Discussion on 13 June 2025

The key goal here is to eventually prove the following results:

- Direct sums of exact sequences of (pre)sheaves on a topological space are exact.
- Direct products of epimorphisms of sheaves on X are not necessarily epimorphisms. (This is still incomplete. Once done, I'll append to the same link)
- Let $\mathcal{T}$ be a triangulated category and $\mathcal{L}$ be a localizing subcategory of $\mathcal{T}$. If $f : x \to y$ be a morphism in $\mathcal{T}$ that is mapped to 0 by the projection $\mathcal{T} \to \mathcal{T}_{/\mathcal{L}}$, then f factors through an object of $\mathcal{L}$.

Notation 1. For a topological space X, $\mathsf{Sh}(X)$ (resp. $\mathsf{PSh}(X)$) denotes the category of sheaves (resp. presheaves) of abelian groups over X.

Proposition 2. *Let X be a topological space and I be a set. If for each $i \in I$,*

$$0 \longrightarrow \mathcal{F}^i \longrightarrow \mathcal{G}^i \longrightarrow \mathcal{H}^i \longrightarrow 0$$

is an exact sequence in $\mathsf{PSh}(X)$, then,

$$0 \longrightarrow \bigoplus_i \mathcal{F}^i \longrightarrow \bigoplus_i \mathcal{G}^i \longrightarrow \bigoplus_i \mathcal{H}^i \longrightarrow 0$$

is also exact.

Proof. By assumption, for any open subset U of X,

$$0 \longrightarrow \mathcal{F}^i(U) \longrightarrow \mathcal{G}^i(U) \longrightarrow \mathcal{H}^i(U) \longrightarrow 0$$

is exact. Since direct sums of exact sequences are exact in the category Ab of abelian groups, the sequence

$$0 \longrightarrow \left(\bigoplus_i \mathcal{F}^i\right)(U) \longrightarrow \left(\bigoplus_i \mathcal{G}^i\right)(U) \longrightarrow \left(\bigoplus_i \mathcal{H}^i\right)(U) \longrightarrow 0$$

is exact. Thus, it follows that

$$0 \longrightarrow \bigoplus_i \mathcal{F}^i \longrightarrow \bigoplus_i \mathcal{G}^i \longrightarrow \bigoplus_i \mathcal{H}^i \longrightarrow 0$$

is exact. □

Proposition 3. *Let X be a topological space and I be a set. If for each $i \in I$,*

$$0 \longrightarrow \mathcal{F}^i \longrightarrow \mathcal{G}^i \longrightarrow \mathcal{H}^i \longrightarrow 0$$

is an exact sequence in $\mathsf{Sh}(X)$, then,

$$0 \longrightarrow \bigoplus_i \mathcal{F}^i \longrightarrow \bigoplus_i \mathcal{G}^i \longrightarrow \bigoplus_i \mathcal{H}^i \longrightarrow 0 \tag{1}$$

is also exact.

Proof. Let $U : \mathsf{Sh}(X) \to \mathsf{PSh}(X)$ denote the forgetful functor and $\_^\dagger : \mathsf{Sh}(X) \to \mathsf{PSh}(X)$ denote the sheafification. Then, the sequence in Eq. (1) is isomorphic to the following sequence

$$0 \to (\bigoplus_i U(\mathcal{F}^i))^\dagger \to (\bigoplus_i U(\mathcal{G}^i))^\dagger \to (\bigoplus_i U(\mathcal{H}^i))^\dagger \to 0$$

For any $x \in X$, since the forgetful and sheafification functors preserve stalks and since direct sums commute with colimits, applying the stalk functor $(\_)_x$ yields (upto isomorphism) the following sequence

$$0 \longrightarrow \bigoplus_i \mathcal{F}^i_x \longrightarrow \bigoplus_i \mathcal{G}^i_x \longrightarrow \bigoplus_i \mathcal{H}^i_x \longrightarrow 0$$

which is exact since direct sums of exact sequences are exact in Ab. □

Proposition 4. *Let $\mathcal{T}$ be a triangulated category, $\mathcal{L}$ a localising subcategory and $\mathcal{T}_{/\mathcal{L}}$ denote the Verdier quotient. A morphism $f : x \to y$ in $\mathcal{T}$ is mapped to the zero morphism by the projection $\mathcal{T} \to \mathcal{T}_{/\mathcal{L}}$ if and only if f factors through an object in $\mathcal{L}$.*

Proof. Suppose that f is taken to 0 by the projection. That is, there exist $z, z' \in \mathcal{T}$, quasi-isomorphisms (relative to $\mathcal{L}$) σ, σ' and morphisms f', g, h in $\mathcal{T}$ making the following diagram commute.

$$\begin{tikzcd}
 & y \arrow[d, "g"] \arrow[dr, equal] & \\
x \arrow[ur, "f"] \arrow[r, "f'"] \arrow[dr, "0"'] & z' & y \arrow[l, "\sigma'"', "\sim"] \arrow[dl, "\sim"', "\sigma"] \\
 & z \arrow[u, "h"] &
\end{tikzcd}$$

Equivalently, there exists a quasi-isomorphism g such that $g \circ f = 0$. Hence, there exists a distinguished triangle of the following form in $\mathcal{T}$

$$y \xrightarrow{g} z \xrightarrow{\alpha} w \xrightarrow{\beta} \Sigma y$$

where $w \in \mathcal{L}$. By one of the axioms defining a (pre)triangulated category, there exists a dotted arrow $x \to \Sigma^{-1} w$ making the following diagram commute, thus yielding the required factorisation.

$$\begin{tikzcd}
\Sigma^{-1} x \arrow[r] \arrow[d, "\Sigma^{-1} f"'] & 0 \arrow[r] \arrow[d, "0"] & x \arrow[r, "\mathrm{Id}"] \arrow[d, dashed] & x \arrow[d, "f"] \\
\Sigma^{-1} y \arrow[r, "\Sigma^{-1} g"] & \Sigma^{-1} z \arrow[r, "\Sigma^{-1} \alpha"] & \Sigma^{-1} w \arrow[r, "-\Sigma^{-1} \beta"] & y
\end{tikzcd}$$

Conversely, suppose that f factors as in the following diagram, where $w \in L$.

$$\begin{tikzcd}
x \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & y \\
 & w \arrow[ur, "\beta"'] &
\end{tikzcd}$$

As previously argued, it is sufficient to show that there exists a quasi-isomorphism g such that $g \circ f = 0$. By one of the axioms defining a (pre)triangulated category, there exists a diagram of the following form, where the rows are distinguished and there is a dotted arrow that completes the morphism of distinguished triangles.

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "\alpha"'] & y \arrow[r] \arrow[d, equal] & \mathrm{cone}(f) \arrow[r] \arrow[d, dashed] & \Sigma x \arrow[d, "\Sigma \alpha"] \\
w \arrow[r, "\beta"] & y \arrow[r] & \mathrm{cone}(\beta) \arrow[r] & \Sigma w
\end{tikzcd}$$

We take the morphism $y \to \mathrm{cone}(\beta)$ in the above diagram to be g. $\square$