A Solver for Hyperbolic Systems using Discontinuous Galerkin Method

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By

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Certificate

This is to certify that the thesis entitled, "*A Solver for Hyperbolic Systems using Discontinuous Galerkin Method*" and submitted by Balavarun Pedapudi ID No. 2013B5AB0462P in partial fulfillment of the requirements of BITS F421T Thesis embodies the work done by him under my supervision.

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Abstract

Master of Science (Hons.) Physics

A Solver for Hyperbolic Systems using Discontinuous Galerkin Method

by Balavarun Pedapudi

The Discontinuous Galerkin method is an efficient way to evolve conservative partial differential equations. In this thesis, we develop a solver for hyperbolic systems using the discontinuous Galerkin method. The solver, at its current capacity can evolve one dimensional and two dimensional hyperbolic equations and is shown to converge exponentially. The code is efficient and has been designed to work on arbitrary computer architectures; both on CPUs and GPUs.

Acknowledgements

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Finally I must thank my family for their support and love since day one.

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Chapter 1

Introduction

1.1 The Advection Equation

To solve the Maxwell's equations a scheme to solve the wave equation is needed. As a first step, the 1D wave advection equation is simulated

The 1D advection equation is given by the equation

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial F(u)}{\partial x} = 0 \tag{1.1}$$

Where,

$u(x,t)$ is the wave function

F is taken to be

$$F(u) = cu \tag{1.2}$$

Where c is the wave speed.

The domain of the wave to be advected is $x\epsilon[-1,1]$ This space is divided into M number of elements using $M+1$ number of equally spaced points.

1.2 Isoparametric mapping

The solutions for the wave equation will be computed for each element separately. For each element, the x space is mapped to $\xi\epsilon[-1,1]$ space and the calculations are done in the ξ space

The ξ space is split into N Legendre-Gauss-Lobatto points(LGL), The LGL points are the roots of the equation.

$$(1-\xi^2)P'_{n-1}(\xi) = 0 \tag{1.3}$$

Where P_n is The Legendre polynomial.

The LGL points are represented as ξ_i

An element defined by the points x_i and x_{i+1} on the x space is mapped onto the $\xi\epsilon[-1,1]$ space using the relation.

$$x = \frac{(1-\xi)}{2}x_i + \frac{(1+\xi)}{2}x_{i+1} \tag{1.4}$$

1.2.1 2D Isoparametric mapping

To map given (x, y) to the (ξ, η) space, Isoparametric mapping is done using 8 nodes.

The nodes in the (ξ, η) space are taken to be at (-1, 1), (-1, 0), (-1, -1), (0, -1), (1, -1), (1, 0) and (1, 1)

Let the nodes of the element in (x, y) coordinates be given by (x^e, y^e)

The mapping relation is obtained as

$$x = F_i(\xi, \eta)x_i^e \tag{1.5}$$

$$y = G_i(\xi, \eta)y_i^e \tag{1.6}$$

Where F_i is a function which returns zero for all x_j^e when j!=i and has the value 1 when i=j, a similar constraint is enforced by G_i

This can be written as

$$F_i(\xi, \eta)x_j^e = \delta_j^i \tag{1.7}$$

$$G_i(\xi, \eta)y_j^e = \delta_j^i \tag{1.8}$$

These constraint relations can be used to map arbitrary points in the (ξ, η) space to the (x, y) space

An example of the isoparametric mapping is shown in the image below

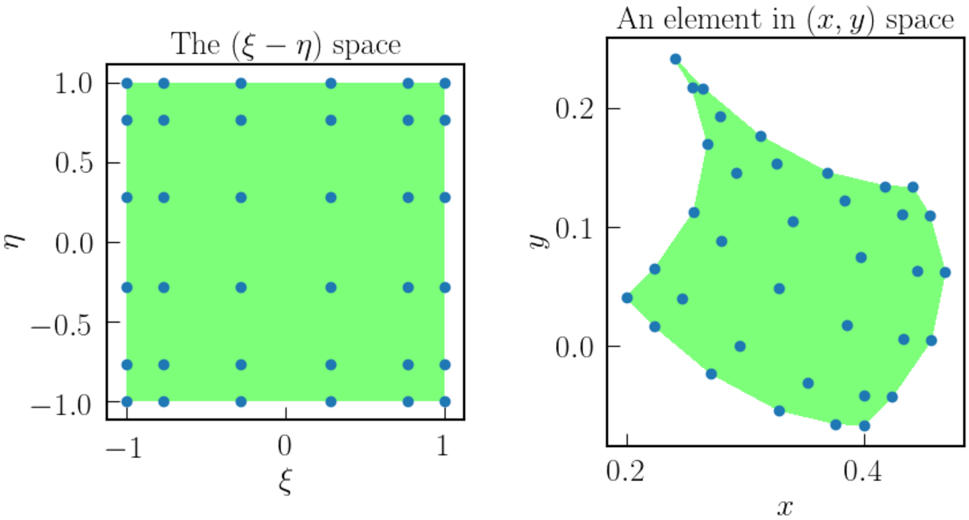

FIGURE 1.1: LGL points from the (ξ, η) space mapped to an element in (x, y) space

Chapter 2

1D Advection

2.1 1D Advection Equation

To solve the $1D$ Wave equation, The strong form (differential form) of the wave equation needs to be converted to the weak form (integral form)[4]. This is done by multiplying a test function $v(x)$ to 1.1 and integrating both sides of the equation.

$$v(x)\frac{\partial u(x,t)}{\partial t} + v(x)\frac{\partial F(u)}{\partial x} = 0 \tag{2.1}$$

$$\int v(x)\frac{\partial u(x,t)}{\partial t}dx + \int v(x)\frac{\partial F(u)}{\partial x}dx = 0 \tag{2.2}$$

$$\int v(x)\frac{\partial u(x,t)}{\partial t}dx + v(x)F(u) - \int F(u)\frac{\partial v(x)}{\partial x}dx = 0 \tag{2.3}$$

$u^n(x)$ denotes the wave function at the n^{th} time step. Given $u^n(x)$, The next step is to find $u^{n+1}(x)$. (i.e.:- a recurrence relation is to be found).

Equation 2.3 can be rewritten as:

$$\frac{\int v(x)u^{n+1}(x)dx - \int v(x)u^n(x)dx}{\Delta t} + v(x)F(u^n) - \int F(u^n)\frac{\partial v(x)}{\partial x}dx = 0 \tag{2.4}$$

$$\int v(x)u^{n+1}(x)dx - \int v(x)u^n(x)dx + \Delta t\{v(x)F(u^n) - \int F(u^n)\frac{\partial v(x)}{\partial x}dx\} = 0 \tag{2.5}$$

$$\int v(x)u^{n+1}(x)dx = \int v(x)u^n(x)dx - \Delta t\{v(x)F(u^n) - \int F(u^n)\frac{\partial v(x)}{\partial x}dx\} \tag{2.6}$$

Coordinate transformation of Eq. 2.7 to the ξ space for a single element,

$$\int v(\xi)u^{n+1}(\xi)\frac{dx}{d\xi}d\xi = \int v(\xi)u^{n}(\xi)\frac{dx}{d\xi}d\xi - \Delta t\{v(\xi)F(u^n) - \int F(u^n)\frac{\partial v(\xi)}{\partial \xi}d\xi\} \tag{2.7}$$

The test function $v(\xi)$ is chosen to be the Lagrange basis L_p polynomials created using the N LGL points,

$$L_p(\xi) = \prod_{m=0, m\neq p}^{m=N-1} \frac{\xi - \xi_m}{\xi_m - \xi_p} \tag{2.8}$$

where,

$L_p(\xi)$ is the Lagrange basis polynomial with index p.

For each element, the wave equation is transformed to the ξ space as a linear combination of the Lagrange basis polynomials created using the LGL points in that space, i.e.,

$$u^n(\xi) = \sum_{i=0}^{N-1} u_i^n L_i(\xi) \tag{2.9}$$

where,

u_i^n are the coefficients for L_i and N is the total number LGL points in the ξ space.

Substituting equation 2.9 and $v(\xi) = L_p(\xi)$ in the equation 2.7,

$$\sum_{i=0}^{N-1} u_i^{n+1} \int L_p(\xi)L_i(\xi)\frac{dx}{d\xi}d\xi = \sum_{i=0}^{N-1} u_i^{n} \int L_p(\xi)L_i(\xi)\frac{dx}{d\xi}d\xi - \Delta t\{L_p(\xi)F(u^n) - \int F(u^n)\frac{\partial L_p(\xi)}{\partial \xi}d\xi\} \tag{2.10}$$

This is the recurrence relation of interest,

This equation can be written in a different way by introducing A_{pi} and b_p, given by

$$A_{pi} = \int L_p(\xi) L_i(\xi) \frac{dx}{d\xi} d\xi \tag{2.11}$$

$$b_p(u^n) = -\{L_p(\xi)F(u^n) - \int F(u^n) \frac{\partial L_p(\xi)}{\partial \xi} d\xi\} \tag{2.12}$$

On varying p from 0 to $N-1$, N such linear equations for u_i^{n+1}, $i\epsilon 0, 1, \ldots, N-1$ is obtained.

Matrix representation of this system of linear equations is

$$\begin{bmatrix} A_{00} & A_{01} & \ldots & A_{0N-1} \\ A_{10} & A_{11} & \ldots & A_{1N-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{(N-1)0} & A_{(N-1)1} & \ldots & A_{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} u_0^{n+1} - u_0^n \\ u_1^{n+1} - u_1^n \\ \vdots \\ u_{N-1}^{n+1} - u_{N-1}^n \end{bmatrix} = \Delta t \begin{bmatrix} b_0(u^0) \\ b_1(u^1) \\ \vdots \\ b_(u^{N-1}) \end{bmatrix} \tag{2.13}$$

or

$$Au^{n+1} = Au^n + b(u^n) \tag{2.14}$$

$$u^{n+1} = u^n + A^{-1}b(u^n) \tag{2.15}$$

2.2 A matrix

A matrix whose elements A_{pi} are described in equation 2.11 is to be obtained

This integral can be evaluated using the Gauss-Lobatto Quadrature.

2.2.1 Lobatto Quadrature

Lobatto quadrature is a method to find integral of the type,

$$\int_{-1}^{1} f(\xi) d\xi \tag{2.16}$$

The integral is then given by,

$$\int_{-1}^{1} f(\xi)d\xi = \frac{2}{N(N-1)}[f(1)+f(-1)] + \sum_{i=1}^{N-1} w_i f(x_i) \tag{2.17}$$

where, x_i are the solution of L'_{N-1}, the LGL points. w_i are the weights calculated at x_i, it is given by,

$$w_i = \frac{2}{N(N-1)[P'_{n-1}(x_i)]^2} \tag{2.18}$$

As seen in equation 1.3

ξ_i which are the LGL points are also the points to be used in the Lobatto quadrature.

2.11 can be rewritten as,

$$\sum_{k=0}^{N-1} w_k L_p(\xi_k) L_i(\xi_k) \frac{dx}{d\xi}|_{\xi_k} \tag{2.19}$$

2.3 B vector

To obtain the b vector, Its elements b_p described in 2.12 need to be calculated

The final b vector would be,

$$b = \begin{bmatrix} b_0(u^0) \\ b_1(u^1) \\ \vdots \\ b_(u^{N-1}) \end{bmatrix}_{N\times 1} \tag{2.20}$$

b_p consists of the terms, $\Delta t L_p(\xi)F(u^n)|_{-1}^{1}$ and $\Delta t \int_{-1}^{1} F(u^n)\frac{\partial L_p(\xi)}{\partial \xi}d\xi$.

2.3.1 Lax-Friedrichs flux

By varying p from 0 to $N-1$ in the first term of the R.H.S of 2.12(i.e. $F(u^n)L_p(\xi)|_{-1}^{1}$), Surface term is obtained.

The surface term is used because of the discontinuities in element boundaries. Since the value of the flux would be different for adjacent elements, There are two values for $F(u)$ at the element boundaries. To resolve this ambiguity of flux, Lax-Friedrichs flux is introduced.

The Lax-Friedrichs flux f_M for an element boundary between element M and $M+1$ at a timestep n is given by,

$$f_M = \frac{F(u^n_{0,M+1}) + F(u^n_{N-1,M})}{2} - \frac{\Delta x}{2\Delta t}(u^n_{0,M+1} - u^n_{N-1,M}) \tag{2.21}$$

In the above equation, $u^n_{0,M+1}$ is the amplitude at the first (leftmost) LGL point of the element with index $M+1$, Similarly $u^n_{N-1,M}$ is the $(N-1)^{th}$ (rightmost) LGL point of the element with index M, Δx is the size of the element M. This is the flux for element boundaries, It is a number.

Shown below is a diagram demonstrating the two different wave amplitudes across an element boundary.

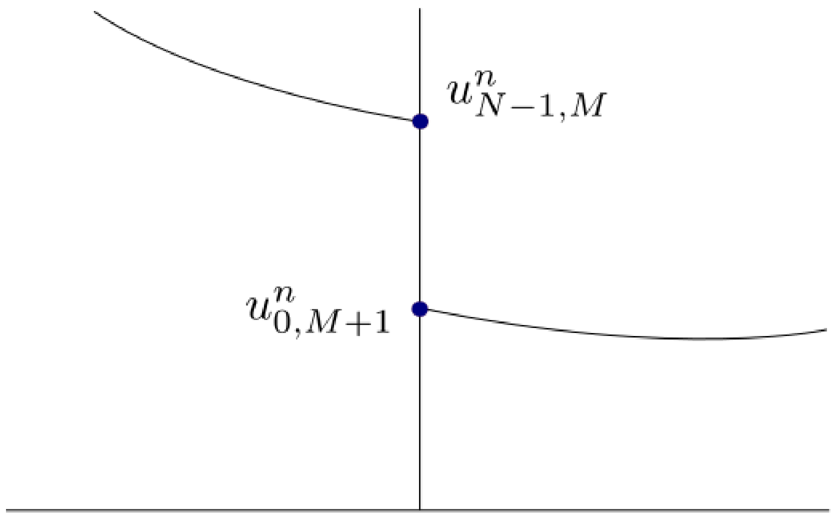

FIGURE 2.1: The discontinuity in the amplitudes across an element boundary

2.3.2 Surface term

The Surface term is obtained by varying p in $F(u^n)L_p(\xi)|^1_{-1}$ from 0 to $N-1$ for an element with an index M

$$F(u^n)L_p(\xi)|^1_{-1} = L_p(1)f_M - L_p(-1)f_{M-1} \tag{2.22}$$

The equation 2.22 has $L_p(1)$ and $L_p(-1)$. These are elements of 1D arrays of shape $N \times 1$. Since the index p varies from 0 to $N-1$ multiplying them with the flux at the element boundaries (a number), A vector of shape $N \times 1$ is obtained.

2.3.3 Volume Integral flux

The third term of the R.H.S of 2.12 is $\int_{-1}^{1} F(u^n) \frac{\partial L_p(\xi)}{\partial \xi} d\xi$. This integral is to be evaluated using Gauss-Lobatto rules.

$$\int_{-1}^{1} F(u^n) \frac{\partial L_p(\xi)}{\partial \xi} d\xi = \Sigma_{k=0}^{N-1} w_k \frac{\partial L_p(\xi)}{\partial \xi}|_{\xi_k} F(u^n) \tag{2.23}$$

By varying p from 0 to $N-1$ along the rows and k from 0 to $N-1$ across the columns, $w_k \frac{\partial L_p(\xi)}{\partial \xi}|_{\xi_k}$ becomes a 2D array of shape $N \times N$

$$w_k \frac{\partial L_p(\xi)}{\partial \xi}|_{\xi_k} = \begin{bmatrix} w_0 \frac{\partial L_0(\xi)}{\partial \xi}|_{\xi_0} & w_1 \frac{\partial L_0(\xi)}{\partial \xi}|_{\xi_1} & \cdots & w_{N-1} \frac{\partial L_0(\xi)}{\partial \xi}|_{\xi_{N-1}} \\ w_0 \frac{\partial L_1(\xi)}{\partial \xi}|_{\xi_0} & w_1 \frac{\partial L_1(\xi)}{\partial \xi}|_{\xi_1} & \cdots & w_{N-1} \frac{\partial L_1(\xi)}{\partial \xi}|_{\xi_{N-1}} \\ \vdots & \vdots & \ddots & \vdots \\ w_0 \frac{\partial L_{N-1}(\xi)}{\partial \xi}|_{\xi_0} & w_1 \frac{\partial L_{N-1}(\xi)}{\partial \xi}|_{\xi_1} & \cdots & w_{N-1} \frac{\partial L_{N-1}(\xi)}{\partial \xi}|_{\xi_{N-1}} \end{bmatrix}_{N \times N} \tag{2.24}$$

While $F(u^n)$ is

$$F(u^n) = cu^n = c \begin{bmatrix} u_0^n \\ u_1^n \\ \vdots \\ u_{N-1}^n \end{bmatrix}_{N \times 1} \tag{2.25}$$

Matrix multiplication of 2.24 with 4.7 gives a vector of shape $N \times 1$

This is to be incorporated into the solver[1]

2.4 Results

To check the implementation of the recurrence relation for a 1D wave advection,

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial F(u)}{\partial x} = 0 \tag{2.26}$$

The recurrence relation 2.15 is evolved using a higher order Runge-Kutta RK6 method.

$u(x,t)$ is chosen to be $sin(2\pi(x-ct))$, the speed of the wave is taken to be 1

The domain is split into 10 elements and each element is further split using 8 LGL points.

at time, $t=0$, the plot of u vs x is

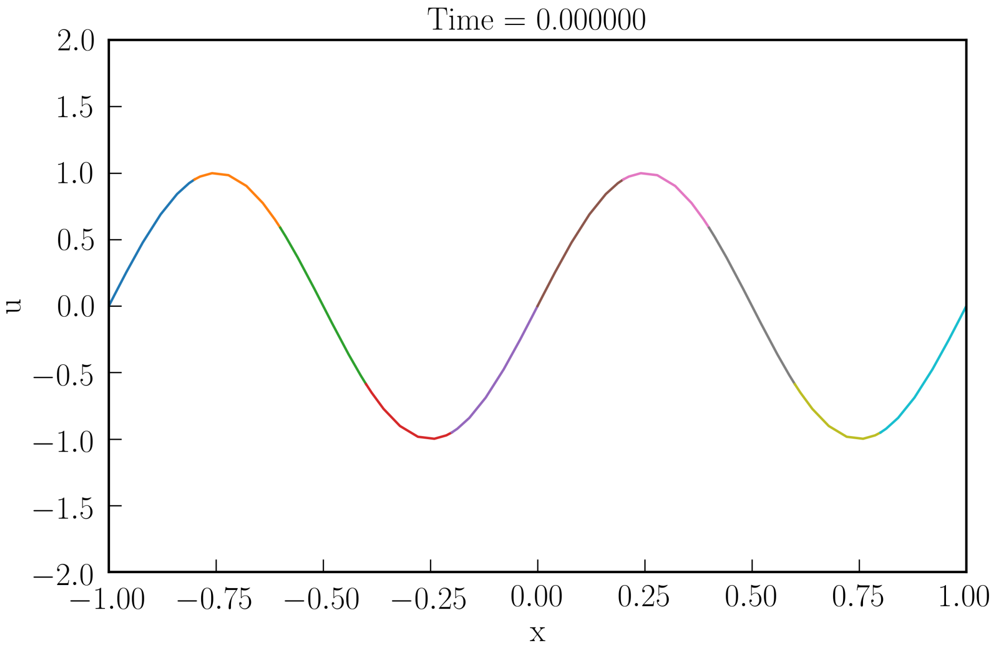

FIGURE 2.2: The waveform at the zeroeth time step

This wave is advected 10 times and the L1 norm of the error is computed.

The L1 norm of the error after approximately 10 advections is $1.240005129210433e-07$.

Shown below is a plot of L1 norm of error after one full advection vs the number of LGL points taken.

As seen, the L1 norm of error falls as $N_{LGL}^{-N_{LGL}}$ till 1e-8. This pattern is following expected theoretical decay of L1 norm of error with LGL points.

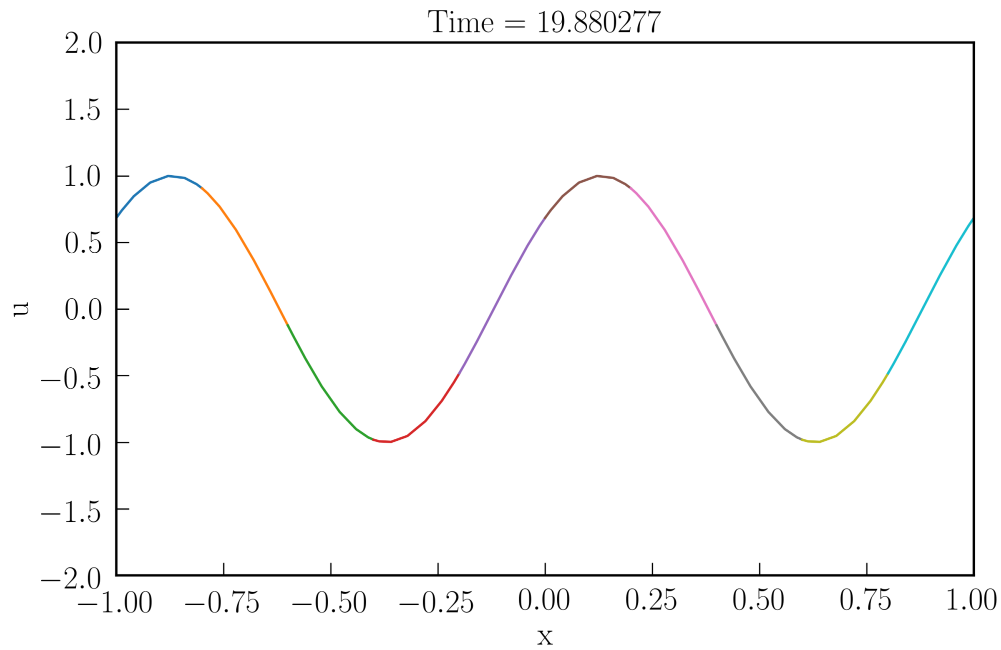

FIGURE 2.3: The waveform after 10 full advections

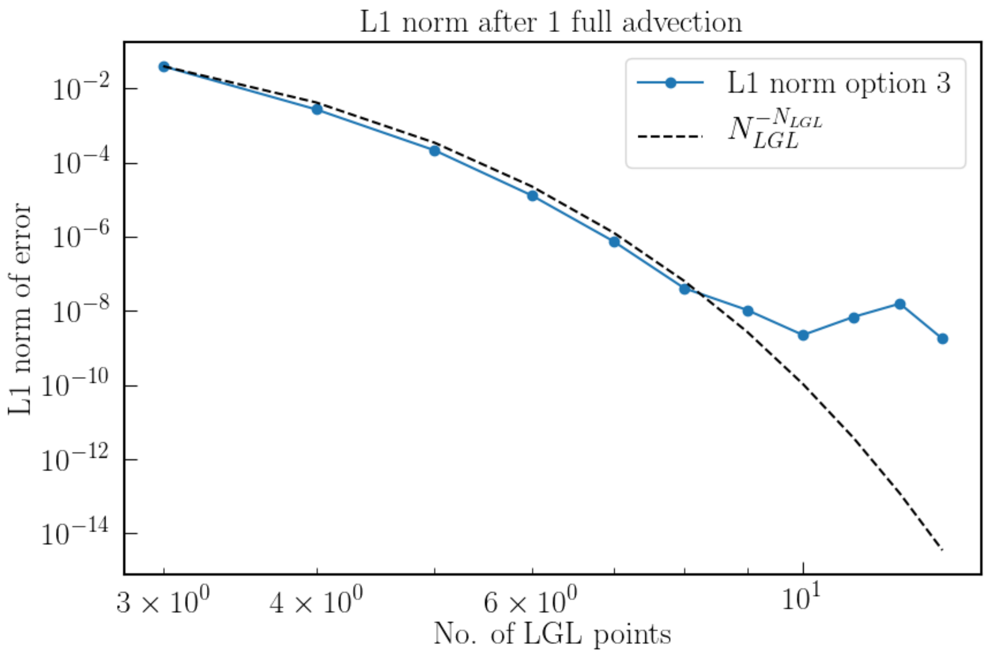

FIGURE 2.4: Convergence of the L1 norm of error after one full 1D advection

Chapter 3

2D Advection

3.1 Weak form of the 2D Advection equation

The 2D wave advection equation is

$$\frac{\partial u}{\partial t} + \bar{\nabla}.\bar{F}(u) = 0 \tag{3.1}$$

This is solved using a similar approach that was used to solve the 1D advection equation, The strong form of the advection equation is converted to the weak form by multiplying a test function test function $v(x, y)$ and integrating with respect to a differential area $d\bar{a} = d\bar{x} \times d\bar{y}$

$$\int v(x, y)\frac{\partial u}{\partial t} d\bar{a} + \int v(\bar{\nabla}.\bar{F})d\bar{a} = 0 \tag{3.2}$$

Using the vector calculus identity [3]

$$\bar{\nabla}.(\varphi\bar{F}) = (\bar{\nabla}\varphi).\bar{F} + \varphi(\bar{\nabla}.\bar{F}) \tag{3.3}$$

The second term of L.H.S of Equation 3.2 becomes

$$\int v(\bar{\nabla}.\bar{F})d\bar{a} = \int \left(\bar{\nabla}.(v\bar{F}) - (\bar{\nabla}v).\bar{F}\right) d\bar{a} \tag{3.4}$$

The weak form of the equation now becomes

$$\int v\frac{\partial u}{\partial t} d\bar{a} + \int \bar{\nabla}.(v\bar{F})d\bar{a} - \int (\bar{\nabla}v).\bar{F}d\bar{a} = 0 \tag{3.5}$$

Using the divergence theorem[3]

$$\int \bar{\nabla}.\bar{P}d\bar{a} = \oint \bar{P}.\hat{n}ds \tag{3.6}$$

The second term of equation 3.5 can be written as a surface integral,

$$\bar{\nabla}.(\bar{F}v) = \oint \bar{F}v.\hat{n}ds \tag{3.7}$$

Using equation 3.7 to simplify the weak for of the 2D advection equation

$$\int v\frac{\partial u}{\partial t}d\bar{a} + \oint \bar{F}v.\hat{n}ds - \int (\bar{\nabla}v).\bar{F}d\bar{a} \tag{3.8}$$

Let the wave function u at the n^{th} timestep be denoted by $u^n(x, y)$

Equation 3.8 can be rewritten as

$$\frac{\int vu^{n+1}(x,y)d\bar{a} - \int vu^n(x,y)d\bar{a}}{\Delta t} + \oint \bar{F}v.\hat{n}ds - \int (\bar{\nabla}v).\bar{F}d\bar{a} = 0 \tag{3.9}$$

Simplifying this expression

$$\left(\frac{u^{n+1} - u^n}{\Delta t}\right)\int vd\bar{a} = \int (\bar{\nabla}v).\bar{F}d\bar{a} - \oint \bar{F}v.\hat{n}ds \tag{3.10}$$

3.1.1 Jacobian

The Jacobian matrix is given by

$$J = \begin{bmatrix} \frac{d\xi}{dx} & \frac{d\xi}{dy} \\ \frac{d\eta}{dx} & \frac{d\eta}{dy} \end{bmatrix} \tag{3.11}$$

For a general case, the determinant of the Jacobian matrix is

$$det[J] = \frac{d\xi}{dx}\frac{d\eta}{dy} - \frac{d\eta}{dx}\frac{d\xi}{dy} \tag{3.12}$$

The area $d\bar{a}$ under this transformation would become

$$d\xi d\eta = det[J]dxdy \tag{3.13}$$

3.1.2 Affine scaled transformation

The transformation between the (x, y) and (ξ, η) spaces is chosen to be affine scaled, meaning that the element would be a scaled down version of the domain. Under this special transformation the determinant of the Jacobian matrix reduces to

$$det[J] = \frac{d\xi}{dx}\frac{d\eta}{dy} \tag{3.14}$$

Since,

$$\frac{d\xi}{dy} = 0 \tag{3.15}$$

$$\frac{d\eta}{dx} = 0 \tag{3.16}$$

Also since there isn't any distortion of shape during transformation, the unit vectors which span the domain and the element should be the same

$$\hat{x} = \hat{\xi} \tag{3.17}$$

$$\hat{y} = \hat{\eta} \tag{3.18}$$

3.2 Advection equation

The test function $v(\xi, \eta)$ can be chosen as

$$v(\xi, \eta) = L_p(\xi)L_q(\eta) \tag{3.19}$$

Where $L_p(\xi)$ and $L_q(\eta)$ are Lagrange basis polynomials (discussed in 2.8), obtained using N_{LGL} LGL points with indices p and q varying from 0 to $N_{LGL} - 1$

The domain is split into E equal elements

The value of the wave function $u(x, y)$ in an element e at a timestep n can be represented in (ξ, η) space using the transformations discussed in 1.5 and 1.6 as $u_e^n(\xi, \eta)$, This $u_e^n(\xi, \eta)$ can be interpolated using Lagrange basis polynomials in the (ξ, η) space as

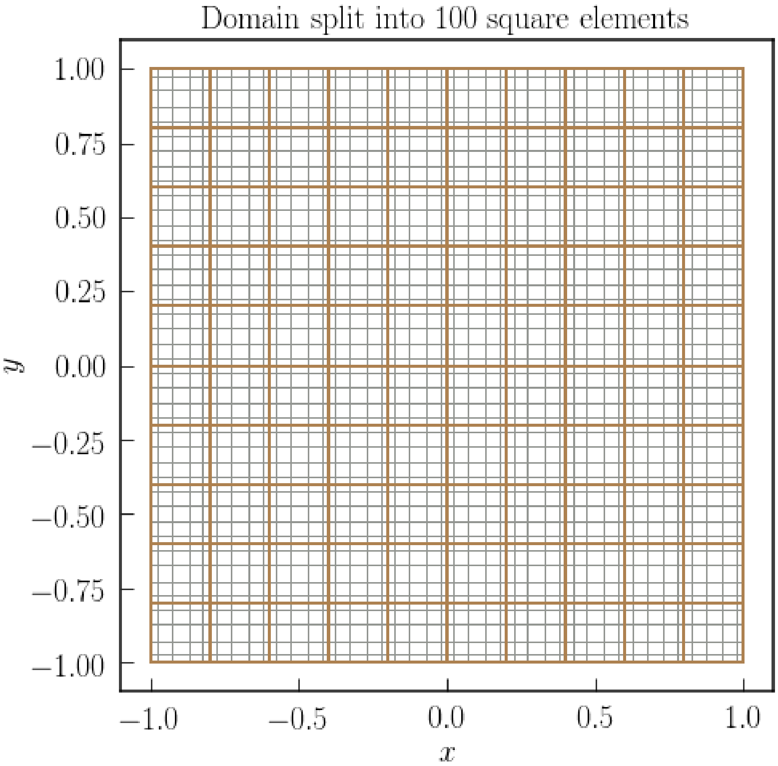

FIGURE 3.1: The domain over which 2D advection occurs

$$u_e^n(\xi,\eta) = u_{eij}^n L_i(\xi) L_j(\eta) \tag{3.20}$$

Where u_{eij}^n is the value of u^n corresponding to $L_i(\xi)$ ($i^{th}\xi$ mapped LGL node) and $L_j(\eta)$ (j^{th} mapped η LGL node)

The weak form of the wave equation now becomes

$$\left(\frac{u_{eij}^{n+1} - u_{eij}^n}{\Delta t}\right) \int L_p(\xi)L_q(\eta)L_i(\xi)L_j(\eta)\frac{d\bar{\xi}d\bar{\eta}}{det[J]} = \int (\bar{\nabla}v).\bar{F}\frac{d\bar{\xi}d\bar{\eta}}{det[J]} - \oint \bar{F}v.\hat{n}ds \tag{3.21}$$

3.2.1 A Matrix

The integral $\int L_p(\xi)L_q(\eta)L_i(\xi)L_j(\eta)\frac{d\bar{\xi}d\bar{\eta}}{det[J]}$ on the R.H.S of equation 3.21 is the A matrix for 2D advection. The indices of this matrix p, q, i and j all vary from 0 to $N_{LGL} - 1$

This matrix needs to be calculated only once as it is time independent, The double integration is carried out using the quadrature method

$$A_{pqij} = \int L_p(\xi)L_q(\eta)L_i(\xi)L_j(\eta)\frac{d\bar{\xi}d\bar{\eta}}{det[J]} = w_i w_j L_p(\xi_i)L_q(\eta_j)L_i(\xi_i)L_j(\eta_j) \tag{3.22}$$

where w_i and w_j are quadrature weights corresponding to the quadrature points ξ_i and η_j

The procedure implemented to calculate the double integral is described in the appendix

3.2.2 Volume Integral

The first term of the R.H.S of the equation 3.21 equation is similar to the volume integral term seen in 2.3.3

$$\int (\bar{\nabla} L_p(\xi) L_q(\eta)).\bar{F}\frac{d\bar{\eta}d\bar{\xi}}{\det[J]} \tag{3.23}$$

The gradient, $\bar{\nabla}$ in the (ξ, η) space is

$$\bar{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} \tag{3.24}$$

$$= \hat{\xi}\left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right) + \hat{\eta}\left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}\right) \tag{3.25}$$

Since scaled transformations are used, 3.15 and 3.16 can be applied to simplify the gradient

$$\bar{\nabla} = \hat{\xi}\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \hat{\eta}\frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} \tag{3.26}$$

This form of the gradient is used to evaluate the volume integral term, Equation 3.23 becomes

$$\int \left(L_q(\eta)\frac{\partial L_p(\xi)}{\partial \xi}\frac{\partial \xi}{\partial x}\hat{\xi} + L_p(\xi)\frac{\partial L_q(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y}\hat{\eta}\right).\bar{F}\frac{d\bar{\eta}d\bar{\xi}}{\det[J]} \tag{3.27}$$

The flux operator $\bar{F}(u)$ can be written as

$$\bar{F}(u) = (c_x u(\xi, \eta))\hat{\xi} + (c_y u(\xi, \eta))\hat{\eta} \tag{3.28}$$

Using this in 3.27

$$\int \left(L_q(\eta)\frac{\partial L_p(\xi)}{\partial \xi}\frac{\partial \xi}{\partial x}\hat{\xi} + L_p(\xi)\frac{\partial L_q(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y}\hat{\eta}\right).(c_x u(\xi, \eta))\hat{\xi} + (c_y u(\xi, \eta)\hat{\eta})\frac{d\bar{\eta}d\bar{\xi}}{\det[J]} \tag{3.29}$$

becomes

$$\int L_q(\eta)\frac{\partial L_p(\xi)}{\partial \xi}\frac{\partial \xi}{\partial x}c_x u(\xi,\eta)\frac{d\bar{\eta}d\bar{\xi}}{\det[J]} + \int L_p(\xi)\frac{\partial L_q(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y}c_y u(\xi,\eta)\frac{d\bar{\eta}d\bar{\xi}}{\det[J]} \tag{3.30}$$

Which is the volume integral term simplified, This term needs to be evaluated numerically at each time step to evolve the system. This is done using integration schemes discussed later in the appendixA.

3.2.3 Surface term

The second term of the R.H.S of 3.21 is the surface term for 2D advection.

$$\oint \bar{F}v.nds \tag{3.31}$$

The surface term can be split into 4 line integrals over the (x, y) domain, Since the elements are square in shape.

$$\begin{aligned}\oint \bar{F}v.nds = &\int_{x_{min}}^{x_{max}} (F(x,\bar{y}_{max})v).\hat{y}dx + \int_{y_{min}}^{y_{max}} (F(\bar{x_m}ax,y)v).\hat{x}dy \\ &- \int_{x_{min}}^{x_{max}} (F(x,\bar{y}_{min})v).\hat{y}dx - \int_{y_{min}}^{y_{max}} (F(\bar{x}_{min},y)v).\hat{x}dy\end{aligned} \tag{3.32}$$

The differential measures dx and dy transform as

$$dx = \frac{dx}{d\xi}d\xi + \frac{dx}{d\eta}d\eta dy = \frac{dy}{d\xi}d\xi + \frac{dy}{d\eta}d\eta \tag{3.33}$$

Using the property that the cross terms in the jacobian matrix, this simplifies to

$$dx = \frac{dx}{d\xi}d\xi \tag{3.34}$$

$$dy = \frac{dy}{d\eta}d\eta \tag{3.35}$$

The surface term now using this integral measure becomes

$$\oint \bar{F}v.nds = \int_{-1}^{1} L_p(1)L_q(\eta)\left(\bar{F}(1,\eta).\hat{\xi}\right)\frac{\partial y}{\partial \eta}d\eta - \int_{-1}^{1} L_p(-1)L_q(\eta)\left(\bar{F}(-1,\eta).(\hat{\xi})\right)\frac{\partial y}{\partial \eta}d\eta + \int_{-1}^{1} L_p(\xi)L_q(1)\left(\bar{F}(\xi,1).(\hat{\eta})\right)\frac{\partial x}{\partial \xi}d\xi - \int_{-1}^{1} L_p(\xi)L_q(-1)\left(\bar{F}(\xi,-1).\hat{\eta}\right)\frac{\partial x}{\partial \xi}d\xi \tag{3.36}$$

This surface term is to be numerically calculated at each timestep and used in the recurrence relation for the 2D advection equation

3.3 Recurrence relation

Taking the difference between the volume integral term to be the b vector for 2D advection

$$b_{ij}^e = \int (\bar{\nabla}v).\bar{F}\frac{d\bar{\xi}d\bar{\eta}}{det[J]} - \oint \bar{F}v.\hat{n}ds \tag{3.37}$$

The weak form of the wave equation reduces to

$$A\frac{u_{eij}^{n+1} - u_{eij}^{n}}{\Delta t} = b_{ij}^e \tag{3.38}$$

Premultiplying the inverse of A matrix and Δt on both sides of the equation

$$u_{eij}^{n+1} - u_{eij}^{n} = A^{-1}\Delta t(b) \tag{3.39}$$

Which is the recurrence relation used to evolve this system. When u^n is numerically known, u^{n+1} can be obtained.

3.4 Results

Advecting a Gaussian wave across the square domain shown in with $c_x = 1.0$ and $c_y = 0$ using periodic boundaries across the domain boundary gives the following result

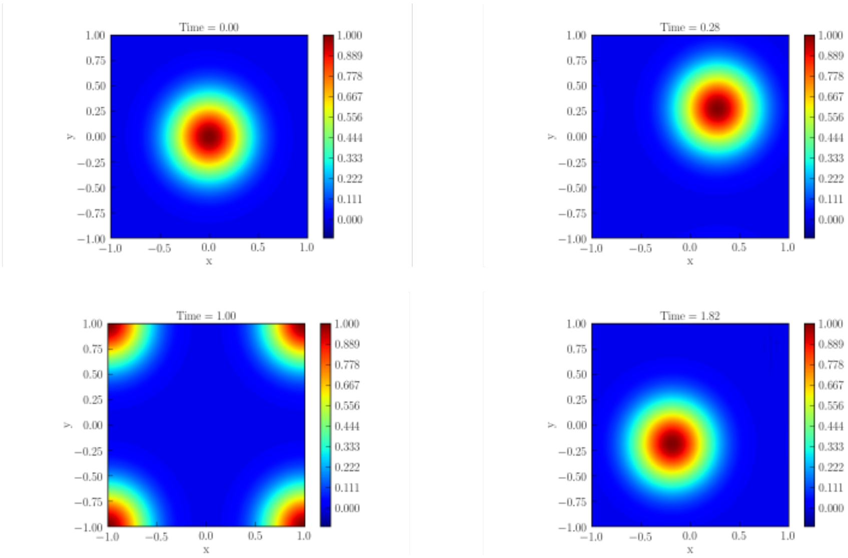

FIGURE 3.2: Gaussian wave advecting across the domain

The L1 norm of error after one full advection is plotted against the number of LGL points taken and the error falls of as N^{-N} as was seen for 1D advection

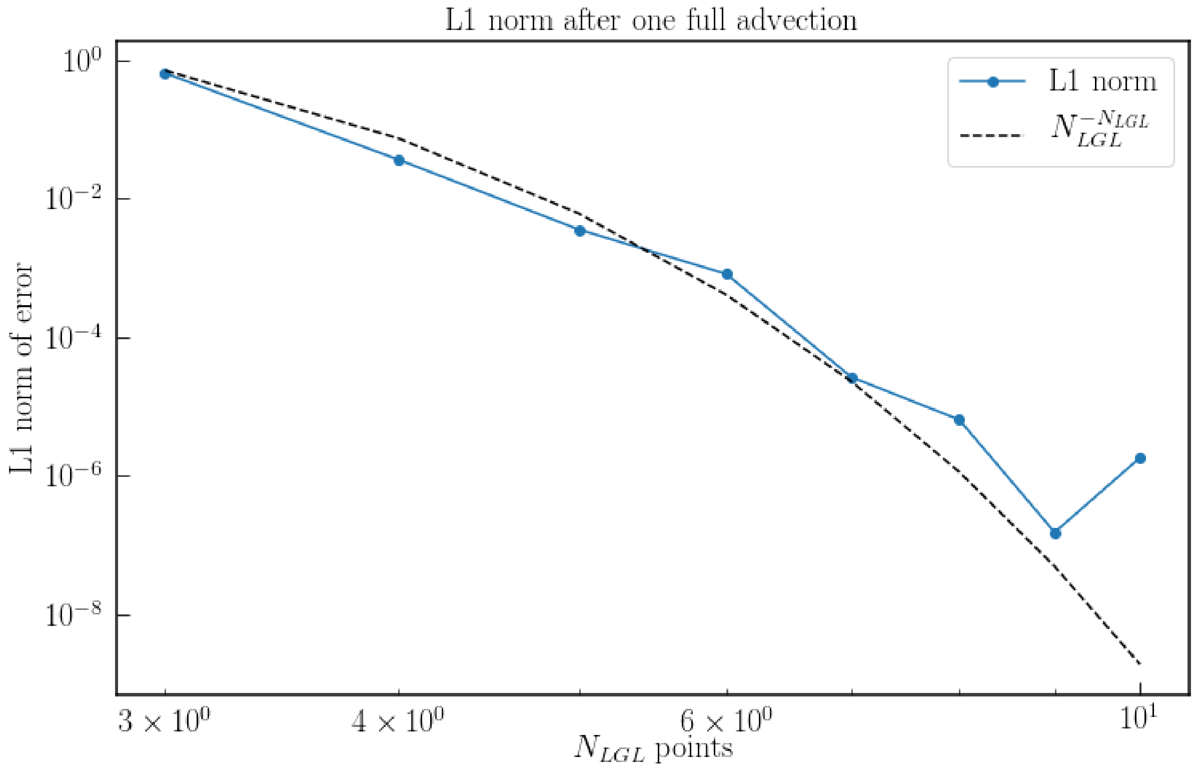

FIGURE 3.3: L1 norm of error v/s N_{LGL} for 2D advection

Chapter 4

Formulation in arbitrary coordinates

In the previous chapter, the 2D advection equation was solved in the by transforming the equation to the (ξ, η) space by means of an isoparametric mapping and the recurrence relation was obtained to evolve the system numerically.

However the elements into which the domain was split was taken to be square to simplify the calculation of the volume integral term and for easy numerical evaluation of the surface term

This chapter deals with solving the same advection equation using a different approach without making any assumptions for the element shape.

4.1 Advection Equation

The weak form of the 2D advection equation obtained from 3.10 is,

$$\left(\frac{u^{n+1} - u^n}{\Delta t}\right) \int v d\bar{a} = \int (\bar{\nabla} v).\bar{F} d\bar{a} - \oint \bar{F} v.\hat{n} ds \tag{4.1}$$

This is to be solved in the (ξ, η) space using the integral measure $\sqrt{g} d\xi d\eta$

Where $\sqrt{g}$ is the root of the determinant of the metric tensor g_{ab}

The weak for of the 2D advection using this integral measure becomes

$$\left(\frac{u_{eij}^{n+1} - u_{eij}^n}{\Delta t}\right) \int v(\xi, \eta) L_i(\xi) L_j(\eta) \sqrt{g} d\xi d\eta = \int (\bar{\nabla} v).\bar{F} \sqrt{g} d\xi d\eta - \oint \bar{F} v.\hat{n} \sqrt{g} ds \tag{4.2}$$

The metric tensor for the transformation between the (x, y) space and the (ξ, η) space is given by

$$g_{ab} = \begin{bmatrix} \frac{dx}{d\xi}^2 + \frac{dy}{d\xi}^2 & \frac{dx}{d\xi}\frac{dx}{d\eta} + \frac{dy}{d\xi}\frac{dy}{d\eta} \\ \frac{dx}{d\xi}\frac{dx}{d\eta} + \frac{dy}{d\xi}\frac{dy}{d\eta} & \frac{dx}{d\eta}^2 + \frac{dy}{d\eta}^2 \end{bmatrix} \tag{4.3}$$

$\sqrt{g}$ is the square root of the determinant of this g_{ab} matrix, This varies for each element.

The test function v can be taken as $v = L_p(\xi)L_q(\eta)$ with the indices p and q varying from 0 to $N_{LGL} - 1$

4.1.1 A matrix

The A matrix, which is the the integral on the R.H.S of 4.2

$$A^e_{pqij} = \int L_p(\xi)L_q(\eta)L_i(\xi)L_j(\eta)\sqrt{g^e}d\xi d\eta \tag{4.4}$$

with the indices p, q, i and j varying from 0 to $N_{LGL} - 1$ and e varying from 0 to E where E is the total number of elements which make up the domain.

This A matrix would vary for each element unlike the previous cases discussed. The integral is evaluated using the scheme to solve multivariable polynomials using the quadrature rule

4.1.2 Volume Integral

The first term of R.H.S of equation 4.2 is the volume integral term

The tangent vectors [2] a $\bar{e}_x$ and $\bar{e}_y$ in the (x, y) space can be represented in terms of $\bar{e}_\xi$ and $\bar{e}_\eta$

$$\bar{e}_x = \frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} \tag{4.5}$$

Similarly

$$\bar{e}_y = \frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} \tag{4.6}$$

$\bar{F}$ can be written as

$$\bar{F} = F_x\bar{e_x} + F_y\bar{e_y} \tag{4.7}$$

Representing $\bar{e}_x$ and $\bar{e}_y$ in terms of $\bar{e}_\xi$ and $\bar{e}_\eta$,

$$\bar{F} = F_x \left(\frac{\partial \xi}{\partial x} \bar{e}_\xi + \frac{\partial \eta}{\partial x} \bar{e}_\eta \right) + F_y \left(\frac{\partial \xi}{\partial y} \bar{e}_\xi + \frac{\partial \eta}{\partial y} \bar{e}_\eta \right) \tag{4.8}$$

Rearranging the terms to make it in the form of $F_\alpha \bar{e}^\alpha$,

$$\bar{F} = \bar{e}_\xi \left(F_x \frac{\partial \xi}{\partial x} + F_y \frac{\partial \xi}{\partial y} \right) + \bar{e}_\eta \left(F_x \frac{\partial \eta}{\partial x} + F_y \frac{\partial \eta}{\partial y} \right) \tag{4.9}$$

Now the flux can be written as

$$\bar{F} = F_\xi \bar{e}_\xi + F_\eta \bar{e}_\eta \tag{4.10}$$

The gradient operator $\bar{\nabla}$ in the (ξ, η) space can be would become

$$\bar{\nabla} = \bar{e}^\xi \frac{\partial}{\partial \xi} + \bar{e}^\eta \frac{\partial}{\partial \eta} \tag{4.11}$$

Where $\bar{e}^\nu$ is the 1 form[2] with the property

$$\bar{e}^\nu . \bar{e}_\mu = \delta^\nu_\mu \tag{4.12}$$

The volume integral term now becomes

$$\int (\bar{\nabla} v).\bar{F} \sqrt{g} d\xi d\eta = \int \left((\bar{e}^\xi \frac{\partial}{\partial \xi} + \bar{e}^\eta \frac{\partial}{\partial \eta})(v) \right) .(F_\xi \bar{e_\xi} + F_\eta \bar{e_\eta}) \sqrt{g} (d\bar{\xi} \times d\bar{\eta}) \tag{4.13}$$

Expanding the integrand and simplifying it

$$\int (\bar{\nabla} v).\bar{F} \sqrt{g} d\xi d\eta = \int \left((\bar{e^\xi}.\bar{e_\xi}) \frac{\partial v}{\partial \xi} F_\xi + (\bar{e^\xi}.\bar{e_\eta}) \frac{\partial v}{\partial \eta} F_\eta + (\bar{e^\eta}.\bar{e_\xi}) \frac{\partial v}{\partial \xi} F_\xi + (\bar{e^\eta}.\bar{e_\eta}) \frac{\partial v}{\partial \eta} F_\eta \right) (\hat{n}(\sqrt{g} d\xi d\eta)) \tag{4.14}$$

Using 4.12, 4.14 becomes

$$\int (\bar{\nabla} v).\bar{F}\sqrt{g}d\xi d\eta = \int \left(\frac{\partial v}{\partial \xi}F_{\xi} + \frac{\partial v}{\partial \eta}F_{\eta}\right)\sqrt{g}d\xi d\eta \tag{4.15}$$

4.15 becomes

The volume integral term now becomes

$$\int \left(L_q(\eta)\frac{\partial L_p(\xi)}{\partial \xi}F^{\xi} + L_p(\xi)\frac{\partial L_p(\eta)}{\partial \eta}F^{\eta}\right)\sqrt{g}d\xi d\eta \tag{4.16}$$

This is the volume integral term for the (ξ, η) formulation, This is to be numerically computed at every timestep and is used in the recurrence relation

4.1.3 Surface Term

The surface term, which is the second term in the R.H.S of equation 4.2 is given by

$$\oint (v\bar{F}).\hat{n}\sqrt{g}ds \tag{4.17}$$

Substituting the value of the test function and expressing the flux $\bar{F}$ using the equation 4.10

$$\oint (v\bar{F}).\hat{n}\sqrt{g}ds = \oint (L_p(\xi)L_q(\eta))(F^{\xi}\bar{e_{\xi}} + F^{\eta}\bar{e_{\eta}})).\hat{n}\sqrt{g}ds \tag{4.18}$$

The domain (ξ, η) is a square, with both ξ and η varying from $(-1, 1)$.

The surface integral term can be split into 4 integrals, across each edge of the (ξ, η) domain.

The four edges of the domain are described by $\eta\epsilon(-1, 1)$ with $\xi = \pm 1$ and $\xi\epsilon(-1, 1)$ with $\eta = \pm 1$

$$
\begin{aligned}
\oint (v\bar{F}).\hat{n}\sqrt{g}ds = &\int_{-1}^{1}((L_p(1)L_q(\eta))(F^{\xi}(1,\eta))\sqrt{g(1,\eta)}d\eta \\
&+\int_{-1}^{1}((L_p(\xi)L_q(1))(F^{\eta}(1,\xi))\sqrt{g(\xi,1)}d\xi \\
&-\int_{-1}^{1}((L_p(-1)L_q(\eta))(F^{\xi}(-1,\eta)))\sqrt{g(-1,\eta)}d\eta \\
&-\int_{-1}^{1}((L_p(\xi)L_q(-1))(F^{\eta}(\xi,-1)))\sqrt{g(\xi,-1)}d\xi
\end{aligned}
\tag{4.19}
$$

This is the surface term which is also to be computed at each timestep and used in the recurrence relation

4.2 Recurrence relation

The recurrence relation for this formulation is obtained in a similar way to the one in equation 3.39

The b vector is obtained by taking the difference between the volume integral term and the surface term

$$
A^e_{pqij}\frac{u^{n+1}_{ij}-u^n_{ij}}{\Delta t} = b^e_i j \tag{4.20}
$$

If u^n_{ij} is known, u^{n+1}_{ij} can be obtained numerically.

Appendix A

Multivariable polynomial integration

A.1 Multivariable polynomials

In many cases, especially for the 2D advection solution, There is a need to evaluate a double integral of the form

$$\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta)d\xi d\eta \tag{A.1}$$

This is to be evaluated using the quadrature rule

$$\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta)d\xi d\eta = w_i w_j F(\xi_i,\eta_j) \tag{A.2}$$

The involves numerical calculation of $F(\xi_i,\eta_j)$, this is done by storing the coefficients of the polynomial $F(\xi,\eta) = \Sigma_{a=0,b=0}^{a=m,b=n} c^{ab}\xi^a\eta^b$ is stored in a 2D array as follows

$$c^{ab} = \begin{bmatrix} c^{mn} & c^{m-1n} & \cdots & c^{0n} \\ c^{mn-1} & c^{m-1n-1} & \cdots & c^{0n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c^{m0} & c^{m-10} & \cdots & c^{00} \end{bmatrix} \tag{A.3}$$

Using this array, the value of the function can be calculated at any arbitrary (ξ,η)

A.2 Lagrange interpolation

In a few cases, the value of a particular function $F(\xi, \eta)$ is known at the mapped LGL points u^e_{ij}, using this information, the analytical form of the function can be recovered by interpolating the function using the Lagrange basis polynomials

$$F(\xi, \eta) = F_{ij} L_i(\xi) L_j(\eta) \tag{A.4}$$

Using this analytical form of the function, the integral can be calculated for a higher number of quadrature points than was previously possible. This gives more accurate results

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