Crank-Nicolson method

$\theta = \frac{1}{2}$ & the source fun is linear (pg No 259 8.10 eqn)

$$b = S_u + \frac{1}{2} S_p T_p + \frac{1}{2} S_p T_p^0$$

$$a_p T_p = a_E \left[\frac{T_E + T_E^0}{2}\right] + a_W \left[\frac{T_W + T_W^0}{2}\right]$$

$$+ \left[a_p^0 - \left(\frac{a_E}{2} + \frac{a_W}{2}\right)\right] T_p^0 + \underbrace{S_u + \frac{1}{2} S_p T_p^0}_{\text{source}(S_u)}$$

$$* \; a_p = \frac{a_W + a_E}{2} + a_p^0 - \frac{1}{2} S_p$$

$$\& \quad a_p^0 = \rho c \frac{\Delta x}{\Delta t}$$

a_W	a_E
$\frac{k_w}{\delta x_{WP}}$	$\frac{k_e}{\delta x_{PE}}$

time step $\quad \Delta t < \rho c \frac{\Delta x^2}{k}$

Given Example problem

Key points

* all coefficients are +ve for realistic & bounded results.
* $$a_p^0 > \left(\frac{a_E + a_W}{2}\right)$$
* Its based on the Central differencing. So ~~its~~ 2nd order accuracy.
* With sufficiently small time steps its is possible to achieve. ~~Cons~~ great accuracy.

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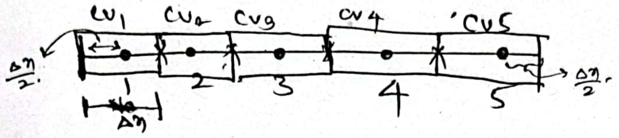
given time step 8s | by C.N. Method $\theta = \frac{1}{2}$

$\boxed{\Delta t = 8\ sec}$

$K = 10\ W/m.K$; $\rho c = 10 \times 10^6\ J/m^3.K$



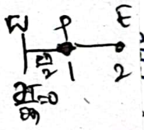
the Governing eqn

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) \longrightarrow (1)$$

Integrate the eqn (1) wr to Control Volume (CV) & time ~~differentials~~ (t) to (t+Δt)

we have:

Now for the ~~Con~~ Node (1) & CV (1) then we have &



$$\rho c \frac{(T_P - T_P^0)}{\Delta t}\Delta x = D(T_E - T_P) + D\left(\frac{T_E^0 - T_P^0}{2}\right)$$

$$= D\left(\frac{T_E + T_E^0}{2}\right) - \frac{D}{2}T_P^0 - \frac{D}{2}T_P$$

$$\frac{\rho c \Delta x}{\Delta t}T_P - \left(\frac{\rho c \Delta x}{\Delta t}\right)T_P^0 =$$

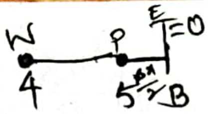
$$\therefore \left(\frac{\rho c \Delta x}{\Delta t} + \frac{D}{2}\right)T_P = D\left(\frac{T_E + T_E^0}{2}\right) + \left(\frac{\rho c \Delta x}{\Delta t} - \frac{D}{2}\right)T_P^0$$

$\therefore a_W = 0$; $a_E = \frac{K}{\Delta x} = D$.

i.e. $\left(a_P^0 + \frac{a_E}{2}\right)T_P = a_E\left(\frac{T_E + T_E^0}{2}\right) + \left(a_P^0 - \frac{a_E}{2}\right)T_P^0$

$$a_P = \frac{a_E}{2} + a_P^0 \;//$$

Node (5) ie for CV(5)



$t>0$
$t=0, T=200°C$

$$\rho C\left(\frac{T_P-T_P^0}{\Delta t}\right)\Delta x = \frac{1}{2}\left[\frac{K}{\frac{\Delta x}{2}}(T_B-T_P)-\frac{K}{\Delta x}(T_P-T_W)\right]$$
$$+\frac{1}{2}\left[\frac{K}{\frac{\Delta x}{2}}(T_B^0-T_P^0)-\frac{K}{\Delta x}(T_P^0-T_W^0)\right]$$
$$=\frac{1}{2}\left[2D(T_B)-2DT_P-DT_P+DT_W\right]$$
$$+\frac{1}{2}\left[2DT_B^0-2DT_P^0-DT_P^0+DT_W^0\right]$$

$$\frac{\rho C\Delta x}{\Delta t}T_P-\frac{\rho C\Delta x}{\Delta t}T_P^0 = DT_B-DT_P-\frac{D}{2}T_P+\frac{D}{2}T_W$$
$$+DT_B^0-DT_P^0-\frac{D}{2}T_P^0+\frac{D}{2}T_W^0$$

$$\left[\frac{\rho C\Delta x}{\Delta t}+\frac{a_W}{2}+D\right]T_P = a_W\frac{(T_W+T_W^0)}{2}+\left[\frac{\rho C\Delta x}{\Delta t}-\frac{a_W}{2}-D\right]T_P^0$$
$$+D(T_B+T_B^0)$$

$$\therefore\left(a_P^0+\frac{a_W}{2}+S_P\right)T_P = a_W\left(\frac{T_W+T_W^0}{2}\right)+\left(a_P^0-\frac{a_W}{2}-S_P\right)T_P^0$$
$$+D(T_B+T_B^0)$$

where $a_P^0=\rho C\frac{\Delta x}{\Delta t}$ | $S_P=D$ | $S_u=D(T_B+T_B^0)$

and $D=\frac{K}{\Delta x}$; coeff of $T_P^0 = a_P^0-\frac{a_W}{2}-S_P$

Now for Nodes 2, 3, 4 ie for CV 2, 3, 4, then,

$$\rho C\left(\frac{T_P-T_P^0}{\Delta t}\right)\Delta x = \frac{1}{2}\left[\frac{K}{\Delta x}(T_E-T_P)-\frac{K}{\Delta x}(T_P-T_W)\right]$$
$$+\frac{1}{2}\left[\frac{K}{\Delta x}(T_E^0-T_P^0)-\frac{K}{\Delta x}(T_P^0-T_W^0)\right]$$

$$\therefore\left(a_P^0+\frac{a_W}{2}+\frac{a_E}{2}\right)T_P = a_E\left(\frac{T_E+T_E^0}{2}\right)+a_W\left(\frac{T_W+T_W^0}{2}\right)$$
$$+\left[a_P^0-\frac{a_W}{2}-\frac{a_E}{2}\right]T_P^0$$

$$\therefore a_E=a_W=\frac{K}{\Delta x}\ ;\quad a_P^0=\rho C\frac{\Delta x}{\Delta t}$$

$$a_P=\left(a_P^0+\left(\frac{a_W+a_E}{2}\right)\right)\qquad \text{Coeff of } T_P^0 = a_P^0-\frac{a_W}{2}-\frac{a_E}{2}$$

Node	a_W	a_E	a_P ($a_P^0 = \rho c \frac{\Delta x}{\Delta t}$)	S_P	S_u
1	0	$\frac{K}{\Delta x}$	$\frac{a_E}{2} + a_P^0$	0	0
2.	$\frac{K}{\Delta x}$	$\frac{K}{\Delta x}$	$\frac{a_E}{2} + \frac{a_W}{2} + a_P^0$	0	0
3.	$\frac{K}{\Delta x}$	$\frac{K}{\Delta x}$	$\frac{a_E}{2} + \frac{a_W}{2} + a_P^0$	0	0
4.	$\frac{K}{\Delta x}$	$\frac{K}{\Delta x}$	$\frac{a_E}{2} + \frac{a_W}{2} + a_P^0$	0	0
5.	$\frac{K}{\Delta x}$	0	$\frac{a_W}{2} + a_P^0 + \underset{ae}{S_P}$	$\frac{K}{\Delta x}$	$\frac{K}{\Delta x}(T_B + T_B^0)$

Node(1)

$$a_P T_P = a_E\left(\frac{T_E + T_E^0}{2}\right) + \left(a_P^0 - \frac{a_E}{2}\right) T_P^0$$

$$a_P^0 = \rho c \frac{\Delta y}{\Delta t}$$

$$a_P = \frac{a_E}{2} + a_P^0;$$

$$D = \frac{k}{\Delta y} = \frac{10}{0.004}$$

$$a_P = \frac{2500}{2} + 20,000 \Rightarrow a_P = 21250$$

$$\boxed{a_E = \frac{k}{\Delta y} = 2500}$$

$$a_P T_P = \frac{a_E}{2} T_E + \frac{a_E T_E^0}{2} + \left(a_P^0 - \frac{a_E}{2}\right) T_P^0$$

$$a_P^0 = \rho c \frac{\Delta y}{\Delta t} = \frac{10\times10^6 \times 0.004}{2}$$

$$\boxed{21250 T_P = 1250 T_E + 18750 T_P^0 + 1250 T_E^0}$$

$$\boxed{a_P^0 = 20,000}$$

$$\frac{a_E}{2} = 1250$$

Node(5)

$$a_P T_P = a_W\left(\frac{T_W + T_W^0}{2}\right) + \left(a_P^0 - \frac{a_W}{2} - S_P\right) T_P^0 + D(T_B + T_B^0)$$

$$a_P = a_P^0 + \frac{a_W}{2} + S_P = 20000 + 1250 + 2500 = 23750$$

$$a_P T_P = \frac{a_W}{2} T_W + \frac{a_W}{2} T_W^0 + \left(a_P^0 - \frac{a_W}{2} - S_P\right) T_P^0 + D(T_B + T_B^0)$$

$$\boxed{23750 T_P = 1250 T_W + 1250 T_W^0 + 16250 T_P^0 + 2500 T_B^0} + 2500 T_B$$

Node(2)(3)(4)

$$a_P T_P = a_E\left(\frac{T_E + T_E^0}{2}\right) + a_W\left(\frac{T_W + T_W^0}{2}\right) + \left[a_P^0 - \left(\frac{a_W}{2} + \frac{a_E}{2}\right)\right] T_P^0$$

$$a_P = a_P^0 + \left(\frac{a_W}{2} + \frac{a_E}{2}\right) = 20000 + 2500 = 22500$$

$$22500 T_P = \frac{a_E}{2} T_E + \frac{a_W T_W}{2} + \frac{a_E}{2} T_E^0 + \frac{a_W T_W^0}{2} + \left[a_P^0 - \left(\frac{a_W + a_E}{2}\right)\right] T_P^0$$

$$\boxed{22500 T_P = 1250 T_E + 1250 T_W + 1250 T_E^0 + 1250 T_W^0 + 17500 T_P^0}$$

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Node(1) $21250 T_P - 1250 T_E = 18750 T_P^0 + 1250 T_E^0$ $\times \begin{array}{cc} P & E \\ 1 & 2 \end{array}$

(B)

~~Node(2)~~ $- \cancel{12500 T_E}$

$$\boxed{2125 T_P - 125 T_E = 1875 T_P^0 + 125 T_E^0}$$

Node(2)

$$-1250 T_W + 22500 T_P - 1250 T_E = 1250 T_E^0 + 1250 T_W^0 + 17500 T_P^0$$

$\begin{array}{ccc} & P & \\ 1 & 2 & 3 \\ W & & E \end{array}$

Node(3) $\rightarrow$ (11) $\rightsquigarrow$ $\begin{array}{ccc} W & P & E \\ 2 & 3 & 1 \end{array}$

Node(4) (11) $\begin{array}{ccc} W & P & E \\ 3 & 4 & 5 \end{array}$

Node(5) $- \cancel{1250}$ $\begin{array}{ccc} W & & B \\ 4 & 5 & | \end{array}$

$$-125 T_W + 2375 T_P = 125 T_W^0 + 1625 T_P^0 + 2500 T_B^0.$$

The Matrix $A\Phi = b \cdot \underline{ix}$.

$$\begin{bmatrix} 21250 & -125 & 0 & 0 & 0 \\ -125 & \cancel{2250} & -125 & 0 & 0 \\ 0 & -125 & 21250 & -125 & 0 \\ 0 & 0 & -125 & 21250 & -125 \\ 0 & 0 & 0 & \cancel{1250}\ -125 & \cancel{2125}\ 2375 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 18750 T_1^0 + 125 T_2^0 \\ 125 T_1^0 + 17500 T_2^0 + 125 T_3^0 \\ 125 T_2^0 + 17500 T_3^0 + 125 T_4^0 \\ 125 T_3^0 + 17500 T_4^0 + 125 T_5^0 \\ 125 T_4^0 + 1625 T_5^0 + 2500 T_B^0 \\ + 2500 T_B \end{bmatrix}$$