The key goal here is to eventually prove the following results:

- Direct sums of exact sequences of (pre)sheaves on a topological space are exact.
- Direct products of epimorphisms of sheaves on X are not necessarily epimorphisms. (This is still incomplete. Once done, I'll append to the same link)
- Let \mathcal{T} is a triangulated category and \mathcal{L} is a localizing subcategory. If $f: x \to y$ be a morphism in \mathcal{T} that is mapped to 0 by the projection $\mathcal{T} \to \mathcal{T}_{/\mathcal{L}}$, then f factors through an object of \mathcal{L} .

Notation 1. For a topological space X, $\mathsf{Sh}(X)$ (resp. $\mathsf{PSh}(X)$) denotes the category of sheaves (resp. presheaves) of abelian groups over X.

Proposition 2. Let X be a topological space and I be a set. If for each $i \in I$,

$$0 \longrightarrow \mathcal{F}^i \longrightarrow \mathcal{G}^i \longrightarrow \mathcal{H}^i \longrightarrow 0$$

is an exact sequence in PSh(X), then,

$$0 \longrightarrow \bigoplus_{i} \mathcal{F}^{i} \longrightarrow \bigoplus_{i} \mathcal{G}^{i} \longrightarrow \bigoplus_{i} \mathcal{H}^{i} \longrightarrow 0$$

is also exact.

Proof. By assumption, for any open subset U of X,

$$0 \longrightarrow \mathcal{F}^{i}(U) \longrightarrow \mathcal{G}^{i}(U) \longrightarrow \mathcal{H}^{i}(U) \longrightarrow 0$$

is exact. Since direct sums of exact sequences are exact in the category Ab of abelian groups, the sequence

$$0 \longrightarrow \left(\bigoplus_{i} \mathcal{F}^{i}\right)(U) \longrightarrow \left(\bigoplus_{i} \mathcal{G}^{i}\right)(U) \longrightarrow \left(\bigoplus_{i} \mathcal{H}^{i}\right)(U) \longrightarrow 0$$

is exact. Thus, it follows that

$$0 \longrightarrow \bigoplus_i \mathcal{F}^i \longrightarrow \bigoplus_i \mathcal{G}^i \longrightarrow \bigoplus_i \mathcal{H}^i \longrightarrow 0$$

is exact.

Proposition 3. Let X be a topological space and I be a set. If for each $i \in I$,

$$0 \longrightarrow \mathcal{F}^i \longrightarrow \mathcal{G}^i \longrightarrow \mathcal{H}^i \longrightarrow 0$$

is an exact sequence in Sh(X), then,

$$(1) 0 \longrightarrow \bigoplus_{i} \mathcal{F}^{i} \longrightarrow \bigoplus_{i} \mathcal{G}^{i} \longrightarrow \bigoplus_{i} \mathcal{H}^{i} \longrightarrow 0$$

is also exact.

Proof. Let $U: \mathsf{Sh}(X) \to \mathsf{PSh}(X)$ denote the forgetful functor and $_^\dagger: \mathsf{Sh}(X) \to \mathsf{PSh}(X)$ denote the sheafification. Then, the sequence in Eq. (1) is isomorphic to the following sequence

$$0 \to (\bigoplus_i U(\mathcal{F}^i))^\dagger \to (\bigoplus_i U(\mathcal{G}^i))^\dagger \to (\bigoplus_i U(\mathcal{H}^i))^\dagger \to 0$$

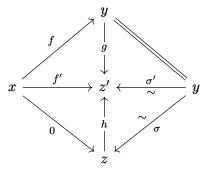
For any $x \in X$, since the forgetful and sheafification functors preserve stalks and since direct sums commute with colimits, applying the stalk functor $(_)_x$ yields (upto isomorphism) the following sequence

$$0 \longrightarrow \bigoplus_{i} \mathcal{F}_{x}^{i} \longrightarrow \bigoplus_{i} \mathcal{G}_{x}^{i} \longrightarrow \bigoplus_{i} \mathcal{H}_{x}^{i} \longrightarrow 0$$

which is exact since direct sums of exact sequences are exact in Ab.

Proposition 4. Let T be a triangulated category, \mathcal{L} a localising subcategory and $T_{/\mathcal{L}}$ denote the Verdier quotient. A morphism $f: x \to y$ in T is mapped to the zero morphism by the projection $T \to T_{/\mathcal{L}}$ if and only if f factors through an object in \mathcal{L} .

Proof. Suppose that f is taken to 0 by the projection. That is, there exist $z, z' \in \mathcal{T}$, quasi-isomorphisms (relative to \mathcal{L}) σ, σ' and morphisms f', g, h in \mathcal{T} making the following diagram commute.

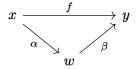


Equivalently, there exists a quasi-isomorphism g such that $g \circ f = 0$. Hence, there exists a distinguished triangle of the following form in \mathcal{T}

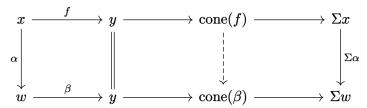
$$y \stackrel{g}{-\!\!\!-\!\!\!-\!\!\!-\!\!\!-} z \stackrel{lpha}{-\!\!\!\!-\!\!\!\!-} w \stackrel{eta}{-\!\!\!\!-\!\!\!\!-} \Sigma y$$

where $w \in \mathcal{L}$. By one of the axioms defining a (pre)triangulated category, there exists a dotted arrow $x \to \Sigma^{-1} w$ making the following diagram commute, thus yielding the required factorisation.

Conversely, suppose that f factors as in the following diagram, where $w \in L$.



As previously argued, it is sufficient to show that there exists a quasi-isomorphism g such that $g \circ f = 0$. By one of the axioms defining a (pre)triangulated category, there exists a diagram of the following form, where the rows are distinguished and there is a dotted arrow that completes the morphism of distinguished triangles.



We take the morphism $y \to \text{cone}(\beta)$ in the above diagram to be g.