Some Functors and Adjunctions involving Complexes and Bicomplexes

Definition/Notation 1. For an abelian category $\mathcal{A}$, by a *complex* over $\mathcal{A}$, we mean unbounded cochain complexes of objects in $\mathcal{A}$. We let $\mathsf{Comp}(\mathcal{A})$ denote the abelian category of complexes over $\mathcal{A}$. Similarly, by a *bicomplex* over $\mathcal{A}$, we mean an unbounded cochain complex over $\mathsf{Comp}(\mathcal{A})$. Equivalently and more conveniently, a bicomplex over $\mathcal{A}$ essentially consists of

- Objects $A^{i,j}$ in $\mathcal{A}$ for all $i, j \in \mathcal{A}$
- Morphisms $d^{i,j}_{\uparrow,A} : A^{i,j} \to A^{i,j+1}$ in $\mathcal{A}$ for all $i, j \in \mathbb{Z}$
- Morphisms $d^{i,j}_{\to,A} : A^{i,j} \to A^{i+1,j}$ in $\mathcal{A}$ for all $i, j \in \mathbb{Z}$

such that

- $d^{i,j+1}_{\uparrow,A} \circ d^{i,j}_{\uparrow,A} = 0$ for all $i, j \in \mathbb{Z}$
- $d^{i,j+1}_{\to,A} \circ d^{i,j}_{\to,A} = 0$ for all $i, j \in \mathbb{Z}$
- $d^{i+1,j}_{\uparrow,A} \circ d^{i,j}_{\to,A} = d^{i,j+1}_{\to,A} \circ d^{i,j}_{\uparrow,A}$ for all $i, j \in \mathbb{Z}$.

We denote the category of bicomplexes over $\mathcal{A}$ by $\mathsf{Bicomp}(\mathcal{A})$.

Remark 2. We will use the notation introduced in Definition/Notation 1 to describe bicomplexes, often without explicit mention. For instance, we simply write "X is a bicomplex over $\mathcal{A}$" to indicate the data of objects $X^{i,j}$ and morphisms $d^{i,j}_{\to,X}, d^{i,j}_{\uparrow,X}$ satisfying the appropriate conditions as in Definition/Notation 1. Furthermore, if the underlying complex and indices are either clear from the context or irrelevant, we abuse notation and write $d_\to$ (resp. $d_\uparrow$) instead of $d^{i,j}_{\to,X}$ (resp. $d^{i,j}_{\to,X}$). Analogously, when we say "Y is a complex", we take for granted the data of objects Y^i in $\mathcal{A}$ and morphisms $d^i_Y : Y^i \to Y^{i+1}$ for all $i \in \mathbb{Z}$, such that $d^{i+1}_Y \circ d^i_Y = 0$ for all $i \in \mathbb{Z}$. Again, we shorten d^i_Y to d_Y or just d if it causes no confusion.

Convention 3. Throughout, we will reserve the symbol $\mathcal{A}$ for an arbitrary abelian category that has all countable products and coproducts. Unless otherwise mentioned, complexes and bicomplexes are assumed to be over $\mathcal{A}$.

Definition 4. Define a functor $\mathcal{D} : \mathsf{Comp}(\mathcal{A}) \to \mathsf{Bicomp}(\mathcal{A})$ in the following manner

- For $A \in \mathsf{Comp}(\mathcal{A})$, $\mathcal{D}(A)$ is the bicomplex given by
  - $\mathcal{D}(A)^{i,j} = A^{i+j}$ for all $i, j \in \mathbb{Z}$
  - $d^{i,j}_{\to,\mathcal{D}(A)} = d^{i,j}_{\uparrow,\mathcal{D}(A)} = d^{i+j}_A$ for all $i, j \in \mathbb{Z}$.

- If $f : A \to B$ is a morphism in $\mathsf{Comp}(\mathcal{A})$, then $\mathcal{D}(f)$ is the morphism of bicomplexes induced by the maps $\mathcal{D}(A)^{i,j} = A^{i+j} \xrightarrow{f^{i+j}} B^{i+j} = \mathcal{D}(B)^{i,j}$

Definition 5. Suppose that $\mathcal{A}$ admits all countable coproducts. Then, we may define a functor $\mathcal{L} : \mathsf{Bicomp}(\mathcal{A}) \to \mathsf{Comp}(\mathcal{A})$ in the following manner.