Symplectic singularities reading group

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We follow [1] sections 1.1.,1.2.,1.4. very closely.

1 Symplectic Geometry

By a manifold M, we mean a complex algebraic manifold (smooth variety over $\mathbb{C}$). A holomorphic 2-form ω on M is *symplectic* if

- $d\omega = 0$,
- The map $\widetilde{\omega}_m : T_m M \to T_m^* M$ given by $X \mapsto \omega(X, -)$ is an isomorphism for all m.

In this case we say (M, ω) is a sympelctic manifold.

Remark 1.1. *The above conditions are actually equivalent to that* $\omega = \sum_{i=1}^n p_i \wedge q_i$ *in a local chart (this is the Darboux theorem). Note that unlike Riemannian geometry, there is no local invariant like curvature.*

We have implicitly used that $\mathbb{C}^{2n}$ with coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ and

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$

is a symplectic vector space, hence a symplectic manifold.

Proposition 1.2. *Let* M *be a manifold, then* T^*M *has a canonical symplectic form* ω*.*

Proof. Let $\pi : T^*M \to M$ be the bundle projection map and $\pi_* : T(T^*M) \to TM$ its differential. Then at any point $(m, \alpha) \in T^*M$,

$$\pi_{*(m,\alpha)} : T_{(m,\alpha)}(T^*M) \to T_m M$$

is a linear map. Define $\lambda \in \Omega^1(T^*M)$ by

$$\lambda_{(m,\alpha)}(X) = \alpha(\pi_{*(m,\alpha)} X)).$$

Let $\omega = d\lambda$. Exercise: check that ω is non-degenerate.

Q.E.D.

2 Poisson Structures

In algebraic geometry, we know $\mathcal{O}(M)$ is more important than M. So we would like to know what structure does $\mathcal{O}(M)$ get if (M,ω) is symplectic. If $f,g \in \mathcal{O}(M)$, then $df, dg \in \Omega^1(M)$. Since $\widetilde{\omega}_m$ is an isomorphism, there exist vector fields X_f, X_g, called *Hamiltonian vector fields* for f,g, such that $\omega(-,X_f) = df, \omega(-,X_g) = dg$ (i.e. $-df = \iota_{X_f}\omega$). These vector fields have the property that

$$\begin{aligned} L_{X_f}\omega &= (di_{X_f} + i_{X_f}d)\omega \\ &= d(\omega(X_f,-)) \\ &= d(-df) \\ &= 0. \end{aligned}$$

If a vector field X satisfies $L_X\omega = 0$, i.e. if flow along it preserves ω, then we say that X is symplectic. So we have a well-defined map $\mathcal{O}(M) \to \{\text{symplectic vector fields on } M\}$. Note that if we have two symplectic vector fields X, Y then $[X,Y]$ is symplectic as

$$L_{[X,Y]}\omega = (L_X L_Y - L_Y L_X)\omega = 0 + 0.$$

We may also define

$$\{f,g\} = \omega(X_f, X_g) = dg(X_f) = -df(X_g) = -X_g(f) = X_f(g) \in \mathcal{O}(M).$$

Called the *Poisson bracket* of f and g. We want to prove that the assignment $f \mapsto X_f$ intertwines the Poisson and Lie brackets, i.e. that

$$X_{\{f,g\}} = [X_f, X_g].$$

Recall that for any two vector fields Y, Z on M, we have

$$[L_Y, i_Z] = i_{[Y,Z]}$$

and for any vector fields Y on M, we have

$$\begin{aligned} X(\omega(Y,Z)) &= L_X(\omega(Y,Z)) \\ &= L_X(i_Z(i_Y(\omega))) \\ &= (i_Z L_X + i_{[X,Z]})(i_Y(\omega)) \\ &= i_Z L_X(i_Y(\omega)) + \omega(Y,[X,Z]) \\ &= i_Z((i_Y L_X + i_{[X,Y]})(\omega)) + \omega(Y,[X,Z]) \\ &= i_Z(i_Y(L_X(\omega))) + i_Z(i_{[X,Y]}(\omega)) \\ &= (L_X\omega)(Y,Z) + \omega(L_X Y, Z) + \omega(Y, L_X Z) \end{aligned}$$

In particular, if X is a symplectic vector field, then

$$X(\omega(Y,Z)) = \omega([X,Y],Z) + \omega(Y,[X,Z]).$$

Since Hamiltonian vector fields are symplectic, for any Y,

$$\begin{aligned} X_f(\omega(X_g,Y)) &= \omega([X_f,X_g],Y) + \omega(X_g,[X_f,Y]) \\ X_f(-dg(Y)) &= \omega([X_f,X_g],Y) - dg([X_f,Y]) \\ -X_f(Y(g)) &= \omega([X_f,X_g],Y) - X_f(Y(g)) + Y(X_f(g)) \\ -Y(X_f(g)) &= \omega([X_f,X_g],Y) \\ -Y(\{f,g\}) &= \omega([X_f,X_g],Y) \\ \omega(X_{\{f,g\}},Y) &= \omega([X_f,X_g],Y) \end{aligned}$$

And since Y was arbitrary, we conclude that

$$X_{\{f,g\}} = [X_f, X_g].$$

Since (symplectic) vector fields with the Lie bracket $[-,-]$ form a *Lie algebra*, i.e. a vector space with an antisymmetric bilinear bracket satisfying the Jacobi identity, we have defined a Lie algebra structure on $\mathcal{O}(M)$. Moreover, since

$$\{fh, g\} = X_g(fh) = fX_g(h) + X_g(f)h = f\{h,g\} + \{f,g\}h,$$

the bracket $\{f,-\}$ is a derivation (it satisfies the Leibniz rule). This algebraic structure is called a *Poisson algebra*.

3 The Moment Map

If a function $f \in \mathcal{O}(M)$ is constant, then since $df = 0$ and ω is nondegenerate, $X_f = 0$. Otherwise, however, $X_f \neq 0$, and in particular, we have the following exact sequence

$$0 \to \mathbb{C} \to \mathcal{O}(M) \to \{\text{Symplectic vector fields on } M\}.$$

However, the final map may not be surjective. By Cartan's magic formula, we know that

$$\begin{aligned} L_X(\omega) &= (di_X + i_X d)(\omega) \\ &= d(i_X\omega) \end{aligned}$$

which is equal to 0 if and only if $i_X\omega$ is closed. For Hamiltonian vector fields,

$$i_{X_f}\omega = -df$$

is exact, so we can complete the above sequence to

$$0 \to \mathbb{C} \to \{\text{Symplectic vector fields on } M\} \to H^{0,1}(M) \to 0$$

(hopefully $H^{0,1}(M)$ is right, in the real case it should be $H^1_{dR}(M)$).

For us, a Lie group G will always be complex. Let

$$\mathfrak{g} = T_e G$$

be the tangent space to the identity. It is a vector space of dimension $\dim(G)$. Using the action of G on itself by left translation, we see that $\mathfrak{g}$ is isomorphic to $\mathrm{Lie}(G)$, the vector space of left-invariant vector fields on G. By this we mean the following: Let $L_g : G \to G$ denote the map $h \mapsto gh$. Then we say that a vector field X is *left-invariant* if for all $g, h \in G$,

$$(dL_{gh^{-1}})(X_h) = X_g.$$

Exercise: Check that the Lie bracket of left-invariant vector fields is left-invariant. Hence $\mathfrak{g}$ is a Lie algebra.

A Lie group action on a manifold is a smooth map

$$G \to \mathrm{Aut}(M)$$

and by differentiating we get

$$\mathfrak{g} \to \{\text{Vector fields on } M\}.$$

If a Lie group G acts on M preserving ω, i.e.

$$G \to \mathrm{Sympl}(M)$$

then

$$\mathfrak{g} \to \{\text{Symplectic vector fields on } M\}.$$

Whenever this map lands inside Hamiltonian vector fields, or, in other words, when we have a lifting

$$\begin{tikzcd} & \mathfrak{g} \arrow[dl, dashed] \arrow[dr] & \\ \mathcal{O}(M) \arrow[rr] & & \{\text{Symplectic vector fields on } M\} \end{tikzcd}$$

we say that the action of G on (M, ω) is Hamiltonian. A choice of a lifting (note that any shift by constants will work too) $\mu^{\#} : \mathfrak{g} \to \mathcal{O}(M)$ is called a comoment map, and we define the *moment map*

$$\mu : M \to \mathfrak{g}^*$$

by

$$\mu(m)(X) = (\mu^{\#}(X))(m)$$

for any $X \in \mathfrak{g}$. The map μ has many wonderful properties, we describe two of them related to our discussion of Poisson structures and moment maps. Consider $\mathbb{C}[\mathfrak{g}^*]$. Linear functions on $\mathfrak{g}^*$ correspond to $\mathfrak{g}$, where we have the Lie bracket

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}. \tag{1}$$

We can extend this bracket to $\mathbb{C}[\mathfrak{g}^*]$ via the Leibniz rule and obtain a Poisson algebra structure on $\mathbb{C}[\mathfrak{g}^*]$. Also, G acts on itself by conjugation, which fixes the identity, hence by differentiating the action map

$$\begin{aligned} (-)^g : G &\to G \\ h &\mapsto ghg^{-1} \end{aligned}$$

at the identity, we get

$$\begin{aligned} Ad_g : \mathfrak{g} &\to \mathfrak{g} \\ h &\mapsto ad_g(h) \end{aligned}$$

or, in other words, we get a representation

$$Ad : G \to GL(\mathfrak{g}).$$

We can take the contragradient (dual) representation

$$Ad^* : G \to GL(\mathfrak{g}^*)$$

to get a G-action on $\mathfrak{g}^*$, called the coadjoint representation. Now we can state the theorem

Theorem 3.1 (Kostant)**.** *1. The map*

$$\mu^* : \mathbb{C}[\mathfrak{g}^*] \to \mathcal{O}(M)$$

induced by μ intertwines the Poisson brackets.

2. *If G is connected then μ is G-equivariant relative to the coadjoint action on $\mathfrak{g}^*$.*

Proof. 1. It suffices to verify this for linear functions. Let $X, Y \in \mathfrak{g}$, we have

$$\begin{aligned} \{\mu^* X, \mu^* Y\} &= \mu^*[X, Y] && \text{as the lifting } \mathfrak{g} \to \mathcal{O}(M) \text{ is a Lie algebra homomorphism} \\ &= \mu^*\{X, Y\} && \text{by (1)} \end{aligned}$$

2. Skipped (see [1], Lemma 1.4.2. if interested).

Q.E.D.

References

[1] Chriss, N.,Ginzburg, V. Representation theory and complex geometry, Birkhäuser, Boston, 1997, x + 495 pp., ISBN 0-8176-3792-3