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Optimization of Plate Fin Arrays with Laminar and Turbulent Forced Convection

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Abstract. The minimum thermal resistance for isothermal plate fin arrays with array volume, number of the fins and either fan power or pressure drop fixed can be found in the literature. We solve the same fundamental problems for non-isothermal fin arrays by using a one-dimensional fin theory. Both laminar and turbulent cases are solved for coolants with Prandtl number equal to 0.7. The ratio of the fin to coolant thermal conductivity is 600, 6000 or 14000. Isothermal boundary condition is used at the fin base. The contraction and expansion losses at the inlet and exit are taken into account in the calculation of the pressure drop. The optimal design is characterized by various non-dimensional variables. The most important non-dimensional variable combines the three criteria and the thermal properties of the coolant and the fin material.

1. Introduction

Plate fin arrays as shown in figure 1 are commonly found, e.g. in the cooling of electronics components and in various heat exchangers. In designing any kind of heat sink or heat exchanger, the need to save energy and reduce costs always sets the following requirements for the design: the total heat transfer rate should be high while the volume and weight of the system should be low. When cooling electronics, the temperature of the components must stay below a fixed value but in heat exchangers it is also advantageous to be able to use a small temperature difference. In forced convection cooling, the power consumed by the fan or pump should be low to reduce the operational costs.

Accurate methods which take into account the coupling of conduction and convection have been developed for single fins [1], but the results found using these types of models are not valid for densely spaced fins. Teertstra et al. showed that the traditional 1D fin theory is also applicable to fin arrays when the heat transfer coefficient is defined for the isothermal channel using the temperature difference between the fin surface and the inlet flow [2]. In the state-of-the-art analytical heat sink model of Lehtinen, the 3D heat equation in the base plate and the 2D heat equation in the fins are solved simultaneously using a truncated Fourier series expression for the temperature. The flow is treated as a hydrodynamically fully developed flow between two parallel plates [3].

Optimization of fin arrays has been studied a lot under the assumption of isothermal fins. Bejan and Sciubba maximized the heat transfer rate of an isothermal fin array with laminar flow when the pressure drop and total fin array width were fixed [4]. Bejan and Morega solved the corresponding turbulent case [5]. Later, Mereu et al. minimized the thermal resistance with fixed width and fan power, but only a numerical solution was given with constant heat flux at the fins [6]. The analytical

solution for this problem with isothermal fins was given by Canhoto and Heitor Reis who also included the effect of the local pressure losses [7]. Lindstedt and Karvinen extended and summarized the above studies by minimizing thermal resistance with either fixed width or fixed volume and either fixed pressure drop or fan power. More accurate solutions for the local pressure losses were used [8].

In order to take into account the decrease in the fin temperature, at least one-dimensional heat conduction must be assumed in the fins. Liu and Garimella [9] and Song et al. [10] have used conjugated fin array models in the optimization. Lindstedt and Karvinen [11] present optimization results which use the 2D solution of Lehtinen [3] for the fins. Thermal resistance was minimized with the volume, the number of the fins and either the fan power or the pressure drop fixed. Isothermal and isoflux boundary conditions at the fin base were dealt with. Optimization with comparable or even more accurate conjugated heat transfer modeling has only been made in CFD studies, such as those by Li and Peterson [12] and Wang et al. [13] among others.

There is very little useful information on the optimization of non-isothermal fin arrays. Very often the optimization results are presented as tabulated values for different designs or as curves representing the effect of a design variable on the thermal resistance or the total heat transfer rate. Such results do not provide any generally applicable information and thus are not useful when either certain initial values, or the coolant, are changed.

Building on the work of previous studies, we present solutions for the following two fundamental problems in fin array design: 1. "minimize thermal resistance with fixed volume, fan power and number of fins", 2. "minimize thermal resistance with fixed volume, pressure drop and number of fins". Both laminar and turbulent cases are solved for gases with Prandtl number 0.7 assuming a constant temperature at the fin base, i.e. the base plate is isothermal. None of the fin or channel dimensions are fixed. The solutions of the problems are presented using non-dimensional variables, thus they are valid for fin arrays of any size.
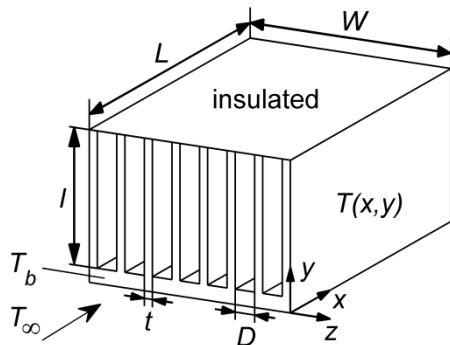

Figure 1. Schematic of typical heat sink used in power electronics cooling. The base plate is at the temperature T_b. There are N channels and fins.

2. Hydrodynamic and thermal models

To circumvent the time-consuming task of solving the full conjugated 3D RANS and energy equations for the fin array in figure 1, empirical correlations are used for hydrodynamics and heat transfer. Due to their symmetry, each channel is assumed to have identical flow and temperature fields. Constant values are assumed for all the thermal properties: fluid thermal conductivity, k_f, density, ρ, specific heat at constant pressure, c_p, viscosity, μ and for the fin thermal conductivity, k_s.

The pressure loss over the fin array, with no by-pass flow, is

$$\Delta p = \frac{1}{2}\rho U^2 \left(K_c + 4L^+ f_{app} \text{Re} + K_e \right), \qquad L^+ = \frac{L}{D_h \text{Re}}, \qquad \text{Re} = \frac{\rho U D_h}{\mu}, \qquad D_h = \frac{4Dl}{2(D+l)} \quad (1)$$

where U is the mean velocity in a channel between fins. The contraction loss coefficient is found by curve fitting from figure 5 in Kays [14], while the expansion loss can be solved analytically from the 1D momentum equation over the outlet section

$$K_c = 0.4(1 - \sigma^{2.1}), \qquad K_e = 1 - 2\beta\sigma + \sigma^2, \qquad \sigma = \frac{D}{D+t} \quad (2)$$

where the momentum correction factor for a developing laminar flow between parallel plates is [8]

$$\beta = 1 + \left[\left(4.224\sqrt{L^+}\right)^{-2.7} + 0.2^{-2.7}\right]^{-1/2.7} \tag{3}$$

For a fully developed laminar flow in a rectangular duct, the friction factor is obtained from equation (340) in Shah and London [15] by taking only the first term of the series solution

$$f\text{Re} = \frac{24}{\phi(\epsilon)}, \qquad \phi(\epsilon) = (1+\epsilon)^2\left[1 - \frac{192\epsilon}{\pi^5}\tanh\left(\frac{\pi}{2\epsilon}\right)\right], \qquad \epsilon = \frac{D}{l} \tag{4}$$

Muzychka and Yovanovich [16] have combined the above solution for the fully developed flow with the asymptotic solution for the entry flow using Churchill and Usagi's method

$$f_{app}\text{Re} = \left[\left(\frac{3.44}{\sqrt{L^+}}\right)^2 + (f\text{Re})^2\right]^{1/2} \tag{5}$$

For a turbulent flow in a rectangular channel the effective Reynolds number is calculated using the method proposed by Jones [17] and the friction factor is then evaluated using the Blasius correlation

$$f_{turb} = 0.0791\left(\frac{2\phi(\epsilon)}{3}\text{Re}\right)^{-1/4}, \qquad 4000 < \frac{2\phi(\epsilon)}{3}\text{Re} < 100000 \tag{6}$$

Correlation for developing turbulent flow is not used since this requires a model for the momentum correction factor. With fully developed value for friction factor and $\beta = 1$, the effect of the velocity profile development is partially cancelled (see equations (1) and (2)).

The calculation of the heat transfer is simplified by neglecting the heat transfer from the base plate between the fins. The simple energy balance in an isothermal channel gives

$$\Phi_i = \rho c_p l D U \Delta T_{max}\left[1 - \exp\left(-4L^*\text{Nu}_{m,T}\right)\right], \qquad L^* = \frac{L}{D_h\text{RePr}}, \qquad \text{Pr} = \frac{\mu c_p}{k_f} \tag{7}$$

where $\Delta T_{max} = T_b - T_\infty$ is the maximum temperature difference. Stephan's correlation, equation (329) in ref. [15], for laminar flow between isothermal parallel plates is used, since the optimal fin arrays have high aspect ratios ($l/D > 10$) in the case of laminar flow

$$\text{Nu}_{m,T} = 7.55 + \frac{0.024(L^*)^{-1.14}}{1 + 0.0358(L^*)^{-0.64}\text{Pr}^{0.17}}, \qquad 0.1 < \text{Pr} < 1000 \tag{8}$$

The length scale $D_h = 2D$ is used in heat transfer calculations with laminar flow. The Gnielinski correlation [18] is often used for turbulent flow in parallel plate channels

$$\text{Nu}_{m,T} = \frac{\frac{f_{turb}}{2}(\text{Re} - 1000)\text{Pr}}{1 + 12.7\sqrt{\frac{f_{turb}}{2}}(\text{Pr}^{2/3} - 1)}\left[1 + \left(\frac{D_h}{L}\right)^{2/3}\right], \qquad 2300 < \text{Re} < 10^6, 0.6 < \text{Pr} < 10^5 \tag{9}$$

The calculation of the total heat transfer rate is easy when the channel surfaces are isothermal. In actual fin arrays the fins are far from isothermal since they are made thin in order to reduce the size of the system. As a result, the fin temperature distribution depends on the convection whereas the local heat flux depends on the temperature distribution. The following section shows how this conjugated heat transfer problem has been solved in an approximate manner.

2.1 Conjugated solution for a fin array

The fin efficiency is defined as the ratio of the real heat transfer rate Φ and the heat transfer rate of a geometrically identical isothermal (ideal) fin Φ_i. In the ideal solution the temperature is taken to be the maximum temperature of the fin base and the flow is assumed to be unchanged. The traditional solution for a plate fin with a constant local heat transfer coefficient and insulated tip can be written in the following non-dimensional form

$$\eta = \frac{\tanh(X)}{X}, \quad X = \sqrt{\mathrm{Nu}_{m,T}^{\infty} Y}, \qquad Y = \frac{k_f l^2}{k_s D t} \tag{10}$$

The real heat transfer rate from the fins can be calculated as $\Phi = \eta \Phi_i$.

In the case of densely spaced fins, the fluid temperature changes in the flow direction. This conflicts with the inherent assumption in the one dimensional fin theory, in which the ambient temperature is assumed to be constant. Teertstra et al. circumvented this by defining the Nusselt number $\mathrm{Nu}_{m,T}^{\infty}$ using the difference between the local fin temperature and the fluid inlet temperature in an isothermal channel [2]. The authors devised a new $\mathrm{Nu}_{m,T}^{\infty}$ correlation for developing laminar flow.

The method of Teertstra et al. [2] can be refined as follows. The Nusselt number for an isothermal fin array with respect to the inlet temperature is first obtained using the energy balance in equation (7)

$$\mathrm{Nu}_{m,T}^{\infty} = \frac{D_h}{k_f} \frac{\Phi_i}{2Ll\Delta T_{max}} = \frac{1}{4L^*} \left(1 - \exp\left(-4L^* \mathrm{Nu}_{m,T}\right)\right) \tag{11}$$

Now it is possible to utilize conventional correlations of the type $\mathrm{Nu}_{m,T}$ which are available for several different internal flow situations, unlike $\mathrm{Nu}_{m,T}^{\infty}$. The equation for the fin efficiency (10) may be put in a non-dimensional form

$$\eta = \frac{\tanh(X)}{X}, \qquad X = \left(\frac{Y}{4L^*} \left[1 - \exp\left(-4L^* \mathrm{Nu}_{m,T}\right)\right]\right)^{1/2} \tag{12}$$

In the laminar case, the fin efficiency is a function of three non-dimensional variables, $\{L^*, \mathrm{Pr}, Y\}$, but in the turbulent case Re and ϵ also have an effect. The channel aspect ratio ϵ has an effect in the laminar case if appropriate correlation, such as (45) in ref. [19], is used. Equation (12) predicts values for fin efficiency, which are within a few percent of the 2D solution of Lindstedt and Karvinen [11].

3. Optimization procedures

In the introduction, two multi-objective optimization problems were introduced. Both optimization problems include the thermal resistance, $R = \Delta T_{max}/(N\Phi_i\eta)$, and the volume, $V = NlL(D + t)$, as a criterion, but a choice between the pressure drop, Δp, and the fan power, $P = NlDU\Delta p$, must be made. Essentially, this comes down to a question of finding the most expensive cost factor. The criteria are functions of the six design variables $\{D, L, U, l, t, N\}$.

Numerical experiments show that in order to find a solution to these problems with isothermal fin base, the number of the fins, N, must be fixed. Doing this actually incorporates the idea that a small number of fins is desirable. Other problems with isothermal fin base are not relevant or they are impossible to solve. For example, using the fin array width as a criterion instead of the volume leads to non-physical solutions. With discrete heat sources, this and many other problems become relevant.

Multi-objective optimization problems are usually solved by first transforming them into single criterion optimizations. With the constraint method, one of the criteria is minimized while the others are turned into fixed constraints or constraints with upper bounds. The problem may then be solved using conventional optimization methods [20].

The problem "minimize thermal resistance, volume and fan power with fixed number of fins" is selected for a detailed analysis while the other problem can be solved in a similar manner. This problem may be transformed into the three problems "minimize thermal resistance with fixed volume, fan power and number of fins", "minimize volume with fixed thermal resistance, fan power and number of fins" or "minimize fan power with fixed thermal resistance, volume and number of fins". In this application, the results of all the three scalarized problems are equivalent when appropriate fixed values are chosen. Solving any of the problems gives Pareto optimal results, i.e. the results are such that none of the criteria can be improved without worsening at least one other criterion. The concept of Pareto optimality is discussed further by Miettinen [20].

In order to present the optimization results in a general manner, suitable non-dimensional variables must be used. Some non-dimensional variables have been found in the optimization of non-isothermal

fin arrays. Equation (26) in Bejan and Sciubba [4], equation (48) in Mereu et al. [6], and the quantity used as an ordinate in figures 2-5 in Canhoto and Heitor Reis [7] are examples of such variables. We have used these and similar variables with isothermal (equations (25)-(29) in ref. [8]) and non-isothermal fin arrays (equations (41)-(42) in ref. [11]). However, none of these variables are well suited for non-isothermal fin arrays, since they are dependent on l, which is a design variable in the non-isothermal case. It is desirable that such variables be found, where only the three criteria and the material properties are included. This kind of definition would also be valid for other geometries.

By using the solution for the total heat transfer rate, (20) in [11], and the equations for volume and fan power, a suitable non-dimensional variable for the non-isothermal was found. The three criteria and the five material properties can be combined into a single non-dimensional variable

$$\Lambda_P = R(VP)^{1/2}k_s k_f^{-1/2}\rho c_p^{1/2}\mu^{-1} \tag{13}$$

which will be called the performance number. With pressure drop as the third criterion, as indicated in the subscript of Λ, the performance number is

$$\Lambda_{\Delta p} = RV\Delta p k_s k_f^{-1}\rho c_p\mu^{-1} \tag{14}$$

When the performance numbers are low, the values of all the criteria are also low.

3.1. Numerical optimization setup
The first optimization problem "minimize thermal resistance with fixed volume, fan power and number of fins" can be formally stated as

$$\text{find} \quad \boldsymbol{x} = \{D, L, U, l, t\}$$
$$\text{to minimize} \quad R \tag{15}$$
$$\text{subject to} \quad V \leq V_0, P \leq P_0, N = N_0$$

The second constraint is replaced by $\Delta p \leq \Delta p_0$ when pressure loss is a criterion. The choices between laminar and turbulent flow are realized by selecting appropriate correlations for $f_{app}\text{Re}$, $\text{Nu}_{m,T}$ and β from the equations presented above. When problem (15) is solved numerically using certain algorithms, it might be necessary to provide lower and upper bounds for the design variables. These bounds must only be given to speed up the convergence, and in the final solution none of the variables should be found at either of its bounds. This guarantees the most general results.

By giving numerical values to the constants P_0 and V_0 in the problem (15) and performing the optimization with, for example, Matlab, a certain optimum configuration for a fin array is found. As an example, with available fan power 15 W, volume 1500 cm^3, and $N = 30$, solving problem (15) for laminar flow with aluminum and air at $T = 333$ K gives the optimum design $\{D = 1.8$ mm, $L = 0.270$ m, $U = 13.3$ m/s, $t = 1.5$ mm, $l = 55.6$ mm$\}$ and performance $R = 1.19$ K/W. That result might be important in a specific power electronics cooling application. In order to find generally applicable results, the problem must be solved with several different values for the fixed constraints and material properties, and the optimal values of the non-dimensional variables must be reported.

4. Results
Solving problem (15) with several values for V_0 and P_0, yields an interesting result. The optimal values of some non-dimensional variables are always found in certain curves, which are presented in figures 2-3. With laminar flow, most of the optimal values are constants. The parameter values $k_s/k_f = \{600, 6000, 14000\}$ correspond to steel, aluminum and copper fins with air as a coolant. Other material combinations can be interpolated from these results. With pressure drop as a criterion, the same approach is used and similar results are obtained. These results are shown in figures 4-5. The results do not change much when the more accurate solution with 2D heat conduction for the fin efficiency [11] is used.

The design of a fin array with minimum thermal resistance, volume and fan power is found in figure 2. The advantage of using the non-dimensional variables σ, L^*, l/L and l/D is in the

universality of the presentation of the results. The variable L^*, for instance, is the channel length with respect to the development of the fluid temperature profile. With laminar flow and fan power criterion, all the results are found at $L^* = 0.04$, which approximately corresponds to the thermal entrance length in a channel. The actual fin array dimensions can be anything from millimeters to meters. The fin arrays with minimum pressure drop have a little bit less thermally developed flow with $L^* = 0.024$.

With laminar flow, the optimal fin arrays are essentially parallel plate channels ($l/D > 10$) where the effects of the base plate can be neglected (see figures 2b and 4b). With turbulent flow, the optimum solutions can have aspect ratios which are as low as $l/D = 2$. This means that the calculation of the heat transfer is not accurate, since the heat transfer coefficient varies in the y-direction of the fin surface. The heat transfer from the base plate should also be taken into account.

The optimal values of the performance numbers Λ in figures 3a and 5a show that reducing the Reynolds number improves the performance, i.e. decreases the thermal resistance when the available volume and either fan power or pressure drop is fixed. This means that the enhancement of heat transfer in turbulent flow cannot compensate for the penalty of increased friction when the available fin array volume is fixed. The results suggest that Λ has a local minimum immediately after the transition from laminar to turbulent flow. It is difficult to find that minimum due to difficulties in the modeling of simultaneously developing transitional flows. The values of Λ_P and $\Lambda_{\Delta p}$ in figures 3a and 5a have been scaled by constants $10^{-4}\left(k_f/k_s\right)^{4/5}$ and $10^{-7}\left(k_f/k_s\right)^{4/5}$ for the sake of clarity.
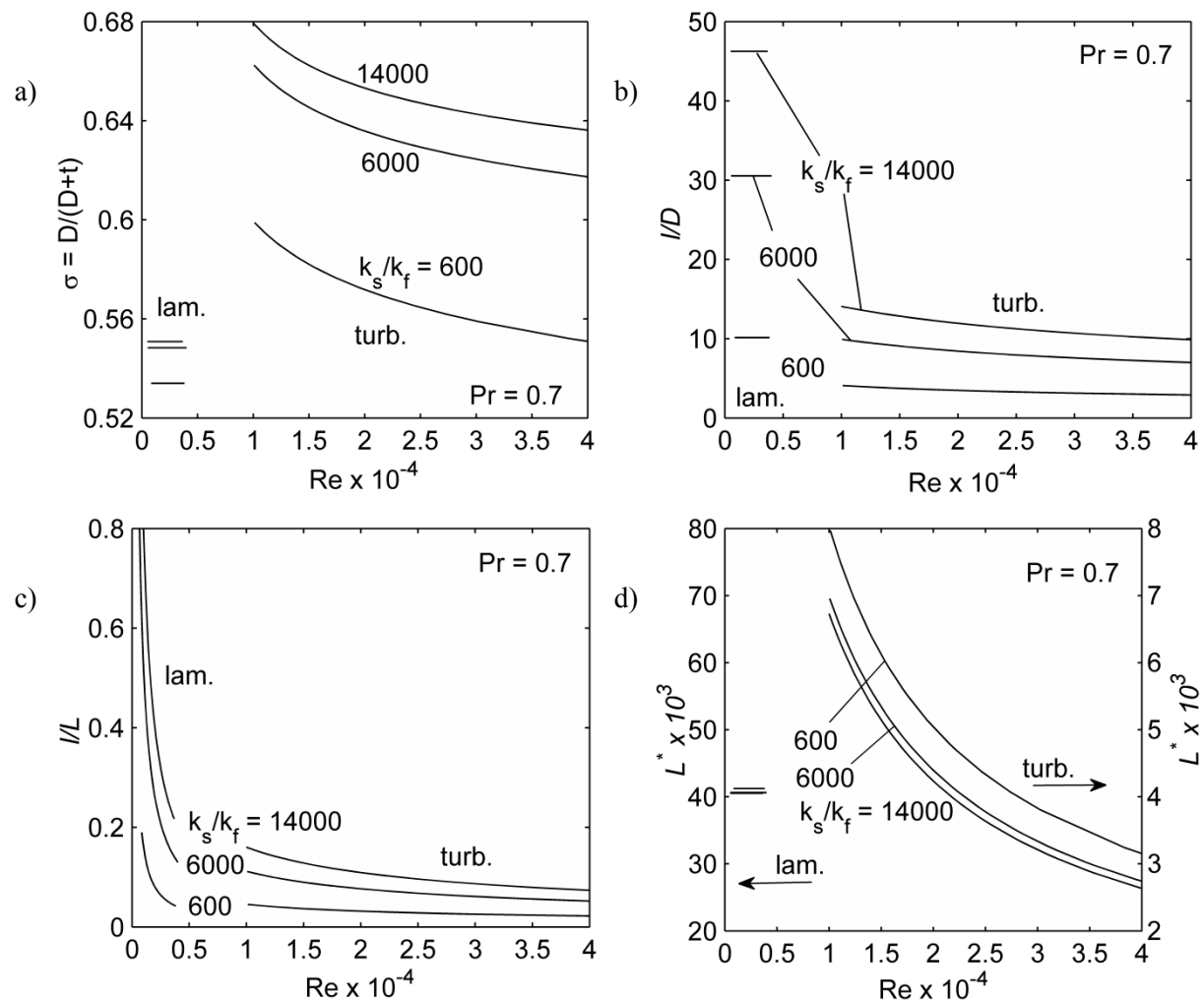

Figure 2. Optimal design in problem "minimize thermal resistance, volume and fan power".
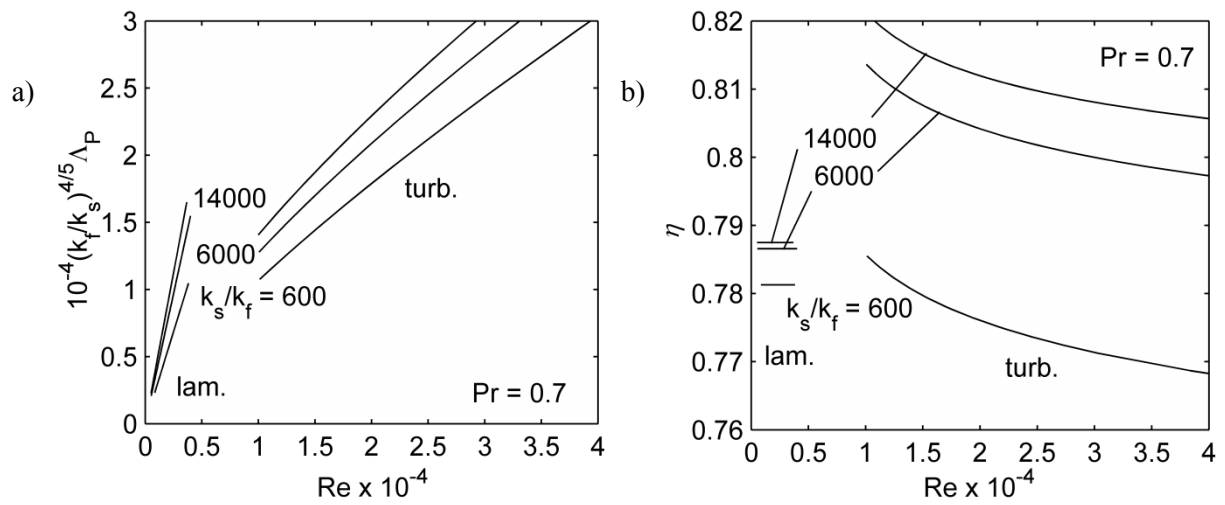
Figure 3. Optimal performance in problem "minimize thermal resistance, volume and fan power".
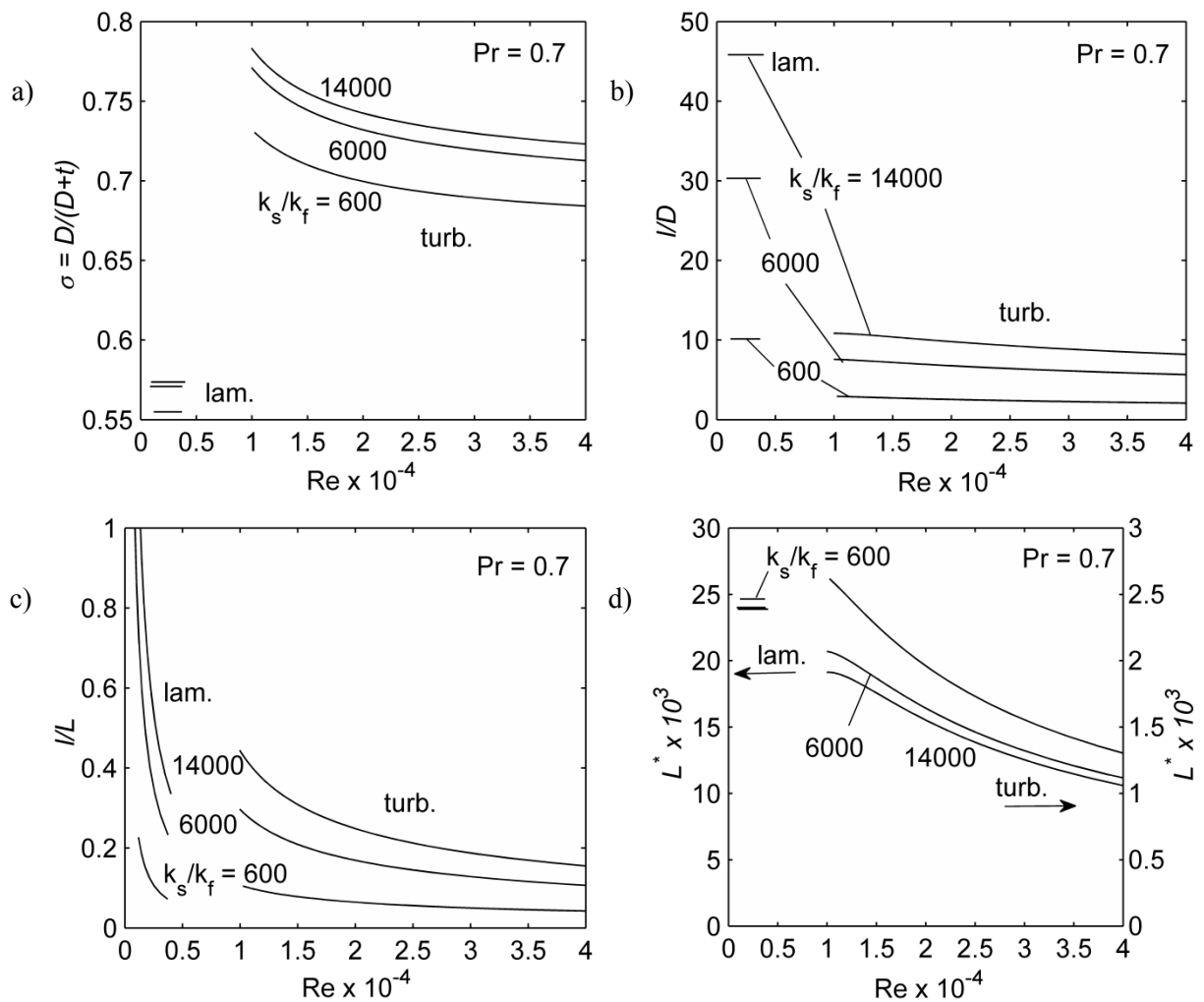

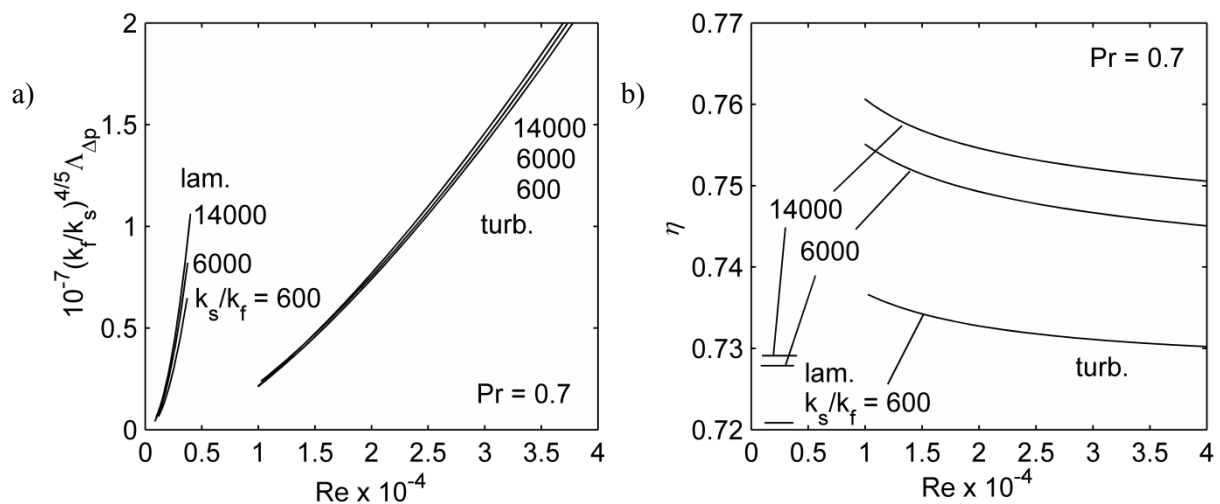**Figure 4.** Optimal design in problem "minimize thermal resistance, volume and pressure drop".

Figure 5. Optimal performance in problem "minimize thermal resistance, volume and pressure drop".

5. Conclusions

"Minimum thermal resistance, volume and fan power" and "minimum thermal resistance, volume and pressure drop" have been found for non-isothermal fin arrays with a fixed number of fins and an isothermal base. Fin efficiency was calculated with the 1D fin solution, where the heat transfer coefficient was defined between the inlet flow and isothermal channel walls. This model of Teertstra et al. [2] was modified so that the calculation is possible also with turbulent flow. The optimal fin efficiency in both problems varies between 72 and 82 %, which means that isothermal fin arrays are always too large or they have too large pressure drop or fan power. Lower Reynolds number gives better values for the criteria but there is a local optimum just after the transition to turbulence.

References

[1]   Karvinen R and Karvinen T 2010 *Int. J. Heat and Mass Transfer* **53** 5380-5
[2]   Teertstra P, Yovanovich M M and Culham J R 2000 *J. Electr. Manufact.* **10** 253-61
[3]   Lehtinen A 2005 Ph.D. Thesis Tampere University of Technology
[4]   Bejan A and Sciubba E 1992 *Int. J. Heat and Mass Transfer* **35** 3259-64
[5]   Bejan A and Morega M 1994 *Int. J. Heat and Mass Transfer* **37** 1045-8
[6]   Mereu S, Sciubba E and Bejan A 1993 *Int. J. Heat and Mass Transfer* **36** 3677-86
[7]   Canhoto P and Heitor Reis A 2011 *Int. J. Heat and Mass Transfer* **54** 1441-7
[8]   Lindstedt M and Karvinen R 2012 *J. Enhanced Heat Transfer* in print
[9]   Liu D and Garimella S V 2005 *Int. J. for Numerical Methods in Heat & Fluid Flow* **15** 7-26
[10] Song G-E, Lee J and Lee D-Y 2012 *Int. J. Heat and Mass Transfer* **55** 2992-8
[11] Lindstedt M and Karvinen R *Proc. 6th BHTC2011*, (Tampere, Finland, 24-26 August 2011) ISBN 978-952-15-2638-1
[12] Li J, Peterson G P 2007 *Int. J. Heat and Mass Transfer* **50** 2895-904
[13] Wang Z-H, Wang X-D, Yan W-M, Duan Y-Y, Lee D-J and Xu J-L 2011 *Int. J. Heat and Mass Transfer* **54** 2811-9
[14] Kays W M 1950 *Trans. ASME* **72** 1067-74
[15] Shah R K and London A L 1978 *Laminar flow forced convection in ducts (Advances in Heat Transfer Supp. 1)* ed T F Irvine and J P Hartnett, (New York: Academic Press)
[16] Muzychka Y S and Yovanovich M M 2009 *J. Fluids Eng.* **131** 111105
[17] Jones O C Jr. 1976 *J. Fluids Eng.* **98** 173-81
[18] Gnielinski V 1975 *Forschung im Ingenieurwesen* **41** 8-16
[19] Muzychka Y S and Yovanovich M M 2004 *J. Heat Transfer* **126** 54-61
[20] Miettinen K 1999 *Nonlinear Multiobjective Optimization* (Dordrecht: Kluwer)