MATH 223 — Final Exam — 150 minutes

18th December 2024

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Cheat sheet

These are the most important definitions that we encountered:

- A list $\vec{v}_1, \ldots \vec{v}_m$ of vectors is **linearly independent** if
$$a_1\vec{v}_1 + \ldots + a_m\vec{v}_m = \vec{0}$$
has only the trivial solution $a_1 = \ldots = a_m = 0$.
- The **span** of a list $\vec{v}_1, \ldots \vec{v}_m$ is the set
$$\{a_1\vec{v}_1 + \ldots + a_m\vec{v}_m \in V \mid a_i \in \mathbb{F}\}$$
- A **basis** of a vector space is a linearly independent spanning list.
- The **dimension** of a vector space is the number of elements in a basis.
- For $T \in \mathcal{L}(V, W)$, the **null space** of T, denoted $\text{null}(T)$ is
$$\text{null}(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}.$$
- For $T \in \mathcal{L}(V, W)$, the **range** of T, denoted $\text{range}(T)$ is
$$\text{range}(T) = \{T(\vec{v}) \mid \vec{v} \in V\}.$$
- A linear map $T \in \mathcal{L}(V, W)$ is called **invertible** if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST is the identity on V and TS is the identity on W.
- Suppose $T \in \mathcal{L}(V)$. A nonzero vector $\vec{v} \in V$ is called an **eigenvector** of T corresponding to the eigenvalue $\lambda \in \mathbb{F}$ if
$$T(\vec{v}) = \lambda\vec{v}.$$

- Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. The **minimal polynomial** of T is the unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree such that $p(T) = 0_V$ is the zero operator.

- Two vectors $\vec{v}$ and $\vec{w}$ are **orthogonal** in an inner product space if

$$\langle \vec{v}, \vec{w} \rangle = 0.$$

- A list of vectors is **orthonormal** if each vector $\vec{v}$ in the list has $\langle vecv, \vec{v} \rangle = 1$ and is orthogonal to all the other vectors in the list.

- Suppose $T \in \mathcal{L}(V)$. A basis of V is called a **Jordan basis** for T if with respect to this basis T has a block diagonal matrix

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix}$$

in which each A_k is an upper-triangular matrix of the form

$$A_k = \begin{pmatrix} \lambda_k & 1 & & 0 \\ 0 & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{pmatrix}.$$

- An m-**linear form** on V is a function $\beta : V^m \to \mathbb{F}$ that is linear in each slot when the other slots are held fixed. The set of m-linear forms on V is denoted by $V^{(m)}$.

- An m-linear form α on V is called **alternating** if $\alpha(\vec{v}_1, \ldots, vecv_m) = 0$ whenever $\vec{v}_1, \ldots, \vec{v}_m$ is a list of vectors in V with $\vec{v}_j = \vec{v}_k$ for some $j \neq k$. The set of alternating m-linear forms is denoted by $V_{\text{alt}}^{(m)}$.

- The **determinant** $\det T$ of an operator $T \in \mathcal{L}(V)$ is the unique scalar in $\mathbb{F}$ such that

$$\alpha(T\vec{v}_1, \ldots, T\vec{v}_m) = (\det T)\alpha(\vec{v}_1, \ldots, \vec{v}_m)$$

for all $\alpha \in V_{\text{alt}}^{(\dim V)}$.

1. Let

$$U = \left\{ p \in \mathcal{P}_3(\mathbb{R}) \;\middle|\; \int_0^1 p(x)\ dx = p(1) \right\}.$$

(a) Prove that U is a subspace.

Solution: We check the conditions:

- For the zero polynomial $\vec{0}(x)$, we have $\int_0^1 \vec{0}(x)\,dx = 0 = \vec{0}(1)$, so $\vec{0} \in U$.
- If $p, q \in U$, then $\int_0^1 (p+q)(x)\,dx = \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx = p(1) + q(1) = (p+q)(1)$, so $p + q \in U$.
- If $p \in U, \lambda \in \mathbb{R}$, we have $\int_0^1 (\lambda p)(x)\,dx = \lambda \int_0^1 p(x)\,dx = \lambda p(1) = (\lambda p)(1)$ so $\lambda p \in U$.

Therefore U is a subspace.

(b) Find a basis for U and compute its dimension.

Solution: Let $p(x) = ax^3 + bx^2 + cx + d$. Then $\int_0^1 p(x)\,dx = \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d$ and $p(1) = a + b + c + d$. To be in U, p must satisfy

$$\frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = a + b + c + d,$$

Consider the list of polynomials

$$x^3 - \frac{9}{8}x^2, x^2 - \frac{4}{3}c, 1.$$

The list is linearly independent since the polynomials are of different degrees. All of the polynomials are in U. We claim that they form a basis for U. Since U is a subspace of $\mathcal{P}_3(\mathbb{R})$, its dimension is at most 4. But $x^3 \notin U$, so U is not the whole $\mathcal{P}_3(\mathbb{R})$. So $\dim U \leq 3$, but we have already found a list of three linearly independent vectors in U, so $\dim U = 3$ and the list above is a basis.

2. Let V, W be vector spaces and $T : V \to W$ a linear map. Prove that if

$$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$$

is linearly dependent, then

$$T\vec{v}_1, T\vec{v}_2, \ldots, T\vec{v}_n$$

is linearly dependent.

Solution: Since $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is LD, we have $c_1, \ldots, c_n \in \mathbb{F}$ not all zero such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}.$$

Apply T to both sides of the equation, we get the following equality in W:

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n) = \vec{0}.$$

Since T is linear, we can rewrite the left-hand side as

$$c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \ldots + c_nT(\vec{v}_n) = \vec{0},$$

and since not all c_i are zero, this implies that

$$T\vec{v}_1, T\vec{v}_2, \ldots, T\vec{v}_n$$

is linearly dependent.

3. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be given by reflection across the line $y = 2x$. Find all eigenvalues and eigenvectors of T.

Solution: Observe that a vector on the line is sent to itself by a reflection, and a vector orthogonal to the line is sent to its negative. So the eigenvalues are 1 and -1, and the eigenvectors are

$$\{(x, 2x) \in \mathbb{R}^2 \mid x \neq 0\}$$

for 1. Notice that $(2, -1) \cdot (1, 2) = 0$, so the orthogonal complement of the line is spanned by $(2, -1)$. Therefore

$$\{(2x, -x) \in \mathbb{R}^2 \mid x \neq 0\}$$

are the eigenvectors for the eigenvalue -1.

4. Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R})$ be the linear transformation given by

$$T(p) = x^2 p.$$

Compute the matrix of T with respect to the bases $(1, x+1, x^2+x+1)$ of $\mathcal{P}_2(\mathbb{R})$ and $(1, x, x^2, x^3, x^4)$ of $\mathcal{P}_4(\mathbb{R})$.

Solution: We compute

$$\begin{aligned} T(1) &= 1(x^2) \\ T(x+1) &= 1(x^3) + 1(x^2) \\ T(x^2+x+1) &= 1(x^4) + 1(x^3) + 1(x^2) \end{aligned}$$

So the matrix is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

5. For each of the following parts, give a concrete example. For this question only, you do not need to justify your answers.

   (a) A nilpotent operator $T \in \mathcal{L}(V)$ that has $\dim \operatorname{range}(T) = \dim V - 1$

   > **Solution:** Let $V = \mathbf{C}^2$ and $T(x, y) = (y, 0)$.

   (b) Two linear maps S, T such that $ST = I$ but S is not invertible.

   > **Solution:** Let $S : \mathbf{C}^2 \to \mathbf{C}$ and $T : \mathbf{C} \to \mathbf{C}^2$ be given by $S(x, y) = x, T(x) = (x, 0)$.

   (c) An operator on $V = \mathbb{C}^2$ whose minimal polynomial is $z - 1$.

   > **Solution:** The identity I_V.

   (d) An inner product on $\mathbb{R}^2$ different from the usual dot product.

   > **Solution:** Let $\langle (x, y), (z, w) \rangle = xz + 2yw$.

   (e) A nonzero vector in $\mathcal{P}_2(\mathbb{R})$ orthogonal to $p(x) = x$ under the inner product $\langle p, q \rangle = \int_0^1 pq$.

   > **Solution:** Let $q(x) = -\frac{3}{2}x + 1$.

6. Let $V = \mathcal{P}_1(\mathbb{R})$, and, for $p \in V$, define $\varphi_1, \varphi_2 \in V'$ by

$$\varphi_1(p) = \int_0^1 p(x)\,dx, \quad \text{and} \quad \varphi_2(p) = \int_0^2 p(x)\,dx.$$

Prove that (φ_1, φ_2) is a basis for V' and find a basis for V for which it is the dual basis.

Solution: First note that $\varphi_2(x-1) = 0$ and $\varphi(1)(x-1) = \int_0^1 x - 1\,dx = -\frac{1}{2}$. Also, $\varphi_1(2x-1) = 0$ and $\varphi_2(2x-1) = \int_0^2 2x - 1\ dx = 3$. This computation shows that neither φ_1, φ_2 is zero, and also they are not scalar multiples of each other. Therefore they are linearly independent. Since $\dim V = 2$, we also have $\dim V' = 2$, and any linearly independent set of size 2 is a basis, therefore φ_1, φ_2 is a basis.

To find the dual basis, note that we already found that $x - 1 \in \text{null}\varphi_2$ and $2x - 1 \in \text{null}\varphi_1$. So if we take

$$-2x + 2, \frac{2x-1}{3}$$

as our list, then we have

$$\begin{aligned}
\varphi_1(-2x+2) &= 1 \\
\varphi_1(-2x+2) &= 0 \\
\varphi_2\left(\frac{x-1}{3}\right) &= 0 \\
\varphi_2\left(\frac{x-1}{3}\right) &= 1
\end{aligned}$$

so the list is the basis for which φ_1, φ_2 is the dual basis.

7. Suppose that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a basis for an inner product space V over $\mathbb{R}$ and that

$$\begin{aligned}
\langle \vec{v}_1, \vec{v}_1 \rangle &= 1 \\
\langle \vec{v}_2, \vec{v}_2 \rangle &= 2 \\
\langle \vec{v}_3, \vec{v}_3 \rangle &= 2 \\
\langle \vec{v}_1, \vec{v}_2 \rangle &= -1 \\
\langle \vec{v}_1, \vec{v}_3 \rangle &= -1 \\
\langle \vec{v}_2, \vec{v}_3 \rangle &= 1
\end{aligned}$$

Find an orthonormal basis for V. Justify your answer.

Solution: We use the Gram-Schmidt procedure. Let $\vec{f}_1 = \vec{v}_1$. Then we let

$$\begin{aligned}
\vec{f}_2 &= \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{f}_1 \rangle}{\langle \vec{f}_1, \vec{f}_1 \rangle} \vec{f}_1 \\
&= \vec{v}_2 + \vec{f}_1 \\
&= \vec{v}_1 + \vec{v}_2 \\
\vec{f}_3 &= \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{f}_1 \rangle}{\langle \vec{f}_1, \vec{f}_1 \rangle} \vec{f}_1 - \frac{\langle \vec{v}_3, \vec{f}_2 \rangle}{\langle \vec{f}_2, \vec{f}_2 \rangle} \vec{f}_2 \\
&= \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 - \frac{\langle \vec{v}_3, \vec{v}_1 + \vec{v}_2 \rangle}{\langle \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 \rangle} (\vec{v}_1 + \vec{v}_2) \\
&= \vec{v}_3 + (1)\vec{v}_1 + (0)(\vec{v}_1 + \vec{v}_2).
\end{aligned}$$

It remains to scale each $\vec{f}_1, \vec{f}_2, \vec{f}_3$ to unit length.

$$\begin{aligned}
\vec{e}_1 &= \frac{\vec{f}_1}{\langle \vec{f}_1, \vec{f}_1 \rangle} = \vec{f}_1 = \vec{v}_1 \\
\vec{e}_2 &= \frac{\vec{f}_2}{\langle \vec{f}_2, \vec{f}_2 \rangle} = \vec{f}_2 = \vec{v}_1 + \vec{v}_2 \\
\vec{e}_3 &= \frac{\vec{f}_3}{\langle \vec{f}_3, \vec{f}_3 \rangle} = \vec{f}_3 = \vec{v}_1 + \vec{v}_3
\end{aligned}$$

So an orthonormal basis is

$$\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_3.$$

8. Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null}(T) = U$ if and only if $\dim U \geq \dim V - \dim W$.

Solution: Assume that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null}(T) = U$. Then by the FTLM,

$$\begin{aligned} \dim \operatorname{null}(T) + \dim \operatorname{range}(T) &= \dim V \\ \dim U &= \dim V - \dim \operatorname{range}(T) \\ \dim U &\geq \dim V - \dim W \end{aligned}$$

For the converse, assume that U is a subspace of V and $\dim U \geq \dim V - \dim W$. Pick a basis $\vec{v}_1, \ldots, \vec{v}_k$ for U and extend to a basis $\vec{v}_1 \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n$ of V. Note that the assumption on dimensions implies that $k \geq n+k-m$, or, equivalently, that $n \leq m$. Pick a basis $\vec{w}_1, \ldots, \vec{w}_m$ of W. Define

$$T(\vec{v}_i) = \begin{cases} \vec{0} & \text{if } i = 1, \ldots, k \\ \vec{w}_{i-k} & \text{if } i = k+1, \ldots, n \end{cases}$$

Since $n \leq m$, this map is well-defined, and by the linear map lemma it defines a linear map $T : V \to W$. Now note that $\operatorname{range}(T) = \operatorname{Span}(\vec{w}_1, \ldots \vec{w}_n)$, so $\dim \operatorname{range}(T) = n$, so $\dim \operatorname{null}(T) = \dim V - \dim \operatorname{range}(T) = n+k-n = k$. By the definition above, we have $U \subseteq \operatorname{null}(T)$, and therefore we have $\operatorname{null}(T) = U$ as required.

9. Suppose that V is finite-dimensional inner product space and U is a subspace of V. Show that

$$P_{U^\perp} = I_V - P_U,$$

where I_V is the identity map on V and P_W denotes the orthogonal projection map onto a subspace $W \subseteq V$.

Solution: Let $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for U and extend it to an orthonormal basis $\vec{v}_1, \ldots, v_k, v_{k+1}, \ldots \vec{v}_n$ of V. Then since $\langle \vec{v}_j, \vec{v}_i \rangle = \delta_i^j$, the list $\vec{v}_{k+1}, \ldots, \vec{v}_n$ is an orthonormal basis for $U^\perp$, since it a is linearly independent list of $\dim V - \dim U$ vectors in $U^\perp$. Recall that for all $\vec{v} \in V$, we can uniquely write

$$c_1\vec{v}_1 + \ldots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1} + \ldots + c_n\vec{v}_n$$

and by the definition of orthogonal projections, we have

$$\begin{aligned} P_U(\vec{v}) &= c_1\vec{v}_1 + \ldots + c_k\vec{v}_k \\ P_{U^\perp}(\vec{v}) &= c_{k+1}\vec{v}_{k+1} + \ldots + c_n\vec{v}_n \\ I_V(\vec{v}) &= c_1\vec{v}_1 + \ldots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1} + \ldots + c_n\vec{v}_n \end{aligned}$$

for all $\vec{v} \in V$, and therefore $P_U + P_{U^\perp} = I_V$, or equivalently,

$$P_{U^\perp} = I_V - P_U.$$

10. Two linear operators $S, T \in \mathcal{L}(V)$ are said to be simultaneously diagonalizable if there exists a basis $(\vec{v}_1, \ldots, \vec{v}_n)$ of V such that the matrices $\mathcal{M}(S, (\vec{v}_1, \ldots, \vec{v}_n))$ and $\mathcal{M}(T, (\vec{v}_1, \ldots, \vec{v}_n))$ of S and T with respect to this basis are both diagonal. Prove that if S and T are simultaneously diagonalizable, then they commute, i.e.

$$ST = TS$$

Solution: Let λ_i be the eigenvalue corresponding for $\vec{v}_i$ for S and let μ_i be the eigenvalue corresponding to $\vec{v}_i$ for T. Then we have

$$ST(\vec{v}_i) = S(\mu_i \vec{v}_i) = \mu_i(S\vec{v}_i) = \mu_i \lambda_i \vec{v}_i = \lambda_i \mu_i \vec{v}_i = \lambda_i S(\vec{v}_i) = TS(\vec{v}_i).$$

Since $\vec{v}_1, \ldots, \vec{v}_n$ form a basis, and $ST(\vec{v}_i) = TS(\vec{v}_i)$, by the linear map lemma, we have that $ST = TS$.

11. Suppose that $T \in \mathcal{L}(\mathbb{C}^3)$ is an operator such that the minimal polynomial of T is

$$z^3 + 2z^2 + z.$$

Find all the possibilities for the matrix of $\mathcal{M}(T)$ with respect to a Jordan basis (up to reordering the basis).

Solution: The minimal polynomial factors as $z(z+1)^2$. So the eigenvalues of T are 0 and -1. Since the $(z+1)$ factors occurs with multiplicity 2, we must have a Jordan block of size 2 corresponding to the eigenvalue -1, and therefore $\dim G(-1, T) \geq 2$. Since 0 is an eigenvalue, we have $\dim G(0, T) \geq 1$. Since $1 + 2 = 3 = \dim \mathbb{C}^3$, these inequalities must be in fact equalities, and then the matrix of T with respect to a Jordan basis must look like

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(up to reordering the basis).

12. Suppose $V = \mathbb{F}^n$ and $T \in \mathcal{L}(V)$ is given by

$$T(x_1, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n).$$

Find the minimal polynomial of T. Justify your answer.

Solution: Let $\vec{e}_i$ denote the standard basis vector with a 1 in position i and 0 everywhere else. Then note that

$$T(\vec{e}_i) = i\vec{e}_i$$

for all i. Therefore $1, 2, \dots, n$ are eigenvalues of T with corresponding eigenvectors $\vec{e}_1, \dots, \vec{e}_n$. So for $i = 1, \dots, n$, we have that $(z - i)$ is a factor of the minimal polynomial. Notice that

$$\deg\left((z-1)(z-2)\cdots(z-n)\right) = n$$

which is $\dim V$, which is the maximum degree of the minimal polynomial. In addition, $(z-1)(z-2)\cdots(z-n)$ is monic, so in fact the minimal polynomial of T must be

$$(z-1)(z-2)\cdots(z-n)$$

13. Suppose $T \in \mathcal{L}(V)$ is nilpotent. Prove that $\det(I+T) = 1$.

Solution: Let $\alpha \in V_{\text{alt}}^{(\dim V)}$ be a nonzero alternating $\dim V$-linear form. Pick a Jordan basis $\vec{v}_1, \ldots, \vec{v}_n$ for T. Then $\det(I+T)$ is the scalar such that

$$\alpha((I+T)\vec{v}_1, \ldots, (I+T)\vec{v}_n) = \det(I+T)\alpha(\vec{v}_1, \ldots, \vec{v}_n).$$

Since $\vec{v}_1, \ldots, \vec{v}_n$ is a Jordan basis for T and T is nilpotent the only eigenvalue of T is zero, and therefore we have that

$$T(\vec{v}_k) = \vec{v}_{k-1} \text{ or } T(\vec{v}_k) = \vec{0}.$$

Then we have

$$\alpha((I+T)\vec{v}_1, (I+T)\vec{v}_2, \ldots, (I+T)\vec{v}_n) = \alpha(\vec{v}_1, (I+T)\vec{v}_2, \ldots, (I+T)\vec{v}_n).$$

As α is multilinear, we have that

$$\begin{aligned}
&\alpha(\vec{v}_1, \ldots, \vec{v}_{k-1}, (I+T)\vec{v}_k, (I+T)\vec{v}_{k+1}, \ldots, (I+T)\vec{v}_n) = \\
&\alpha(\vec{v}_1, \ldots, \vec{v}_{k-1}, \vec{v}_k, (I+T)\vec{v}_{k+1}, \ldots, (I+T)\vec{v}_n) + \\
&\alpha(\vec{v}_1, \ldots, \vec{v}_{k-1}, T\vec{v}_k, (I+T)\vec{v}_{k+1}, \ldots, (I+T)\vec{v}_n)
\end{aligned}$$

Since α is alternating and $\vec{v}_1, \ldots \vec{v}_{k-1}, T\vec{v}_k$ is always linearly dependent, we also have for any $1 \leq k \leq n$,

$$\begin{aligned}
&\alpha(\vec{v}_1, \ldots, \vec{v}_k, (I+T)\vec{v}_{k+1}, \ldots, (I+T)\vec{v}_n) = \\
&\qquad = \alpha(\vec{v}_1, \ldots \vec{v}_k, \vec{v}_{k+1}, \ldots, (I+T)\vec{v}_n).
\end{aligned}$$

so by induction,

$$\alpha((I+T)\vec{v}_1, \ldots, (I+T)\vec{v}_n) = \alpha(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n),$$

and therefore

$$\det(I+T) = 1.$$

14. Suppose V is finite-dimensional and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by

$$\mathcal{A}(T) = ST$$

for $T \in \mathcal{L}(V)$.

(a) Prove that $\dim \operatorname{null}(\mathcal{A}) = (\dim V)(\dim \operatorname{null}(S))$.

Solution: We will show that $\operatorname{null}(\mathcal{A}) \cong \mathcal{L}(V, \operatorname{null}(S))$. Given $T \in \mathcal{L}(V, \operatorname{null}(S))$, we have

$$\begin{aligned}
\operatorname{null}(A) &= \{T \in \mathcal{L}(V) \mid ST = 0\} \\
&= \{T \in \mathcal{L}(V) \mid ST(\vec{v}) = \vec{0} \text{ for all } \vec{v} \in V\} \\
&= \{T \in \mathcal{L}(V) \mid S(T(\vec{v})) = \vec{0} \text{ for all } \vec{v} \in V\} \\
&= \{T \in \mathcal{L}(V) \mid \operatorname{range}(T) \subseteq \operatorname{null}(S)\} \\
&\cong \mathcal{L}(V, \operatorname{null}(S))
\end{aligned}$$

Therefore, we have $\dim \mathcal{A} = \mathcal{L}(V, \operatorname{null}(S)) = (\dim(V))(\dim \operatorname{null}(S))$

(b) Prove that $\dim \operatorname{range}(\mathcal{A}) = (\dim V)(\dim \operatorname{range}(S))$.

Solution: By the fundamental theorem of linear maps, we have $\dim \operatorname{null}(\mathcal{A}) + \dim \operatorname{range}(\mathcal{A}) = \dim(\mathcal{L}(\mathcal{L}(V)) = (\dim V)^2$. Also by the fundamental theorem, $\dim \operatorname{null}(S) + \dim \operatorname{range}(S) = \dim V$. Combining these two, we get

$$\begin{aligned}
\dim \operatorname{range}(\mathcal{A}) &= (\dim V)^2 - \dim \operatorname{null}(\mathcal{A}) \\
&= (\dim V)^2 - (\dim V)(\dim \operatorname{null}(S)) \\
&= (\dim V)(\dim V - \dim \operatorname{null}(S)) \\
&= (\dim V)(\dim \operatorname{range}(S))
\end{aligned}$$