Hints and Solution Sketches to Problems in [Har10] & Random Musings

Notation and Conventions:

- All rings are assumed to be commutative with unity.
- Given a topological space X, $\mathrm{Top}(X)$ denotes the preorder category of open subsets, with the preorder given by inclusion.
- Set, Ring, Ab, Top, RS, LRS, Sch, AffSch, Var_k denote respectively the categories[1] of sets, (commutative) rings, Abelian groups, topological spaces, ringed spaces, locally ringed spaces, schemes, affines schemes and varieties over a field k. For a space X and a bicomplete category $\mathcal{C}$, let $\mathsf{Psh}_{\mathcal{C}}(X)$ and $\mathsf{Sh}_{\mathcal{C}}(X)$ denote presheaves and sheaves over X valued in $\mathcal{C}$ respectively. If $\mathcal{C} = \mathsf{Set}$, we simply write $\mathsf{Psh}(X)$ and $\mathsf{Sh}(X)$ for $\mathsf{Psh}_{\mathsf{Set}}(X)$ and $\mathsf{Sh}_{\mathsf{Set}}(X)$

Chapter II : Schemes

Miscellaneous Remarks.

(1) **Pullback-Pushforward Adjunction**

(2) **Stalk-Skyscraper Adjunction**
Let $X \in \mathsf{Top}$, $p \in X$ and $\mathcal{C}$ be a bicomplete category. Let $\mathrm{St}_p : \mathsf{Sh}_{\mathcal{C}}(X) \to \mathcal{C}$ denote the stalk (at p) functor and $\mathrm{Sky}_p : \mathcal{C} \to \mathsf{Sh}_{\mathcal{C}}(X)$ the skyscraper. That is,

$$(\mathrm{Sky}_p a)(U) := \begin{cases} a & \text{if } p \in U \\ * & \text{if } p \notin U \end{cases}$$

Identifying $\mathcal{C}$ with $\mathsf{Sh}_{\mathcal{C}}(*)$, the stalk and skyscraper functors are respectively the pullback and push-forward along $\{p\} \hookrightarrow X$ and hence, $\mathrm{St}_p : \mathsf{Sh}_{\mathcal{C}}(X) \rightleftarrows \mathcal{C} : \mathrm{Sky}_p$.

(3) **Abstract Nonsense: Colimits in Ring**
The category of Ring is both complete and cocomplete (see [Bor95, Chapter 3] for some generalities). We record a construction of colimits as this is useful, for instance, while computing fibre products of affine schemes and stalks of sheaves of rings.
Let I be a small category and $F : I \to \mathsf{Ring}$ be a functor. Let $G := \bigsqcup_{i\in I} F(i)$ as sets. Define $\mathfrak{a}$ to be the ideal of $\mathbb{Z}[\{x_g : g \in G\}]$ generated by the union of the following collections of elements:
 (i) $x_{a+b} - x_a - x_b$ for all $a, b \in F(i)$ and for all $i \in I$.
 (ii) $x_{ab} - x_a x_b$ for all $a, b \in F(i)$ and for all $i \in I$.
 (iii) $1_{\mathbb{Z}[\{x_g : g\in G\}]} - x_{1_{F(i)}}$ for all $i \in I$.
 (iv) $x_a - x_{Ff(a)}$ for all $a \in F(i)$ and for all $f : i \to j$ in $\mathrm{Mor}(I)$.
Let $R := \mathbb{Z}[\{x_g : g \in G\}]/\mathfrak{a}$. By construction, the maps $F(i) \to R$, $a \mapsto x_a$ are ring maps and make R a cocone of F. Further, for every other cocone $(R', \phi_i : F(i) \to R')$, the association $x_{a\in F(i)} \mapsto \phi_i(a)$ is the unique morphism of cocones from R to R'. Thus, $R \cong \mathrm{colim}_I F$.

(4) **Abstract Nonsense: Limits in RS, LRS and Sch**

(5) **(Locally of finite type/finite type/finite) morphisms are closed under base change**
Suppose that $X \to Y$ is finite (resp. of finite type) and $Y' \to Y$ be a morphism of schemes. We claim that $f' : X \times_Y Y' \to Y'$, the base change along g, is finite (resp. of finite type) as well.
 Pick a point $y' \in Y'$ and let $y = g(y')$. Choose an affine open subscheme $y \in \mathrm{Spec}(R_y)$ of Y such that for a finite (resp. finitely generated) R_y-algebra $h : R_y \to S$, the following holds true.

[1] The "smallness assumption" is suppressed throughout

$$
\begin{tikzcd}
\mathrm{Spec}(S) \arrow[drr, "\mathrm{Spec}(h)"'] & \cong & X \times_Y \mathrm{Spec}(R_y) \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& & \mathrm{Spec}(R_y) \arrow[r, "g"] & Y
\end{tikzcd}
$$

Let $y' \in \mathrm{Spec}(R_{y'})$ be an affine open subscheme of Y' contained in the open set $g^{-1}(\mathrm{Spec}(R_y))$. Then, we have the following diagram where the dotted arrows are given by universality.

$$
\begin{tikzcd}
& & \mathrm{Spec(S)} \arrow[d, "\mathrm{Spec(h)}"] \arrow[drr] \arrow[dr, phantom, "\lrcorner", very near start] & & \\
\mathrm{Spec}(R_{y'}) \times_Y X \arrow[urr, dashed] \arrow[dr] \arrow[drrr, dashed] & & \mathrm{Spec}(R_y) \arrow[drr, hook] & & X \arrow[d, "f"] \\
& \mathrm{Spec}(R_{y'}) \arrow[ur, "g"] \arrow[drr, hook] & & X \times_Y Y' \arrow[ur] \arrow[d] & Y \\
& & & Y' \arrow[ur, "g"] &
\end{tikzcd}
$$

Observe that by universality, the following subdiagrams are pullback squares too.

$$
\begin{tikzcd}
\mathrm{Spec}(R_{y'}) \times_Y X \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}(S) \arrow[d] \\
\mathrm{Spec}(R_{y'}) \arrow[r, "g"'] & \mathrm{Spec}(R_y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Spec}(R_{y'}) \times_Y X \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \times_Y Y' \arrow[d] \\
\mathrm{Spec}(R_{y'}) \arrow[r, hook] & Y'
\end{tikzcd}
$$

So, $\mathrm{Spec}(R_{y'}) \times_Y X \cong R_{y'} \otimes_{R_y} S$, which is a finite (resp. finitely generated) $R_{y'}$-algebra.

Section 1. Sheaves.

Problem 1.3

(a)

(b) Let X be a topological space whose underlying set is $\{a, b, c\}$ with $U := \{a, b\}$ $V := \{b, c\}$ and $U \cap V$ exactly being the proper non-trivial open subsets. Then, the obvious map

$$\mathrm{Hom}_{\mathrm{Top}(X)}(\_, U) \coprod \mathrm{Hom}_{\mathrm{Top}(X)}(\_, V) \to \mathrm{Hom}_{\mathrm{Top}(X)}(\_, X)$$

is surjective at the level of stalks but not on global sections. Note that contravariant representable functors are sheaves and sheafification, being a left adjoint, preserves colimits. Finally, apply the free functor if one instead wants a sheaf valued in Ring, R-Mod etc.

Problem 1.17 See Remark (2).

Problem 1.18 See Remark (1)

Section 2. Schemes.

Problem 2.2

Straightforward. Follows from the fact that distinguished open subsets of affine schemes with the restricted structure sheaf are again affine.

Problem 2.3 (Reduced Schemes)

(a) The sheaf condition in terms of the equaliser implies that a scheme $(X, \mathcal{O}_X)$ is reduced if and only if all its affine open subschemes are reduced. Hence, consider $\operatorname{Spec}(R)$ for a ring R. If there is a non-zero nilpotent $(x_\mathfrak{p})_{\mathfrak{p}\in U}$ in $\mathcal{O}_{\operatorname{Spec}(R)}(U)$ for some open $U \subseteq \operatorname{Spec}(R)$, then for some $\mathfrak{p} \in U$, $x_\mathfrak{p}$ is non-zero and hence, a nilpotent in $R_\mathfrak{p}$. Conversely suppose that the stalk at some $\mathfrak{p} \in \operatorname{Spec}(R)$ (which is $R_\mathfrak{p}$) has a non-zero nilpotent. Then, R has a non-zero nilpotent and subsequently so does the ring of global sections.

(b)

Section 3. First Properties of Schemes.

Problem 3.1

(1) One direction is obvious. For the other, we have the adjunction $\mathsf{LRS} : \Gamma(\cdot, \mathcal{O}.) \rightleftarrows \operatorname{Spec} : \mathsf{Ring}^{\mathrm{op}}$ which is an equivalence upon restriction to affine schemes. Hence, for a ring R and non-zero $f \in R$, $R \hookrightarrow R_f$ corresponds (up to isomorphism) to the inclusion $D_f \hookrightarrow \operatorname{Spec}(R)$.

(2) Cover a given affine open subset with finitely many (by compactness) "nice" affine open subsets which exist by the locally finite type assumption. Work with distinguished open subsets on both sides using (1).

Problem 3.2

Problem 3.3

Problem 3.4

Problem 3.9

(a) Spec is a right adjoint. So,

$$\operatorname{Spec}(k[x]) \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k[x]) \cong \operatorname{Spec}(k[x] \coprod_k k[x]) \cong \operatorname{Spec}(k[x] \otimes_k k[x]) \cong \operatorname{Spec}(k[x, y])$$

The underlying point sets are the union of two intersecting lines and the plane and hence, distinct.

(b) Similar to (a), the fibred product is $\operatorname{Spec}(k(s) \otimes_k k(t))$. Letting $X := \{fg \in k[s,t] | f \in k[s] \setminus 0 \text{ and } f \in k[t] \setminus 0\}$, $k(s) \otimes_k k(t) \cong X^{-1}k[s,t]$. This is clearly a domain but not a field and hence its spectrum contains more than one point.

Problem 3.13

(a)
(b)
(c)
(d)
(e)
(f)
(g) By Hilbert basis theorem and since quotients of Noetherian rings are Noetherian, if Y is Noetherian and X is of finite type, then X is locally Noetherian. By Problem 3.2 and Problem 3.3 (a), X is also quasicompact.

Problem 3.14

(Pedantic Remark: by saying "closed points are dense", Hartshorne means that the set of closed points is dense)
By Problem 3.4, we may assume WLOG that $X = \operatorname{Spec}(R)$ for a finite R-algebra k. Since k is algebraically closed, this means that $k = R$.
Take ???

Problem 3.16

Vacuously, $\emptyset$ satisfies $\mathcal{P}$. Suppose towards contradiction that $X_0 := X$ does not. Then, one can find $\emptyset \subsetneq X_1 \subsetneq X$ that does not satisfy $\mathcal{P}$. Suppose for $1 \leqslant i \leqslant k$, $X_i \subsetneq X_{i-1}$ and X_i does not satisfy $\mathcal{P}$. Then, there exists $\emptyset \subsetneq X_{i+1} \subsetneq X_i$ not satisfying $\mathcal{P}$. By induction, this implies that X is not Noetherian.

Problem 3.17

(a) The generic point condition follows from Problem 2.9. Write X as a finite union of $\operatorname{Spec}(R_i)$ with R_i being Noetherian and hence, $\operatorname{Spec}(R_i)$ being Noetherian as well. Suppose towards contradiction that there exists a strictly decreasing chain of closed subsets of X. Take intersections with the complements $\operatorname{Spec}(R_i)^{\mathrm{c}}$. All of these stabilise and these complements cover X, leading to the required contradiction.

(b) A minimal closed subset $A = \bigcap_{a \in A} \overline{a}$. If A contains more than one element, then it is irreducible with more than one generic point.

(c) Given $x \in X$, the intersection of all closed subsets containing x is a minimal non-empty closed subset of X. T_0 easily follows.

(d) Trivial, even to me.

4. Separated and Proper Morphisms.

Problem 4.1

By Problem 3.5 (b) and Remark (5), finite morphisms are universally closed. Finite morphisms are separated by [Har10, Chapter 2, Proposition 4.1, Corollary 4.6 (f)].

5. Sheaves of Modules.

Problem 5.8

References

[Bor95] Francis Borceaux. Handbook of Categorical Algebra, Volume 2: Categories and Structures.
[Har10] Robin Hartshorne. *Algebraic Geometry.*