AIS - COHOMOLOGY OF COMMUTATIVE ALGEBRAS

Convention 0.1. Throughout, "ring" means a commutative ring with unity. If R is a ring, then the category of R-modules is denoted by R-Mod.

1. Kähler differentials

Definition 1.1. Let k be a ring and R a k-algebra. Let $M \in R$-Mod. A *k-derivation* of R in M is a k-linear map $d : R \to M$ such that $d(ab) = ad(b) + bd(a)$ for all $a, b \in R$.

Notation 1.2. For an R-module M. $\mathrm{Der}_k(R, M)$ is the collection of k-derivations of R in M. If $M = R$, we will write $\mathrm{Der}_k(R)$ for $\mathrm{Der}_k(R, R)$.

Example 1.3.
(1) Let $U \subseteq \mathbb{R}^n$ be open and $x_1, \dots, x_n$ be coordinates. Let $R = C^\infty(U)$ be the ring of smooth functions. Have derivations $\frac{\partial}{\partial x_i} \in \mathrm{Der}_{\mathbb{R}}(U)$.
(2) Let R be the ring of smooth functions again. For $x \in U$, define $\mathfrak{m}_x$ to be the maximal ideal at x. Then, the maps $d_i : R \to \mathbb{R} = R/\mathfrak{m}_x$, $f \mapsto \frac{\partial f}{\partial x_i}(x)$ are $\mathbb{R}$-derivations of R in $\mathbb{R}$.
(3) Formal derivatives in polynomial rings.
(4) Let $R = k[x_1, \dots, x_n]$ and $M = \bigoplus_{i=1}^n R dx_i$, where dx_i are symbols. Define $d : R \to M$, $f \mapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$. This is a derivation of R in M.
(5) Let R be a k-algebra. Consider the map $\mu : R \otimes_k R \to R$, $a \otimes b \mapsto ab$. Let $I = \ker(\mu)$. Exercise: Show that this kernel is generated by $\langle a \otimes 1 - 1 \otimes a | a \in R \rangle$. Note that $R \otimes_k R$ has two R-module structures (multiplication on the left and on the right). Check that these two structures coincide on I/I^2 (use the fact that $(r \otimes 1 - 1 \otimes r)(a \otimes 1 - 1 \otimes a) \in I^2$). The map $\delta : R \to I/I^2$, $a \mapsto (a \otimes 1 - 1 \otimes a)$ is a k-derivation.

Definition 1.4. For a k-algebra R, let F be the free R-module generated by the set $\{dr | r \in R\}$. Let N be the submodule generated by $\langle d(rr') - rdr' - r'dr,\ d(ar + a'r') - adr - a'dr' | r, r' \in R \rangle \subseteq F$. Let $\Omega_{R/k} := F/N$. By definition, the map $\phi : R \to \Omega_{R/k}$, $r \mapsto dr$ is a derivation. In fact, for any $M \in R$-Mod, there is a natural isomorphism $\mathrm{Der}_k(R, M) \cong \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(\Omega_{R/k}, M)$.

Remark 1.5. The functor $\mathrm{Der}_k(R, \_) : R$-$\mathsf{Mod} \to \mathsf{Set}$ is represented by $\Omega_{R/k}$.

Exercise 1.6. The object (in our case $\Omega_{R/k}$) representing a functor is unique up to unique isomorphism.

Example 1.7. Let $R = k[x_1, \dots, x_n]$. Then, $\Omega_{R/k}$ is generated by $dx_1, \dots, dx_n$. We claim that $dx_1, \dots, dx_n$ are also linearly independent. That is, $\Omega_{R/k} = \bigoplus R dx_i$. Consider the free R-module $R^{\oplus n}$ with basis $e_1, \dots, e_n$. To see this, use the isomorphism $\mathrm{Der}_k(R, M) \cong \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(\Omega_{R/k}, M)$ for when $M = R^{\oplus n}$ and the differential $\delta : R^{\oplus n} \to R$, $f \mapsto \sum \frac{\partial f}{\partial x_i} e_i$.

Notation 1.8. $R \ltimes M$ is the R-module $R \oplus M$ with multiplication defined as $(r, m) \cdot (r', m') = (rr', rm' + r'm)$.

Proposition 1.9. *Let $I = \ker(R \otimes_k R \to R)$ and $\delta : R \to I/I^2$, $r \mapsto r \otimes 1 - 1 \otimes r$. Then, for $M \in R$-Mod, for all $\phi \in \mathrm{Der}_k(R, M)$, there exists a unique R-linear map $\tilde{\phi} : I/I^2 \to M$ such that $e = \tilde{e} \circ \delta$. In particular, there is a unique isomorphism $(I/I^2, \delta) \cong (\Omega_{R/k}, d)$.*

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Proof. The first statement says that I/I^2 represents the functor $\mathrm{Der}_k(R, \_)$. As representing objects are unique up to unique isomorphism, the second statement follows. For the first statement, consider $\phi \in \mathrm{Der}_k(R, M)$. Define $\check{\phi} : R \to R \ltimes M$, $r \mapsto (r, \phi(r))$, which is a k-algebra homomorphism. There is also an inclusion homomorphism $i : R \to R \ltimes M$, $r \mapsto (r, 0)$. Thus, we have a canonical k-algebra homomorphism $h : R \otimes_k R \to R \ltimes M$, $r \otimes r' \mapsto \check{\phi}(r)i(r') = (rr', r'e(r))$. Note that $h(r \otimes 1 - 1 \otimes r) = (0, e(r))$. Thus, $h(I^2) = 0$ and h descends to a map $\tilde{\phi} : I/I^2 \to M$. □

Theorem 1.10 (First fundamental sequence)**.** *Let $k \to R \to S$ be maps of rings. Then, there is an exact sequence of S-modules*

$$S \otimes_R \Omega_{R/k} \xrightarrow{\alpha} \Omega_{S/k} \xrightarrow{\beta} \Omega_{S/R} \to 0$$

where $\alpha(s \otimes d_{R/k}r) = sd_{S/k}r$ and $\beta(d_{S/k}(s)) = d_{S/R}(s)$

Proof. Clearly, β is a surjection (same generators) with kernel (exercise: check this) $\langle \{d_{S/k}r | r \in R\} \rangle$. This is clearly the image of α. □

Theorem 1.11 (Second Fundamental Sequence)**.** *As before, R is a k-algebra. Let $I \subseteq R$ be an ideal and $S = R/I$. There is an exact sequence of S-modules*

$$I/I^2 \xrightarrow{\gamma} S \otimes_R \Omega_{R/k} \xrightarrow{\gamma'} \Omega_{S/k} \to 0$$

where $\gamma : r \mapsto 1 \otimes d_{R/k}r$ and $\gamma' : s \otimes d_{R/k}r \mapsto sd_{S/k}r$

Proof. Note that γ' is the same as α. As $R \to S$ is surjective, $\Omega_{S/R} = 0$. We claim (proof left as an exercise) that $\ker(\Omega_{R/k} \to \Omega_{S/k}) = I\Omega_{R/k} + R\{d_{R/k}r | r \in I\}$. Assuming this, we see

$$\ker(S \otimes_R \Omega_{R/k} \to \Omega_{S/R}) = \langle 1 \otimes d_{R/k}r | r \in I \rangle = \mathrm{Im}(\gamma)$$

Check that γ is S-linear. □

Regularity.

Proposition 1.12. *Let k be a field and $(R, \mathfrak{m})$ a local k-algebra such that $K = R/\mathfrak{m}$ is a finite separable extension of k and $\mathfrak{m}$ has finite embedding dimension. Then, $\Omega_{K/k} = 0$.*

Proof. Let $a \in K$ with $f \in k[x]$ being the minimal polynomial of a over k. Then, for any k-derivation, $0 = df(a) = f'(a)da$. As K/k is separable, $f'(a) \neq 0$ which means that $da = 0$. From the second fundamental sequence, we get a K-linear surjection

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} K \otimes_R \Omega_{R/k} \to \Omega_{K/k}$$

We claim that δ is an isomorphism. If true, we conclude that $\dim_K(\mathfrak{m}/\mathfrak{m}^2) = \dim_K(K \otimes_R \Omega_{R/k}) \geqslant \dim(R)$. To prove this, it suffices to show that

$$\delta^\vee : (K \otimes_R \Omega_{R/k})^\vee \to (\mathfrak{m}/\mathfrak{m}^2)^\vee$$

is surjective. Let $f : \mathfrak{m}/\mathfrak{m}^2 \to K$. Extend f to R using any k-linear splitting (by composition with $R \twoheadrightarrow R/\mathfrak{m}^2$)

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to R/\mathfrak{m}^2 \to K \to 0$$

Set $\tilde{f}(K) = 0$. Then (exercise) $\tilde{f} \in \mathrm{Der}_k(R, K)$. Let $g \in \mathrm{Hom}_R(\Omega_{R/k}, K)$ be the corresponding linear map. Then, since $\mathrm{Hom}_R(\Omega_{R/k}, K) = \mathrm{Hom}_R(K \otimes_R \Omega_{R/k}, K) = \mathrm{Hom}_K(K \otimes_R \Omega_{R/k}, k)$, for all $r \in \mathfrak{m}$, $f(r) = \tilde{f}(r) = g(d_{R/k}(r))$. So, $f = g\delta$. □

2. Smoothness

Notation 2.1. Let k be a commutative ring and R is a commutative k-algebra.

Definition 2.2. Let $M \in R\text{-}\mathsf{Mod}$. A *square-zero extension* of R by M is a short exact sequence of k-modules

$$0 \to M \xrightarrow{\theta} E \to R \to 0$$

where E is a k-algebra, $E \to R$ is a k-algebra morphism and $\theta(M) \subseteq E$ is an ideal such that $\theta(M)^2 = 0$.

Remark 2.3. In the above case, identify M with $\theta(M)$, an ideal in E. Hence, E acts k-linearly on M. So, $M^2 = 0$ essentially means that this action factors as follows.

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & \mathrm{End}_k(M) \\ & E/M = R \arrow[ur, dashed] & \end{tikzcd}$$

Example 2.4. For a field k,

$$0 \to k \equiv k\varepsilon \to k[\varepsilon]/(\varepsilon^2) \to k \to 0$$

is a square-zero extension of k by itself.

Definition 2.5. An extension E of R by M

$$0 \to M \to E \to R \to 0$$

is *trivial* if it is isomorphic to the extension

$$0 \to M \to R \ltimes M \to R \to 0$$

That is, we have

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\text{k-alg isom.}"] & R \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & M \ltimes R \arrow[r] & R \arrow[r] & 0 \end{tikzcd}$$

Observation 2.6. An extension

$$0 \to M \xrightarrow{\alpha} E \xrightarrow{\beta} R \to 0$$

is trivial if and only if there is a k-algebra section $\sigma : R \to E$ of β. Check that $E \cong R \ltimes M$ by writing $e = \sigma\beta(e) - (\sigma\beta(e) - e)$, where the first term belongs to R and the second to M.

Definition 2.7. Let R be a k-algebra. We say that R is *smooth* over k if for every square-zero extension

$$0 \to M \to E \to T \to 0$$

of commutative k-algebras and every k-algebra map $u : R \to T$, there exists a k-algebra lifting $v : R \to E$ of u.

Example 2.8. $k[x_1, \dots, x_n]$ is free over k and is hence smooth.

Remark 2.9. Let R be a smooth k-algebra and

$$0 \to J \to E \to R \to 0 \tag{1}$$

a nilpotent extension. That is, $J^m = 0$ for some $m \geqslant 1$. Then, smoothness implies that Eq. (1) is split. The idea is to induct using the following diagram

$$0 \to J^n/J^{n+1} \to E/J^{n+1} \to E/J^n \to 0$$

which is a square-zero extension.

Proposition 2.10. *Let $k \to R \xrightarrow{f} S$ be maps of rings*

(1) *If S is smooth over R, then the first fundamental exact sequence*

$$0 \to \Omega_{R/k} \otimes_R S \xrightarrow{\alpha} \Omega_{S/k} \to \Omega_{S/R} \to 0$$

is split exact.

(2) *If $S = R/I$ and suppose S is smooth over k, then the second fundamental exact sequence is split exact*

$$0 \to I/I^2 \to \Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to 0$$

Proof. As $\mathrm{Hom}_S(\Omega_{S/k}, \Omega_{R/k} \otimes_R S) \cong \mathrm{Der}_k(S, \Omega_{R/k} \otimes_R S)$, for a splitting, we need a suitable derivation. Before proceeding, we introduce a general construction.

Let $N \in S\text{-}\mathsf{Mod}$ and $\delta \in \mathrm{Der}_k(R, N)$. This defines a k-algebra map $\phi : R \to S \ltimes N$, $r \mapsto (f(r), \delta r)$. Since S is smooth over R, the square-zero extension of R-algebras

$$0 \to N \xrightarrow{i_2} S \ltimes N \xrightarrow{\pi_1} S \to 0$$

has an R-algebra splitting σ. Note that $\sigma \circ f = \phi$ as σ is an R-algebra morphism. Let $\delta' = \pi_2 \sigma \in \mathrm{Der}_R(S, N)$, where $\pi_2 : S \ltimes N \to N$ is the projection to the second coordinate. Check that $\delta' f = \delta$. Now set $N = \Omega_{R/k} \otimes_R S$ and $\delta = d_{R/k} \otimes 1$. Let $\gamma \in \mathrm{Hom}_S(\Omega_{S/k}, \Omega_{R/k} \otimes_R S)$ be the map corresponding to δ'. That is, $\delta' = \gamma d_{S/k}$. Then, for $r \in R$ and $s \in S$,

$$d_{R/k} r \otimes s \overset{\alpha}{\mapsto} s d_{S/k}(f(r)) \overset{\gamma}{\mapsto} s(d_{R/k} r \otimes 1) = d_{R/k} r \otimes s$$

Thus, γ is a splitting. Proof of the second statement is left as an exercise. □

Proposition 2.11. *Let R be a smooth k-algebra, then $\Omega_{R/k}$ is projective.*

Proof. Note that every map $f : M \to N$ of R-modules induces a map of R-algebras

$$R \ltimes M \xrightarrow{\mathrm{Id}, f} R \ltimes N$$

Conversely, every map of R-algebras $\phi : R \ltimes M \to R \ltimes N$ yields an R-linear map $\phi|_{0 \oplus M} : M \to N$. Let $I = \ker(R^e := R \otimes_k R \to R)$. Then, $I/I^2 = \Omega_{R/k}$. As R/k is smooth, the extension

$$0 \to \Omega_{R/k} \to R^e/I^2 \xrightarrow{\mu} R \to 0$$

is trivial with an isomorphism $\gamma : R^e/I^2 \to R \ltimes \Omega_{R/k}$. Let $p : R^e \to R^e/I^2$ be the surjection. Give $R^e = R \otimes_k R$ the R-module structure given by left multiplication. Then, p, μ, γ are all R-algebra homomorphisms. Let $M, N \in R\text{-}\mathsf{Mod}$ and consider the diagram

$$\begin{tikzcd} & \Omega_{R/k} \arrow[d, "h"] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

of R-module homomorphisms. This gives a diagram of R-algebras

$$\begin{tikzcd}
 & I \arrow[r, phantom, "\subseteq"] & R \otimes_k R \arrow[r, phantom, "\cong"] & R^e \arrow[r, "\gamma p"] \arrow[d, dashed, "\exists \phi"'] & R \ltimes \Omega_{R/k} \arrow[d, "{(1,h)}"] & \\
0 \arrow[r] & \ker(1, f) \arrow[rr] & & R \ltimes M \arrow[r, "{(1,f)}"] & R \ltimes N \arrow[r] & 0
\end{tikzcd}$$

The bottom extension is square-zero. As an exercise, prove that $R^e = R \otimes_k R$ is smooth over R. So, there exists ϕ as in the diagram so that it commutes. By the commutativity of the above diagram, $\gamma p(I) \subseteq 0 \oplus \Omega_{R/k}$. So, $((1, h) \circ \gamma p)(I) \subseteq 0 \oplus M$. Thus, we get an R-algebra map $\tilde{\phi} : R \ltimes \Omega_{R/k} \to R \ltimes M$ with $\mathrm{Im}(\Omega_{R/k}) \subseteq 0 \oplus M$. This gives a lifting of h to an R-linear map $\pi_2 \circ \tilde{\phi} \circ i_2 : \Omega_{R/k} \to M$ $\square$

We'll prove the following result next lecture.

Proposition 2.12. *If R is a noetherian local ring containing a field k that is smooth over k, then R is a regular local ring.*

3. More on Smoothness & Prelude to André-Quillen

Proposition 3.1. *Let $(R, \mathfrak{m})$ be a noetherian local containing a field. If R is smooth over k, then R is a regular local ring.*

Proof. We may assume that k is a prime field. Any bigger field is smooth over the prime field. This ensures that $K = R/\mathfrak{m}$ is smooth over k. Let $d = \dim_K(\mathfrak{m}/\mathfrak{m}^2)$. Let $S = K[x_1, \dots, x_d]_{(x_1,\dots,x_d)}$ and $M \subseteq S$ the maximal ideal. Since K/K is smooth, the following square-zero extension of K-algebras is trivial.

$$0 \to M/M^2 \to S/M^2 \to K \to 0$$

So, $S/M^2 \cong K \ltimes M/M^2$ as K-algebras. As K/k is smooth, similarly, $R/\mathfrak{m}^2 \cong K \ltimes \mathfrak{m}/\mathfrak{m}^2$ as k-algebras. Since $M/M^2 \cong \mathfrak{m}/\mathfrak{m}^2$ as K-algebras, we see that $R/\mathfrak{m}^2 \cong S/M^2$. As R is smooth over k, the surjection $R \twoheadrightarrow S/M^2$ can be lifted to a surjective (Nakayama) map $R \twoheadrightarrow S/M^n$ for all $n \geqslant 2$.

$$\begin{tikzcd}
 & & & R \arrow[dl, dashed, two heads] \arrow[d, two heads] & \\
0 \arrow[r] & \mathfrak{m}^2/\mathfrak{m}^3 \arrow[r] & S/M^3 \arrow[r] & S/M^2 \arrow[r] & 0
\end{tikzcd}$$

We claim that $\ker(R \to S/M^n) = \mathfrak{m}^n$. In fact it is sufficient to see that $\mathfrak{m}^n$ is contained in $\ker(R \to S/M^n) = \mathfrak{m}^n$.

Let H_R and H_S be the Hilbert Samuel polynomials of R and S respectively. Recall that

(1) $H_R(n) = \text{length}(R/\mathfrak{m}^n)$ and $H_S(n) = \text{length}(S/M^n)$.
(2) $\deg(H_R) = \dim(R)$ and $\deg(H_S) = \dim(S) = d$

Then, for all $n \geqslant 2$, $H_R(n) \geqslant H_S(n)$. Hence, $\dim(R) \geqslant d$. However, we know that $\dim(R) \leqslant d = \dim_K(\mathfrak{m}/\mathfrak{m}^2)$. Thus, $\dim(R) = \dim_K(\mathfrak{m}/\mathfrak{m}^2)$. □

Definition 3.2. Let k be a field and R be a k-algebra. R is said to be *geometrically regular* if for every finite extension $k \hookrightarrow l$, the ring $R \otimes_k l$ is regular.

Fact 3.3. A k-algebra R is smooth if and only if R is geometrically regular.

We now turn to the construction and properties of the T^i-functors (following Hartshorne's Deformation Theory).

Construction 3.4. Let A be a ring and B an A-algebra. Let $e_0 : R \to B$ be a surjective A-algebra homomorphism, where A is a polynomial ring over A (we can always choose such e_0). Let $I = \ker(e_0)$ and let

$$0 \to Q \xrightarrow{\alpha} F \xrightarrow{j} I \subseteq R \to 0$$

be an exact sequence of R-modules such that F is free over R. More elaborately, you pick a generating set of I, a sufficiently large free module F over the polynomial ring, construct a surjective R-module map j and define α to be the kernel of j. Let (you should be reminded of the Koszul complex here)

$$F_0 = \langle \{ j(x)y - xj(y) | x, y \in F \} \rangle \subseteq F$$

We note that $F_0 \subseteq Q$ is a submodule and $IQ \subseteq F_0$. We have a complex of B-modules

$$
\begin{tikzcd}
 & L_2 \arrow[d, equal] & L_1 \arrow[d, equal] & L_0 \arrow[d, equal] \\
\mathbb{L}_\bullet := & Q/F_0 \arrow[r, "\overline{\alpha}"] & (F/F_0) \otimes_R B \arrow[r, "\delta \circ \overline{j}"] \arrow[d, phantom, "\cong"] & \Omega_{R/A} \otimes_R B \\
 & & F/IF \arrow[r, "\overline{j}"'] & I/I^2 \arrow[u, "\delta"']
\end{tikzcd}
\tag{2}
$$

$$
\text{Degree :} \qquad (2) \qquad (1) \qquad (0)
$$

Note that L_1, L_0 are free B-modules. For any $M \in B\text{-}\mathsf{Mod}$, define

$$T^i(B/A, M) := H^i(\mathrm{Hom}(\mathbb{L}_\bullet, M))$$

for $i = 0, 1, 2$.

There are many unnatural choices we've made here but the following lemma says that these do not matter.

Lemma 3.5. *The modules $T^i(B/A, M)$ are independent of the choices of R, F.*

Theorem 3.6. *Let B be an A-algebra and*

$$0 \to M' \to M \to M'' $$

be a short exact sequence of B-modules. Then, there is a long exact sequence

$$0 \to T^0(B/A, M') \to T^0(B/A, M) \to T^0(M/M'') \to T^1(B/A, M') \to \ldots \to T^2(B/A, M'')$$

Proof. As L_0 and L_1 are free B-modules,

$$0 \to \mathrm{Hom}_B(L_\bullet, M') \to \mathrm{Hom}_B(L_\bullet, M) \to \mathrm{Hom}_B(L_\bullet, M'') \to 0$$

is exact except for $i = 2$ (where it is left exact). Easy calculation of homology. $\square$

Theorem 3.7 (Jacobi-Zariski)**.** *Let $A \to B \to C$ be homomorphisms of rings. Let $M \in C\text{-}\mathsf{Mod}$. Then, there is an exact sequence of C-modules*

$$0 \to T^0(C/B, M) \to T^0(C/A, M) \to T^0(B/A, M) \to T^1(C/B, M) \to \ldots \to T^2(C/A, M) \to T^2(B/A, M)$$

Proof. Omitted $\square$

Proposition 3.8. *For any map of rings $A \to B$ and $M \in B\text{-}\mathsf{Mod}$, $T^0(B/A, M) = \mathrm{Hom}(\Omega_{B/A}, M) = \mathrm{Der}_A(B, M)$*

Proof. Write B as a quotient

$$0 \to I \to R \to B \to 0$$

where R is a polynomial ring. Then, there is an exact sequence

$$I/I^2 \to \Omega_{R/A} \otimes_A B \to \Omega_{B/A} \to 0$$

From the construction of $\mathbb{L}_\bullet$, we have a surjective map $L_1 \to I/I^2$. Thus, the sequence $L_1 \to L_0 \to \Omega_{B/A} \to 0$ is exact. Taking $\mathrm{Hom}(\mathbb{L}_\bullet, M)$, we get $T^0(B/A, M) = \mathrm{Hom}(\Omega_{B/A}, M)$. $\square$

Proposition 3.9. *If B is a polynomial ring over A, then $T^i(B/A, M) = 0$ for $i = 1, 2$ for all M.*

Proof. Take $R = B$ in the construction. Then, $I = 0$ and $F = 0$. So, $L_2 = L_1 = 0$. $\square$

Proposition 3.10. *If $A \twoheadrightarrow B$ is a surjective ring homomorphism with kernel I, then $T^0(B/A, M) = 0$ for all M and $T^1(B/A, M) = \mathrm{Hom}_B(I/I^2, M)$.*

Proof. The first statement follows from Proposition 3.9 and Problem 2 of Tutorial 1 (which says that $S^{-1}\Omega_{R/k} \cong \Omega_{S^{-1}R/k}$). Take $R = A$. Then, $L_0 = 0$. Further, the exact sequence

$$0 \to Q \to F \to E \to 0$$

gives

$$\begin{tikzcd}
Q \otimes_A B \arrow[rr] & & F \otimes_A B \arrow[r] & I/I^2 \arrow[r] & 0 \\
Q/IQ \arrow[u, equal] \arrow[dr, two heads] & & F/IF = L_1 \arrow[u, equal] & & \\
& Q/F_0 = L_2 \arrow[ur] & & &
\end{tikzcd}$$

Taking $\mathrm{Hom}_B(\_, M)$ and cohomology, we get

$$T^1(B/A, M) = \mathrm{Hom}_B(I/I^2, M)$$

□

Theorem 3.11. *Let k be an algebraically closed field and B be a finite type k-algebra. Then, B is smooth over k if and only if $T^1(B/k, M) = 0$ for all M.*

Proof. Let $B = A/I$, where $A = k[x_1, \dots, x_n]$. Then, B is smooth over k if and only if the conormal sequence

$$0 \to I/I^2 \to \Omega_{A/k} \otimes_A B \to \Omega_{B/k} \to 0$$

is split exact, and $\Omega_{B/k}$ is projective. We will prove the theorem modulo the following claim: There is an exact sequence

$$0 \to T^0(B/k, M) \to \mathrm{Hom}_A(\Omega_{A/k}, M) \xrightarrow{\eta} \mathrm{Hom}_B(I/I^2, M) \to T^1(B/k, M) \to 0$$

Note that $\mathrm{Hom}_A(\Omega_{A/k}, M) = \mathrm{Hom}_B(\Omega_{A/k} \otimes_A B)$, and η is the map $\mathrm{Hom}_B(\Omega_{A/k} \otimes_A B, M) \to \mathrm{Hom}_B(I/I^2, M)$. Note that η is surjective if and only if $T^1(B/k, M) = 0$ for all M. Suppose that B/k is smooth. Then, the conormal sequence is split exact and hence, η is surjective.

Conversely, suppose $T^1(B/k, M) = 0$ for all M. By the claim, η is surjective. Hence, there exists $\sigma \in \mathrm{Hom}_B(\Omega_{A/k} \otimes_A B, M)$ such that $\eta(\sigma) = \mathrm{Id}_{I/I^2}$. This gives a splitting of the conormal exact sequence. Moreover, by Theorem 3.6, $T^0(B/k, \_) = \mathrm{Hom}_B(\Omega_{B/k}, \_)$ is an exact functor. Hence, $\Omega_{B/k}$ is a projective B-module. Hence, B is smooth over k.

□

Remark 3.12. We will prove next time that splitting and projective implies smoothness.

4. Proofs of some statements from last time

Proposition 4.1. *Let $A = k[x_1, \dots, x_n]$, $B = A/I$. Then, B is smooth over k if (in fact if and only if) the conormal sequence*

$$0 \to I/I^2 \to \Omega_{A/k} \otimes B \to \Omega_{B/k} \to 0 \tag{3}$$

is split exact.

Proof. Consider a square-zero extension

$$0 \to M \to E \to T \to 0$$

of k-algebras, and let $f : B \to T$ be a map of k-algebras. We will show that f extends to a map $B \to E$. Note that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, dashed, "\exists h"] & B \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & M \arrow[r] & E \arrow[r] & T \arrow[r] & 0 \end{tikzcd}$$

As A/k is smooth, there exists a map $h : A \to E$. The map $h|_I : I \to M$ induces a map $\overline{h} : I/I^2 \to M$ (as $M^2 = 0$) Applying $\mathrm{Hom}_B(\_, M)$ to Eq. (3), we get

$$0 \to \mathrm{Hom}_B(\Omega_{B/k}, M) \to \mathrm{Hom}_B(\Omega_{A/k} \otimes_A B, M) = \mathrm{Hom}_k(\Omega_{A/k}, M) \to \mathrm{Hom}_B(I/I^2, M) \to 0$$

Let θ be a lift of $\overline{h} \in \mathrm{Hom}_B(I/I^2, M)$. We may regard θ as a derivation $\tilde{\theta} \in \mathrm{Der}_k(A, M)$. Check, as an exercise, that $h' = h - \tilde{\theta}$ is a left inverse $h' : B \to E$ of f. $\square$

Proposition 4.2. *Suppose $A = k[x_1, \dots, x_n]$, $B = A/I$. Then, for any $M \in B\text{-}\mathsf{Mod}$, there is an exact sequence*

$$0 \to T^0(B/k, M) \to \mathrm{Hom}(\Omega_{A/k}, M) \to \mathrm{Hom}(I/I^2, M) \to T^1(B/k, M) \to 0$$

Further, $T^2(B/A, M) = T^2(B/k, M)$

Proof. We have maps of rings $k \to A \to B$. There is an exact sequence (by Theorem 3.7, known as the Jacobi Zariski Sequence)

$$\begin{tikzcd} 0 \arrow[r] & T^0(B/A, M) \arrow[r] & T^0(B/k, M) \arrow[r] & T^0(A/k, M) \arrow[dll] \\ & T^1(B/A, M) \arrow[r] & T^1(B/k, M) \arrow[r] & T^1(A/k, M) \arrow[dll] \\ & T^2(B/A, M) \arrow[r] & T^2(B/k, M) \arrow[r] & T^2(A/k, M) \end{tikzcd}$$

The assertion follows from the following observations.

(1) $T^0(B/A, M) = 0$ since $A \twoheadrightarrow B$.
(2) $T^1(B/A, M) = \mathrm{Hom}_B(I/I^2, M)$ by Proposition 3.10.
(3) $T^0(A/k, M) = \mathrm{Hom}(\Omega_{A/k}, M)$
(4) $T^2(A/k, M) = T^1(A/k, M) = 0$ as A is a polynomial ring over k.

$\square$

Proposition 4.3. *Let A be a local ring and $B = A/I$, where I is generated by a regular sequence $a_1, \dots, a_n$. Then, $T^2(B/A, M) = 0$ for all M.*

Proof. Examine the construction of $\mathbb{L}_\bullet$ in this case.

$$0 \to I \to A = R \to B \to 0$$
$$0 \to Q \to F = A^{\oplus n} \xrightarrow{\theta} I \to 0$$

□

Proposition 4.4. *The construction of the T^i functors is compatible with localisation.*

Proof. Left as an exercise. □

From last time,

Theorem 4.5. *Let k be an algebraically closed field and B a finite type k-algebra. Then, B is smooth over k if and only if $T^1(B/k, M) = 0$ for all $M \in B\text{-}\mathsf{Mod}$. Further, if B is smooth over k, then also $T^2(B/k, M) = 0$ for all $M \in B\text{-}\mathsf{Mod}$.*

Proof. Suppose B is smooth over k and let $B = k[x_1, \dots, x_n]/I$. By Proposition 4.2, $T^2(B/k, M) = T^2(B/A, M)$. Consider the conormal sequence

$$0 \to I/I^2 \to \Omega_{A/k} \otimes B \to \Omega_{B/k} \to 0$$

Localise at any prime $\mathfrak{p}$ of B

$$0 \to (I/I^2)_\mathfrak{p} \to \Omega_{A/k} \otimes B_\mathfrak{p} \to \Omega_{B/k,\mathfrak{p}} \to 0$$

We see that $I_\mathfrak{p}$ is generated by $n - 1 = \dim(A) - \dim(B)$ elements in the regular local ring $A_\mathfrak{p}$. These generators form a regular sequence. This implies that $T^2(B_\mathfrak{p}, M)$ □

Theorem 4.6. *Let A be a regular local k-algebra with residue field k with $k = \overline{k}$ and let $B = A/I$. Then, B is a local complete intersection in A if and only if $T^2(B/k, M) = 0$ for all $M \in B\text{-}\mathsf{Mod}$.*

Proof. As A is regular and $k = \overline{k}$, A is geometrically regular and hence, smooth over k. So $T^i(A/k, M) = 0$ for all M and for $i = 1, 2$. So, by the Jacobi-Zariski sequence (Theorem 3.7), get that $T^2(B/k, M) = T^2(B/A, M)$ for all M. If B is a complete intersection in A, then localising, we get the vanishing $T^2(B_\mathfrak{p}/k, M) = 0$ for any prime $\mathfrak{p}$ of B. Conversely, suppose $T^2(B/k, M) = 0$ for all M. As above, $T^2(B/A, M) = 0$ for all M. We look at $\mathbb{L}_\bullet$.

$$0 \to I \to A \to B \to 0$$

We may assume F maps to a minimal set of generators of I.

$$0 \to Q \to F \to I \to 0$$

And,

$$\mathbb{L}_\bullet = \left(Q/F_0 \xrightarrow{d_2} F/IF \xrightarrow{d_1} \Omega_{A/k} \otimes_A B \right)$$

$T^2(B/A, M) = 0$ implies that $\mathrm{Hom}_B(F/IF, M) \to \mathrm{Hom}_B(Q/F_0, M)$ is surjective for all M. Take $M = Q/F_0$ and get a splitting $p : F/IF \to Q/F_0$ of d_2. Since $a_1, \dots, a_r$ is a minimal set of generators of I, $Q \subseteq mF$. Thus, by Nakayama, $Q = F_0$ if and only if $H_1(K_\bullet(a_1, \dots, a_n))$ is a regular sequence. Hence, B is a local complete intersection.

□