**SUPPLEMENT TO** [BN93]

**Notation 1.**

(1) For an abelian category $\mathcal{A}$, $\mathsf{Comp}(\mathcal{A})$ denotes the category of unbounded cochain complexes over $\mathcal{A}$ and $K(\mathcal{A})$ denotes the corresponding homotopy category. For a morphism $f$ in $\mathsf{Comp}(\mathcal{A})$, we write $[f]$ to denote its image in $K(\mathcal{A})$ under the projection $\mathsf{Comp}(\mathcal{A}) \to K(\mathcal{A})$.

## 1. Direct Sums in Triangulated Categories

**Proof of Lemma 1.1.** Let $\{A_i\}_{i\in I}$ be an indexed collection of objects in $\mathsf{Comp}(\mathcal{A})$. Let $C := \bigoplus_{i\in I} A_i$ along with structure maps $\iota_j : A_j \to C$ be their coproduct in $\mathsf{Comp}(\mathcal{A})$. Let $D$ be an object in

**On Remark 1.4.**

*(1.3.2)⇒(1.3.1).* Suppose that $A$ and $B$ are objects in a triangulated category $(\mathcal{T}, \Sigma)$ with all countable direct sums. Let $\mathcal{L}$ be a triangulated subcategory that is closed under countable direct sums. More precisely, we require not merely that countable direct sums in $\mathcal{T}$ of objects in $\mathcal{L}$ exist in $\mathcal{L}$ but that *any* direct sum in $\mathcal{T}$ of a collection of objects in $\mathcal{L}$ is contained in $\mathcal{L}$. Thus $\mathcal{L}$ is necessarily a replete triangulated subcategory of $\mathcal{T}$. Let

$$C := (B \oplus A) \oplus (B \oplus A) \oplus \dots$$
$$C' := (A \oplus B) \oplus (A \oplus B) \oplus \dots$$

Then, $C, C'$ are contained in $\mathcal{L}$ and $C' \cong A \oplus C$. Further, the triangle

$$C \longrightarrow C' \longrightarrow A \longrightarrow \Sigma C$$

is distinguished in $\mathcal{T}$. Thus, $A \in \mathcal{L}$.

*On Rickard's criterion.*

**Definition 2** (Épaisse subcategories, [?])**.** Let $\mathcal{T}$ be a triangulated category. A triangulated and full subcategory $\mathcal{E}$ of $\mathcal{T}$ is said to be *épaisse* if for all $f : X \to Y$ in $\mathcal{T}$ that factors through an object in $\mathcal{E}$ and is part of a distinguished triangle $(X, Y, Z, f, g, h)$ in $\mathcal{T}$ with $Z$ contained in $\mathcal{E}$, both $X$ and $Y$ are contained in $\mathcal{E}$.

**Proposition 3** (Rickard)**.** *Let $\mathcal{E}$ be a triangulated and full subcategory of a triangulated category $\mathcal{T}$. Then, $\mathcal{E}$ is épaisse if and only if every direct summand of an object in $\mathcal{E}$ is contained in $\mathcal{E}$.*

The following is proof is a slightly more detailed version of [**?**, Criterion 1.3]

*Proof.* Suppose first that $\mathcal{E}$ is épaisse. Let $A, B$ be objects in $\mathcal{T}$ such that $A \oplus B$ is contained in $\mathcal{E}$. Then, the following triangle is distinguished

$$\Sigma^{-1}B \xrightarrow{0} A \longrightarrow A \oplus B \longrightarrow B$$

Since $\mathcal{E}$ is épaisse, $A$ is contained in $\mathcal{E}$.
Conversely, suppose that every direct summand of an object in $\mathcal{E}$ is contained in $\mathcal{E}$. □



*Date*:

## References


[BN93] Bökstedt, Marcel, and Amnon Neeman. "Homotopy limits in triangulated categories." *Compositio Mathematica* 86, no. 2 (1993): 209-234.

[Ric89] Rickard, Jeremy. "Derived categories and stable equivalence." *Journal of pure and applied Algebra* 61, no. 3 (1989): 303-317.