1. APPLICATION OF DIFFERENTION EQUATION

A tank contains 100 liters of fresh water 2 liters per minute of brine, run in each containing 1gram of salt and the mixture runs at 1 liter per minute. Find the amount of salt present when the tank contains 150 liters of water?

Sol:-

Let the amount of brine the tank at a liter'‘t' liters

Mixture of fresh water=100+t liters

Input in Δt time after time‘t’= 2 Δt

Output in Δt time after time‘t’= $\frac{x}{100+t}$ Δt

Accumulation=Input-output

$\Delta x = 2\Delta t - \frac{x}{100+t}\Delta t$

$\Delta x = (2 - \frac{x}{100+t})\Delta t$

$\frac{\Delta x}{\Delta t} = 2 - \frac{x}{100+t}$

Proceeding the limit when Δt→0

$\frac{dx}{dt} = 2 - \frac{x}{100+t}$

$\frac{dx}{dt} + \frac{x}{100+t} = 2$

This is linear differential equation in ’x’

Here Formula:-

$$\frac{dx}{dt} + py = q$$

$p = \frac{1}{100+t}$, Q=2 y.$e\int pdx = \int Q.e\int pdx\, dx + c$

Find $e\int pdt = e\int \frac{1}{100+t}\, dt$

$= e^{\log(100+t)}$

= 100+t

Hence the sol:-

$$X\,(100+t) = \int 2.(100+t)\ dt + c$$

$$X\,(100+t) = \frac{2(100+t)^2}{2} + c$$

$$X\,(100+t) = (100+t)^2 + c \rightarrow (1)$$

Initially t=0, x=0 in (1)

$$0 = 100^2 + c$$

$$-c = 100^2$$

$$c = -(100)^2$$

Subs

$$c = -(100)^2 in\ (1)$$

$$X\,(100+t) = (100+t)^2 - 100^2$$

$$x = \frac{(100+t)^2 - 100^2}{(100+t)}$$

$$= \frac{100^2 + t^2 + 200t - 100^2}{100+t}$$

$$= \frac{200t + t^2}{100+t}$$

$$= \frac{t(200+t)}{100+t}$$

When the tank contains 150 liters of brine t=50, since the tank contains 100 liters of fresh water at t=0 and the accumulation per minute in the tank is on liter

When t=50

$$x = \frac{50(200+50)}{100+50}$$

$$= 83\tfrac{1}{3}\text{Grams}$$

2. AN APPLICATION INVOLVING SNOWPLOW

One early morning it starts to snow at 7A.M. A snow plow sets off to clear the road, by 8 A.M it has gone 2miles, It takes an additional 2 hours, for the plow to go denotes the distance travelled by the plow at a time't'.Assuming the snowplow clears snow at a constant rate in cubic meters/hour.

(i) Find the differential equation modeling the value of 'x'?
(ii) When did not start snowing

Sol:-

(i) Let k_1 be the rate (height/hour) of snow fall and k_2 is the rate of

Snow clearance. Then the height of snow is

$$= k_1 t$$

The differential equation

$$\frac{\Delta x}{\Delta t} = \frac{k_2}{k_1 t}$$

$\frac{\Delta x}{\Delta t} = \frac{k}{t}$, where $k = \frac{k_2}{k_1}$

Assuming $\Delta t \to 0$, then

$$\frac{dx}{dt} = \frac{k}{t}$$

$$dx = \frac{k}{t}.dt$$

Integrating

$$\int dx = \int \frac{k}{t} dt$$

X (t) =k log t +c → (1)

(ii)

Let $t = t_1$ at 7a.m.

Then t=$t_1 + 1$ at 8a.m. for 2miles

And t=$t_1 + 3$at 10a.m. for 2miles between 8a.m. to 10a.m.

Equation (1) using, we get

$$x(t_1 + 1) - x(t_1) = 2$$

$$k \log\left(\frac{t_1 + 1}{t_1}\right) = 2 \rightarrow (2)$$

For

4 miles between 7a.m. and 10a.m.

$$x(t_1 + 3) - x(t_1) = 4$$

$$k \log\left[\frac{t_1+3}{t_1}\right] = 4 \rightarrow (4)$$

Using (3) & (4) becomes

$$\text{K} \log\left(\frac{t_1+3}{t1}\right) = (2)(2)$$

$$= 2.\ \text{K} \log\left(\frac{t_1+1}{t_1}\right)$$

$$\text{K} \log \frac{t_1+3}{t_1} = k \log\left(\frac{t_1+1}{t_1}\right)^2$$

Taking antilog on both sides

$$\frac{t_1+3}{t_1} = \frac{(t_1+1)^2}{{t_1}^2}$$

$$\frac{t_1 + 3}{t_1} = \frac{t^2 + 1 + 2t_1}{{t_1}^2}$$

$${t_1}^2(t_1 + 3) = t_1({t_1}^2 + 1 + 2t_1)$$

$${t_1}^2 + 3t_1 = {t_1}^2 + 1 + 2t_1$$

$$3t_1 - 2t_1 = 1$$

$$t_1 = 1$$

Hence the snow started at 6A.M.

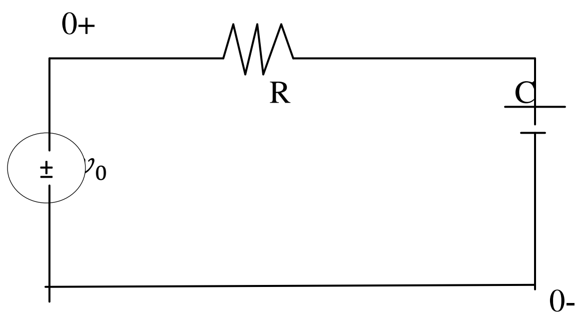
3.FIRST ORDER R-C SERIES CIRCUITS

The voltage drop across a capacitor with capacitance C is given by $\frac{q(t)}{c}$, where q is the charge on two capacitors. Hence for the serious circuits shown in the figure we get the following equation.

By applying kirchh off's second law,

$$Ri + \frac{q}{c} = E(t) \rightarrow (1)$$

v_s



Since,

$$i = \frac{\mathrm{dq}}{\mathrm{dt}}$$

Equation (1) can be written as,

$$R\left(\frac{dq}{dt}\right) + \frac{q}{c} = E(c) \rightarrow (2)$$

Problem:-

A <u>100 volt</u> electromotive force is applied to an R-C series circuit in which the resistance in 200 ohms and the<u> Capacitor is 10^{-4} Farads.</u> Find the change q(t) on the capacitor if q(0)=0.Find the current $i(t)$.

Sol:-

$$(2) => R\left(\frac{dq}{dt}\right) + \frac{q}{c} = E(t)$$ Where,

E=100 volt, R=200 ohms, C=$10^{-4} Farad$

$$200\left(\frac{dq}{dt}\right) + \frac{q}{10^{-4}} = 100$$

$$\frac{dq}{dt} + \left(\frac{10^{-4}}{200}\right) q = \frac{100}{200}$$

$$\frac{dq}{dt} + 50\text{q} = \frac{1}{2}$$

This is linear differential equation of 1st order,

I.F is $P = 50, Q = \frac{1}{2}$

$$e^{\int 50dt} = e^{50t}$$

Solution is $q(t)e^{50t} = \frac{1}{2}(\int e^{50t}dt + c)$

$$q(t)e^{50t} = \frac{1}{2}\left(\frac{e^{50t}}{50}\right) + c$$

$$\div\, e^{50t}$$

$$q(t) = \left(\frac{1}{100}\right) + ce^{-50t}$$

Initial time, t=0, given initial charge q (0) =0

$$\therefore \left(\frac{1}{100}\right) + ce^{-50(0)} = 0$$

$$c = \left(\frac{-1}{100}\right)$$

Hence,

$$q(t) = \left(\frac{1}{100}\right) - \left(\frac{e^{-50t}}{100}\right)$$

$$i(t) = \frac{dq}{dt}$$

$$= \left(\frac{50}{100}\right) e^{-50t}$$

Current $i(t) = \left(\frac{1}{2}\right) e^{-50t}$

4. DRUG DISTRIBUTION IN HUMAN BODY

Definition:-

To combat the infection to a human body an appropriate dose of medicine is essential. Because the amount of the drug in human body decreases with time, medicine must be given in multiple doses.

The rate at which the level of the drug(y) in patient blood decays can be modeled by the decay equation

$$\frac{dy}{dx} = ky(\because \text{ decrease})$$

$$\frac{dy}{dx} + ky = 0$$

$$e^{\int pdx} = e^{\int k.dx} = e^{kx}$$

Then the complete solution is

$$y.e^{\int pdx} = \int Q.e^{\int pdx}.dx + c$$

$$y.e^{\int kx} = \int Q.e^{kx}.dx + c$$

$$y.e^{kx} = c$$

$$y = ce^{-kx}$$

Problem

A representative of a pharamedical company recommends that a new drug of his company be given every 'T' hours in doses of quantity y_0 for an extended period of time.

Find the stedy state drug in the patients body.

Solution:-

Since the initial dose isy_0.The drug concentration at any time $t \geq 0$,is of the equation.

$$y(t) = y_0 e^{-kt}$$

At the second stage t=T,

When the dose y_0 is taken, which is increases the drug level t=T

Then $y(T) = y_0 + y_0 e^{-kt}$

$= y_0\left(1 + e^{-kt}\right)$

And then the drug level

Immediately begins to decay

Find mathematics expression we solve the initial value problem,

$\frac{dy}{dx} = -ky$

$\Rightarrow y(T) = y_0\left(1 + e^{-kt}\right)$

Solving I.V.P

We get

$y = y_0\left(1 + e^{-kt}\right)e^{-k(t.T)}$

This equation gives for the drug level it t>T

At the third t=2T ofy_0.

Then the drug level just decreases

$y = y_0\left(1 + e^{-kt}\right)e^{-k}(2T - T)$

$y = y_0\left(1 + e^{-kt}\right)e^{-kT}$

The dosage $y_0$taken t=2T raises, the drug level

$y(2T) = y_0 + y_0\left(1 + e^{-kt}\right)e^{-kT}$

$= y_0\left(1 + e^{-k(T)} + e^{-k(2T)}\right)$

Continue is this way find after $(n+1)^{th}$ T does is taken that the drug level is,

$y((n+1)T) = y_0\left(1 + e^{-k(T)} + e^{-k(2T)} \ldots \ldots .. + e^{-k(n+1)T}\right)$

The drug level $(n+1)^{th}$T dose is the sum of first ‘n’ terms of a geometric series, with the first term y_0, and the common term e^{-kt}

$$y(nT) = \frac{y_0\left(1 + e^{-(n+1)kT}\right)}{1 - e^{-kt}}$$

As n becomes large, the drug level approaches a steady state value say y_s

Given by $y_s = \lim_{n\to\infty} y(nT)$

$$= \lim_{n\to\infty} \frac{y_0\left(1+\mathrm{e}^{-(\mathrm{n}+1)\mathrm{kT}}\right)}{1-e^{-kt}}$$

$$= \frac{y_0}{1-e^{-kt}}$$

The steady state y_s is called satuation level of the drug.

5. THE BRACHISTO CHORNE USING SECOND ORDER DIFFERENTIAL EQUATION

Let a point A be joined to a lower point B by a straight wire and a bead be allowed to slide along the wire AB without friction. We can also consider the case where the wire is bent into a are of a circle so that the motion of the bead resembles that of the bob of a simple pendulums. We can have even the curve bent into the form of an arbitrary curve. Then there is infinity of ways of joining A to B. The question posted is "Which of the many curves will give the shortest possible time of descent for the bead". Galileo believed that the circular path would give the least time. The curve giving the least time of descant from A to B is called a Brachisto Chorne bernoulle offered a very beautiful solution to this problem.

Join A to b by an arbitrary curve. Choose the horizontal and downward vertical throught A as the X and Y axes. Let P be the position of the bead at time t after start from A beings let, the tangent PT make Ø with the X-axis and 'α'with the y-axis.

By conservation of energy $V = \sqrt{2gy} \rightarrow (1)$

By Snell's law of refraction, $\sqrt{2g}\,\frac{\sin\alpha}{v} = constant \rightarrow (2)$

Sin α =sin (90^0-Ø) = cos Ø=$\frac{1}{\sec Ø} = \frac{1}{\sqrt{1+y'^2}}$

Where y'=$\frac{dy}{dx}$

Hence (2) gives $\frac{\sqrt{2g}}{\sqrt{1+y'^2}}.\frac{1}{v} = c$

$=> \frac{1}{\sqrt{y}\sqrt{1+y'^2}} = c$ By (1) $\qquad \left(\therefore V = \sqrt{2gy} => \frac{v}{\sqrt{2g}} = \sqrt{y}\right)$

$y(1 + y'^2) = c \rightarrow (3)$

$sec^2Ø - tan^2Ø = 1$

$sec^2Ø = 1 + tan^2Ø$

$secØ = \sqrt{1 + tan^2Ø}$

$secØ = \sqrt{1 + y'^2}$

Thus (3) is the differential equation of the bachisto chorne

$Y\left[1+\left(\frac{dy}{dx}\right)^2\right]=C$

$Y+Y\left(\frac{dy}{dx}\right)^2=C$

$Y+\left(\frac{dy}{dx}\right)^2=C-Y$

$\left(\frac{dy}{dx}\right)^2=\frac{C-Y}{Y}$

$\left(\frac{dx}{dy}\right)^2=\frac{Y}{C-Y}$

$\frac{\mathrm{dx}}{\mathrm{dy}}=\left(\frac{Y}{C-Y}\right)^{1/2}$

$dx=\left(\frac{Y}{C-Y}\right)^{1/2}dy\rightarrow(4)$

Put, $tan\emptyset=\left(\frac{Y}{C-Y}\right)^{1/2}$

∴ dx =tanØ.dy----> (5)

Here we take,

$y=c\ sin^2\emptyset$

dy =2c sinØ cosØ dØ------> (6)

subs to (6) to (5)

dx =tanØ.2c sinØ cosØ dØ

$=\frac{sin\emptyset}{cos\emptyset}\ 2c\ sin\emptyset\ cos\emptyset\ d\emptyset$

dx = 2c sinØ dØ

$dx=2c(sin^2\emptyset)d\emptyset$

$dx=2c\left(\frac{1-\cos 2\emptyset}{2}\right)d\emptyset$

$dx=c(1-\cos 2\emptyset)d\emptyset$

We integrate, we get

$x = c\left(\emptyset - \frac{\sin 2\emptyset}{2}\right) + c$

$= \frac{c}{2}(2\emptyset - \sin 2\emptyset) + c$

As the curve is to pass through

The origin x=y=0;

When Ø=0

∴c =0

Hence $x = \frac{c}{2}(2\emptyset - \sin 2\emptyset) \rightarrow (7)$

$y = sin^2\emptyset$

$= c\left(\frac{1-\cos 2\emptyset}{2}\right)$

$y = \frac{c}{2}(1 - \cos 2\emptyset) \rightarrow (8)$

Now, we take

${}^{c}/_{2} = a,\ \ and\ \ 2\emptyset = \theta$

We have,

$x = a(a - sin\theta)$

$y = a(1 - cos\theta)$

These two equations are given cycloid, the generating circle with radius 'a'.

6.SECOND ORDER DIFFERENTIAL EQUATION DAMPED VIBRATION

Consider the motion of a spring that is subjected to a frictional force. Horizontal spring or a damping force vertical spring. An example is the damping force supplied by a shock absorber in a car or a bicycle.

We assume that the damping force is proportional to the velocity to the mass and acts in the direction opposite to the motion.

damping force = -c dx/dt

where c is a positive constant, called the damping constant.

Thus in this case, Newton's second law gives.

m $\frac{d^2x}{dt^2}$ = restoring force+damping force

$= \frac{-kx-cdx}{dt}$

$\frac{Md^2x}{dt^2}+\frac{cdx}{dt}$+kx=0

Auxiliary equation:-

mr^2+cr+k=0

$$r_1 = \frac{-c+\sqrt{c^2-4mk}}{2m},\ r_2 = \frac{-c-\sqrt{c^2-4mk}}{2m}$$

we need to discuss three cases.

Case 1:-

c^2-4mk>0

In this case r and r are distinct real roots and

X=c_1e^{r1t}+c_2e^{r2t}.

since c_1m and k are all positive, we have

$$\sqrt{c^2} - 4mk < c$$

so the roots r and r given by negative. This shows that X—>0

as t—>∞.

Case 2:-

c^2-4mk=0

This case corresponds to equal roots.

r_1=r_2= - $\frac{c}{2m}$

and the solution is given by

X=$(c_1$+$c_2t)$ $e^{(-\frac{c}{2m})}$t.

Case 3:-

c^2-4mk<0

Here the roots are complex.

$r_1 = r_2$=$\frac{-c}{2m}$±wi

w = $\frac{\sqrt{4mk}-c^2}{2m}$

where the solution is given by

X= $e^{-\left(\frac{c}{2m}\right)t}$ ($c_1$cos wt+$c_2$sin wt)

we see that there are oscillations that are damped by the factor,

$$e^{-\left(\frac{c}{2m}\right)t}$$

since c>0 and m>0,we have -($\frac{c}{2m}$)<o

so

$e^{-\left(\frac{c}{2m}\right)t}$—>0 as t—>∞. This implies that X—>0 as t—>∞, that is, the motion decays to 0 as time increases.

7. ELECTRIC CIRCUITS

We are in a position to analyze the shown in figure. It contains an electromotive force E (supplied by a battery or generator), a resistor R, an inductor L, and a Capacitor C in series. If the charge on the capacitor at time t is $Q = Q(t)$, then the current is the rate of change of Q with respect to t. $I = \frac{dQ}{dt}$, that the voltage drops across the resistor, inductor and capacitor are,

$$RI, L\frac{dI}{dt}, \frac{Q}{C}$$

Respectively. Kirchhoff"s Voltage law says that the sum of these voltage drops is equal to the supplied voltage:

$$h\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$$

Since $I = \frac{dQ}{dt}$, this equation becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t) \text{----- (1)}$$

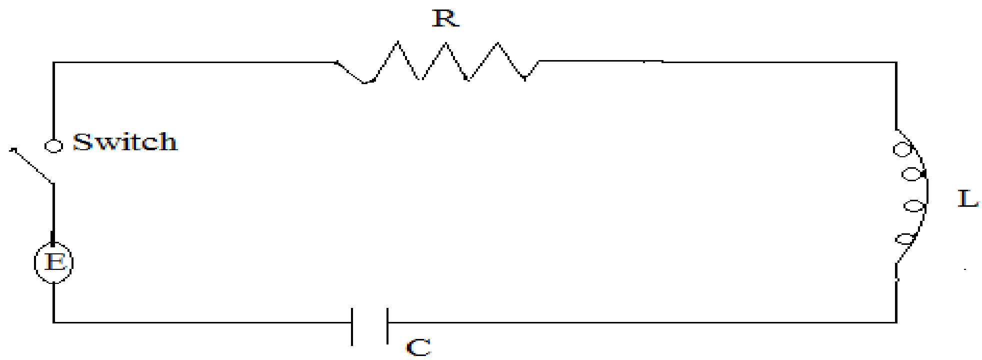
Which is a second order linear differential equation with constant co-efficient. If the charge $Q(t)$ and the current $I(t)$ are known at time 0, then we have the initial conditions. $Q(t) = Q(0)$

$$Q'(t) = Q'(0) = I(0) = I(t)$$

A differential equation for the current can be obtained by differentiating equation 1 with respect to t and remembering that,

$$I = \frac{dQ}{dt}$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E'(t)$$



Problem:-

A series circuit consist of a resistor with 20Ω an inductor for L=1Hz a capacitor C=0.002F and 12V battery if the initial charge and current both 0. Find charge and current at a time t.

Solution:-

The second order differential equation with consist of resistor, capacitor and inductor. Then,

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \left(\frac{1}{C}\right)Q = E(t)$$

$$(1) * \left(\frac{d^2Q}{dt^2}\right) + 20\frac{dQ}{dt} + \left(\frac{1}{0.002}\right)Q = E(t)$$

$$\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + \left(\frac{1000}{2}\right)Q = E(t)$$

$$\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + 500Q = 12$$

The auxiliary equation is,

$$m^2 + 20m + 500 = 0. \; where, m = \frac{dQ}{dt}$$

$$m = \frac{(-20 \pm \sqrt{400 - 2000})}{2}$$

$$m = \frac{(-20 \pm \sqrt{-1600})}{2}$$

$$m = \frac{-20 \pm 40i}{2}$$

$$m = -10 \pm 20i$$

Therefore, C.F of Q is,

$$Q = e^{-10t}(A \cos 20t + B\, sin20t)$$

Particular Integral:-

$$P.I = \frac{12e^{0t}}{D^2+20D+500}$$

$$P.I = \frac{12e^{0t}}{500}$$

$$P.I = \frac{3}{125}e^{0t}[since, \propto = D = 0]$$

The complete solution is,

$$Q(t) = C.F + P.I$$

$\therefore Q(t) = e^{-10t}(A\, cos20t + B\, sin20t) + \frac{3}{125}$ ----- (2)

Since, the initial charge $Q(0) = 0$ At a time, t=0

Then, $Q(0) = e^{-10(0)}\big(A\, cos20(0) + B\, sin20(0)\big) + \frac{3}{125}$

$$0 = A + \frac{3}{125}$$

$$A = \frac{-3}{125}$$

Differentiate with respect to t from (2),

$$Q'(t) = e^{-10t}(-20A\ sin20t + 20B\ cos20t)$$
$$+ (A\ cos20t + B\ sin20t)(-10)e^{-10t} + 0$$

At the time t=0,

$$0 = (0 + 20B) + (A + 0)(-10)$$

$$0 = 20B - 10A$$

$$10A = 20B$$

$$-\frac{30}{125} = 20B$$

$$B = \frac{-30}{125 * 20}$$

$$B = \frac{-3}{250}$$

$$Q(t) = e^{-10t}\left[\left(\frac{-3}{125}\right)A\ cos20t - \left(\frac{3}{250}\right)B\ sin20t\right] + \frac{3}{125}$$

Which is the charge at a time t and the current is,

$$I(t) = e^{-10t}\left(-\left(\frac{3}{250}\right)B\ sin20t\right)$$

8. SECOND ORDER RLC ELECTRIC CIRCUITS

Problem:

First the charge and Current at time t in the Circuits. If R=40, l=1h, C=16*10^{-4}F, E(t)=100 cos 10 t and the initial charge and Current are both 0.

Solution:

With the given value of L,R,C and E(t) the second order differential equation is

$$\frac{d^2Q}{dt^2}+R\frac{dQ}{dQ}+(\frac{I}{c})Q=E(t)$$

$$\frac{d^2Q}{dt^2}+40\frac{dQ}{dt}+(\frac{1}{16})10=100 \cos 10 t$$

$$\frac{d^2Q}{dt^2}+40\frac{dQ}{dt}+(\frac{10{,}000}{16})Q=100 \cos 10 t$$

$$\frac{d^2Q}{dt^2}+40\frac{dQ}{dt}+625Q=100\cos 10 t \text{ -------------- } 1$$

The auxilary equation,

$(m^2 + 40m + 625)$Q=100cos 10 t,

Where m=$\frac{dQ}{dt}$

m^2+40m+625=0

m=$\frac{4\pm\sqrt{(40^2)-(4*625)}}{2*1}$

m=-20±15i

The C,F of Q is =e^{-20t}(C1cos 15 t+C2sin 15 t)

To find the particular solution for the Method Undetermind Coefficient toke

Qp(t)=A cos 10 t + B sin 10 t

Qp'(t)=-10A sin 10 t + 10B cos 10 t

Qp''(t)=-100A cos 10 t-100B sin 10t

Substituing in -------1

$\frac{d^2Q}{dt^2}$+40$\frac{dQ}{dt}$+625=100cos 10t

(-100A cos 10t-100B sin 10t)+40(-10A sin 10t+10B cos 10t)+625(A cos 10t +B sin 10t)=100cos 10 t(-100A+400B+625)+sin 10t(-100B-400A+625B)=100cos 10t

Cos 10t(525A+400B)+sin 10t(-400A+525B)=100cos 10t

Equating cos 10t,sin 10t we have

525A+400B=100

÷25 => 21A+16B=4----------------2

525B-400A=0

-16A+21B=0------------------------3

B=$\frac{64}{697}$

A=$\frac{84}{697}$

The particular solution is

Qp(t)=A cos10t+$\frac{64}{697}$sin 10t

=$\frac{1}{697}$(84 cos 10t+64 sin 10t)

Since

Intial condition Q(0)=0 at time t=0 the complete solution is Qp(t)=C.F+P.I

=$e^{-20(0)}$(C1 cos 15t+C2 sin 15t)+$\frac{1}{164}$(84 cos 10t+64 sin 10t)----------------4

At intial time t=0

Q(0)=$e^{-20(0)}$(C1 cos 15(0)+C2sin 15(0))+$\frac{1}{697}$(84 cos(0)+C2 sin(0))

0=C1+$\frac{84}{697}$

$C1=-\frac{84}{697}$

Differrentiate "t" in ---------- 4

$Qp'(t)=i=\frac{dQ}{dt}=e^{-20t}$(-15C1 sin 15t+15C2 cos 15t)

(C1 cos 15t+C2 sin 15t)(-20e^{-20t})+$\frac{1}{697}$(-840 sin 10t+640 cos 10t)+(84 cos 10t+64 sin 10t)(0)

I(t)=e^{-20t}(cos 15t (-20C1+15C2)+(-15C1-20C2 sin 15t)+$\frac{1}{697}$(-840 sin 10t + 640 cos 10t))

t=0

I(0)=$e^{-20(0)}$(-20C1+15C2)+$\frac{640}{697}$

I(0)=[-20($\frac{-84}{697}$)+15C2] +$\frac{690}{697}$

-15C2 =$\frac{1680+640}{697}$

=$\frac{2320}{697}$

C2=$\frac{-2320}{697*15}$=$\frac{-464}{2091}$

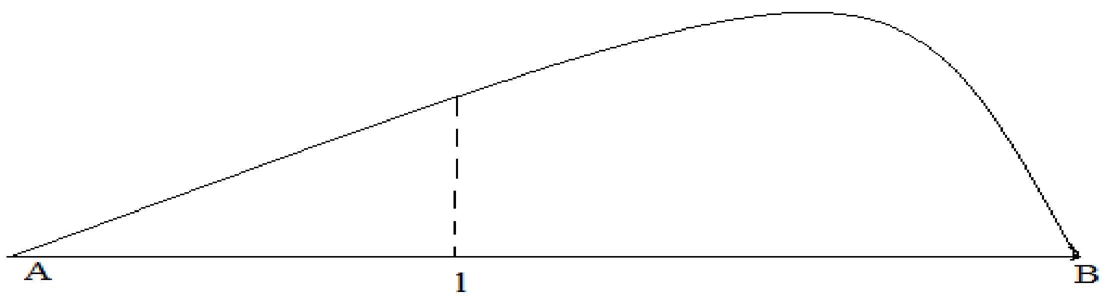
C2=$\frac{-464}{2091}$

And the expression for the current at a time 't' is

I(t)=e^{-20t}\{cos 15t[-20(-$\frac{-84}{697}$)+15($\frac{-464}{2091}$)+[-20($\frac{-464}{2091}$)-15($\frac{-84}{697}$)sin 15t+$\frac{1}{697}$[-840 sin 10t+640cos 10t]\}

9. WAVE EQUATION IN VIBRATING STRING

A string is stretched and fastened to two points L a part motion is started into the form $y = y_0 \frac{sin\pi x}{l}$from which it is released at time $t = 0$, find the displacement at time t.



Let the fixed point A, B and let the position of a point at a distance X from $A\ to\ P$ at a time t and a distance from $AB = y$ then Y satisfies the differential equation.

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2} \text{------ (1)}$$

The boundary are,

$x = 0, y = 0\ (i.e)\ y(0, t) = 0$

When, $x = t, y = 0\ (i.e)\ y(l, t) = 0$

Initial conditions are

$\frac{\partial y}{\partial t} = 0att = 0$, when $t = 0$ is released.

From rest in $y(x, 0) = y_0 \frac{sin\pi x}{l}$.

Let $y = x(X)T(t)$ be a solution at

$\frac{\partial^2 y}{\partial t^2}$=a$^2\frac{\partial^2 y}{\partial x^2}$

Hence partial differentiate with respect to X & t.

$\frac{\partial^2 y}{\partial x^2}$=$x''(X)T(t)$----- (2)

$\frac{\partial^2 y}{\partial t^2}$=$x(X)T''(t)$----- (3)

Subs. (2), (3) in (1),

$x(X)T''(t) = a^2x''(X)T(t)$

$$\frac{x''(X)}{x(X)} = \frac{1}{a^2}\frac{T''(t)}{T(t)}$$

As the member is a function of X alone and the second member is function of t alone.

<u>Case (i):</u>

$k = 0$

Then, $y_0 = (AX + B)(X + 1)e^{a\lambda(t)}$

The boundary conditions are

When $x = 0, y = 0$ then $B = 0$

When $x = 1, y = 0$ then $A = 0$

$y$has the solution is trivial.

<u>Case(ii):</u>

If $k > 0$

$(i.e)k = \lambda^2$ (positive)

Then, $Y = (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{a\lambda t} + De^{-a\lambda t})$

The boundary conditions are

When $x = 0, y = 0$ gives $A + B = 0$

When $x = 1, y = 0$ gives $Ae^{\lambda t} + Be^{-\lambda t} = 0$

<u>Case (iii):</u>

$k < 0$

Let $k = -\lambda^2$

The solution is $y = (A\,cos \lambda\, x + B\,sin \lambda\, x)(C\,cos \lambda\, at + Dsin \lambda\, at)$

From the boundary conditions.

<u>Case (i):</u>

$x = 0, y = 0$

$A = 0, B$ sin$\lambda\, l = 0$

As,$B \neq 0, sin \lambda\, l = 0$

$\lambda\, l = n\pi\ (n = 1,2,3, \ldots.)$

$\lambda = \frac{n\pi}{l}$

$Y(x,t) = B\,sin\frac{n\pi x}{l}(C\,cos\frac{n\pi}{l} + D\,sin\frac{n\pi at}{l})$

From condition (iii),

$t = 0, \frac{dy}{dt} = 0$ Gives $D = 0$

$y(x,t) = BC\,sin\frac{n\pi x}{l}cos\frac{n\pi at}{l}.where, n = 1,2,3,$ ....

$y(x,t) = \sum_{n=1}^{\infty} A_n\ sin\frac{n\pi x}{l}cos\frac{n\pi at}{l}$ ----- (4)

From (4),

$$y(x,0) = \sum_{n=1}^{\infty} A_n\,sin\frac{n\pi x}{l}$$

$$= y_0 \, sin \frac{n\pi x}{l}$$

$A_1 = y_0, A_2 = 0, \dots \dots A_n = 0$

The required solution,

$$y(x,t) = y_0 \, sin \frac{\pi x}{l} \, cos \frac{\pi a t}{l}$$

10. HEAT FLOW IN A RECTANGULAR PLATE

An infinitely long rectangular metal plate is enclosed between the lines y=0and y=10 for positive value of x.

The temperature is zero along the edges y=0 and y=10 and the edges at infinitely.If the edges x=0 is kept at temperature $U=4k \sin^{3\pi y/10}$.find the steady state temperature x, y of the plate.

Solution:

Let u(x, y) be the temperature in steady state at any point p(x, y) in the plate.

Then U(X, Y) satisfies laplace equation

$U_{xx} + U_{yy}=0 \quad \rightarrow (1)$

The solution U(x, y) can be taken as

$U(x, y) = (Ae^{-px}+Be^{-px}) (C \cos py+D \sin py) \rightarrow (2)$

The boundary conditions are

U(x, 0) =0

U(x, l) =0

U (α, y) =0

U (0, y) =0

Using the boundary conditions are sub (i) in (2)

$U(x, 0) = (Ae^{-px}+Be^{px}) \quad C=0$

Since $Ae^{-px}+Be^{px} \neq 0$

Then C=0

Sub C=0 in (2)

$U(x, y) = (Ae^{-px}+Be^{-px}) D \sin py=0\rightarrow (3)$

Boundary condition (ii)

$U(x, l) = 0$ using in (3)

$U(x, l) = (Ae^{-px}+Be^{-px})\, D \sin py = 0$

Since $Ae^{-px}+Be^{py} \neq 0$

Then $D\sin pl = 0$

$D\sin pl = 0$

$Pl = n\pi$ since $n = 1, 2, 3, \ldots$

$P = n\pi/l$ since $n = 1, 2, 3, \ldots$

Boundary conditions (iii)

$U(\alpha, y) = 0$ in (4)

$U(\alpha, y) = (Ae^{-n\pi\alpha/l}+Be^{n\pi\alpha/l})\, D\sin^{n\pi y/l} = 0$

⇨ $B = 0$

Since $B = 0$ in (4)

$U(x, y) = ADe^{-n\pi x/l}\sin^{n\pi y/l}$

Using the boundary condition,

The most general solution is

$U(x, y) = \sum_{n=1}^{\alpha} Ane^{-n\pi x/l}\sin^{n\pi y/l} \rightarrow (5)$

Boundary condition (iv)

$U(D, y) = \sum_{n=1}^{\alpha} An\sin^{n\pi y/l} \rightarrow (6)$

$= 4k\sin^{3n\pi y} \rightarrow (7)$

Comparing (6) and (7)

$4[k\sin^{3n\pi y/l}] = \sum_{n=1}^{\alpha} An\sin^{n\pi y/l}$

$4[3\sin^{\pi y/l} - \sin^{3\pi y/l}] = \sum_{n=1}^{\alpha} An\sin^{n\pi y/l}$

Here $A1=3k_1$

$A_2=0$, $A3=-k$

$A_4=A_5=A_6=\ldots.=0$

The required temperature distribution

$U(x, y) = \sum_{n=1}^{\alpha} A \sin^{n\pi y/l} e^{-n\pi x/l}$

$U(x, y) = A_n \sin^{n\pi y/l} e^{-\pi x/l} + A_2 \sin^{2\pi/l} e^{-2\pi x/l} + A_3 \sin^{3\pi y/l} e^{-3\pi x/l} + \ldots.$

$= -3k \sin^{\pi y/l} e^{-\pi x/l} + 0 - k \sin^{3\pi y/l} e^{-3\pi x/l} + \ldots$

11. SOLVE CIRCUIT EQUATION BY USING LAPLACE TRANSFORM.

Problem:

Solve for I(t) for the circuit given that v(t) = 10v, R = 4Ω and L = 2μ, I(0) = 0using Laplace transformation.

The differential equation is $L\frac{dI}{dt} + kI = V$ ----------(1)

Given: V = 10v, R = 4Ω ,L = 2μ

From (1)

$$2\frac{dI}{dt} + 4I = 10$$

Apply the Laplace transformation we get,

$$L\left(2\frac{dI}{dt} + 4I\right) = L(10)$$

2L(I) + 4 L(I) = L(10)

$$2sL(y) - I(0) + 4L(I) = \frac{10}{s}$$

$$2sL(I) - 2I(0) + 4L(I) = \frac{10}{s}$$

$$L(I)(2s+4) = \frac{10}{s}$$

$$L(I) = \frac{10}{s(2s+4)} = \frac{5}{s(s+2)}$$

$$I = L^{-1}\left(\frac{5}{s(s+2)}\right) \qquad \text{--------------------------- (2)}$$

Consider,

$$\frac{5}{s(s+2)} = \frac{A}{s} + \frac{B}{(s+2)} \qquad \text{---------------------------- (3)}$$

$$\frac{5}{s(s+2)} = \frac{A(s+2)+Bs}{s(s+2)}$$

$$5 = A(s+2)+B$$

Case (i): when s=0.

5 = 2A

A = 5/2

Case (ii) when s+2 = 0.

5 = A(0)+B(-2)

B = -5/2

Substitute A and B in (3)

$$\mathbf{I} = L^{-1}\left(\frac{5}{s(s+2)}\right)$$

$$= L^{-1}\left(\frac{A}{s}+\frac{B}{s+2}\right)$$

$$= L^{-1}\left(\frac{5/2}{s}-\frac{5/2}{s+2}\right)$$

$$= 5/2\ L^{-1}\left(\frac{1}{s}\right) - 5/2\ L^{-1}\left(\frac{1}{s+2}\right)$$

$$= 5/2\ (1) \quad - 5/2(e^{-2t})$$

$I(t) = 5/2\ (1 - e^{-2t})$ at a time t.

12. SOLVING A SPRING MASS SYSTEM THAT IS CRITICALLY DAMPED

Problem:

An 98 Newton weight is attached to a spring constant k of 40 N/m. The spring is stretched 4m and its rest and then equilibrium position. It is then released from rest with an initial upward velocity of 2m/s. Find the equation if the system contains damping force of 40 times the initial velocity.

Given $a=9.81\ N/m^2$ (acceleration due to gravity)

$F=ma$

$F/a=m$

$98/9.8=m$

Spring constant $k=40$ N/M

Damping constant $C=40$

The initial condition

$X(0)=0$

(In opposite direction to rest)

$x'(0)=-4$

Standard differential equation

$md^2x / dt^2+cdx/dt + kx=0$

$98/9.81d^2x/dt^2 + 40\ dx /dt +40x=0$

$10d^2x/dt^2 +40\ dx/dt +40x=0$

/by 10 $\Rightarrow d^2 x/dt^2 +4dx/dt+4x=0$

Applying Laplace transformation on both sides

$L(x'') +4L(x') +L(x)=L(0)$

$[S^2L(x)-Sx(0)-x'(0)] +4[SL(x)-x(0)] + 4L(x)=0$

$L(x)[S^2+4S+4]+4=0$

$L(x)[S^2+4S+4]=-4$

$L(x) = -4/(S+2)^2$

$x=L^{-1}(-4/(S+2)^2$

$=e^{-2t}L^{-1}(-4/S^2)$

$=-4e^{-2t}L^{-1}(1/S^{2)}$

$x=-4te^{-2t}$

Which is the equation of a critically damped spring mass system.