Toric surfaces, Pizzas and Kazhdan-Lusztig atlases

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- Let H be a semisimple algebraic group (e.g. $SL_n(\mathbb{C})$).
- Let P a parabolic subgroup (e.g. a subgroup containing all upper triangular matrices).
- Then H/P is a projective variety, known as a flag variety (e.g. $Fl = \{(V_1 \subset V_2 \subset \dots V_k \subset \mathbb{C}^n)\}$).
- The subgrop B (upper triangular matrices) acts on H/P with finitely many orbits, $H/P = \bigsqcup_{w \in W^P} BwP/P$, with $BwP/P \cong \mathbb{C}^{l(w)}$ (e.g. $\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \dots \sqcup \{\mathrm{pt}\}$).
- The closures $X^w := \overline{BwP/P}$ are called **Schubert varieties.** Their classes $[X^w]$ form an additive basis of $H^*(H/P)$.

Since the $[X^w]$ are a basis of $H^*(H/P)$, the class $[V]$ of any subvariety can be written as $[V] = \sum c_w[X^w]$ with $c_w \in \mathbb{N}$.

Definition 1

(Brion, [1]) Let $V \subseteq H/P$ be a subvariety. Write $[V] = \sum c_w[X^w]$. Then V is **multiplicity-free** if $c_w \in \{0, 1\}$.

Theorem 2

(Brion, [1]) Let $V \subseteq H/P$ be multiplicity-free. Then V admits a flat degeneration to a (reduced, C-M) union of Schubert varieties.

We are interested in finding multiplicity-free subvarieties of full flag varieties H/B_H. Generically, they receive the following structure:

Definition 3

(He, Knutson, Lu, [2]) An **equivariant Bruhat atlas** on a stratified T_M-manifold $(M, \mathcal{Y})$ is the following data:

1. A Kac-Moody group H with $T_M \hookrightarrow T_H$,
2. An atlas for M consisting of affine spaces U_f around the minimal strata, so $M = \bigcup_{f \in \mathcal{Y}_{\min}} U_f$,
3. A ranked poset injection $w : \mathcal{Y}^{\mathrm{opp}} \hookrightarrow W_H$ whose image is a union of Bruhat intervals $\bigcup_{f \in \mathcal{Y}_{\min}} [e, w(f)]$,
4. For $f \in \mathcal{Y}_{\min}$, a stratified T_M-equivariant isomorphism $c_f : U_f \xrightarrow{\sim} X_o^{w(f)} \subset H/B_H$,
5. A T_M-equivariant degeneration $M \rightsquigarrow M' := \bigcup_{f \in \mathcal{Y}_{\min}} X^{w(f)}$ of M into a union of Schubert varieties, carrying the anticanonical line bundle on M to the $\mathcal{O}(\rho)$ line bundle restricted from H/B_H.

Some remarkable families of stratified varieties possess *(equivariant?)* Bruhat atlases:

Theorem 4

(He, Knutson, Lu, [2]) Let G be a semisimple linear algebraic group. There are equivariant Bruhat atlases on every G/P, and for the wonderful compactification $\widehat{G}$ of a group G.

A rather interesting fact about the Bruhat atlases on the above spaces related to G is that the Kac-Moody group H is essentially never finite, or even affine type, although H's Dynkin diagram is constructed from G's.

Equivariant Bruhat atlases put the families G/P and $\overline{G}$ in the same basket, so one naturally wonders what other spaces could have this structure. Let $(H, \{c_f\}_{f\in\mathcal{Y}_{\min}}, w)$ be an equivariant Bruhat atlas on $(M, \mathcal{Y})$. We would like to understand what sort of structure a stratum $Z \in \mathcal{Y}$ inherits from the atlas. Each Z has a stratification,

$$Z := \bigcup_{f\in\mathcal{Y}_{\min}} U_f \cap Z, \quad \text{with} \quad U_f \cap Z \cong X_o^{w(f)} \cap X_{w(Z)}$$

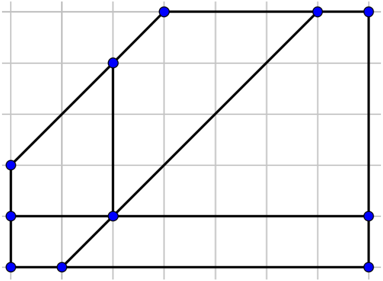
since by (3), the isomorphism $U_f \cong X_o^{w(f)}$ is stratified. Therefore Z has an “atlas” composed of Kazhdan-Lusztig varieties.

Definition 5

A **Kazhdan-Lusztig atlas** on a stratified T_V-variety $(V, \mathcal{Y})$ is:

1. A Kac-Moody group H with $T_V \hookrightarrow T_H$,
2. A ranked poset injection $w_M : \mathcal{Y}^{\mathrm{opp}} \to W_H$ whose image is
$$\bigcup_{f \in \mathcal{Y}_{\min}} [w(V), w(f)],$$
3. An open cover for V consisting of affine varieties around each $f \in \mathcal{Y}_{\min}$ and choices of a T_V-equivariant stratified isomorphisms
$$V = \bigcup_{f \in \mathcal{Y}_{\min}} U_f \cong X_o^{w(f)} \cap X_{w(V)},$$
4. A T_V-equivariant degeneration $V \rightsquigarrow V' = \bigcup_{f \in \mathcal{Y}_{\min}} X^{w(f)} \cap X_{w(V)}$ carrying some ample line bundle on V to $\mathcal{O}(\rho)$.

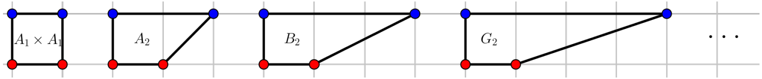
Let M be a smooth toric surface with an equivariant Kazhdan-Lusztig atlas. Part (4) of definition 5 gives us a decomposition of M's moment poly**gon** into the moment polytopes of the Richardson varieties $X^{w(f)} \cap X_{w(V)}$, or, more pictorially, a slicing of the polytope into pizza slices:



The definition is a big package, so we summarize what we are after as a checklist. To put an equivariant Kazhdan-Lusztig atlas on a smooth toric surface M, we need:

- ☐ A subdivision of M's moment polygon into a pizza.
- ☐ A Kac-Moody group H with $T_M \hookrightarrow T_H$.
- ☐ An assignment w of elements of W_H to the vertices of the pizza.
- ☐ A point $m \in H/B_H$ such that $\overline{T_M \cdot m} \cong M$.

It turns out that the Bruhat case is not very interesting, largely because the moment polytopes of the pizza slices must be moment polytopes of Schubert varieties (labeled by the rank 2 groups where they appear):

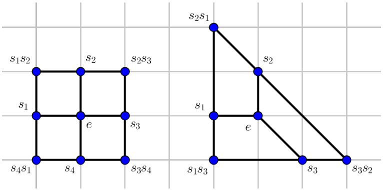


which must attach to the center of the pizza at one of the red vertices.

Theorem 6

The only smooth toric surfaces admitting equivariant Bruhat atlases are $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2$.

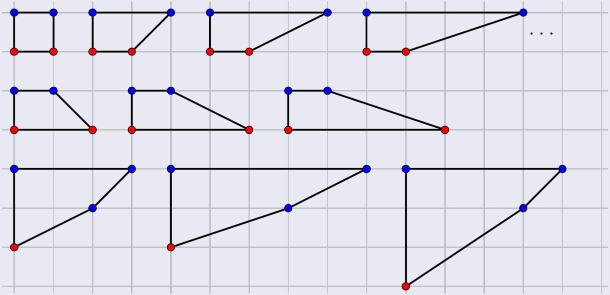
The corresponding pizzas are:



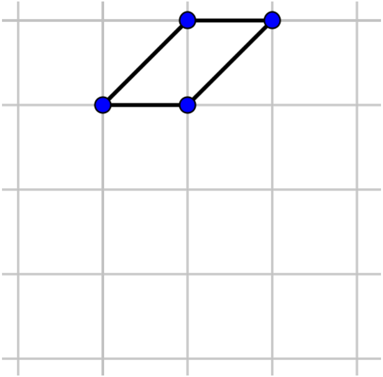
with $H = (SL_2(\mathbb{C}))^4, \widehat{SL_2(\mathbb{C})}$, respectively.

Proposition 7

The moment polytope of a Richardson surface in any H appears in a rank 2 Kac-Moody group, and the following is a complete list of the ones who are smooth everywhere except possibly where they attach to the center of the pizza (possible center locations in red).



How does one go about baking a pizza? We could just start putting pieces together:



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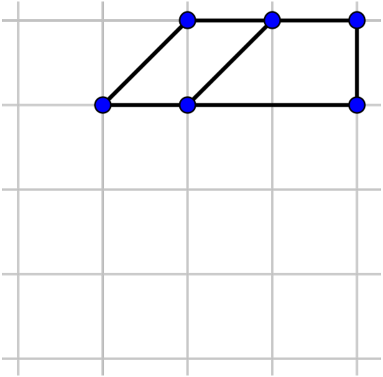

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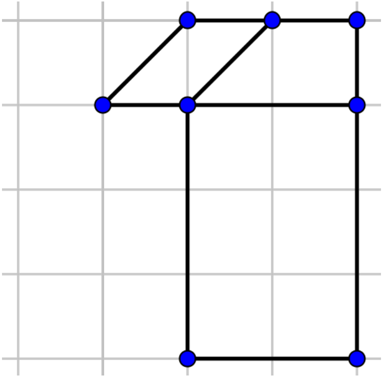

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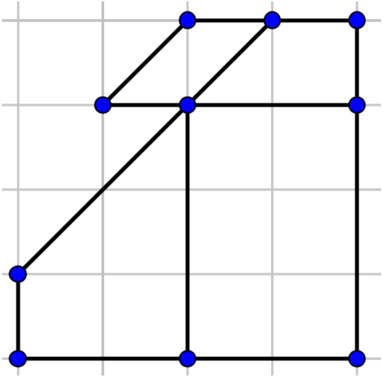
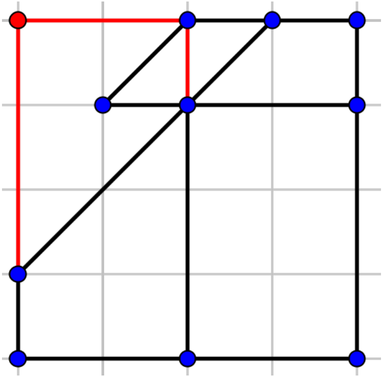
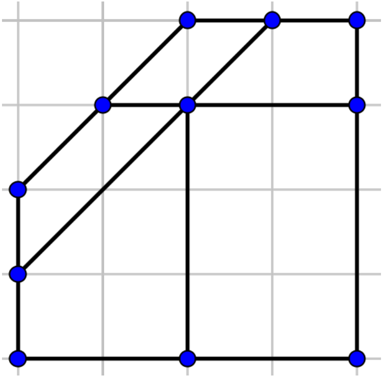


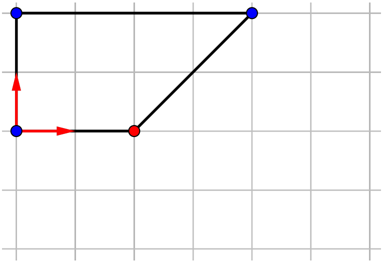
How does one go about baking a pizza? We could just start putting pieces together:



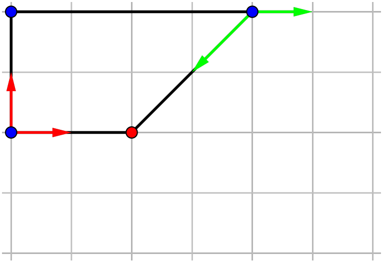
How does one go about baking a pizza? We could just start putting pieces together:



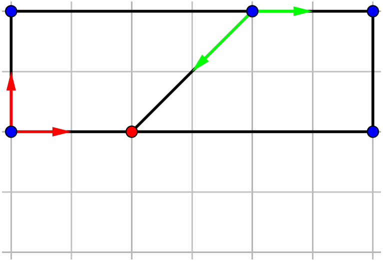
To do it more systematically, start with a single pizza slice sheared in a way that the bottom left basis of $\mathbb{Z}^2$ is the standard basis:



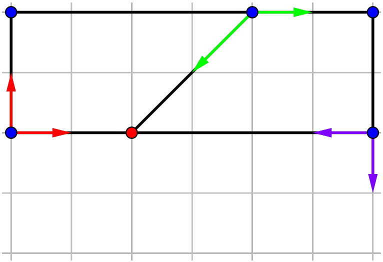
We know that the (clockwise) next slice will have to attach to the green basis



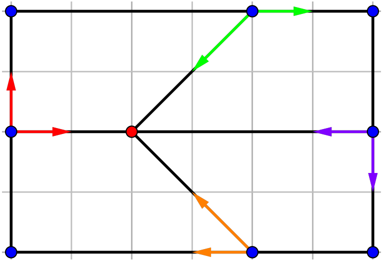
For instance,



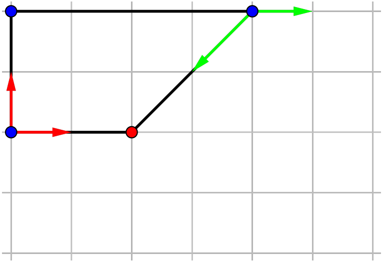
And the next slice will have to attach to the purple basis:



And if a pizza is formed, we must get back to the standard basis after some number of pizza slices

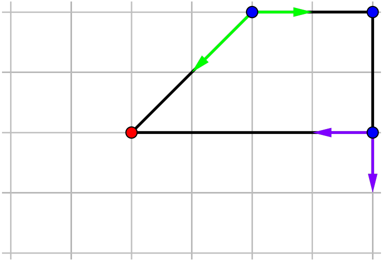


So we assign a matrix (in $SL_2(\mathbb{Z})$) for each pizza slice that records how it transforms the standard basis, for example



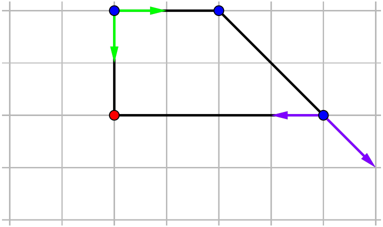
is assigned the matrix $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$.

And the second pizza slice



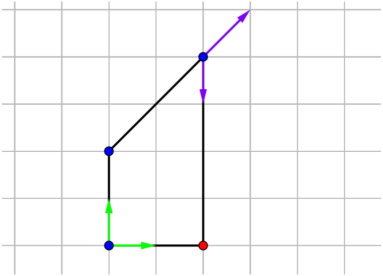
is assigned the matrix $\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$.

And the second pizza slice



is assigned the matrix $\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$.

And the second pizza slice



is assigned the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

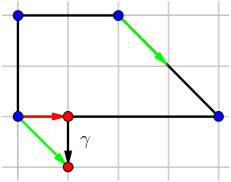
So if the first pizza slice changes the standard basis to M and the second one to N, then the two pizza slices consecutively change it to $(MNM^{-1})M = MN$.

Theorem 8

Let $M_1, M_2, \ldots, M_l$ be the matrices associated to a given list of pizza slices. If they form a pizza, then $\prod_{i=1}^{l} M_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

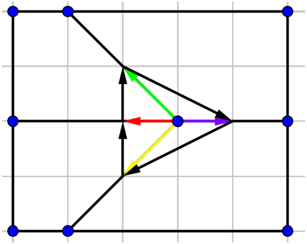
One unfortunate thing that the converse is not true, at least as long as we want to bake single-layered pizzas.

To make sure our pizza is single-layered, we want to think of pizza slices not living in $SL(2,\mathbb{R})$ but in its universal cover $\widetilde{SL_2(\mathbb{R})}$. We will represent this by assigning the slice its matrix and the straight line path connecting $\begin{pmatrix}1\\0\end{pmatrix}$ to $M\begin{pmatrix}1\\0\end{pmatrix}$, i.e.



and we think of multiplication in $\widetilde{SL_2(\mathbb{R})}$ as multiplication of the matrices and appropriate concatenatenation of paths.

Then for a pizza, we will have a closed loop around the origin. Also, as this path is equivalent to the path consisting of following the primitive vectors of the spokes of the pizza, its winding number will coincide with the number of layers of our pizza, as demonstrated by the following picture:



Some fun facts about this lifting of pizza slices to $\widetilde{SL_2(\mathbb{R})}$:

Theorem 9

(Wikipedia) The preimage of $SL_2(\mathbb{Z})$ inside $\widetilde{SL_2(\mathbb{R})}$ is Br_3, the braid group on 3 strands.

So the pizza slices should be thought of as braids, but we would prefer to work with matrices:

Lemma 10

The map $Br_3 \to SL_2(\mathbb{Z}) \times \mathbb{Z}$, with second factor ab given by abelianization, is injective.

This integer should be compatible with the abelianization maps:

Lemma 11

([4]) The abelianization of $SL_2(\mathbb{Z})$ *is* $\mathbb{Z}/12\mathbb{Z}$. *Moreover, for*

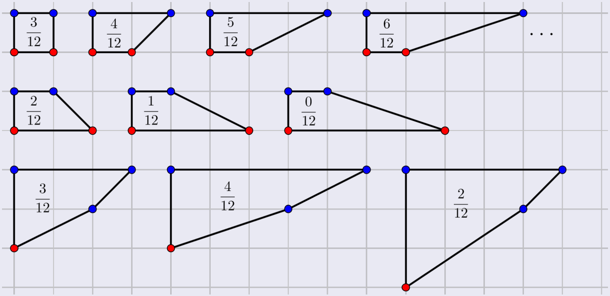
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

the image in $\mathbb{Z}/12\mathbb{Z}$ *can be computed by taking*

$$\chi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ((1-c^2)(bd+3(c-1)d+c+3)+c(a+d-3))/12\mathbb{Z}.$$

Definition 12

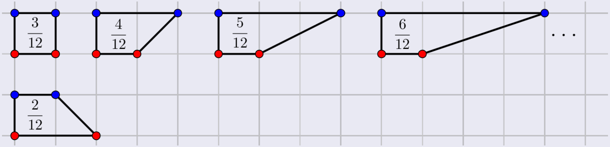
The **nutritive value** ν of a pizza slice S is the rational number $\frac{\mathrm{ab}(S)}{12}$. They are given by



Now we can make sure our pizza is bakeable in a conventional oven by requiring that the sum of the nutritive values of the slices in the pizza is $\frac{12}{12}$. This almost reduces the classification to a finite problem. If we insist, however, that our atlas is with H simply laced (but not necessarily finite or affine type), then the problematic pizza slice (with $\nu(S) = 0$) can not appear.

Lemma 13

The only Richardson quadrilaterals appearing in H/B_H for H simply laced are:



Then we force this through a computer to obtain all possible pizzas.

- ☑ A subdivision of M's moment polygon into a pizza.
- ☐ A Kac-Moody group H with $T_M \hookrightarrow T_H$.
- ☐ An assignment w of elements of W_H to the vertices of the pizza.
- ☐ A point $m \in H/B_H$ such that $\overline{T_M \cdot m} \cong M$.

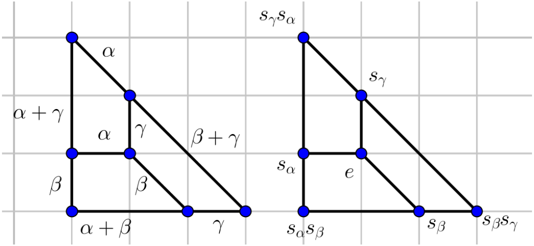
Having made the dough, we must not forget about toppings.

Recall that in order to have a Kazhdan-Lusztig atlas on a toric surface, we need a Kac-Moody group H and a map $w : \mathcal{Y}^{\text{opp}} \to W$, i.e. we need a map from the vertices of the pizza to W, where vertices should be adjacent when there is a covering relation between them.

Lemma 14

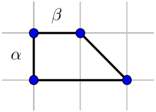
All covering relations $v \lessdot w$ are of the form $vr_\beta = w$ for some positive root β, and we will label the edges in the lattice pizza by these positive roots of H. The lattice length of an edge in a lattice pizza equals the height of the corresponding root.

Consider the example of $\mathbb{CP}^2$:

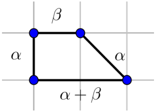


with α, β, γ the simple roots of $H = \widehat{SL_2(\mathbb{C})}$.

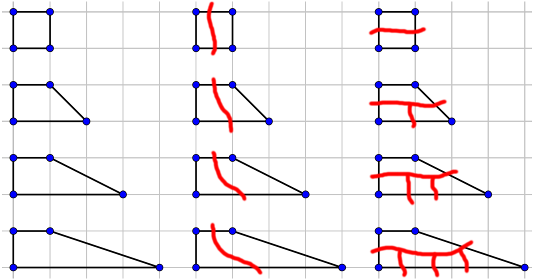
Note that the covering relations in W correspond to T-invariant $\mathbb{CP}^1$'s in H/B_H, and the edge labels are determined by the cohomology classes of these. For instance, if we know the labels on two edges of a pizza slice:



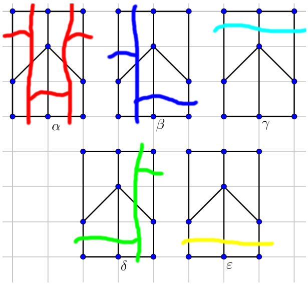
Then we can deduce the other two:



To figure out what H should be in general, we should look at all possible ways a simple root can appear in the edge labels of the pizza. We first look at how a simple root can label edges of individual pizza slices, which we will represent by drawing topping, that is, a curve which passes through all edges labeled by the given simple root. The possible toppings on the individual pizza slices are:

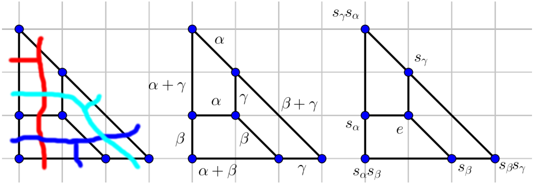


Then we find all possible topping arrangements on a given pizza, for example:

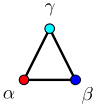


Then we try to find a group H whose simple roots correspond to (some subset of the) the toppings, and an element w to label the center.

For $\mathbb{CP}^2$, the compatible topping arrangement leading to this atlas is:



with H's diagram being



So considering the toppings on the pizzas, we can find potential H's.

- ☑ A subdivision of M's moment polygon into a pizza.
- ☑ A Kac-Moody group H with $T_M \hookrightarrow T_H$.
- ☐ An assignment w of elements of W_H to the vertices of the pizza.
- ☐ A point $m \in H/B_H$ such that $\overline{T_M \cdot m} \cong M$.

For a given H, finding W_H-elements labeling the vertices of the pizza is not very difficult.

- ☑ A subdivision of M's moment polygon into a pizza.
- ☑ A Kac-Moody group H with $T_M \hookrightarrow T_H$.
- ☑ An assignment w of elements of W_H to the vertices of the pizza.
- ☐ A point $m \in H/B_H$ such that $\overline{T_M \cdot m} \cong M$.

Having the labels on the vertices, for H finite type, we may use the map $H/B_H \twoheadrightarrow H/P_{\alpha_i^c}$ for simple roots α_i to find which Plücker coordinates should vanish on a potential m.

- ☑ A subdivision of M's moment polygon into a pizza.
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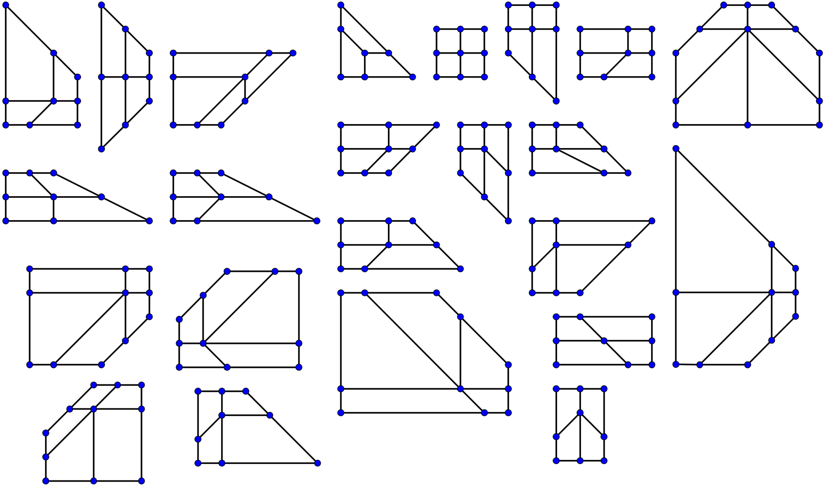
- ☑ A point $m \in H/B_H$ such that $\overline{T_M \cdot m} \cong M$.

Our main result is the following:

Theorem 15

There are 23 non-equivalent pizzas made of simply laced pizza slices, and each of those have Kazhdan-Lusztig atlases (i.e. there is also a compatible labeling of the vertices by W-elements). Moreover, in each of the cases where H is of finite type, the degeneration of definition 5 can be carried out inside H/B_H.

Relaxing the simply laced assumption to "doubly laced" still leaves a finite problem, but with more than 400 non-equivalent pizzas, and until the process of finding H can also be automated, this is not feasible. Other future directions could be relaxing any/all of the toric, smooth, 2-dimensional assumptions.



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