Balancing Social Interactions and Access to Resources

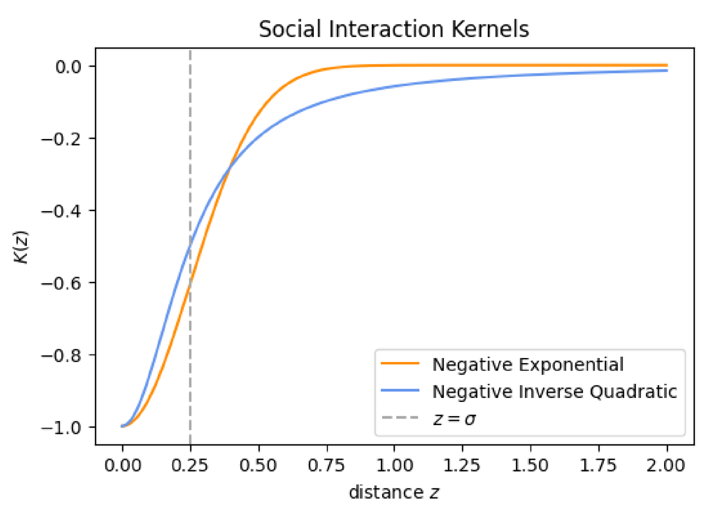
Carnegie Mellon University

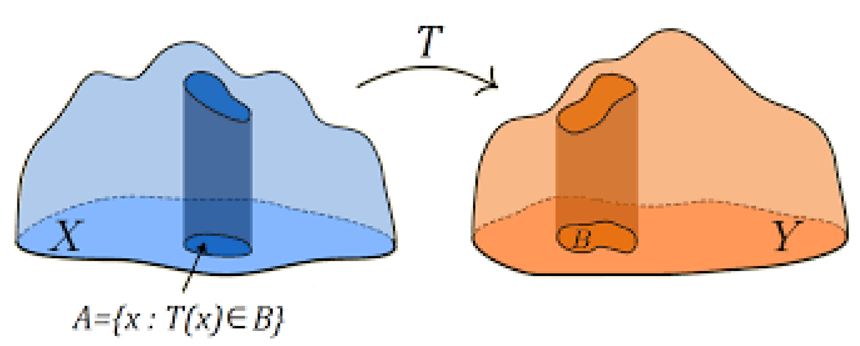
Justin T. Fung   Noah Kim   Peter Panov

Department of Mathematical Sciences

Introduction

Whether studying the emergence of human settlements or the development of distribution hubs, the tendency of interacting agent systems to cluster depends on two factors: the benefits (and disadvantages) of local social interactions and the costs of transporting resources. We propose an energy functional which combines these two factors and study this functional analytically and numerically: **of particular interest is understanding what conditions precipitate the formation of "cities," or clusters of agents.**





Our Model

Given a distribution of resources, we define the energy E of a configuration of n agents in terms of pairwise interactions between agents and the transport of resources to those agents. We define the following terms:

- X_i is the ith agent
- K quantifies social interactions
- ρ is a distribution of resources
- λ scales transport costs
- $\mathcal{T}_i$ is the territory belonging to X_i
- p is the power of the cost kernel

$$E(X_1, \ldots, X_n) = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} K(|X_i - X_j|) + \frac{\lambda}{2} \sum_{i=1}^{n} \int_{\mathcal{T}_i} |X_i - x|^p d\rho(x) \tag{1}$$

Mathematical Analysis

Before running numerical experiments, we proved the existence of a minimizing configuration of agents, and we characterized the minimizer analytically:

- We show under mild assumptions that E is lower semi-continuous on a compact set, satisfying the Weierstrass criterion for the existence of minimizers [3].
- Introducing a flow map $\Phi(x_0, t) = x(t)$ and the Continuity Equation allows us to take the derivative of $E(\vec{X})$ with respect to time and evaluate it at $t = 0$ [4].
- An energy minimizing configuration must have a derivative equal to zero, allowing us to characterize minimizing configurations explicitly.

The Social Interaction Kernel

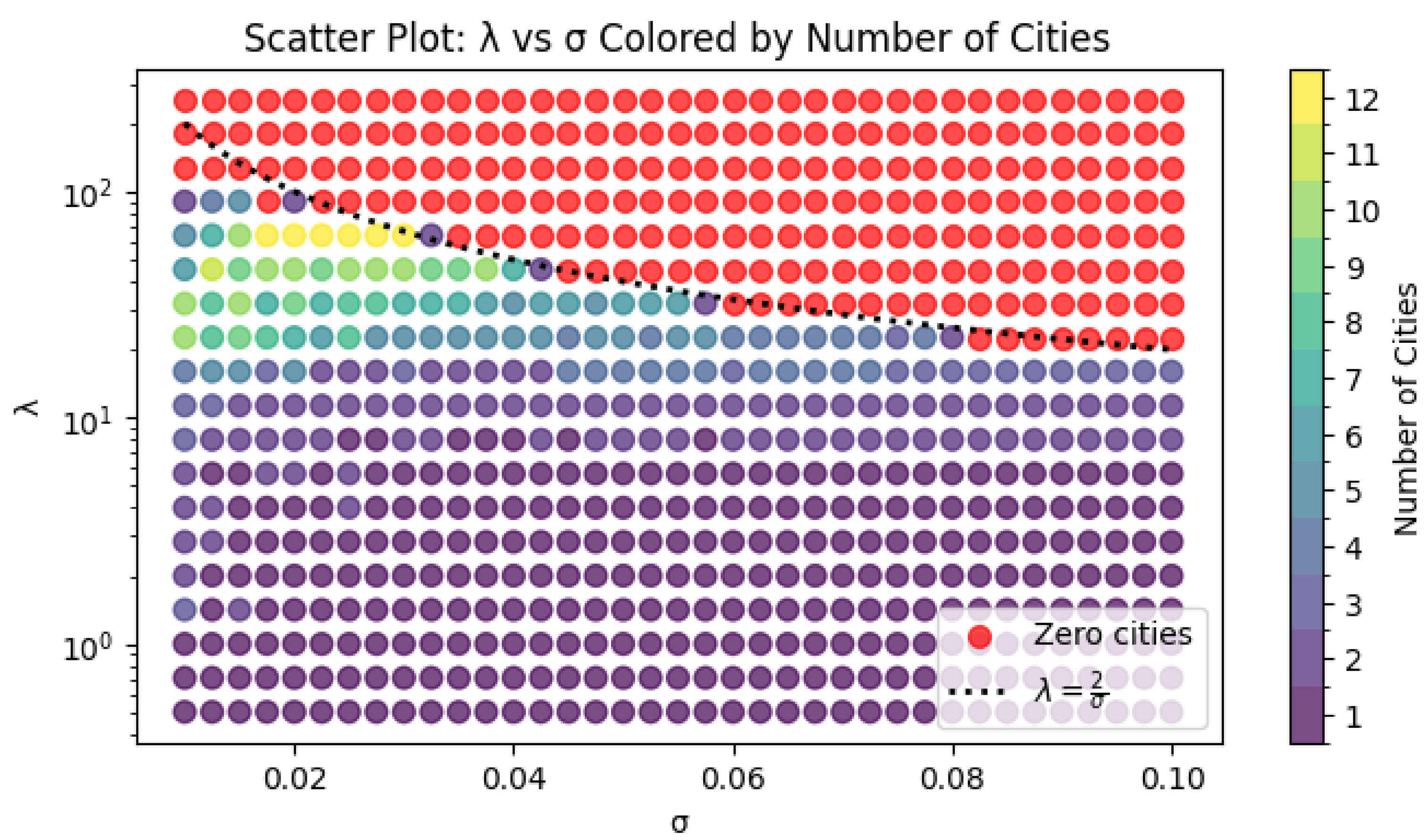
To quantify the benefit of local social interactions, we chose a negative inverse quadratic kernel with parameter σ, which controls the "radius" of social interactions. When agents are spaced more than σ apart, the mutual benefit they experience drastically decreases.

$$K(z) = \frac{-1}{1 + \frac{z^2}{\sigma^2}}$$

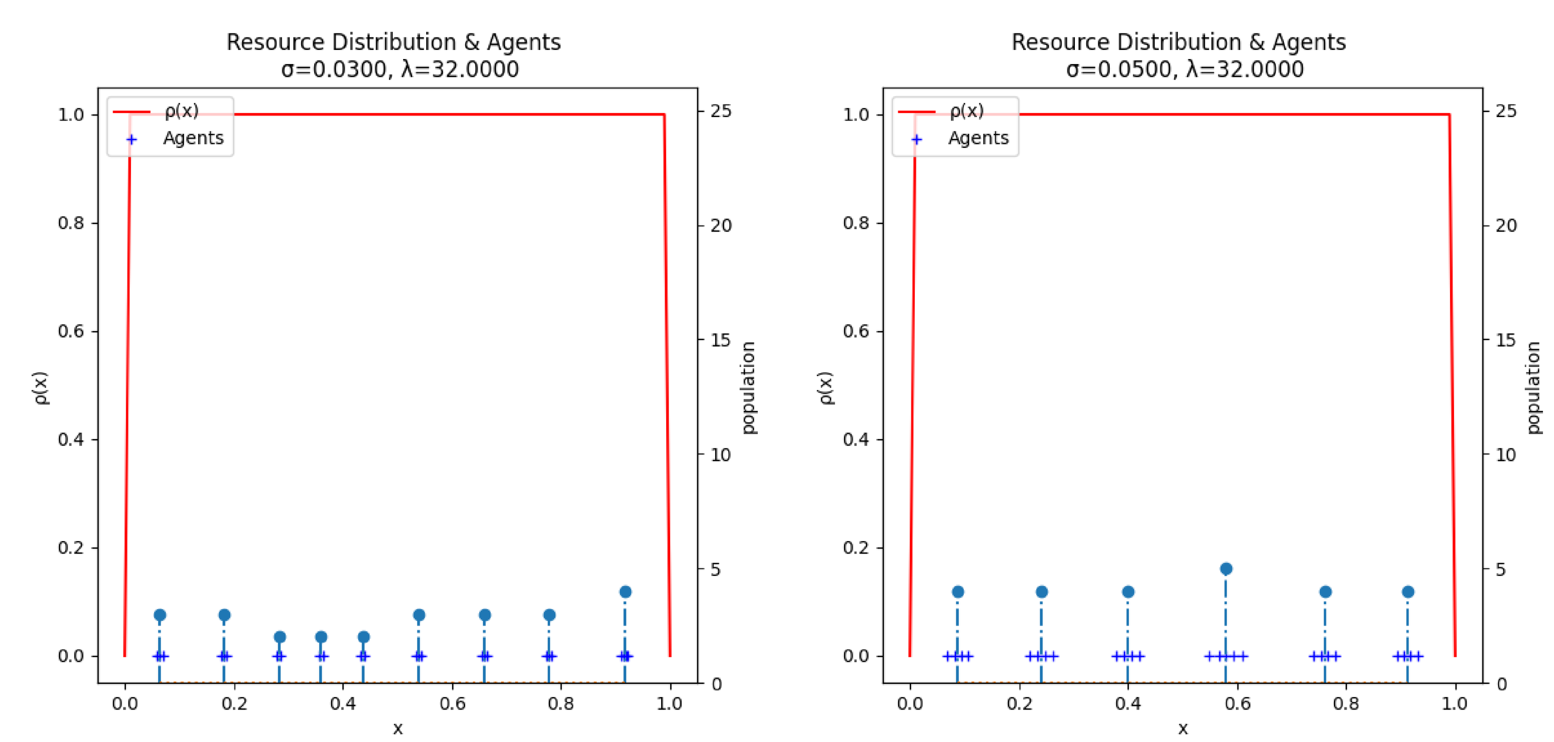
Experimental Setup

We wrote code in python and used SciPy's optimize.minimize function with the limited memory, bounded Broyden–Fletcher–Goldfarb–Shanno algorithm, which falls under the quasi-Newton family of optimization algorithms [5].

Results

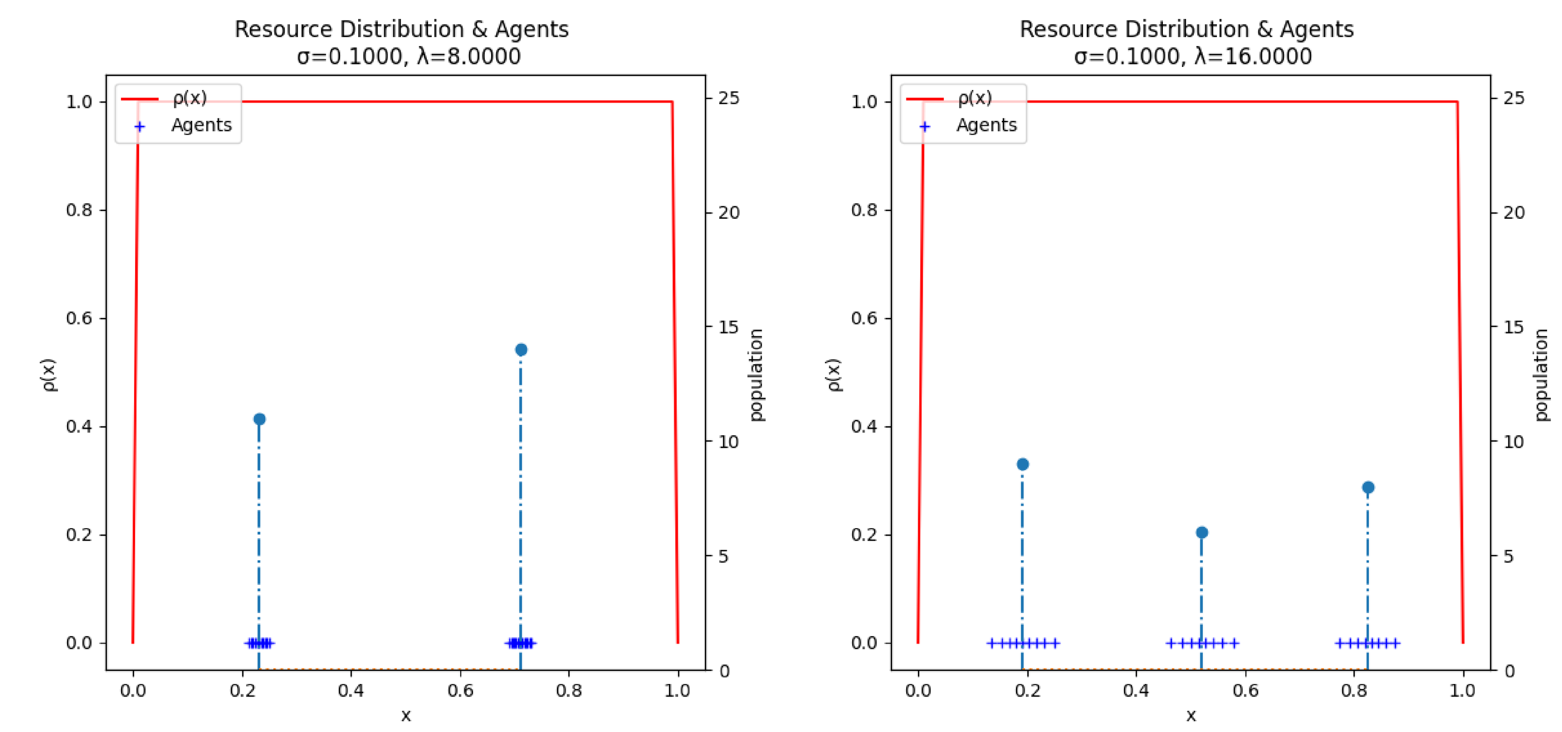


We hypothesized that a phase transition would occur when λ is greater than or equal to $\frac{c}{\sigma}$, where c is some positive constant. Indeed, we verified this numerically as shown above. In particular, we note that zero cities form when λ is above this decision boundary, suggesting that if costs of transporting resources to agents are too high, then cities will not form.



We note that as σ increases, the maximum number of cities that form decreases. This is caused by the fact that σ determines the "radius" of cities. For larger values of σ, the transition to no cities can occur when fewer cities are present, as individual agents within those cities are able to spread out more.

Results Continued



We see that increasing λ, the weight of the transportation cost, while holding σ constant causes a greater number of cities to form. We also see that agents tend to "spread out" within the city radius σ for sufficiently large λ.

Conclusions

Thus, we found that agents will form cities over a uniform distribution of resources when the weight of the transport cost λ is greater than $\frac{2}{\sigma}$. Heuristically, this implies that if the radii of the cities multiplied by the weight of the transportation cost is more than twice of the diameter of the resource distribution, we will see no cities because transporting resources to cities becomes too costly.

Future Work

We plan to continue studying the energy functional analytically in $\mathbb{R}^d$ and computationally in two dimensions. We also plan to implement bespoke gradient descent algorithms for one and two dimensions.

We are interested in attraction-repulsion interaction kernels. One such interaction kernel is the Lennard-Jones potential [1], given by: $K(z) = A|z|^{-12} - B|z|^{-6}$

Acknowledgements

We thank Prof. Dejan Slepčev and Sangmin Park for their guidance.

References

[1] Adams, J.B.: Bonding Energy Models. Version: 2001. http://dx.doi.org/https://doi.org/10.1016/B0-08-043152-6/00146-7. In: Buschow, K.H. J. (Hrsg.) ; Cahn, Robert W. (Hrsg.) ; Flemings, Merton C. (Hrsg.) ; Ilschner, Bernhard (Hrsg.) ; Kramer, Edward J. (Hrsg.) ; Mahajan, Subhash (Hrsg.) ; Veyssière, Patrick (Hrsg.): *Encyclopedia of Materials: Science and Technology*. Oxford : Elsevier, 2001. – DOI https://doi.org/10.1016/B0-08-043152-6/00146-7. – ISBN 978-0-08-043152-9, 763-767

[2] Kolouri, Soheil ; Park, Se R. ; Thorpe, Matthew ; Slepcev, Dejan ; Rohde, Gustavo K.: Optimal Mass Transport: Signal processing and machine-learning applications. In: *IEEE Signal Processing Magazine* 34 (2017), Nr. 4, S. 43–59. http://dx.doi.org/10.1109/MSP.2017.2695801. – DOI 10.1109/MSP.2017.2695801

[3] Santambrogio, Filippo: *Progress in Nonlinear Differential Equations and their Applications*. Bd. 87: *Optimal transport for applied mathematicians*. Birkhäuser/Springer, Cham, 2015. – xxvii+353 S. doi.org/10.1007/978-3-319-20828-2. – ISBN 978-3-319-20827-5; 978-3-319-20828-2. – Calculus of variations, PDEs, and modeling

[4] Santambrogio, Filippo: *A Course in the Calculus of Variations: Optimization, Regularity, and Modeling*. Cham, Switzerland : Springer Nature Switzerland AG, 2023 (Universitext). http://dx.doi.org/10.1007/978-3-031-45036-5. http://dx.doi.org/10.1007/978-3-031-45036-5. – ISBN 978-3-031-45035-8, 978-3-031-45036-5 (eBook)

[5] Zhu, Ciyou ; Byrd, Richard H. ; Lu, Peihuang ; Nocedal, Jorge: Algorithm 778: L-BFGS-B: Fortran subroutines for large-scale bound-constrained optimization. In: *ACM Trans. Math. Softw.* 23 (1997), Dezember, Nr. 4, S. 550–560. http://dx.doi.org/10.1145/279232.279236. – DOI 10.1145/279232.279236. – ISSN 0098-3500