Central Differencing Scheme (CDS) - Detailed Explanation

1. Definition

The **Central Differencing Scheme (CDS)** is a numerical method used to approximate derivatives in **convection-diffusion problems**. It works by calculating the value at a given point based on the **average** of the neighboring points. CDS is widely used in finite volume methods (FVM) and finite difference methods (FDM) to solve partial differential equations (PDEs) in computational fluid dynamics (CFD).

CDS is especially popular in problems where **diffusion** effects are significant, but it can have limitations when convection dominates.

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2. Mathematical Formulation

In a **1D convection-diffusion problem**, where you have a node P surrounded by its neighboring nodes W (west) and E (east), the central differencing scheme estimates the values at the faces between the control volumes.

For example:

- **At the east face (e)**, the value is approximated as:

$$\phi_e = \frac{\phi_P + \phi_E}{2}$$

- **At the west face (w)**, the value is approximated as:

$$\phi_w = \frac{\phi_W + \phi_P}{2}$$

These values are then used to compute fluxes across the control volume faces in convection and diffusion terms.

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3. Application in Convection-Diffusion Problems

In the context of **convection-diffusion problems**, CDS is used to discretize the **advection** (convection) and **diffusion** terms in the governing equation.

For example:

- The **diffusion term** is discretized using:

$$\left(\frac{d\phi}{dx}\right)_e \approx \frac{\phi_E - \phi_P}{\Delta x}$$

- The **convection term** is discretized using:

$$(\rho u \phi)_e = \rho u \cdot \frac{\phi_P + \phi_E}{2}$$

This scheme is **second-order accurate** in space, meaning it has a higher accuracy than first-order schemes like **upwind** for diffusion-dominated problems.

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4. Peclet Number (Pe)

The **Peclet number** (Pe) is a dimensionless number used to characterize the relative importance of convection versus diffusion in a fluid flow. It is defined as:

$$Pe = \frac{\rho u \Delta x}{\Gamma}$$

Where:

- ρ is the density of the fluid (kg/m^3)
- u is the velocity of the fluid (m/s)
- Δx is the grid size (m)
- Γ is the diffusion coefficient (kg/m·s)

Interpretation of Pe:

- **Low Pe (Pe < 1)**: Diffusion dominates, and the flow is **smooth**. CDS is stable and accurate.
- **Pe ≈ 1**: Convection and diffusion are balanced. CDS is still effective.
- **High Pe (Pe > 2)**: Convection dominates, and CDS becomes **unstable**, often leading to **non-physical oscillations**.

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5. Advantages of Central Differencing Scheme (CDS)

1. **Second-order accuracy**: CDS is a **second-order accurate method** in space, meaning it provides better accuracy compared to first-order methods like upwind.
2. **Simplicity**: The formulation is straightforward and easy to implement.
3. **Symmetric formulation**: Unlike other schemes, CDS treats both the forward and backward fluxes symmetrically, making it ideal for diffusion-dominated flows.

4. **Effective in smooth flows**: CDS works well in situations where the flow is smooth, and diffusion effects are stronger than convection effects.

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6. Limitations of Central Differencing Scheme (CDS)

1. **Unstable in convection-dominated flows**: When the Peclet number is high (Pe > 2), CDS can become **unstable** and produce **non-physical oscillations**. This instability arises due to the scheme's symmetric nature, which doesn't account for the directional bias of convection.
2. **Loss of boundedness**: CDS can violate the **boundedness** of the solution, meaning that values like temperature or concentration may exceed physically possible limits (e.g., becoming negative or larger than expected).
3. **Not suitable for sharp gradients**: CDS can struggle to handle sharp gradients in the solution, leading to spurious oscillations.

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7. Case Observations

Let's now examine three different cases that illustrate the behavior of the CDS scheme under varying conditions:

Case 1: u = 0.1 m/s, 5 cells

In this case, the flow is **diffusion-dominated**, with a low Peclet number (Pe ≈ 0.2). The grid size $\Delta x = 0.2$ m.

- **Observation**: CDS performs accurately and stably in this case, with the numerical solution closely matching the **analytical solution**. The flow is dominated by diffusion, and the scheme remains stable without oscillations.

Case 2: u = 2.5 m/s, 5 cells

Here, the velocity is increased to $u = 2.5\,\mathrm{m/s}$, and the Peclet number rises to Pe ≈ 5.0, indicating that convection dominates.

- **Observation**: CDS becomes unstable because the Peclet number is too high. The numerical solution starts to show **oscillations**, which is a sign of instability. The scheme struggles with convection-dominated flows, producing **non-physical results**.

Case 3: u = 2.5 m/s, 20 nodes

In this case, the velocity remains $u = 2.5\,\mathrm{m/s}$, but the number of grid cells is increased to 20, reducing the grid size Δx and lowering the Peclet number to Pe ≈ 1.315.

- **Observation**: With grid refinement, the Peclet number becomes **marginally balanced** between convection and diffusion, and CDS **regains stability**. The numerical solution is **physically**

meaningful and closely matches the analytical solution.

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8. Conclusion

- **Low Peclet number** (Case 1): CDS is stable and accurate for diffusion-dominated flows.
- **High Peclet number** (Case 2): CDS becomes **unstable** and produces oscillations due to convection dominance.
- **Grid refinement** (Case 3): Reducing the grid size (increasing the number of nodes) lowers the Peclet number, allowing CDS to regain stability and produce accurate results.

Recommendation:

For **convection-dominated problems**, use **upwind schemes** or **QUICK schemes** instead of CDS, or refine the grid to lower the Peclet number and stabilize the results with CDS.