The key goal here is to eventually prove the following results:

- Direct sums of exact sequences of (pre)sheaves on a topological space are exact.
- Direct products of epimorphisms of sheaves on X are not necessarily epimorphisms. (This is still incomplete. Once done, I'll append to the same link)
- Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{L}$  be a localizing subcategory of  $\mathcal{T}$ . If  $f: x \to y$  be a morphism in  $\mathcal{T}$  that is mapped to 0 by the projection  $\mathcal{T} \to \mathcal{T}_{/\mathcal{L}}$ , then f factors through an object of  $\mathcal{L}$ .

**Notation 1.** For a topological space X,  $\mathsf{Sh}(X)$  (resp.  $\mathsf{PSh}(X)$ ) denotes the category of sheaves (resp. presheaves) of abelian groups over X.

**Proposition 2.** Let X be a topological space and I be a set. If for each  $i \in I$ ,

$$0 \longrightarrow \mathcal{F}^i \longrightarrow \mathcal{G}^i \longrightarrow \mathcal{H}^i \longrightarrow 0$$

is an exact sequence in PSh(X), then,

$$0 \longrightarrow \bigoplus_{i} \mathcal{F}^{i} \longrightarrow \bigoplus_{i} \mathcal{G}^{i} \longrightarrow \bigoplus_{i} \mathcal{H}^{i} \longrightarrow 0$$

is also exact.

*Proof.* By assumption, for any open subset U of X,

$$0 \longrightarrow \mathcal{F}^{i}(U) \longrightarrow \mathcal{G}^{i}(U) \longrightarrow \mathcal{H}^{i}(U) \longrightarrow 0$$

is exact. Since direct sums of exact sequences are exact in the category Ab of abelian groups, the sequence

$$0 \longrightarrow \left(\bigoplus_{i} \mathcal{F}^{i}\right)(U) \longrightarrow \left(\bigoplus_{i} \mathcal{G}^{i}\right)(U) \longrightarrow \left(\bigoplus_{i} \mathcal{H}^{i}\right)(U) \longrightarrow 0$$

is exact. Thus, it follows that

$$0 \longrightarrow \bigoplus_i \mathcal{F}^i \longrightarrow \bigoplus_i \mathcal{G}^i \longrightarrow \bigoplus_i \mathcal{H}^i \longrightarrow 0$$

is exact.

**Proposition 3.** Let X be a topological space and I be a set. If for each  $i \in I$ ,

$$0 \longrightarrow \mathcal{F}^i \longrightarrow \mathcal{G}^i \longrightarrow \mathcal{H}^i \longrightarrow 0$$

is an exact sequence in Sh(X), then,

$$(1) 0 \longrightarrow \bigoplus_{i} \mathcal{F}^{i} \longrightarrow \bigoplus_{i} \mathcal{G}^{i} \longrightarrow \bigoplus_{i} \mathcal{H}^{i} \longrightarrow 0$$

is also exact.

*Proof.* Let  $U: \mathsf{Sh}(X) \to \mathsf{PSh}(X)$  denote the forgetful functor and  $\_^\dagger: \mathsf{Sh}(X) \to \mathsf{PSh}(X)$  denote the sheafification. Then, the sequence in Eq. (1) is isomorphic to the following sequence

$$0 \to (\bigoplus_i U(\mathcal{F}^i))^\dagger \to (\bigoplus_i U(\mathcal{G}^i))^\dagger \to (\bigoplus_i U(\mathcal{H}^i))^\dagger \to 0$$

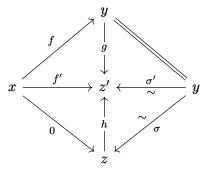
For any  $x \in X$ , since the forgetful and sheafification functors preserve stalks and since direct sums commute with colimits, applying the stalk functor  $(\_)_x$  yields (upto isomorphism) the following sequence

$$0 \longrightarrow \bigoplus_{i} \mathcal{F}_{x}^{i} \longrightarrow \bigoplus_{i} \mathcal{G}_{x}^{i} \longrightarrow \bigoplus_{i} \mathcal{H}_{x}^{i} \longrightarrow 0$$

which is exact since direct sums of exact sequences are exact in Ab.

**Proposition 4.** Let T be a triangulated category,  $\mathcal{L}$  a localising subcategory and  $T_{/\mathcal{L}}$  denote the Verdier quotient. A morphism  $f: x \to y$  in T is mapped to the zero morphism by the projection  $T \to T_{/\mathcal{L}}$  if and only if f factors through an object in  $\mathcal{L}$ .

*Proof.* Suppose that f is taken to 0 by the projection. That is, there exist  $z, z' \in \mathcal{T}$ , quasi-isomorphisms (relative to  $\mathcal{L}$ )  $\sigma, \sigma'$  and morphisms f', g, h in  $\mathcal{T}$  making the following diagram commute.

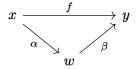


Equivalently, there exists a quasi-isomorphism g such that  $g \circ f = 0$ . Hence, there exists a distinguished triangle of the following form in  $\mathcal{T}$ 

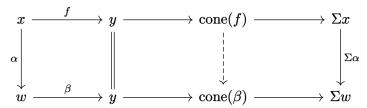
$$y \stackrel{g}{-\!\!\!-\!\!\!-\!\!\!-\!\!\!-} z \stackrel{lpha}{-\!\!\!\!-\!\!\!\!-} w \stackrel{eta}{-\!\!\!\!-\!\!\!\!-} \Sigma y$$

where  $w \in \mathcal{L}$ . By one of the axioms defining a (pre)triangulated category, there exists a dotted arrow  $x \to \Sigma^{-1} w$  making the following diagram commute, thus yielding the required factorisation.

Conversely, suppose that f factors as in the following diagram, where  $w \in L$ .



As previously argued, it is sufficient to show that there exists a quasi-isomorphism g such that  $g \circ f = 0$ . By one of the axioms defining a (pre)triangulated category, there exists a diagram of the following form, where the rows are distinguished and there is a dotted arrow that completes the morphism of distinguished triangles.



We take the morphism  $y \to \text{cone}(\beta)$  in the above diagram to be g.