}(E(R/\mathfrak{p}))$ intersects $R/\mathfrak{p}$ non-trivially and contains a non-zero element, say x. Then, $rx = 0$ if and only if $r \in \mathfrak{p}$ if and only if $r \in \mathfrak{q}$. Thus, $\mathfrak{p} = \mathfrak{q}$. Therefore, $E(R/\mathfrak{p})$ is $\mathfrak{p}$-coprimary and every element of $E(R/\mathfrak{p})$ is killed by a power of $\mathfrak{p}$ (See [Eis13, Proposition 3.9])

(8) LOCALISATION DOES NOT PRESERVE INJECTIVENESS IN GENERAL

[Hoc1, Theorem 2.8 (b)] does not hold if you don't assume that the underlying ring is noetherian. See [Rot09, p.201].

(9) Reference for the coefficient field claim in §4

In the beginning lines of [Hoc1, §4], Hochster mentions the existence of a coefficient field in an Artinian local ring containing a field. This is necessary for the following statement (in [Hoc1]) R is a finite dimensional k-vector space to make sense. The existence of the coefficient field ensures that there is always a residue field action on an R-module – regardless of whether $\mathfrak{m}$ kills it. But if you know that $\mathfrak{m}$ kills the module (that is, the module is naturally an $(R/\mathfrak{m})$-module), you also know that the coefficient field acts the "same way" as $R/\mathfrak{m}$. Hochster proves the existence of the coefficient field in any complete local ring in [Hoc2] with either of the following additional assumptions – when the residue field has characteristic 0 or when the residue field is perfect and of prime characteristic.

(10) A Bunch of remarks concerning §11

(a) Hochster seems to define the dimension of an R-module M as the dimension of $R/\mathrm{Ann}_R(M)$. When M is finite as an R-module, this is equal to the dimension of $\mathrm{Supp}(M)$ as a topological space.

(b)

Questions

(1) Can the construction of the injective hull be made functorial?
(2) Is the homomorphic image of an injective module injective? Probably not true.

References

[Eis13] Eisenbud, David. *Commutative algebra: with a view toward algebraic geometry*. Vol. 150. Springer Science & Business Media, 2013.

[Hoc1] Hochster, Melvin. *Local Cohomology*. (`https://dept.math.lsa.umich.edu/~hochster/615W11/loc.pdf`)

[Hoc2] Hochster, Melvin. *The Structure Theory of Complete Local Rings*. `https://dept.math.lsa.umich.edu/~hochster/615W14/Struct.Compl.pdf`)

[Har77] Hartshorne, Robin. *Algebraic Geometry*

[Rot09] Rotman, Joseph. *An Introduction to Homological Algebra*