The Fundamental Theorem of Calculus

Theorem 1 (MVT). Assume that s is continuous on the closed interval $[a, b]$ and differentiable on (a, b). Then there exists at least one value c in (a, b) such that

$$s'(c) = \frac{s(b) - s(a)}{b - a}.$$

Exercise 1. Recall the functions $s(t) = \frac{-1}{3}\cos(3t) + 3t + \frac{1}{3}$ and $v(t) = \sin(3t) + 3$. These were the position and velocity functions of a person walking in an earlier class.

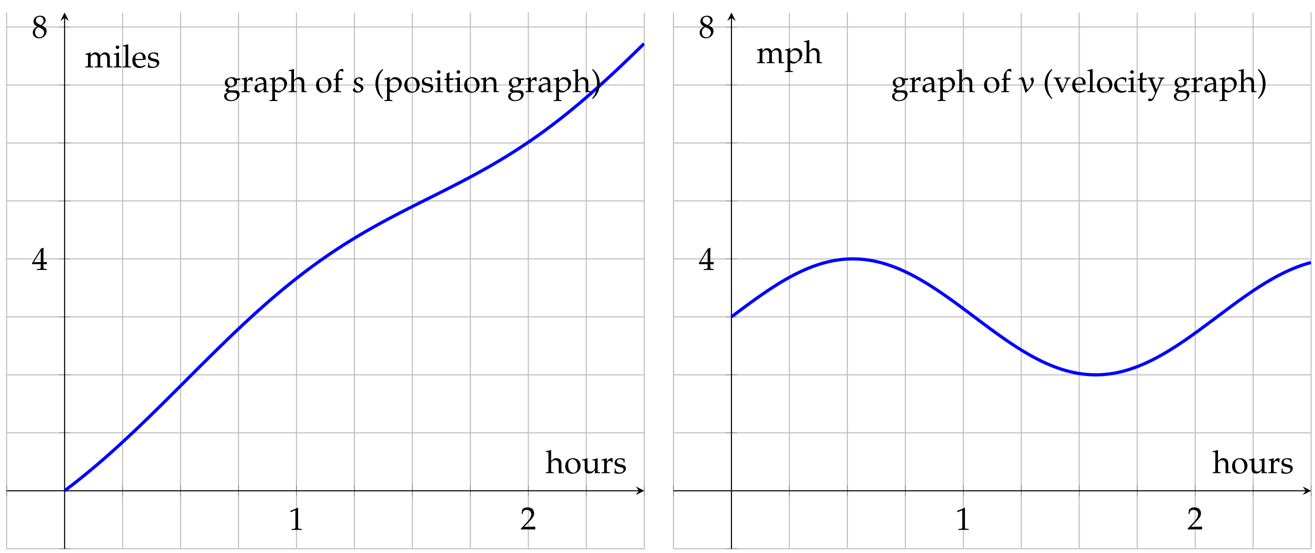

Figure 1: Graphs of s and v

We reconstructed (an approximation of) the graph of s from that of v using rectangular approximations. For the following parts, feel free to draw on Figure 1.

(a) Subdivide the interval $[0, 2]$ on both graphs into half hour blocks. On the position graph, mark where the person is at the beginning and the end of each block.

(b) Connect neighboring marks on the position graph by a straight line. What does the MVT (Theorem 1) tell you when you apply it to one of the 30 minute blocks? (**Hint:** Try to interpret both sides of the equation as slopes)

(c) Note that we can rearrange the two sides of the MVT to obtain that there exists at least one c in the interval (a, b) such that

$$s'(c)(b - a) = s(b) - s(a).$$

For each of the 30-minute blocks, find the corresponding quantity $s(b) - s(a)$ on the graph of s (**Hint:** look on the vertical axis).

(d) For each of the 30-minute blocks, find the quantity $s'(c)(b-a)$ on the graph of v (**Hint:** look for a rectangle with this area).

(e) Add up the results for the four blocks on both graphs. Does anything simplify on the $s(b) - s(a)$ side? Do you recognize the sum on the $s'(c)(b - a)$ side?

(f) Explain what would happen if we instead divided the interval into n equal subintervals and let $n \to \infty$.

Theorem 2 (FTC I)**.** Assume that $a < b$ and that f is continuous on $[a, b]$. If F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$