The key goal here is to eventually prove the following results:

- Direct sums of exact sequences of (pre)sheaves on a topological space are exact. (Propositions 2 and 3)
- ullet Direct products of epimorphisms of sheaves on X are not necessarily epimorphisms. (Construction 4)
- Let \mathcal{T} be a triangulated category and \mathcal{L} be a localizing subcategory of \mathcal{T} . If $f: x \to y$ be a morphism in \mathcal{T} that is mapped to 0 by the projection $\mathcal{T} \to \mathcal{T}_{/\mathcal{L}}$, then f factors through an object of \mathcal{L} . (Proposition 5)

Notation 1. For a topological space X, $\mathsf{Sh}(X)$ (resp. $\mathsf{PSh}(X)$) denotes the category of sheaves (resp. presheaves) of abelian groups over X.

Proposition 2. Let X be a topological space and I be a set. If for each $i \in I$,

$$0 \longrightarrow \mathcal{F}^i \longrightarrow \mathcal{G}^i \longrightarrow \mathcal{H}^i \longrightarrow 0$$

is an exact sequence in PSh(X), then,

$$0 \longrightarrow \bigoplus_{i} \mathcal{F}^{i} \longrightarrow \bigoplus_{i} \mathcal{G}^{i} \longrightarrow \bigoplus_{i} \mathcal{H}^{i} \longrightarrow 0$$

is also exact.

Proof. By assumption, for any open subset U of X,

$$0 \longrightarrow \mathcal{F}^i(U) \longrightarrow \mathcal{G}^i(U) \longrightarrow \mathcal{H}^i(U) \longrightarrow 0$$

is exact. Since direct sums of exact sequences are exact in the category Ab of abelian groups, the sequence

$$0 \longrightarrow \left(\bigoplus_{i} \mathcal{F}^{i}\right)(U) \longrightarrow \left(\bigoplus_{i} \mathcal{G}^{i}\right)(U) \longrightarrow \left(\bigoplus_{i} \mathcal{H}^{i}\right)(U) \longrightarrow 0$$

is exact. Thus, it follows that

$$0 \longrightarrow \bigoplus_{i} \mathcal{F}^{i} \longrightarrow \bigoplus_{i} \mathcal{G}^{i} \longrightarrow \bigoplus_{i} \mathcal{H}^{i} \longrightarrow 0$$

is exact.

Proposition 3. Let X be a topological space and I be a set. If for each $i \in I$,

$$0 \longrightarrow \mathcal{F}^i \longrightarrow \mathcal{G}^i \longrightarrow \mathcal{H}^i \longrightarrow 0$$

is an exact sequence in Sh(X), then,

$$0 \longrightarrow \bigoplus_{i} \mathcal{F}^{i} \longrightarrow \bigoplus_{i} \mathcal{G}^{i} \longrightarrow \bigoplus_{i} \mathcal{H}^{i} \longrightarrow 0$$

is also exact.

Proof. Let $U: \mathsf{Sh}(X) \to \mathsf{PSh}(X)$ denote the forgetful functor and $_^{\dagger}: \mathsf{Sh}(X) \to \mathsf{PSh}(X)$ denote the sheafification. Then, the sequence in Eq. (1) is isomorphic to the following sequence

$$0 \to (\bigoplus_i U(\mathcal{F}^i))^\dagger \to (\bigoplus_i U(\mathcal{G}^i))^\dagger \to (\bigoplus_i U(\mathcal{H}^i))^\dagger \to 0$$

For any $x \in X$, since the forgetful and sheafification functors preserve stalks and since direct sums commute with colimits, applying the stalk functor (_) $_x$ yields (upto isomorphism) the following sequence

$$0 \longrightarrow \bigoplus_{i} \mathcal{F}_{x}^{i} \longrightarrow \bigoplus_{i} \mathcal{G}_{x}^{i} \longrightarrow \bigoplus_{i} \mathcal{H}_{x}^{i} \longrightarrow 0$$

which is exact since direct sums of exact sequences are exact in Ab.

Construction 4. Define

$$X := \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \subseteq \mathbb{R}$$

and

$$\tau := \{ [0, m) \cap X \mid m \in \mathbb{R}_{\geq 0} \} \cup \{ (0, m) \cap X \mid m \in \mathbb{R}_{\geq 0} \}$$

It is easy to see that (X, τ) is a topological space. For each $n \in \mathbb{N}$, define a presheaf \mathcal{F}_n on X as follows

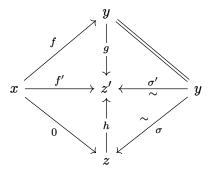
$$\mathfrak{F}_n(U) := egin{cases} \mathbb{Z} & ext{if} & \emptyset
eq U \subseteq [0, 1/n) \\ 0 & ext{otherwise} \end{cases}$$

and the restriction map $\mathcal{F}_n(U) \to \mathcal{F}_n(V)$ is defined to be $\mathrm{Id}_{\mathbb{Z}}$ if $\emptyset \neq V \subseteq U \subseteq [0, \frac{1}{n})$ and 0 otherwise. In fact, \mathcal{F}_n is further a sheaf. Base identity is clear and it is sufficient to check base gluability on open covers of the form $[0, \frac{1}{m'}) = [0, \frac{1}{m}) \cup (0, \frac{1}{m'})$ where m' < m.

Let $\mathcal{G} := \mathsf{Sky}_0(\mathbb{Z})$, the skyscraper sheaf at 0 over X with stalk \mathbb{Z} at 0. By adjointness, the identity map $(\mathcal{F}_n)_0 = \mathbb{Z} \to \mathbb{Z}$ induces a surjective map of sheaves $\phi_n : \mathcal{F}_n \to \mathcal{G}$ for each $n \in \mathbb{N}$. We claim that $\prod_i \phi_i : \prod_i \mathcal{F}_i \to \prod_i \mathcal{G}$ is not surjective. It is easy to see that $(\prod_i \mathcal{G})_0 \cong \prod_i \mathbb{Z}$ which is uncountable. However, for any given open neighbourhood U of $0, \mathcal{F}_i(U)$ is 0 for large enough i. Thus, every abelian group appearing in the countable colimit defining the stalk $(\prod_i \mathcal{F}_i)_0$ is countable and consequently, the stalk $(\prod_i (\mathcal{F}_i))_0$ is itself countable. This proves that $\prod_i \phi_i$ is not surjective. 1.

Proposition 5. Let \mathbb{T} be a triangulated category, \mathcal{L} a localising subcategory and $\mathbb{T}_{/\mathcal{L}}$ denote the Verdier quotient. A morphism $f: x \to y$ in \mathbb{T} is mapped to the zero morphism by the projection $\mathbb{T} \to \mathbb{T}_{/\mathcal{L}}$ if and only if f factors through an object in \mathcal{L} .

Proof. Suppose that f is taken to 0 by the projection. That is, there exist $z, z' \in \mathcal{T}$, quasi-isomorphisms (relative to \mathcal{L}) σ, σ' and morphisms f', g, h in \mathcal{T} making the following diagram commute.



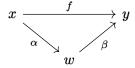
Equivalently, there exists a quasi-isomorphism g such that $g \circ f = 0$. Hence, there exists a distinguished triangle of the following form in \mathcal{T}

$$y \stackrel{g}{-\!\!\!-\!\!\!-\!\!\!-\!\!\!-} z \stackrel{lpha}{-\!\!\!\!-\!\!\!\!-} w \stackrel{eta}{-\!\!\!\!-\!\!\!\!-} \Sigma y$$

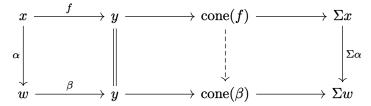
¹The countability argument to simplify the proof of $\prod_i \phi_i$ not being surjective is due to a friend, Atharva Raje, who listened to an earlier and dirtier version of the proof

where $w \in \mathcal{L}$. By one of the axioms defining a (pre)triangulated category, there exists a dotted arrow $x \to \Sigma^{-1} w$ making the following diagram commute, thus yielding the required factorisation.

Conversely, suppose that f factors as in the following diagram, where $w \in L$.



As previously argued, it is sufficient to show that there exists a quasi-isomorphism g such that $g \circ f = 0$. By one of the axioms defining a (pre)triangulated category, there exists a diagram of the following form, where the rows are distinguished and there is a dotted arrow that completes the morphism of distinguished triangles.



We take the morphism $y \to \text{cone}(\beta)$ in the above diagram to be g.