CHERN-CONNES-KAROUBI CHARACTER ISOMORPHISMS AND ALGEBRAS OF SYMBOLS OF PSEUDODIDIFFERENTIAL OPERATORS

ALEXANDRE BALDARE, MOULAY BENAMEUR, AND VICTOR NISTOR

Abstract. We introduce a class of algebras for which the Chern-Connes-Karoubi character is an isomorphism after tensoring with $\mathbb{C}$. We provide several examples of such algebras, such as crossed products with finite groups and certain algebras of pseudodifferential operators. Our main application is to the index of invariant pseudodifferential operators.

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Contents

1. Introduction and statement of main results

1.1. **Motivation: invariant operators and their index.** Let G be a compact Lie group acting on a smooth manifold M (without boundary). Let $E \to M$ be an equivariant bundle and P be a G-invariant pseudodifferential operator acting on the sections of E. It will be convenient to assume that P is of order zero. Let α be an irreducible representation of G and $\pi_\alpha(P)$ be the restriction of P to the isotypical component corresponding to α. In BCLN1, BCLN2, BCN [?, ?, ?], two of the authors and their collaborators have constructed a suitable algebra of symbols $\mathfrak{A}_\alpha$ as well as an "α-principal symbol" $\sigma_m^\alpha(P) \in \mathfrak{A}_\alpha$ such that $\pi_\alpha(P)$ is Fredholm if, and only if, $\sigma_m^\alpha(P)$ is invertible. An immediate consequence of this property is that the index of $\pi_\alpha(P)$ depends only on $\sigma_m^\alpha(P)$ and, more precisely, on $[\sigma_m^\alpha(P)] \in K_1(\mathfrak{A}_\alpha)$ (its class in K_1). One of the *main motivations* for the results that we present in this paper is to *study the index* of $\pi_\alpha(P)$ in case it is Fredholm. One of the *main motivations* for this paper is to *study the index* of P using methods of non-commutative geometry.

Let $\mathfrak{A} := \mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$ be the algebra of G-invariant symbols. It is known ConnesIHES [?] that the index can be expressed as the pairing $\phi_*([\sigma_0(P)])$ between a cyclic cocycle ϕ on $\mathfrak{A}$ and the class of P in $K_1(\mathfrak{A})$. We are thus lead to study the periodic cyclic homology of the symbol algebra $\mathfrak{A} := \mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$. For suitable algebras A and a suitable K-theory functor, let

eq.def.CCC
$$Ch : K_j(A) \to \mathrm{HP}_j(A)\,, \quad j = 0, 1\,, \tag{1}$$

be the Chern-Connes-Karoubi character, where $\mathrm{HP}_j(A)$ is a periodic cyclic homology of A defined using a suitable completion for the cyclic complex. A related natural question then is whether $Ch : K_j(\mathfrak{A}) \otimes \mathbb{C} \to \mathrm{HP}_j(\mathfrak{A})$ is an isomorphism ($j = 0, 1$). Let $\overline{\mathfrak{A}}$ be the C^*-completion of $\mathfrak{A}$. We are also lead to study whether the inclusion $\mathfrak{A} \subset \overline{\mathfrak{A}}$ leads to isomorphisms $K_j(\mathfrak{A}) \to K_j(\overline{\mathfrak{A}})$. We prove that the answer to both these questions is affirmative. The techniques for proving these isomorphisms turn out to apply to a more general setting than that of the algebra $\mathfrak{A}$. We thus introduce and study the class of "Connes algebras," which is, roughly, the class of algebras for which both maps (induced by the character and by inclusion) are isomorphisms.

1.2. **Connes algebras and statement of main results.** Let us consider a category of cyclic complexes (this is, roughly, a category of topological algebras for which a definite choice of completion of the cyclic complex has been made, Definition def.ex.CCC 2.2). We assume that we are given a suitable K_i-theory functor, $i = 0, 1$ that is close to the algebraic K-theory functor and is such that the Chern-Connes-Karoubi character (Equation eq.def.CCC 1) extends to this category. A *Connes algebra* $\mathcal{A}$ (in the given category and for the given K-functor) is an algebra $\mathcal{A}$ in the given category, together with a continuous Banach algebra norm $\|\cdot\|_0$ on $\mathcal{A}$ with the following properties:

(i) The inclusion $\mathcal{A} \to \overline{\mathcal{A}}$ into its norm closure with respect to the norm $\|\cdot\|_0$ induces an isomorphism in K-theory:

eq.iso.K
$$K_i(\mathcal{A}) \xrightarrow{\sim} K_i(\overline{\mathcal{A}})\,, \quad i = 0, 1\,. \tag{2}$$

(ii) The Chern-Connes-Karoubi character induces an isomorphism "after tensoring with $\mathbb{C}$," that is:

eq.iso.HP
$$Ch : K_i(\mathcal{A}) \otimes \mathbb{C} \xrightarrow{\sim} \mathrm{HP}_i(\mathcal{A}) \,, \quad i = 0, 1 \,. \tag{3}$$

Let $\mathcal{K}$ be the algebra of compact operators. With this terminology, our main results about the algebras introduced in the previous subsection is that they are Connes algebras:

thm.main0 **Theorem 1.1.** *The algebra $\mathfrak{A} \simeq \Psi^0(M;E)^G / \big(\Psi^0(M;E)^G \cap \mathcal{K}\big)$ is a Connes algebra. It may not be true for $\Psi^0(M;E)^G$ unless we restrict to an isotypical component.*

To be able to work with the concept of a Connes algebra, we need to make some assumptions that are satisfied in the case of main interest in this paper (that of the algebra $\mathfrak{A}$). First, we assume that the K-theory functor is homotopy invariant and satisfies a six term exact sequence for *admissible* short exact sequences of algebras. We also assume that this category satisfies excision in periodic cyclic homology (again for *admissible* short exact sequences) and that the Chern-Connes-Karoubi character is a natural transformation between the exact sequence in K-theory and periodic cyclic homology (associated to, again, an *admissible* short exact sequence of algebras). We then have the following result that will play an important role in our study of the algebras $\mathfrak{A}$.

Can we say in general that our K-functor is a suitable quotient of K_0^{alg} such that $[e_1]-[e_0] \in \ker(K_0^{\mathrm{alg}}(A)) \to K_0(A)$ whenever e_0 and e_1 are homotopic projections in some matrix algebra over A? M2ALL: I agree with Victor here, but then it must as well satisfy the usual axioms.

Topological algebras $\mathcal{A}$ together with a continuous Banach algebra norm $\|\cdot\|_0$ and the compatible morphisms, define a category that is denoted $\mathfrak{B}$.

thm.main1 **Theorem 1.2.** *Let $\mathfrak{C}$ be a category of topological cyclic complexes that satisfies excision in periodic cyclic homology. Let us assume that the category of topological algebras underlying $\mathfrak{C}$ satisfies excision in K-theory and that it is a subcategory of $\mathfrak{B}$. Let $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{A}/\mathcal{I} \to 0$ be an* admissible *short exact sequence in $\mathfrak{C}$ such that its Banach completion is exact as well. Assume that two of the algebras $\mathcal{I}$, $\mathcal{A}$, and $\mathcal{A}/\mathcal{I}$ are Connes algebras. Then the third one is a Connes algebra as well.*

Possible choices of the K-groups and categories $\mathfrak{C}$ are

- The category of m-algebras and K-groups of Cuntz with the admissible sequences being the linear-split exact sequences CuntzGerman, CuntzBook [?, ?].
- The sub category of *Fréchet* m-algebras, in which all exact sequences are admissible MeyerExcision [?]. This is the case that is relevant for the applications in this paper, and hence this is the case that we shall consider in detail. Moreover, in this case, the K-theory groups of Cuntz coincide with the representable K-theory groups introduced earlier by Phillips PhillipsKThr [?].

Our result for the algebra $\mathfrak{A}$ is obtained by iterating Theorem thm.main1 1.2 for something that we call a C-stratification of ideals (Definition def.Csmooth.S 3.9). More precisely, we have the following result.

Le reste c'est surtout des envies :-) M2ALL: J'ai l'impression que le theoreme suivant est maintenant OK, non?

m.main.strat **Theorem 1.3.** *The algebra* $\mathfrak{A} = \Psi^0(M;E)^G / \big(\Psi^0(M;E)^G \cap \mathcal{K}\big)$ *is a Fréchet m-algebra with a composition series*

$$0 = I_N \subset I_{N-1} \subset \ldots \subset I_0 = \mathfrak{A}$$

consisting of nuclear, Fréchet m-algebra ideals with the following property:

For each $j = 1, \ldots, N$, *there exists a compact manifold with corners* Y_j, *a smooth algebra bundle* $\mathcal{E}_j \to Y_j$ *and an algebra morphism*

$$\phi_j : I_{j+1}/I_j \to \mathcal{C}_0^\infty(Y_j, \partial Y_j; \mathcal{E}_j) := \{f : Y_j \to \mathcal{E}_j \mid f \text{ smooth and } f|_{\partial Y_j} = 0\}$$

that is a homeomorphism onto the image which contains $\cap_{n\in\mathbb{N}} \mathcal{C}_0^\infty(Y_j, \partial Y_j; \mathcal{E}_j)^n$.

In the following, if $k \in \mathbb{Z}/2\mathbb{Z}$, by $H^k(X,Y)$ we shall understand the direct sum of all singular cohomology groups of the pair (X,Y) of the same parity as k and with complex coefficients. In particular, we have

cor.isomo **Corollary 1.4.** *Using the notation of Theorem* thm.main.strat *1.3, the algebra morphisms* ϕ_j *induce isomorphisms in periodic cyclic homology, and we further have* $\mathrm{HP}_*(I_j/I_{j-1}) \simeq H^*(Y_j, \partial Y_j)$).

As usual, this leads to a spectral sequence convergent to $\mathrm{HP}_*(\mathfrak{A})$, see Theorem thm.main.specral ??.

1.3. **Acknowledgements.** We thank ...

2. Background material

For us, a *topological algebra* is a complex algebra $\mathcal{A}$ endowed with a compatible locally convex topology such that the multiplication $\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ is continuous. (The multiplication is thus assumed to be continous jointly in both variables.) In other words, for any continuous seminorm p on $\mathcal{A}$ there exists a continuous seminorm p' such that $p(ab)\le p'(a)p'(b)$, $\forall a,b\in\mathcal{A}$, see ConnesBook [?, Chap 3, Appendix B].

2.1. **Review of periodic cyclic homology.** We briefly recall the topological periodic cyclic homology for topological algebras that will be used in this paper, see for instance Connes?, Connes?, Karoubi, Tsygan [?, ?, ?, ?] for the standard material used in this section. The topological cyclic homology considered in this paper is such that it is the same for an algebra and for its completion.

def.ex.CC **Definition 2.1.** Let $\mathcal{A}$ be a topological algebra. *A topological cyclic complex of* $\mathcal{A}$ is a graded vector space $\mathcal{C}(\mathcal{A}) := \big(\mathcal{C}_n(\mathcal{A})\big)_{n\ge 0}$, where, for all n, $\mathcal{C}_n(\mathcal{A})$ is a suitable completion of the (algebraic) tensor product $\mathcal{A}^+\otimes\mathcal{A}^{\otimes n}$. We also require that these completions be such that the usual differentials b and B extend by continuity to maps denoted by the same symbols: $b:\mathcal{C}_n(\mathcal{A})\to\mathcal{C}_{n-1}(\mathcal{A})$ and $B:\mathcal{C}_n(\mathcal{A})\to\mathcal{C}_{n+1}(\mathcal{A})$. The Hochschild, cyclic, and periodic cyclic homologies of a topological cyclic complex $(\mathcal{A},\mathcal{C}(\mathcal{A}))$ are the corresponding groups defined using the mixed complex $(\mathcal{C}(\mathcal{A}),b,B)$.

The topological cyclic complexes form a category.

def.ex.CCC **Definition 2.2.** A *category of topological cyclic complexes* $\mathfrak{C}$ is a category whose objects are pairs $(\mathcal{A},\mathcal{C}(\mathcal{A}))$ consisting of topological algebra $\mathcal{A}$ together with a topological cyclic complex $\mathcal{C}(\mathcal{A})$ of $\mathcal{A}$. The morphisms in $\mathfrak{C}$ are continuous, linear maps $f=(f_a,f_c):(\mathcal{A},\mathcal{C}(\mathcal{A}))\to(\mathcal{B},\mathcal{C}(\mathcal{B}))$ such that $f_a:\mathcal{A}\to\mathcal{B}$ is an algebra morphism and $f_c(a_0\otimes a_1\otimes\ldots\otimes a_n) = f_a(a_0)\otimes f_a(a_1)\otimes\ldots\otimes f_a(a_n)$.

By abuse of notation, we shall denote by $\mathrm{HP}_*^{\mathrm{top}}(\mathcal{A})$ the periodic cyclic homology groups of the mixed complex $(\mathcal{C}(\mathcal{A}),b,B)$ associated to an object $(\mathcal{A},\mathcal{C}(\mathcal{A}))$ of a category $\mathfrak{C}$ of topological cyclic complexes.

2.2. **Review of topological K-theory.** In order to define a topological K-functor satisfying the properties which are necessary for our study of Connes algebras, such as Bott periodicity for instance, some further conditions need to be imposed on the chosen category of locally convex algebras, see for instance Cuntz, Phillips, others? [?, ?, ?]. Although we have stated our results for a wide category of complete locally convex algebras, the Cuntz category of m-algebras will encompass all the applications we have in mind. In fact, Fréchet m-algebras will be sufficient for the applications that are carried out in the present paper. We have therefore devoted this brief review to the topological K-functor for Fréchet m-algebras, as introduced and studied by Phillips in Phillips [?]. For m-algebras, we refer the interested reader to the excellent survey CuntzSurvey [?].

If $\mathcal{A}$ is a locally convex (complete) topological algebra, we denote by $\mathcal{A}^+$ its unitalization with the semi-norms given for instance by $p_\alpha(a,\lambda) := p_\alpha(a)+|\lambda|$. Recall that any continuous homomorphism $\varphi:\mathcal{A}\to\mathcal{B}$ of locally convex topological algebras extends to the unitalizations as well as to a homomorphism of locally convex topological algebras $M_N(\mathcal{A})\to M_N(\mathcal{B})$, $N\in\mathbb{N}$, and $\mathcal{R}\mathcal{A}\to\mathcal{R}\mathcal{B}$.

Recall that a given complete topological algebra $\mathcal{A}$ as above is an m-algebra when its topology can be defined by a family of submultiplicative semi-norms Michael, Cuntz [?, ?]. A Fréchet m-algebras is an m-algebra which is Fréchet, or is a complete topological algebra whose topology is defined by a countable family of submultiplicative semi-norms. Notice that each of these two categories is stable under projective tensor products. A first important example of a Fréchet m-algebra is the algebra $\mathcal{R}$ of infinite complex matrices with rapidly decreasing entries, see ConnesBook, CuntzG, Phillips [?, ?, ?]. An element $R = (Rij)_{(i,j)\in\mathbb{N}^2}$ in $\mathcal{R}$ satisfies:

$$p_n(R) := \sum_{i,j} (1+i+j)^n |R_{ij}| \; < \; +\infty, \quad \forall n.$$

So $\mathcal{R}$ is the space of rapidly decreasing function on $\mathbb{Z}^2$ and it is a nuclear space.

Given any complete topological algebra $\mathcal{A}$, the completed projective tensor product algebra $\mathcal{R}\hat{\otimes}\mathcal{A}$ will be denoted $\mathcal{R}\mathcal{A}$. It is an m-algebra when $\mathcal{A}$ is an m-algebra Cuntz [?], and it is a Fréchet algebra when $\mathcal{A}$ is a Fréchet algebra Phillips [?]. We shall denote by $\iota : \mathcal{A} \hookrightarrow \mathcal{R}\mathcal{A}$ the inclusion of $\mathcal{A}$ as the top-left entry, $\iota(a) = a\hat{\otimes}E_{00}$.

We also consider the topological algebra $\mathcal{C}^\infty([a,b])\hat{\otimes}\mathcal{A} \simeq \mathcal{C}^\infty([a,b],\mathcal{A})$ of restrictions to $[a,b]$ of smooth $\mathcal{A}$-valued functions on the real line. For $t \in [a,b]$, evaluation at t is then a well defined continuous map $ev_t : \mathcal{C}^\infty([a,b],\mathcal{A}) \to \mathcal{A}$. The algebra of complex smooth functions on $[0,1]$ which vanish with all their derivatives at 0 and 1 will be denoted $\mathcal{C}^\infty(0,1)$, and the smooth suspension $\mathcal{S}\mathcal{A}$ of the topological algebra $\mathcal{A}$ is $\mathcal{S}\mathcal{A} := \mathcal{C}^\infty(0,1)\hat{\otimes}\mathcal{A}$. Again, if $\mathcal{A}$ is an m-algebra (resp. a Fréchet m-algebra) then so are all the above tensor products.

Given a smooth closed manifold X, we shall also consider the Fréchet m-algebra $\mathcal{C}^\infty(X)$ and the complete topological algebra $\mathcal{C}^\infty(X)\hat{\otimes}\mathcal{A}$.

$\mathcal{S}\mathcal{A} \; : \; +\mathcal{C}^\infty_\infty([a,b],\mathcal{A})$. Finally, $\mathcal{C}^\infty_0([a,b])$ is composed of those smooth functions from $\mathcal{C}^\infty([a,b])$ which vanish at a and b.

In Phillips [?], Phillips extended the usual topological K-theory functor on Banach algebras to the category of Fréchet m-algebras. We shall use these groups. More precisely, he defined a functor RK_0, the *representable K-functor*, as well as an odd version RK_1, from the category of Fréchet m-algebras to the catagory of abelian groups which extends the topological K-functors K_0 and K_1 for Banach algebras, and still satisfies the important properties, such as Bott periodicity, homotopy invariance, stability, excision, etc.

Definition 2.3. Let $\mathcal{A}$ be a locally convex topological algebra. Two idempotents $e_0, e_1 \in \mathcal{A}$ will be smoothly homotopic if there exists an idempotent $e \in \mathcal{C}^\infty([0,1],\mathcal{A})$ such that $e(0) = e_0$ and $e(1) = e_1$.

Definition 2.4. Phillips:90 [?] Let $\mathcal{A} \in \mathfrak{E}$.

(1) Denote by $\bar{P}(\mathcal{A})$ the set of idempotents $e \in M_2((\mathcal{R}\mathcal{A})^+)$ such that $e - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathcal{R}\mathcal{A})$. Then the set of smooth homotopy classes in $\bar{P}(\mathcal{A})$ is denoted $RK_0(\mathcal{A})$.
(2) Denote by $\bar{U}(\mathcal{A})$ the set of invertible elements $u \in (\mathcal{R}\mathcal{A})^+$ such that $u - 1 \in \mathcal{R}\mathcal{A}$. Then the set of smooth homotopy classes in $\bar{U}(\mathcal{A})$ is denoted $RK_1(\mathcal{A})$.

Direct sum of idempotent, or invertibles, induces well defined additions on $RK_0(\mathcal{A})$ and $RK_1(\mathcal{A})$. More precisely, for $RK_0(\mathcal{A})$ one needs to identify

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ with } 1 \oplus 0.$$

and hence the direct sum of idempotents is followed by conjugation by the invertible matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M_4(\mathbb{C}) \subset M_4((\mathcal{RA})^+).$$

Moreover, in both cases, we identify $M_2(\mathcal{R})$ with $\mathcal{R}$ in the standard way. We obtain in this way two abelian semi-groups, $RK_0(\mathcal{A})$ with the zero element given by the class of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $RK_1(\mathcal{A})$ with the zero element given by the class of 1. It is then easy to check for $RK_1(\mathcal{A})$ that $[u] + [v] := [uv]$, see again Phillips [?]. Moreover, it was proved in PhillipsToeplitz, Phillips:90 [?, ?] that for any Fréchet m-algebra $\mathcal{A}$, the semi-groups $RK_0(\mathcal{A})$ and $RK_1(\mathcal{A})$ are abelian groups and we end up with well defined covariant functors RK_0 and RK_1 on the category of Fréchet m-algebras, which satisfy the important relation $RK_1(\mathcal{A}) \simeq RK_0(\mathcal{SA})$.

The following theorem gathers the main results about the Phillips representable K-theory that will be needed in the sequel.

Theorem 2.5. Phillips [?] *Representable K-theory for Fréchet m-algebras satisfies stability, Bott periodicity, and excision.*

Explicitly, the following properties of the Phillips representable functors are satisfied:

- There exists a well defined Bott map $\beta : RK_0(\mathcal{A}) \to RK_1(\mathcal{SA})$ extending the classical one for Banach algebras. Moreover, this map is a group isomorphism.
- The canonical inclusion $\iota : \mathcal{A} \hookrightarrow \mathcal{RA}$ induces a group isomorphism on RK_0 and on RK_1.
- For any exact sequence of Fréchet m-algebras $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \to 0$, there are well defined boundary group morphisms $RK_1(\mathcal{B}) \to RK_0(\mathcal{I})$ and $RK_0(\mathcal{B}) \to RK_1(\mathcal{I})$, such that the following is a periodic six-term exact sequence:

$$\begin{tikzcd} RK_0(\mathcal{I}) \arrow[r] & RK_0(\mathcal{A}) \arrow[r] & RK_0(\mathcal{B}) \arrow[d] \\ RK_1(\mathcal{B}) \arrow[u] & RK_1(\mathcal{A}) \arrow[l] & RK_1(\mathcal{I}) \arrow[l] \end{tikzcd}$$

rem.Chern **Remark 2.6.** If $\mathfrak{C}$ is a category of topological cyclic complexes and $(\mathcal{A}, \mathcal{C}(\mathcal{A}))$ is an object of $\mathfrak{C}$. Then the natural map of complexes $\mathcal{C}^{\text{alg}}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$ will induce a natural group morphism $\text{HP}_*^{\text{alg}}(\mathcal{A}) \to \text{HP}_*^{\text{top}}(\mathcal{A})$. As a consequence, we obtain a Chern-Connes-Karoubi character Ch : $RK_i(\mathcal{A}) \to \text{HP}_i^{\text{top}}(\mathcal{A})$ extending the classical Chern-Connes-Karoubi character on algebraic K-theory: Ch : $(K_i^{\text{alg}}(A)/\sim) \to \text{HP}_i^{\text{alg}}(A)$, $i = 0, 1$ where $\sim$ denotes the diffeotopy of projections or of invertible elements Ça marche? M2ALL: On peut probablement le faire et les fleches que Victor propose pour la compatibilite avec les m-Frechet semblent bien definies, mais il faudra apres imposer les proprietes dont on a besoin, Bott, etc. ConnesNCG, ConnesBook, Karoubi? [?, ?, ?]

$$RK_i(\mathcal{A}) \to \big(K_i^{\mathrm{alg}}(\mathcal{R}\widehat{\otimes}\mathcal{A})/\sim\big) \xrightarrow{\mathrm{Ch}} \mathrm{HP}_i^{\mathrm{alg}}(\mathcal{R}\widehat{\otimes}\mathcal{A}) \xrightarrow{Tr} \mathrm{HP}_i^{\mathrm{top}}(\mathcal{A})\,. \tag{4}$$

3. Connes' principle and main results

We introduce a class of algebras for which the Chern-Connes-Karoubi character is an isomorphism (after tensoring with $\mathbb{C}$). Several examples of such algebras will be provided in Section Examples ?? in relation with algebras of symbols of certain pseudodifferential operators.

3.1. **Excisable categories.**

We shall treat excision from an abstract view point.

def.ex.hp **Definition 3.1.** We shall say that a category $\mathfrak{C}$ of topological cyclic complexes has the *excision property in periodic cyclic homology* if for any split should not be necessary if working with Frechet m-algebras admissible short exact sequence $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \to 0$ in $\mathfrak{C}$, the natural inclusion $\mathcal{C}(\mathcal{I}) \to \ker\big(\mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})\big)$ induces an isomorphism of the corresponding periodic cyclic homology groups.

The notion of admissible short exact sequence will depend on the category that we use. For m-algebras, admissible will mean linear-split while all short exact sequences of Fréchet m-algebras will be admissible. It is known that excision is satisfied in many categories of topological algebras including these two categories MeyerExcision, Cuntz1, Cuntz2, CuntzQuillen1, CuntzQu [?, ?, ?, ?, ?].

We shall assume that the (monoidal) categories of topological algebras considered in the present paper, always contain the category of m-algebras (with the completed projective tensor product) as a subcategory.

rem.obvious **Remark 3.2.** A consequence of the excision property of Definition def.ex.hp 3.1 is that there exist boundary maps $\mathrm{HP}_0^{\mathrm{top}}(\mathcal{B}) \to \mathrm{HP}_1^{\mathrm{top}}(\mathcal{I})$ and $\mathrm{HP}_1^{\mathrm{top}}(\mathcal{B}) \to \mathrm{HP}_0^{\mathrm{top}}(\mathcal{I})$, which together with the functorial morphisms yield a six-term exact sequence in periodic cyclic homology:

$$\begin{tikzcd}
\mathrm{HP}_0^{\mathrm{top}}(\mathcal{I}) \arrow[r] & \mathrm{HP}_0^{\mathrm{top}}(\mathcal{A}) \arrow[r] & \mathrm{HP}_0^{\mathrm{top}}(\mathcal{B}) \arrow[d] \\
\mathrm{HP}_1^{\mathrm{top}}(\mathcal{B}) \arrow[u] & \mathrm{HP}_1^{\mathrm{top}}(\mathcal{A}) \arrow[l] & \mathrm{HP}_1^{\mathrm{top}}(\mathcal{I}) \arrow[l]
\end{tikzcd} .$$

We shall need also the analogous excision concept in K-theory.

def.ex.K **Definition 3.3.** Let $\mathfrak{C}$ be a category consisting of topological algebras $\mathcal{A}$ with topological K-functors K_0 and K_1. We shall say that $\mathfrak{C}$ has the *K-excision property* if for any admissible short exact sequence $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \to 0$ in $\mathfrak{C}$, there exist boundary maps $K_0(\mathcal{B}) \to K_1(\mathcal{I})$ and $K_1(\mathcal{B}) \to K_0(\mathcal{I})$, which together with the functorial morphisms yield a six-term exact sequence:

$$\begin{tikzcd}
K_0(\mathcal{I}) \arrow[r] & K_0(\mathcal{A}) \arrow[r] & K_0(\mathcal{B}) \arrow[d] \\
K_1(\mathcal{B}) \arrow[u] & K_1(\mathcal{A}) \arrow[l] & K_1(\mathcal{I}) \arrow[l]
\end{tikzcd}$$

Of course, the results of Phillips state that, if $\mathfrak{C}$ is the category of *Fréchet* m-algebras with $K_* := RK_*$, then this category satisfies the K-excision property. In the same way, the results of Cuntz state that if $\mathfrak{C}$ is the larger category of m-algebras with $K_* := kk_*(\mathbb{C}, \bullet)$, the Cuntz functor, then this category also satisfies the K-excision property.

We shall need the following extension of a result in nistorDocumenta [?].

prop.natural **Proposition 3.4.** *Let $\mathfrak{C}$ be a category of topological cyclic complexes that satisfies excision in periodic cyclic homology. Let us also assume that the underlying category of topological algebras satisfies excision in K-theory. V2All: not necessary if using only Frechet m-algebras. M2ALL: see Corollary below Let $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \to 0$ be an admissible short exact sequence in $\mathfrak{C}$. Assume that for two of the algebras $\mathcal{I}$, $\mathcal{A}$, and $\mathcal{B}$, the character $\mathrm{Ch} : K_i \otimes \mathbb{C} \to \mathrm{HP}_i^{top}$ is an isomorphism, then it is an isomorphism also for the third algebra.*

Proof. The results in nistorDocumenta [?] or Cuntz CuntzG [?] imply that Ch is a natural transformation from the six-term exact sequence of Definition def.ex.K 3.3 to the six-term exact sequence of Remark rem.obvious 3.2. The result then follows from the Five Lemma Atiyah [?]. ☐

See also Solleveld [?]. V2All: En fait, j'aimerais avoir des references precises ou inclure une preuve, mais on peut faire après.

ural.Frechet **Corollary 3.5.** *Let $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \to 0$ be a short exact sequence of Fréchet m-algebras. Assume that for two of the algebras $\mathcal{I}$, $\mathcal{A}$, and $\mathcal{B}$, the character $\mathrm{Ch} : RK_i \otimes \mathbb{C} \to \mathrm{HP}_i^{top}$ is an isomorphism, then it is an isomorphism also for the third algebra.*

Proof. As already observed, the category of Fréchet m-algebras satisfies excision in K-theory and considering the complete projective cyclic complexes, we also know that excision holds for topological periodic cyclic theory. In K-theory the Phillips functors RK_* does satisfy excision, this was proved in Phillips [?], and in topological periodic cyclic homology, $\mathrm{HP}^{\mathrm{top}}$ also satisfies excision for all short exact sequences of Fréchet m-algebras since they are automatically admissible, see Cuntz [?]. Hence Theorem prop.natural 3.4 can be applied to complete the proof. ☐

3.2. **Connes algebras.**

def.maB **Definition 3.6.** A *Banach algebra-norm* $\|\cdot\|_0$ on a topological algebra $\mathcal{A}$ is a continuous norm on $\mathcal{A}$ such that $\|ab\|_0 \le \|a\|_0 \|b\|_0$.

nnes.algebra **Definition 3.7.** Let $(\mathcal{A}, \mathcal{C}(\mathcal{A}))$ be a topological cyclic complex on $\mathcal{A}$, where $\mathcal{A}$ is an algebra endowed with a Banach norm $\|\cdot\|_0$. Let A_0 be the completion of $\mathcal{A}$ in the norm $\|\cdot\|_0$. Then $(\mathcal{A}, \mathcal{C}(\mathcal{A}), \|\cdot\|_0)$ is called a *Connes algebra* if

(i) $K_n(\mathcal{A})) \to K_n(A_0) \simeq K_n(A_0)$ is an isomorphism for all n
(ii) the map $\mathrm{Ch} : K_n(\mathcal{A}) \otimes \mathbb{C} \to \mathrm{HP}_n^{\mathrm{top}}(\mathcal{A})$ is an isomorphism for all n.

This isomorphism and the Chern-Connes-Karoubi character

$$\mathrm{Ch} : K_n(\mathcal{A}) \longrightarrow \mathrm{HP}_n^{\mathrm{top}}(\mathcal{A}) \tag{5}$$

then yield Connes' character

$$\widetilde{\mathrm{Ch}} : K_n\left(A_0\right) \longrightarrow \mathrm{HP}_n^{\mathrm{top}}(\mathcal{A})\,. \tag{6}$$

We are interested in Connes algebras because of "Connes' principle", which can be stated as follows:

Theorem 3.8 (Connes' principle). *If $\mathcal{A}$ is a Connes algebra with $A_0 := \overline{\mathcal{A}}$, then the Connes' character $\widetilde{\mathrm{Ch}} : K_i\left(A_0\right) \otimes \mathbb{C} \longrightarrow \mathrm{HP}_i(\mathcal{A})$ is an isomorphism, for all i (i.e. $i \in \mathbb{Z}/2\mathbb{Z}$).*

Connes' principle permeates his earlier works on cyclic homology and it is what sets them appart from other related works on cyclic homology. It is, of course, due to Connes. It is a useful principle since the K-groups of C^*-algebras are notoriously difficult to compute.

It is an obvious observation that the class of Connes algebras is stable under finite direct sums. In the case of a smooth closed manifold, the topological periodic cyclic homology of the Fréchet m-algebra $\mathcal{C}^\infty(M)$ of complex valued smooth functions on M was shown by Connes to be isomorphic to the de Rham cohomology of M ConnesIHES [?]. Moreover, in this case, the Chern-Connes-Karoubi character coincides precisely with the Chern-Weil realization of the underlying topological Chern character, a rational isomorphism. Since the inclusion $\mathcal{C}^\infty(M) \hookrightarrow C(M)$ obviously yields an isomophism in K-theory, we conclude that $\mathcal{C}^\infty(M)$ is a Connes algebra. This is in some sense one of the building blocks of the class of Connes algebras. Another observation is that, for any (Fréchet) Connes m-algebra $\mathcal{A}$, the (Fréchet) m-algebra $\mathcal{C}^\infty(M, \mathcal{A})$ is a Connes algebra for the sup-norm associated with the given Banach norm on $\mathcal{A}$. In the same way, the (Fréchet) m-algebra $\mathcal{RA}$ is a Connes algebra when $\mathcal{A}$ is a Connes algebra, one can use many Banach completions of $\mathcal{RA}$, see CuntzSurvey [?]. See also Phillips [?] for the case of Fréchet m-algebras.

The main result of this paper states that the category of Connes algebras behaves well with respect to exact sequences. We let again $\mathfrak{B}$ denote the category of pairs $(\mathcal{A}, \|\cdot\|_0)$ consisting of a topological algebra $\mathcal{A}$ and a fixed Banach norm $\|\cdot\|_0$ on $\mathcal{A}$. The morphisms are the continuous algebra morphisms that are continuous also in the given Banach norms.

ef.Csmooth.S **Definition 3.9.** ~~Let $\mathfrak{C}$ be as in Theorem thm.main1 1.2.~~ Let $\mathcal{A}$ be a topological algebra in $\mathfrak{B}$.

- A *C-smooth stratification* of $\mathcal{A}$ is a sequence of two-sided ideals

$$0 = \mathcal{I}_0 \subset \mathcal{I}_1 \subset \ldots \mathcal{I}_{N-1} \subset \mathcal{I}_N = \mathcal{A}$$

  such that $\mathcal{I}_{k+1}/\mathcal{I}_k$ are Connes algebras for the Banach norms induced from $\mathcal{A}$.
- The topological algebra $\mathcal{A}$ is a C-smooth algebra if it admits such a C-smooth stratification.

We shall also write $I_j := \overline{\mathcal{I}}_j$. We are ready now to state the main result of this section. (in a form seemingly more general than the one in the Introduction, but, in fact, an equivalent one).

cor.Csmooth **Theorem 3.10.** *Every C-smooth algebra $\mathcal{A}$, in a category that satisfies excision in K-theory and in periodic cyclic homology is a Connes algebra. In particular, the Connes character*

$$\widetilde{\mathrm{Ch}} : K_i\left(\overline{\mathcal{A}}\right) \otimes \mathbb{C} \longrightarrow \mathrm{HP}_i^{\mathrm{top}}(\mathcal{A}) \text{ is an isomorphism.}$$

Proof. We shall proceed by induction on N. If $N = 0$ or 1, there is nothing to prove. Assume then that $N \geq 2$ and that the result is known for $\mathcal{I}_{N-1}$ and let us prove it for $\mathcal{A} = \mathcal{I}_N$. The completion of the exact sequence

$$0 \to \mathcal{I}_{N-1} \to \mathcal{I}_N \to \mathcal{I}_N/\mathcal{I}_{N-1} \to 0 \tag{7}$$

with respect to the given Banach space norms is exact since we have assumed that the norms are induced from $\mathcal{A}$. We know that $\mathcal{I}_N/\mathcal{I}_{N-1}$ is a Connes algebra by the hypothesis. We also know that $\mathcal{I}_{N-1}$ is a Connes algebra by the induction hypothesis. Theorem 1.2 then gives that $\mathcal{I}_N$ is also a Connes algebra. □

V2All: I removed the section on spectral invariance

4. Some examples of Connes algebras

We already mentionned many standard examples of Connes algebras and we now list some others that will be used in the next section in the study of the algebra of G-equivariant symbols on manifolds. The standard Goodwillie argument plays, as it is customary in cyclic homology, an important part in the applications and computations. Recall that this argument is based on the vanishing of the periodic cyclic homology for the class of (topologically) nilpotent algebras. These algebras show up naturally when dealing with appropriate quotients of ideals, and is, in some sense, a class of "negligible" Connes algebras, since their K-theory groups as well as their periodic cyclic spaces are trivial. Let us restrict ourselves again to the category of Fréchet m-algebras.

lem.top.nil **Definition 4.1.** A Fréchet m-algebra $\mathcal{N}$ will be called a topologically nilpotent algebra, if for any seminorm p there is $k \geq 1$, such that for any $f_1, \cdots, f_k \in \mathcal{N}$, $p(f_1 \cdots f_k) = 0$.

Notice that such topologically nilpotent algebra cannot be unital. It is well known that the topological periodic cyclic homology of any topologically nilpotent algebra $\mathcal{N}$ is trivial, and the same is also true for K-theory, i.e.

$$RK_*(\mathcal{N}) = \{0\} \text{ and } \mathrm{HP}_*^{\mathrm{top}}(\mathcal{N}) = \{0\}.$$

For the topological periodic cyclic homology, we refer for instance to [?, ?] (Meyer.book, Meyer.excision). As for K-theory, this is easy to check. Notice indeed that the algebra $\mathcal{RN}$ is also topologically nilpotent. Moreover, if we set $p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and let $e \in M_2((\mathcal{RN})^+)$ be any idempotent which satisfies that $e - p_0 \in M_2(\mathcal{RN})$. Then $u := ep_0 + (1-e)(1-p_0)$ is an invertible element in $M_2((\mathcal{RN})^+)$ since $1-u = (e-p_0)(1-2p_0) \in M_2(\mathcal{RN})$, and it satisfies $eu = up_0$. Notice that the Neumann series (as well as all the power series) converges in topologically nilpotent algebras. Therefore, using the smooth homotopy $u_t = \mathrm{Id}_2 - t(e-p_0)(1-2p_0)$ of invertible elements, we conclude that the class of e in $RK_0(\mathcal{N})$ is trivial. That $RK_1(\mathcal{N})$ is also trivial can be proved similarly, but notice that this also follows immediately from the isomorphism $RK_1(\mathcal{N}) \cong RK_0(\mathcal{SN})$ proved in [?] (Phillips), since the smooth suspension $\mathcal{SN}$ is then a topologically nilpotent algebra.

We thus proved the following

Proposition 4.2. *Let $\mathcal{N}$ be a topologically nilpotent Fréchet m-algebra with a Banach norm $||\bullet||_0$. Then $\mathcal{N}$ is a Connes algebra, which is spectrally invariant its Banach completion with respect to $||\bullet||_0$.*

Proof. If z is not in the spectrum of $x \in \mathcal{N}$ then $(z-x)$ is invertible in $\mathcal{N}^+$ by Lemma 4.1 (lem.top.nil), thus $\mathcal{N}$ is spectrally invariant in its Banach completion. We conclude that the algebra $\mathcal{N}$ is a Connes algebra since both K-theory and periodic cyclic homology are trivial, as explained above. □

Let us deduce some consequences for the algebras of smooth functions on compact manifolds with corners. Let X be a compact manifold with corners. For a closed subspace $Y \subset X$, $\mathcal{C}_0^\infty(X, Y)$ is the ideal in $\mathcal{C}^\infty(X)$ of those complex valued smooth functions that vanish on Y. We also consider the smaller ideal $\mathcal{C}_\infty^\infty(X, Y)$ of those smooth complex valued functions that *vanish of infinite order* on Y. We shall always assume for simplicity that Y is a CW-complex. M2ALL: IS THIS REALLY NECESSARY?

ma.isom.RKHP **Lemma 4.3.** *Let $Y \subset X$ be a closed subset as above and let I be a closed subalgebra of $\mathcal{C}_0^\infty(X,Y)$ which contains $\mathcal{C}_\infty^\infty(X,Y)$. Then*

(1) $RK_i(\mathcal{C}_\infty^\infty(X,Y)) \to RK_i(I)$ and $RK_i(I) \to K_i(\mathcal{C}_0(X \smallsetminus Y))$ are isomorphisms for all $i \in \mathbb{Z}/2\mathbb{Z}$.

(2) $\mathrm{HP}_i(\mathcal{C}_\infty^\infty(X,Y)) \to \mathrm{HP}_i(I)$ and $\mathrm{HP}_i(I) \to \mathrm{HP}_i(\mathcal{C}_0^\infty(X,Y))$ are isomorphisms for all $i \in \mathbb{Z}/2\mathbb{Z}$.

Proof. The Fréchet m-algebra $\mathcal{C}_0^\infty(X,Y)$ can be endowed with the uniform norm and is an object in the category $\mathfrak{B}$ whose completion is precisely the C^*-algebra $\mathcal{C}_0(X \smallsetminus Y)$. The same observation holds for the smaller subalgebra $\mathcal{C}_\infty^\infty(X,Y)$ which is also a Fréchet m-algebra. Moreover, it is an obviously observation that $\mathcal{C}_0^\infty(X,Y)$ is stable under holomorphic functional calculus in $\mathcal{C}_0(X \smallsetminus Y)$ and hence the inclusion induces an isomorphism $RK_i(\mathcal{C}_0^\infty(X,Y)) \to K_i(\mathcal{C}_0(X \smallsetminus Y))$.

Now notice that the quotient Fréchet m-algebras $I/\mathcal{C}_\infty^\infty(X,Y)$ and $\mathcal{C}_0^\infty(X,Y)/I$ are topologically nilpotent, when endowed with the quotient topologies. Indeed, the semi-norms on the quotient $\mathcal{C}_0^\infty(X,Y)/I$ for instance can be taken to be a sequence induced by the standard (semi-)norms on $\mathcal{C}_0^\infty(X,Y)/(I \cap \mathcal{C}_\ell^\infty(X,Y))$, where $\mathcal{C}_\ell^\infty(X,Y)$ is the algebra of smooth functions vanishing of order $\leq \ell$ on Y. Therefore since any such semi-norm on $\mathcal{C}_0^\infty(X,Y)/I \cap \mathcal{C}_\ell^\infty(X,Y)$ vanishes on ℓ-products, we conclude that $\mathcal{C}_0^\infty(X,Y)/I$ is topologically nilpotent as claimed. Applying the same argument to $I = \mathcal{C}_\infty^\infty(X,Y)$ allows to conclude.

Now, the proof is complete, since we may use excision in K-theory and in topological periodic cyclic homology for Fréchet m-algebras, for the short exact sequences

$$0 \to I \hookrightarrow \mathcal{C}_0^\infty(X,Y) \longrightarrow \mathcal{C}_0^\infty(X,Y)/I \to 0 \text{ and } 0 \to \mathcal{C}_\infty^\infty(X,Y) \hookrightarrow I \longrightarrow I/\mathcal{C}_\infty^\infty(X,Y) \to 0$$

□

We can now state the following

comm.Connes1 **Proposition 4.4.** *Recall that X is a compact manifold with corners and that Y is a closed subspace of X which is a CW-complex. Let I be a closed subalgebra of $\mathcal{C}_0^\infty(X,Y)$ containing $\mathcal{C}_\infty^\infty(X,Y)$. We endow I with the C^*-norm of $\mathcal{C}_0(X \smallsetminus Y)$ and we complete the cyclic complex with respect to the projective tensor product. Then I is a Connes algebra. In particular, both $\mathcal{C}_0^\infty(X,Y)$ and $\mathcal{C}_\infty^\infty(X,Y)$ are Connes algebras.*

Proof. M2ALL: STILL NEEDS TO BE REVIEWED. In view of Lemma lemma.isom.RKHP 4.3, we may assume that $I = \mathcal{C}(X,Y)$.

A result of Meyer's MeyerWhich? [?] states that V2V: to check. if not, if Y is empty, this is what we have dicussed and then we need the case when Y is a union of submanifolds with corners, where we could proceed by induction using the case $Y = \emptyset$ as well $\mathrm{HP}_*(\mathcal{C}_\infty^\infty(X,Y)) \simeq H^*(X,Y)$ ($* \in \mathbb{Z}/2\mathbb{Z}$, so this is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded theories, with $H^i := \oplus_{j \in i} H^j$ for all $i \in \mathbb{Z}/2\mathbb{Z}$, and complex coefficients used everywhere). We also know that the topological Chern character induces an isomorphism $K^*(X,Y) \otimes \mathbb{C} \to H^*(X,Y)$, since X has the homotopy type of a finite CW-complex and Ch is a natural transformation of cohomology theories that is an isomorphism for spheres. We also have that $RK_i(\mathcal{C}_\infty^\infty(X,Y)) \simeq K^i(X,Y)$. These isomorphisms then give that the Chern-Connes-Karoubi character $Ch : RK_i(\mathcal{C}_\infty^\infty(X,Y)) \otimes \mathbb{C} \to \mathrm{HP}_i(\mathcal{C}_\infty^\infty(X,Y))$ is an isomorphism. □

Recall that the algebra of smooth Whitney functions on the closed subspace Y of X is the quotient Fréchet m-algebra

$$\mathcal{C}^{\infty}_{\infty}(Y \subset X) := \mathcal{C}^{\infty}_{\infty}(X)/\mathcal{C}^{\infty}_{\infty}(X,Y).$$

Corollary 4.5. *The Fréchet m-algebra $\mathcal{C}^{\infty}_{\infty}(Y \subset X)$ is a Connes algebra.*

We now extend the previous results to involve Azumaya bundles over compact manifolds with corners. Let us introduce some additional structures on our compact manifold with corners X.

Notation 4.6.
(i) $\mathcal{A}, \mathcal{A}_i \to X$ are smooth, bundles of finite dimensional, *simple* algebras.
(ii) $\mathcal{F} \to X$ is a smooth bundle of finite dimensional, *semi-simple* algebras.
(iii) $E \to X$ is a smooth vector bundle on X.
(iv) $\mathcal{C}^{\infty}(X; E)$ is the vector space of smooth sections of E and $\mathcal{C}(X; \mathcal{F})$ is the vector space of *continuous* sections of $\mathcal{F}$.
(v) Let $Y \subset X$ be a closed subset, then $\mathcal{C}_0(X, Y; E)$ is the vector space of *continuous* sections of $\mathcal{F}$ that *vanish* on Y, $\mathcal{C}^{\infty}_0(X, Y; E)$ is the vector space of smooth sections of E that *vanish* on Y, and $\mathcal{C}^{\infty}_{\infty}(X, Y; E)$ is the vector space of smooth sections of E that *vanish of infinite order* on Y.

So $\mathcal{C}^{\infty}(X)$, $\mathcal{C}^{\infty}_0(X, Y)$, and $\mathcal{C}^{\infty}_{\infty}(X, Y)$ coincide respectively with $\mathcal{C}^{\infty}(X; E)$, $\mathcal{C}^{\infty}_0(X, Y; E)$, and $\mathcal{C}^{\infty}_{\infty}(X, Y; E)$ for $E = \underline{\mathbb{C}}$, the one dimensional trivial bundle.

The fibers of $\mathcal{A}$ and $\mathcal{A}_i$ are matrix algebras (one block), so they have natural C^*-norms. The fibers of $\mathcal{F}$ are direct sums of matrix algebras, so they also have natural C^*-norms. In particular, locally we have $\mathcal{F} \simeq \oplus_{i=1}^{k} \mathcal{A}_i$, but *not* globally. Similarly, we have locally that $\mathcal{A} \simeq \mathrm{End}(E)$, but not globally. In fact, every point x of X has an open neighborhood U such that $\mathcal{A}|_U \simeq U \times M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. In particular, this allows us to define natural C^*-norms on the spaces of sections, which by completion give rise to continuous sections, which also have natural C^*-norms.

We can state the following generalization of Proposition 4.3.

Proposition 4.7. *Let $Y \subset X$ be a closed subspace which is a CW-complex, and I be a closed subalgebra of $\mathcal{C}^{\infty}_0(X, Y; \mathcal{F})$ which contains $\mathcal{C}^{\infty}_{\infty}(X, Y; \mathcal{F})$. Then*

(1) $RK_i(\mathcal{C}^{\infty}_{\infty}(X, Y; \mathcal{F})) \to RK_i(I)$ and $RK_i(I) \to RK_i(\mathcal{C}_0(X \smallsetminus Y; \mathcal{F}))$ are isomorphisms for all $i \in \mathbb{Z}/2\mathbb{Z}$.

(2) $\mathrm{HP}_i(\mathcal{C}^{\infty}_{\infty}(X, Y; \mathcal{F})) \to \mathrm{HP}_i(I)$ and $\mathrm{HP}_i(I) \to \mathrm{HP}_i(\mathcal{C}^{\infty}_0(X, Y; \mathcal{F}))$ are isomorphisms for all $i \in \mathbb{Z}/2\mathbb{Z}$.

Proof. The Fréchet m-algebra $\mathcal{C}^{\infty}_0(X, Y; \mathcal{F})$ can be endowed with the uniform norm by using the natural C^*-norm on the fibers of $\mathcal{F}$. Its completion is then the C^*-algebra $\mathcal{C}_0(X \smallsetminus Y; \mathcal{F})$. The same observation holds for the smaller subalgebras $\mathcal{C}^{\infty}_{\infty}(X, Y; \mathcal{F})$ and I. Moreover, it is an obviously observation that $\mathcal{C}^{\infty}_0(X, Y; \mathcal{F})$ is spectrally invariant in $\mathcal{C}_0(X \smallsetminus Y; \mathcal{F})$. Hence the inclusion induces an isomorphism $RK_i(\mathcal{C}^{\infty}_0(X, Y; \mathcal{F})) \to K_i(\mathcal{C}_0(X \smallsetminus Y; \mathcal{F}))$.

The proof now is totally similar to the proof of Proposition 4.3 and is omitted. It is again a consequence of excision for Fréchet m-algebras and the observation that the quotient Fréchet m-algebras $I/\mathcal{C}^{\infty}_{\infty}(X, Y; \mathcal{F})$ and $\mathcal{C}^{\infty}_0(X, Y; \mathcal{F})/I$ are topologically nilpotent, when endowed with the quotient topologies, so that $RK_i(I/\mathcal{C}^\infty_\infty(X,Y;\mathcal{F}))=0$ and $\mathrm{HP}_i(I/\mathcal{C}^\infty_\infty(X,Y;\mathcal{F}))=0$ for $i\in\mathbb{Z}/2\mathbb{Z}$.

□

The algebra $\mathcal{C}^\infty(X,\mathcal{F})$ is a prototype of an Azumaya algebra (over its center) and it is then a Connes algebra as are its ideals defined above. Recall that a unital Fréchet m-algebra $\mathcal{A}$ is an Azumay algebra over its center Z if $\mathcal{A}$ is a finitely generated projective module over Z such that

$$\mathcal{A}\otimes_Z\mathcal{A}^{op}=\mathrm{End}_Z(\mathcal{A}),$$

where $\mathcal{A}^{op}$ is the algebra $\mathcal{A}$ with the opposite product and $\mathrm{End}_Z(\mathcal{A})$ is the algebra of continuous Z-linear map on $\mathcal{A}$.

Lemma 4.8. *Let X be a manifold with corners and let $\mathcal{F}\to X$ be a locally trivial bundle of semisimple algebras as above. Let Y be a closed subspace of X which is a CW complex. Set $\mathcal{A}:=\mathcal{C}^\infty(X,\mathcal{F})$ and $\mathcal{I}:=\mathcal{C}^\infty_0(X,Y;\mathcal{F})$. Then*

1. *The center $\mathcal{Z}$ of $\mathcal{A}$ is $\mathcal{C}^\infty(\mathfrak{X})$, where $\mathfrak{X}$ is a compact manifold with corners . Moreover, $\mathcal{Z}\cong\mathcal{C}^\infty(X,Z(\mathcal{F}))$, where $Z(\mathcal{F})\to X$ is the center of $\mathcal{F}$.*
2. *The algebras $\mathcal{A}$ and $\mathcal{A}/\mathcal{I}$ are Azumaya algebras over Z.*

Furthermore, the same is true replacing $\mathcal{Z}$ and $\mathcal{A}$ by their C^*-completions Z and A.

Proof. We may assume X connected. On each trivialization U of $\mathcal{F}$, we have $\mathcal{F}|_U\cong U\times\bigoplus_{k=1}^p M_{n_k}(\mathbb{C})$. This implies that the primitive ideal spectrum $\mathfrak{X}:=\mathrm{Prim}(A)$ of the C^*-completion $A:=\mathcal{C}(X,\mathcal{F})$ of $\mathcal{A}$ is a covering of X.

Indeed, above each trivialization U, we have $\mathfrak{X}|_U\cong U\times\{1,\cdots,k\}$, where we associate to $M_{k_j}(\mathbb{C})$ the integer j. This is a trivial consequence of the fact that any irreducible representation of A factors through $A/\ker(\mathrm{ev}_x)A$ because $\mathcal{C}(X)\subset Z$. If (U_i) is a finite open cover of X of trivializations of $\mathcal{F}$, we can then write $\mathfrak{X}=\bigsqcup U_i\times\{1,\cdots,p\}/\sim$, where $(x,\alpha)\sim(y,\beta)$ if $x\in U_i$, $y=x\in U_j$ and $\phi_j(x)\circ\phi_i(x)^{-1}(M_{n_\alpha}(\mathbb{C}))\subset M_{n_\beta}(\mathbb{C})$, with $\phi_k:\mathcal{F}|_{U_k}\to U_k\times\bigoplus_{k=1}^p M_{n_k}(\mathbb{C})$ the trivializations.

Now, the center $\mathcal{Z}$ of $\mathcal{A}$ coincides with the bundle over X with fibers given by the centers of the fibers of $\mathcal{F}$. It is isomorphic to the algebra $\mathcal{C}^\infty(\mathfrak{X})$ as can be checked locally. because locally on each trivialization U of $\mathcal{F}$, we have $\mathcal{C}^\infty(U,Z(\mathcal{F}|_U))\cong\bigoplus_{k=1}^p\mathcal{C}^\infty(U)\cong\mathcal{C}^\infty(U\times\{1,\cdots,p\})$.

For the second item, notice that the $\mathcal{Z}$-module $\mathcal{A}$ is projective and finitely generated, more specifically $\mathcal{A}\cong\mathcal{C}^\infty(\mathfrak{X},\mathcal{V})$, where $\mathcal{V}\to\mathfrak{X}$ is the vector bundle given by $\mathcal{V}|_{U\times\{k\}}\cong U\times\{k\}$ and glued using the transition functions induced by $\mathcal{F}$.

The proof is completed by a local inspection. Recall that if we denote by $\mathbb{C}^p$ the center of $\bigoplus_{k=1}^p M_{n_k}(\mathbb{C})$, then we have

$$\left(\oplus_{k=1}^p M_{n_k}(\mathbb{C})\right)\otimes_{\mathbb{C}^p}\left(\oplus_{k=1}^p M_{n_k}(\mathbb{C})\right)^{op}\cong\mathrm{End}_{\mathbb{C}^p}\left(\oplus_{k=1}^p M_{n_k}(\mathbb{C})\right).$$

The statement for $\mathcal{A}/\mathcal{I}$ is a direct consequence. □

umaya.Connes **Corollary 4.9.** *The algebras $\mathcal{A}=\mathcal{C}^\infty(X,\mathcal{F})$, $\mathcal{I}=\mathcal{C}^\infty_0(X,Y,\mathcal{F})$ and $\mathcal{A}/\mathcal{I}$ are Connes algebras.*

Proof. The proof for $\mathcal{A}$ and $\mathcal{A}/\mathcal{I}$ being similar, we shall only give the proof for $\mathcal{A}$. Notice thatonce we know that $\mathcal{A}$ and $\mathcal{A}/\mathcal{I}$ are Connes algebras, the result for $\mathcal{I}$ will follow readily from Theorem thm.main ??.

First notice that the algebras $\mathcal{Z}$ and $\mathcal{A}$ are spectrally invariant in their completions $Z = \mathcal{C}(\mathfrak{X})$ and $A = \mathcal{C}(X, \mathcal{F})$ thus

$$RK_i(\mathcal{Z}) \cong K_i(Z) \text{ and } RK_i(\mathcal{A}) \cong K_i(A).$$

Since $\mathcal{A}$ (respectively A) is a projective finitely generated $\mathcal{Z}$-module (respectively Z-module), there is a $\mathcal{Z}$-module $\mathcal{B}$ (respectively Z-module B) such that $\mathcal{A} \oplus \mathcal{B} \cong \mathcal{Z}^N$ (respectively $A \oplus B = Z^N$). Moreover, we have natural inclusions

$$\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{A} \overset{j}{\hookrightarrow} \mathrm{End}_Z(\mathcal{A}) \overset{k}{\hookrightarrow} M_N(\mathcal{Z}),$$

and the same is true with for C^*-completions. We thus end up with the corresponding map in K-theory

$$k_* \circ j_* \circ i_* : RK_i(\mathcal{Z}) \longrightarrow RK_i(M_N(\mathcal{Z})) \overset{\mathrm{tr}_*}{\cong} RK_i(\mathcal{Z}),$$

and similarly for the completions.

Now, the previous composite morphisms become isomorphisms after tensoring with $\mathbb{Q}$. Indeed, if $e = e^2 \in Z = \mathcal{C}(X, Z(\mathcal{F}))$ then if U is an open set trivializing $\mathcal{F}$, we have

$$k \circ j \circ i(e)|_U = e \otimes 1_A|_U \cong e_1|_U \otimes \mathrm{Id}_{n_1} \oplus \cdots \oplus e_k|_U \otimes \mathrm{Id}_{n_k}.$$

Tensoring by $\mathbb{Q}$ kills the torsion therefore, the image of e in $K_i(M_N(Z)) \otimes \mathbb{Q} \cong K_i(Z) \otimes \mathbb{Q}$ is equivalent to the element $[(n_1 e_1, \cdots, n_k e_k)]$. It follows that the map

$$k_* \circ j_* \circ i_* : K_i(Z) \otimes \mathbb{Q} \to RK_i(M_N(Z)) \otimes \mathbb{Q} \cong RK_i(Z) \otimes \mathbb{Q}$$

is invertible with inverse $\mathrm{diag}(\frac{1}{n_1}, \cdots, \frac{1}{n_k})$. It thus follows that

$$RK_i(\mathcal{Z}) \cong K_i(Z) \cong K_i(A) \cong RK_i(\mathcal{A}).$$

Applying Proposition cor.comm.Connes1 4.10, we deduce that

$$\mathrm{Ch} : RK_i(\mathcal{Z}) \otimes \mathbb{C} \cong K^i(\mathfrak{X}) \otimes \mathbb{C} \longrightarrow \mathrm{HP}_i^{\mathrm{top}}(\mathcal{Z})$$

is an isomorphism. This implies in turn that the composite morphism

$$k_* \circ j_* \circ i_* : \mathrm{HP}_i^{\mathrm{top}}(\mathcal{Z}) \longrightarrow \mathrm{HP}_i^{\mathrm{top}}(M_N(\mathcal{Z})) \cong \mathrm{HP}_i^{\mathrm{top}}(\mathcal{Z}),$$

is also an isomorphism, whence $\mathrm{HP}_i^{\mathrm{top}}(\mathcal{Z}) \cong \mathrm{HP}_i^{\mathrm{top}}(\mathcal{A})$. Moreover, the Chern character is a natural transformation, it therefore follows that

$$\mathrm{Ch} : RK_i(\mathcal{A}) \otimes \mathbb{C} \cong K_i(A) \otimes \mathbb{C} \to \mathrm{HP}_i^{\mathrm{top}}(\mathcal{A}).$$

is an isomorphism. ∎

comm.Connes1 **Corollary 4.10.** *Let I be a closed subalgebra of $\mathcal{C}_0^\infty(X, Y; \mathcal{F})$ containing $\mathcal{C}_\infty^\infty(X, Y; \mathcal{F})$. We endow I with the C^*-norm of $\mathcal{C}_0(X, Y; \mathcal{F})$ and we complete the cyclic complex with respect to the projective tensor product. Then I is a Connes algebra. In particular, both $\mathcal{C}_0^\infty(X, Y; \mathcal{F})$ and $\mathcal{C}_\infty^\infty(X, Y; \mathcal{F})$ are Connes algebras.*

Proof. M2ALL: DUE TO THE PREVIOUS ADDED MATERIAL FROM ALEX FILE, WE MAY OMIT THIS PROOF AND JUST SAY.

In view of Proposition lemma.isom.RKHP.Azumaya 4.7 and Corollary Azumaya.Connes 4.9, the proof is obvious, and it is omitted here.

In view of Lemma lemma.isom.RKHP 4.3, we may assume that $I = \mathcal{C}(X, Y; \mathcal{F})$. Let $\tilde{X}$ be the primitive ideal spectrum of $\mathcal{C}(X; \mathcal{F})$, which is an Azumaya algebra (with possible different dimensions of the fibers on different components of $\tilde{X}$). Then $\tilde{X} \to X$ is a finite covering map, so $\tilde{X}$ aquires a natural manifold structure from X. Let $\tilde{Y}$ be the preimage of Y in $\tilde{X}$. We can then replace X with $\tilde{X}$, Y with $\tilde{Y}$, and $\mathcal{F}$ with an Azumaya bundle $\mathcal{A}$.

We then have that the inclusion $\mathcal{C}^\infty_\infty(X, Y) \to \mathcal{C}^\infty_\infty(X, Y; \mathcal{A})$ induces isomorphisms $\mathrm{HP}_*(\mathcal{C}^\infty_\infty(X, Y)) \to \mathrm{HP}_*(\mathcal{C}^\infty_\infty(X, Y; \mathcal{A}))$ and $RK_*(\mathcal{C}^\infty_\infty(X, Y)) \otimes \mathbb{C} \to RK_*(\mathcal{C}^\infty_\infty(X, Y; \mathcal{A})) \otimes \mathbb{C}$. This is the matrix trick $Z \to A$ and $A \to \mathrm{End}_Z(A)$. This allows us to further simplify our setting by assuming that $\mathcal{A} = \underline{\mathbb{C}}$, so that $\mathcal{C}^\infty_\infty(X, Y) = \mathcal{C}^\infty_\infty(X, Y; \mathcal{A})$

A result of Meyer's MeyerWhich? [?] states that V2V: to check. if not, if Y is empty, this is what we have dicussed and then we need the case when Y is a union of submanifolds with corners, where we could proceed by induction using the case $Y = \emptyset$ as well $\mathrm{HP}_*(\mathcal{C}^\infty_\infty(X, Y)) \simeq H^*(X, Y)$ ($* \in \mathbb{Z}/2\mathbb{Z}$, so this is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded theories, with $H^i := \oplus_{j \in i} H^j$ for all $i \in \mathbb{Z}/2\mathbb{Z}$, and complex coefficients used everywhere). We also know that the Chern character induces an isomorphism $K^*(X, Y) \otimes \mathbb{C} \to H^*(X, Y)$, since X has the homotopy type of a finite CW-complex and Ch is a natural transformation of cohomology theories that is an isomorphism for spheres. We also have that $RK_i(\mathcal{C}^\infty_\infty(X, Y)) \simeq K^i(X, Y)$. These isomorphisms then give that the Chern-Connes-Karoubi character $Ch : RK_i(\mathcal{C}^\infty_\infty(X, Y)) \otimes \mathbb{C} \to \mathrm{HP}_i(\mathcal{C}^\infty_\infty(X, Y))$ is an isomorphism. $\square$

5. Group actions on Azumaya bundles

The goal of this section is to prove Theorem 5.9.

5.1. **G-manifolds with corners.** We shall use the notation in [?] (page 4 and pages 49-50) for transformation groups and the notation from [?] for manifolds with corners and blow-ups. See also [?, ?]. The reader should also consult these references for the missing definitions or proofs.

As in [?], we let X be a compact manifold with corners with a smooth G-action. Recall that G is a compact Lie group. We may assume that X is endowed with a smooth Riemannian metric and that the action of G is isometric. We shall consider *submanifolds* of X in the strong sense that they have tubular neighborhoods. Thus, if $Y \subset X$ is a *submanifold with corners* of X, then Y is closed and there is a neighborhood U of Y in X such that U is G-diffeomorphic to the normal vector bundle $N^X Y := TX|_Y/TY \to Y$ (the normal vector bundle to Y in X) to via a diffeomorphism of manifolds with corners mapping the zero section of $N^X Y$ to Y. This is the concept of manifolds with corners considered in [?]. In particular, a submanifold with corners of X is also a p-submanifold of X [?, ?, ?], but the converse is not true. A submanifold with corners is called *an interior p-submanifold* in [?].

If $x \in X$, G_x denotes the isotropy group of x, namely,

$$G_x := \{\gamma \in G \mid \gamma(x) = x\} . \tag{8}$$

We know that there exist only a finitely many conjugacy classes $C_j = (H_j) = (G_{x_j})$, $j = 1, \ldots, N$, of isotropy groups G_x, $x \in X$. Then $C_j \leq C_k$ if there is a subgroup in C_j that is contained in a subgroup of C_k. We may assume the numbering to be such that,

$$C_j \leq C_k \Longrightarrow j \geq k.$$

Given a subgroup $H \subset G$, we let (H) denote the set of subgroups of G conjugated to H. If $K \subset G$ is another subgroup, we hence write $(H) \leq (K)$ if H is conjugated to a subgroup of K. In the following, H and K will denote subgroups of G. We shall use the following standard notations

$$\begin{aligned} X_H &:= \{x \in M \mid G_x = H\} , \\ X(H) &:= \{x \in X \mid (G_x) = (H)\} , \\ X(\geq H) &:= \{x \in X \mid\mid (G_x) \geq (H)\} , \quad \text{and} \\ X(\leq H) &:= \{x \in X \mid\mid (G_x) \leq (H)\} . \end{aligned} \tag{9}$$

As usual, X^H denotes the set of fixed points by H. It is known then that

$$GX^H = X(\geq H) \ \text{ and } \ GX_H = X(H) . \tag{10}$$

We also denote by $N(H) := \{g \in G \mid g^{-1}Hg = H\}$ the normalizer of H in G. Moreover, $N(H)$ acts on X^H and we have the following diffeomorphism which will be used later on

$$X(H) \simeq (G/H) \times_{N(H)/H} X_H , \tag{11}$$

with the induced free action of $N(H)/H$ on X_H (Proposition 1.91 and Corollaries 1.92 and 1.94 of [?]). In particular, the principal orbit bundle of the action of $N(H)$ on X^H is precisely X_H.

..dense.alex1 **Lemma 5.1.** *We have $X(\geq H) \subset \overline{X(H)}$.*

Proof. M2A: I HAVE TRIED TO SIMPLIFY THIS PROOF. PLEASE DOUBLE-CHECK I DIDN'T MAKE ANY CONFUSION. Since X_H is the principal orbit bundle of the action of $N(H)$ on X^H, we have $\overline{X_H} = X^H$. Therefore,

$$X(\geq H) = GX^H = G\overline{X_H} = \overline{GX_H} = \overline{X(H)}.$$

Let $x \in X(\geq \Gamma)$ and assume that $x \in X \smallsetminus \overline{M_{(\Gamma)}}$. Then by G-invariance of the open set $M \smallsetminus \overline{M_{(\Gamma)}}$ there is a tube W_x around x contained in $M \smallsetminus \overline{M_{(\Gamma)}} \subset M \smallsetminus \overline{M_\Gamma}$. But because It follows that the orbit Gx is disjoint from M^Γ. This is a contradiction with Proposition 1.77 of [?], which states that $x \in M(\geq \Gamma) = GM^\Gamma$. □

As in [?], we shall say that the action of G on X is *bounday intersection free* if, given a closed face F of M and $g \in G$, we have either $gF = F$ or $gF \cap F = \emptyset$. Notice that if the action of G on X is boundary intersection free, then so is the action of any subgroup $H \subset G$.

The first part of the following statement is a result from [?].

lemma.AM **Lemma 5.2.** *Assume that the action of G on X is boundary intersection free and let H be a subgroup of G. Then X^H is a (closed) submanifold with corners of X. In particular, $\partial[X^H] = [\partial X]^H = X^H \cap \partial X$. Similarly, $X(H)$ is a manifold with corners (but not closed, in general, and hence not a submanifold with corners in general) and $\partial[X(H)] = [\partial X](H) = X(H) \cap \partial X$.*

Proof. Let us fix a G-invariant metric on X. Let $x \in X^H$. Let us suppose that x belongs to an open face F and let us choose a G-invariant neighborhood of x in F. By using the exponential map in directions normal to F, we obtain then that x has a H-invariant neighborhood of the form $U \times [0,1)^j$ with the action of H being diagonal and trivial on $[0,1)^j$ since G is compact and its action on X is intersection free. Hence $(U \times [0,1)^j)^H = U^H \times [0,1)^j$ and U^H is a smooth manifold without boundary. This proves the last statement. By choosing a tubular neighborhood of U^H in U, we obtain a tubular neighborhood of $U^H \times [0,1)^j$ in $U \times [0,1)^j$ and hence that X^H is a submanifold with corners. The last equation is proved in a completely similar way, but choosing U to be G invariant. □

Lemma 5.2 allows us to unambigously write

$$\partial X^H = \partial[X^H] = [\partial X]^H = X^H \cap \partial X \text{ and } \partial X(H) = \partial[X(H)] = [\partial X](H) = X(H) \cap \partial X.$$

That is "$\partial(\cdot)$ commutes with $\cdot^H$ and with $\cdot(H)$."

Remark 5.3. The statement of the last lemma is a local statement, so no assumption of having embedded corners is required.

We formulate the following result as a lemma, for the purpose of further referencing it. Except the statement about X_1, it is Lemma 1.81 of [?].

lemma.closed **Lemma 5.4.** *Assume again that the action of G on compact manifold with corners X is boundary intersection free, and let $\{H_1, H_2, \ldots, H_N\}$ be a complete set of representatives of* *conjugacy classes of isotropy groups with the previous ordering, i.e.* $(H_i) \geq (H_j)$ *implies* $i \leq j$. *Set for* $1 \leq k \leq N$

$$X_k := X(H_1) \cup X(H_2) \cup \ldots \cup X(H_k). \tag{12}$$

Then X_1 *is a closed p-submanifold of* X*, and for* $2 \leq k \leq N$*,* X_k *is always a closed subset of* X *which is not a manifold in general.*

The subspaces $X(H_j)$, $j = 1, \ldots, N$ define a stratification of X.

5.2. **G-equivariant bundles and algebras.** Recall the notations not.algebras1 4.6 which will be now endowed with an action of our compact Lie group G. So, $\mathcal{A}, \mathcal{A}_i \to X$ are G-equivariant bundles of finite dimensional *simple* algebras, and $\mathcal{F} \to X$ is a G-equivariant bundle of finite dimensional, *semi-simple* algebras. Given a closed CW-subspace $Y \subset X$ with $GY = Y$, recall also the spaces $\mathcal{C}_0^\infty(X, Y; E)$ and $\mathcal{C}_\infty^\infty(X, Y; E)$ which are now G-spaces.

comm.Connes2 **Proposition 5.5.** *With the above notations, let* I *be as in Proposition* cor.comm.Connes1 4.10 *such that* $GI = I$. *Assume also that the action of* G *on* X *has a single isotropy type* H. *Then* I^G *is a Connes algebra. In particular, both*

$$\mathcal{C}_0^\infty(X, Y; \mathcal{F})^G \simeq \mathcal{C}_0^\infty(X^H/N(H), Y^H/N(H); \mathcal{F}^H/N(H)) \quad \textit{and}$$
$$\mathcal{C}_\infty^\infty(X, Y; \mathcal{F})^G \simeq \mathcal{C}_\infty^\infty(X^H/N(H), Y^H/N(H); \mathcal{F}^H/N(H))$$

are Connes algebras.

Proof. Let $\Gamma := N(H)/H$. The assumptions combined with the diffeomorphism of Equation (eq.diff.oneH 11) give $X = X(H) \simeq (G/H) \times_\Gamma X_H = (G/H) \times_\Gamma X^H$, and hence

$$\mathcal{C}_0^\infty(X, Y; \mathcal{F})^G \simeq \mathcal{C}_0^\infty(X^H, Y^H; \mathcal{F}^H)^\Gamma \quad \text{and} \quad \mathcal{C}_\infty^\infty(X, Y; \mathcal{F})^G \simeq \mathcal{C}_\infty^\infty(X^H, Y^H; \mathcal{F}^H)^\Gamma .$$

Since the action of $\Gamma := N(H)/H$ on X^H is free, the quotient X^H/Γ is also a manifold with corners and $\mathcal{F}^H$ descends to a bundle of algebras $\mathcal{F}^H/\Gamma \to X^H/G$ such that

$$\mathcal{C}_0^\infty(X^H, Y^H; \mathcal{F}^H)^\Gamma \simeq \mathcal{C}_0^\infty(X^H/\Gamma, Y^H/\Gamma; \mathcal{F}^H/\Gamma) \quad \text{and}$$
$$\mathcal{C}_\infty^\infty(X^H, Y^H; \mathcal{F}^H)^\Gamma \simeq \mathcal{C}_\infty^\infty(X^H/\Gamma, Y^H/\Gamma; \mathcal{F}^H/\Gamma) .$$

The result then follows from Corollary cor.comm.Connes1 4.10 applied to $(X^H/\Gamma, Y^H/\Gamma, \mathcal{F}^H/\Gamma)$ and I^G. □

Note that, even if $\mathcal{F}$ is a trivial bundle, $\mathcal{F}^H/\Gamma$ need not be so, so the extra generality afforded by our setting of sections in algebra bundles is necessary.

5.3. **Blow-ups of singular strata.** See Appendix AppendixA A for some notations and properties of blow-ups. Recall that Y is a closed submanifold of the compact manifold with corners X and that $[X : Y]$ denotes the blow-up manifold. Assume that a Lie group Γ acts smoothly on X such that $\Gamma Y = Y$. Then the action of Γ on X lifts to a smooth action of Γ on $[X : Y]$ (this was proved in the case Γ compact in AlbinMelrose [?] and in general in mougel1 [?]).

The topology on the blow-up$[X : Y]$ is best understood by restricting to a tubular neighborhood U of Y diffeomorphic to N^XY. Let B_1 be the set of vectors of lenght < 1. Then the neighoborhood $\beta^{-1}(U)$ of SN^XY in $[X : Y]$ is identifies with $N^XY \smallsetminus B_1$ such that $\beta : N^XY \smallsetminus B_1 \to N^XY$ is given by the formula $\beta(x) = \frac{\|x\|-1}{\|x\|}x$. Details qui peuvent être enlevés.

Let $\{H_1, H_2, \dots, H_N\}$ be a complete set of representatives of conjugacy classes of isotropy groups of G acting on X. Let

$$V_k := X(H_1) \cup X(H_2) \cup \dots \cup X(H_k),$$

$k = 1, \dots, N$, which is a closed subset of X for each k, by Lemma lemma.closed 5.4. Let $\mathcal{F}$ be a G-equivariant bundle of finite dimensional algebras on X.

not.Ik **Notation 5.6.** We let $I_0 := \mathcal{C}^\infty_\infty(X, \partial X; \mathcal{F})$ and $I_k := \mathcal{C}^\infty_0(X, V_k; \mathcal{F}) \cap I_0$, for $k \geq 1$.

In particular,

$$0 = I_N \subset I_{N-1} \subset \dots \subset I_1 \subset I_0 := \mathcal{C}^\infty_\infty(X, \partial X; \mathcal{F})\,.$$

Intuitively, we have

$$\mathcal{C}^\infty_0(V_k, V_{k-1}; \mathcal{F}) \cap \mathcal{C}^\infty_\infty(V_k, \partial V_k; \mathcal{F}) := I_{k-1}/I_k := \mathcal{C}^\infty_0(X, V_{k-1}; \mathcal{F}) \cap I_0 / \mathcal{C}^\infty_0(X, V_k; \mathcal{F}) \cap I_0\,,$$

which, for the moment, is an abuse of notation since we do not have smooth structures on the spaces V_k, $k \geq 2$, in general. However, soon we will G-equivariantly identify $V_k \smallsetminus (V_{k-1} \cup \partial V_k)$ with the interior of a manifold with corners Y_k following AlbinMelrose [?] (see also KBook [?]).

Let $X_1 := X$ and let us define by induction X_k to be a compact manifold with corners with isotropy types $\{H_k, H_{k+1}, \dots, H_N\}$, $k \leq N$. Assume $X_1, X_2, \dots, X_k$ have been defined with the required property. Then let $Y_k := X_k(H_k)$ (which we will eventually see to be a compact manifold with corners with interior $V_k \smallsetminus V_{k-1}$, as desired). Then $Y_k := X_k(H_k)$ is a closed submanifold of X_k since H_k is a maximal isotropy group of X_k, see AlbinMelrose, KBook [?, ?]. Hence we may define

$$X_{k+1} := [X_k : Y_k] := [X_k : X_k(H_k)]\,. \tag{13}$$

As in AlbinMelrose [?], the only isotropy types of X_{k+1} are as claimed (i.e. $\{H_{k+1}, H_{k+2}, \dots, H_N\}$). Let us recall how this is proved. First, the group H_k will have no more fixed points on the blow-up X_{k+1}, since the action of H_k on the normal bundle $N^{X_k}Y_k$ has no non-zero fixed points (otherwise there would be points with isotropy H_k outside Y_k, which is a contradiction since V_k was defined as the set of all points with isotropy conjugated to H_k and $Y_k := X_k(H_k) = GX_k^{H_k}$, since H_k is a maximal isotropy in X_k). Second, all the isotropy groups of $[X_k : Y_k]$ are contained in isotropy groups of X. Hence all the isotropy types of $[X_k : Y_k]$ are in $\{H_{k+1}, H_{k+2}, \dots, H_N\}$. All these isotropy types appear, since Y_k has minimal isotropy type in X_k. Maybe not necessary?

We next want to identify the effect of these blow-ups on the sets $X_{k+1}(H_j)$, $j \geq k+1$, which we know are manifolds with corners in view of Lemma lemma.AM 5.2 (which the reader should review now since it will be used again below). First of all, the disjoint union decomposition defining X_{k+1} as the blow-up of X_k along its subset $Y_k := X_k(H_k)$ of points of isotropy of type H_k, namely $X_{k+1} = \big[X_k \smallsetminus Y_k\big] \cup SN^{X_k}Y_k$, gives for $j \geq k+1$

$$X_{k+1}(H_j) = \big[X_k \smallsetminus Y_k\big](H_j) \cup \big[SN^{X_k}Y_k\big](H_j)\,. \tag{14}$$

Since β is a diffeomorphism outside $SN^{X_k}Y_k := SN^{X_k}X_k(H_k)$, we further have

$$\big[X_{k+1} \smallsetminus SN^{X_k}Y_k\big](H_j) = \big[X_k \smallsetminus Y_k\big](H_j) = X_k(H_j)\,, \tag{15}$$

since Y_k has isotropy type H_k and $j \geq k+1$. Lemma 5.2 allows us to give an unambigous sense to $\partial X(H) = [\partial X](H) = \partial[X(H)]$ and, together with the fact that $SN^{X_k}Y_k$ is contained in the boundary of X_{k+1} gives that the blow-down map induces for $j \geq k+1$ a diffeomorphism

eq.bd.iso.Xk
$$\beta : X_{k+1}(H_j) \smallsetminus \partial X_{k+1}(H_j) \simeq X_k(H_j) \smallsetminus \partial X_k(H_j) \simeq \ldots \simeq X_1(H_j) \smallsetminus \partial X_1(H_j)\,. \tag{16}$$

(Thus, in case $X_1 = X$ does not have a boundary, $Y_k := X_k(H_k)$ will be a compact manifold with corners with interior diffeomorphic to the stratum $X_1(H_k)$, that is, the desired compactification of $X_1(H_k)$.)

Recall that $V_k := X(H_1) \cup X(H_2) \cup \ldots \cup X(H_k)$. We obtain then (with $Y_{k+1} := X_{k+1}(H_{k+1})$ and $X_1 = X$, as before) that

eq.needed?
$$\beta(Y_{k+1}) \subset V_{k+1} \quad \text{and} \quad \beta(\partial Y_{k+1}) \subset V_k \cup \partial X_1\,. \tag{17}$$

Consequently, given a smooth section f of $\mathcal{F}$ on V_{k+1} that vanishes on $V_k \cup \partial X$, then $\beta^*(f) := f \circ \beta$ will be a smooth section of (the pull-back of) $\mathcal{F}$ on the compact manifold $Y_{k+1} := X_{k+1}(H_{k+1})$ that vanishes on its boundary. This proves the second inclusion of the following proposition, the proof of the first one (by induction on N) being relegated to the next subsection.

op.essential
Proposition 5.7. *Let $\beta : X_{k+1} \to X_1 := X$ be the composition of all the blow-down maps and $J_{k+1} := \cancel{\beta^*(\mathcal{C}_0^\infty(V_{k+1}, V_k; \mathcal{F})) =} \beta^*(I_k)/\beta^*(I_{k+1})$. Let $0 \leq k \leq N-1$. Then $I_k/I_{k+1} \to J_{k+1}$ is an isomorphism of topological algebras and*

$$\mathcal{C}_\infty^\infty(Y_{k+1}, \partial Y_{k+1}; \mathcal{F}) \subset J_{k+1} \subset \mathcal{C}_0^\infty(Y_{k+1}, \partial Y_{k+1}; \mathcal{F})\,.$$

As we have already mentioned, it follows right away (from definitions) that $J_{k+1} \subset \mathcal{C}_0^\infty(Y_{k+1}, \partial Y_{k+1}; \mathcal{F})$. It is also easy to prove that $\mathcal{C}_c^\infty(Y_{k+1}, \partial Y_{k+1}; \mathcal{F}) \subset J_{k+1}$, but we have not been able to use this observation to prove our result. In any case, this adds credibility to our statement and justifies to postpone its proof.

cor.1.main0
Corollary 5.8. *We use the notation of Proposition 5.7. Then the algebras J_{k+1} and J_{k+1}^G are Connes algebra.*

Proof. The fact that J_{k+1} is a Connes algebra is an immediate consequence of Corollary 4.10. Since $Y_{k+1} := X_{k+1}(H_{k+1})$ has a single isotropy type, Corollary 5.5 also yields right away that $\mathcal{C}_\infty^\infty(Y_{k+1}, \partial Y_{k+1}; \mathcal{F})^G$ is a Connes algebra. Since $J_{k+1}^G / \mathcal{C}_\infty^\infty(Y_{k+1}, \partial Y_{k+1}; \mathcal{F})^G$ is a topologically nilpotent algebra as can be checked easily, using the argument in the proof of Lemma 4.3, DETAILS! we obtain that J_{k+1}^G is a Connes algebra as well. □

For the previous sections, to prove that a direct sum of Connes algebras is again Connes. M2ALL: ADDED A SENTENCE ON DIRECT SUMS.

As a corollary, we are now in position to state the following.

main.thm
Theorem 5.9. *We keep the notations of Corollary 5.8. Then*

1. *The algebras $\mathcal{C}_\infty^\infty(X, \partial X; \mathcal{F})^G$ and $\mathcal{C}_0^\infty(X, \partial X; \mathcal{F})^G$ are Connes algebras.*
2. *Assume furthermore that X has embedded faces, then the algebra $\mathcal{C}^\infty(X; \mathcal{F})^G$ is a Connes algebra.*

Proof.
The fact that $\mathcal{C}^\infty_\infty(X, \partial X; \mathcal{F})^G$ is a Connes algebra follows from Theorem ?? and the fact that it has a stratification with subquotients J^G_{k+1}. The fact that $\mathcal{C}^\infty_0(X, \partial X; \mathcal{F})^G$ is a Connes algebra follows from the fact that $\mathcal{C}^\infty_0(X, \partial X; \mathcal{F})^G / \mathcal{C}^\infty_\infty(X, \partial X; \mathcal{F})^G$ is topologically nilpotent.

For the second item, notice that $\mathcal{C}^\infty(X; \mathcal{F})^G$ has a composition series with subquotients $\mathcal{C}^\infty_0(F, \partial F; \mathcal{F})^G$, where F ranges through the set of closed faces of X. Details, this works if X has embedded faces, that is, when each face has a defining function. ☐

5.4. **Proof of Proposition 5.7.** The proof of Proposition 5.7 is an induction on N, the number of isotropy types of X and will be split into a sequence of lemmas. We shall freely use the notation of the previous subsection and Appendix A (but we also recall some of the most important ones from time to time). Since the result is local, we may assume that $\mathcal{F} = \underline{\mathbb{C}}$, thus drop it from the notation.

If $N = 1$, $J_1 = I_0 = \mathcal{C}^\infty_\infty(X_1, \partial X_1)$, $Y_1 = X = X_1$ and there is nothing to prove.

Let us now turn to the induction step, by assuming the result to be true if there are $N-1$ isotropy types and prove it if there are N isotropy types. Let $\{H_1, H_2, \ldots H_N\}$ the isotropy types of $X =: X_1$. To simplify notation, let $Y := Y_1$, $\tilde{X} := [X : Y_1] = [X : Y_1]$ where, we recall $Y_1 := X(H_1)$. Similarly, let $\tilde{Y} := \beta^{-1}(Y)$. We let $\beta^*(Y_k) := \overline{Y_k \smallsetminus Y_1} \subset \tilde{X}$, the closure being in $\tilde{X}$.

We are ready now to takle the induction step. We only need to prove that $\mathcal{C}^\infty_\infty(Y_{k+1}, \partial Y_{k+1}) \subset J_{k+1}$ since the inclusion $J_{k+1} \subset \mathcal{C}^\infty_0(Y_{k+1}, \partial Y_{k+1})$ is obvious. Let $\tilde{Y}_k := \beta^*(Y_k) := \overline{\beta^{-1}(Y_k \smallsetminus Y)}$ (closure in $\tilde{X}$). Let us consider the composition series of 5.6 for $\tilde{X}$, but we shift the indices to account for the fact that $\tilde{X}$ has only $N-1$ isotropy types:

def.tilde.Ik
$$\tilde{I}_k := \mathcal{C}^\infty_0(\tilde{X}, \tilde{Y}_k) \cap \mathcal{C}^\infty_\infty(\tilde{X}, \partial \tilde{X}), \tag{18}$$

with $\tilde{I}_1 := \mathcal{C}^\infty_\infty(\tilde{X}, \partial \tilde{X})$. (Thus the definition of I_k is obtained from the definition of $\tilde{I}_k$ by removing all the symbols ˜s.)

Recall from Lemma A.1 that The pull back $\beta^*(f) := f \circ \beta$ defines an isomorphism $\beta^* : \mathcal{C}_0(X, Y) \to \mathcal{C}_0(\tilde{X}, \tilde{Y})$. Moreover, Lemma A.3 shows that it also defines an isomorphism

$$\beta^* : \mathcal{C}^\infty_\infty(X, Y) \longrightarrow \mathcal{C}^\infty_\infty(\tilde{X}, \tilde{Y}).$$

lemma.four **Lemma 5.10.** *Let $\beta^{*-1} : \mathcal{C}^\infty_\infty(\tilde{X}, \tilde{Y}) \to \mathcal{C}^\infty_\infty(X, Y)$ be the map of Lemma A.3. Then $\beta^{*-1}(\tilde{I}_k) \subset I_k$, $k = 1, \ldots, N$.*

Proof. We have that $\partial \tilde{X} = \tilde{Y} \cup \beta^{-1}(\partial X)$. Hence, with $\tilde{Y}_1 = \tilde{Y}$, we have that Hence $\tilde{I}_1 := \mathcal{C}^\infty_\infty(\tilde{X}, \partial \tilde{X}) = \mathcal{C}^\infty_\infty(\tilde{X}, \tilde{Y}_1) \cap \mathcal{C}^\infty_\infty(\tilde{X}, \beta^{-1}(\partial X))$. Since $\tilde{Y}_1 \subset \tilde{Y}_k$ for all k, we obtain that, for all k, we have

$$\tilde{I}_k := \mathcal{C}^\infty_0(\tilde{X}, \tilde{Y}_k) \cap \mathcal{C}^\infty_\infty(\tilde{X}, \tilde{Y}) \cap \mathcal{C}^\infty_\infty(\tilde{X}, \beta^{-1}(\partial X)), \tag{19}$$

for all k (that is, $k = 1, \ldots, N$). Let then $f \in \tilde{I}_k$. By Lemma A.3, $\beta^{*-1}(f) \in \mathcal{C}^\infty_\infty(X, Y)$. It is obvious that $\beta^{*-1}(f)$ also vanishes of infinite order on ∂X since $Y = Y_1$ is an (interior) submanifold with corners (so $\partial X \smallsetminus Y$ is dense in ∂X). It $(\beta^{*-1}(f))$ also vanishes on Y_k

since f vanishes (even of infinite order) on $\tilde{Y}_1$ and it vanishes on $\tilde{Y}_k \smallsetminus \tilde{Y}_1$, which is mapped bijectively onto its image by β. Hence $\beta^{*-1}(f) \in \mathcal{C}_\infty^\infty(X,Y) \cap \mathcal{C}_0^\infty(X,Y_k) =: I_k$. $\square$

We can now complete the proof. By the induction hypothesis (and since the space Y_{k+1} is the same for both X and $\tilde{X}$, by construction), we have

$$\mathcal{C}_\infty^\infty(Y_{k+1}, \partial Y_{k+1}; \mathcal{F}) \subset \tilde{I}_k/\tilde{I}_{k+1} \xrightarrow{\beta^{*-1}} I_k/I_{k+1}\,. \tag{20}$$

5.5. **Crossed product with finite groups.** We keep the notations of the previous subsections. Then we can state:

Theorem 5.11. *M2ALL: DO WE STATE THE STATEMENT RELATIVE TO ∂X HERE? Assume that G is a finite group. Let X be a compact boundary intersection free G-manifold with corners and let $E \to X$ be a complex G-bundle over X. Then the crossed product algebra $\mathcal{C}^\infty(X, \mathrm{End}(E)) \rtimes G$, is a Connes Fréchet m-algebra, which is spectrallly invariant in its C^*-completion $\mathcal{C}(X, \mathrm{End}(E)) \rtimes G$. In particular, the Chern-Connes-Karoubi character induces an isomorphism*

$$\widetilde{\mathrm{Ch}} : K_n\left(\mathcal{C}(X, \mathrm{End}(E)) \rtimes G\right) \xrightarrow{\simeq} \mathrm{HP}_n^{\mathrm{top}}\left(\mathcal{C}^\infty(X, \mathrm{End}(E)) \rtimes G\right), \quad n = 0, 1.$$

Proof. We denote $\mathcal{A} := \mathcal{C}^\infty(X, \mathrm{End}(E))$, $A = \mathcal{C}(X, \mathrm{End}(E))$, $\mathcal{B} := \mathcal{A} \rtimes G$ and $B := A \rtimes G$. The algebra $\mathcal{B}$ can be identified with the algebra $(\mathcal{A} \otimes \mathrm{End}(V))^G$ where V is the finite dimensional G-representation $V = \ell^2 G$. In the same way the algebra B can be identified with the C^*-algebra $(A \otimes \mathrm{End}(V))^G$. Indeed, an algebra isomorphism is obtained by identifying any kernel $G \times G \to I$ which is G-equivariant with a function of a single variable in G, and it is easy to check that this is a topological identification for the smooth functions and a C^*-algebra isomorphism for the completions. Therefore, it remains to show that the Fréchet m-algebra $(\mathcal{A} \otimes \mathrm{End}(V))^G$ is a Connes algebra which is spectrally invariant in its C^*-completion $(A \otimes \mathrm{End}(V))^G$. Now, this is a consequence of Theorem 5.9. $\square$

5.6. **Pseudodifferential operators.** We still need to prove that $\Psi^0(M;E)^G$ is a Connes algebra. We have that $\Psi^{-\infty}(M;E)^G$ is a direct sum of the algebra $\mathcal{R}$ of rapidly decreasing matrices, so it is a Connes algebra (here we have a problem, in fact, since the direct sum may be infinite and infinite direct sums do not behave well with respect to being Connes, this problem does not arise for π_α of these algebras.s)

That gives that it is a Connes algebra. Then $\Psi^{-1}(M;E)^G/\Psi^{-\infty}(M;E)^G$ is topologically nilpotent. I will repeat this statement in the section on applications.

5.7. **Alexandre's second lemma in general.** We shall need the following lemma.

Pour $J = \cap_{n\geq 1} I^n$ je n'ai pas encore trouvé de référence. Peut-être qu'avec la bonne hypothése sur les semi-normes de I,

on peut répéter l'argument de Meyer pour dire $HP(\mathcal{C}_0^\infty(X,Y)/\mathcal{C}_\infty^\infty(X,Y)) = 0$ (Theorem 6.3 page 30) avec topologiquement nilpotent:

"We claim that the algebra N is topologically nilpotent: if p is any continuous semi-norm on N, then there is $k \in N$ such that p vanishes on all products of k elements in N. Since functions in N vanish on Y , products of k functions in N vanish on Y to order k. These are annihilated by p for sufficiently high k because any continuous semi-norm on N only involves finitely many derivatives"

AppendixA

Appendix A. Smooth functions and blow-ups

Recall that if X is a manifold with corners and $Y \subset X$ is a submanifold with corners, then the normal bundle N^XY of Y in X is diffeomorphic to an open neighborhood U of Y in X by a diffeomorphism ϕ that maps the zero section of the normal bundle N^XY to Y. Let SN^XY be the unit sphere bundle of N^XY. (So its fibers are spheres, as the name indicates it.) Then MelroseBook, KottkeLin, mougel1 [?, ?, ?] the *blow-up of X along* Y (or with respect to Y) is the *disjoint union*

eq.def.blow-up

$$[X : Y] := (X \smallsetminus Y) \cup SN^XY\,. \tag{21}$$

It comes equipped with a *blow-down* map $\beta : [X : Y] \to X$ that is the identity on $X \smallsetminus Y$ and is the bundle projection $SN^XY \to Y$ on SN^XY. We shall identify without further comment $X \smallsetminus Y$ with $[X : Y] \smallsetminus \beta^{-1}(Y)$ in what follows.

Let us denote $[X : Y]$ by $\tilde X$ and set $\tilde Y := \beta^{-1}(Y)$.

lemma.one **Lemma A.1.** *The pull back $\beta^*(f) := f \circ \beta$ defines an isomorphism $\beta^* : \mathcal{C}_0(X, Y) \to \mathcal{C}_0(\tilde X, \tilde Y)$.*

Proof. This is because the blow-down map $\beta : \tilde X \to X$ is such that X has the quotient topology and $\beta(\tilde Y) = Y$. □

Let us denote by $\mathrm{Diff}(X)$, respectively, $\mathrm{Diff}(\tilde X)$ the algebra of differential operators with smooth coefficients on X, respectively, $\tilde X$. Let r_Y be a smooth function on X such that, close to Y it is the distance to Y_1 and otherwise it is > 0 outside Y. Then it is known that r_Y lifts to a smooth function $r_Y \circ \beta$ on $\tilde X$ whose zero set is exactly $\tilde Y$.

Notice that

eq.needed.sp

$$\begin{aligned}
\mathcal{C}^\infty_\infty(X, Y) &:= \{u \in \mathcal{C}_0(X, Y) \mid Pu \in \mathcal{C}_0(X, Y) \ \text{ for all } P \in \mathrm{Diff}(X)\}\,,\\
\mathcal{C}^\infty_\infty(\tilde X, \tilde Y) &:= \{u \in \mathcal{C}_0(\tilde X, \tilde Y) \mid Pu \in \mathcal{C}_0(\tilde X, \tilde Y) \ \text{ for all } \tilde P \in \mathrm{Diff}(\tilde X)\}\,,\\
& r_Y^{-1}\mathcal{C}^\infty_\infty(X, Y) \subset \mathcal{C}^\infty_\infty(X, Y)\,, \quad \text{and}\\
& r_Y^{-1}\mathcal{C}^\infty_\infty(\tilde X, \tilde Y) \subset \mathcal{C}^\infty_\infty(\tilde X, \tilde Y)\,.
\end{aligned} \tag{22}$$

lemma.two **Lemma A.2.** *We have the following equalities as operators on $\mathcal{C}^\infty_c(X \smallsetminus Y) = \mathcal{C}^\infty_c(\tilde X \smallsetminus \tilde Y)$:*

(1) $\mathrm{Diff}(X) \subset \cup_{k=1}^\infty r_Y^{-k}\,\mathrm{Diff}(\tilde X)$ *and, similarly,*
(2) $\mathrm{Diff}(\tilde X) \subset \cup_{k=1}^\infty r_Y \circ \beta^{-k}\mathcal{C}^\infty(\tilde X)\,\mathrm{Diff}(X)$.

Proof. This is a standard fact about blow-ups. For the first relation, since r_Y is smooth on $\tilde X$, $\cup_{k\in\mathbb{N}} r_Y^{-k}\,\mathrm{Diff}(\tilde X)$ is an algebra. It is enough then to prove our statement for a system of generators of $\mathrm{Diff}(X)$. This is clear if P is a multiplication operator, since a smooth function on X lifts to a smooth function on $\tilde X$. Let v be a vector field on X. Then it is known (see, for instance acn [?]) that there exists a vector field $\tilde v$ on $\tilde X$ that restricts to v in the interior. Hence $\tilde X \in \mathrm{Diff}(\tilde X)$ and $X = r_Y \circ \beta^{-1}\tilde X \in \cup_{k\in\mathbb{N}} r_Y^{-k}\,\mathrm{Diff}(\tilde X)$, as desired.

It is easy to see in local coordinates that the algebra $\mathrm{Diff}(\tilde X)$ is generated by $\mathcal{C}^\infty(\tilde X)$, $r_Y \circ \beta^{-1}$, and the lifts of vector fields $r_Y v$, with v a vector field on X. more details here? □

We now come to the following lemma which is used in the proof of Proposition prop.essential 5.7, but which is obviously of independent interest.

This is the only lemma that is not a formal manipulation and required some ideas. We did it with Alexander for $Y = \{0\}$ and $X = \mathbb{R}^2$.

lemma.three **Lemma A.3.** *The pull back $\beta^*(f) := f \circ \beta$ defines an isomorphism*

$$\beta^* : \mathcal{C}^\infty_\infty(X, Y) \to \mathcal{C}^\infty_\infty(\tilde{X}, \tilde{Y})\,.$$

Proof. This follows from the previous two lemmas and Equation (22). Indeed, let $f \in \mathcal{C}^\infty_\infty(X, Y)$ and $\tilde{P} \in \mathrm{Diff}(\tilde{X})$. We want to prove that $\tilde{P}(f \circ \beta) \in \mathcal{C}_0(\tilde{X}, \tilde{Y})$. Then, by the second part of Lemma A.2 we may assume that $\tilde{P} = ar_Y^{-k}P$, with $P \in \mathrm{Diff}(X)$. Then $r_Y^{-k}Pf \in \mathcal{C}^\infty_\infty(X, Y)$ and hence $\tilde{P}(f \circ \beta) = a(r_Y^{-k}Pf) \circ \beta \in \mathcal{C}_0(\tilde{X}, \tilde{Y})$, by Lemma A.1. This shows that the map $\beta^* : \mathcal{C}^\infty_\infty(X, Y) \to \mathcal{C}^\infty_\infty(\tilde{X}, \tilde{Y})$ is well defined. It is obviusly injective since $X \smallsetminus Y$ is dense in both X and $\tilde{X}$. Let us prove that it is onto. Let $g \in \mathcal{C}^\infty_\infty(\tilde{X}, \tilde{Y})$. Then $g = f \circ \beta$ with some $f \in \mathcal{C}_0(X, Y)$, again by lemma A.1. Let $P \in \mathrm{Diff}(X)$. We similarly want to prove that $Pf \in \mathcal{C}_0(X, Y)$. We have that $P = r_Y^{-k}Q$ for some $Q \in \mathrm{Diff}(\tilde{X})$. Then $Pf \circ \beta = r_Y^{-k}Qg$. Since $r_Y^{-k}Qg \in \mathcal{C}_0(\tilde{X}, \tilde{Y})$ by the assumption on g, we have that $Pf \in \mathcal{C}_0(X, Y)$, again by Lemma A.1. ☐

HP.basic

Appendix B. Periodic homology versus basic cohomology

M2ALL: NOT REALLY NEEDED, MAYBE REMOVE EVENTUALLY.

We briefly review in this appendix a folklore theorem relating periodic cyclic homology of G-invariant functions with G-basic cohomology. Let G be a compact Lie group which acts smoothly by diffeomorphisms on the smooth compact manifold M. For simplicity we assume here that M is a closed manifold. If Z is a closed G-subspace of M which is a G-CW-subcomplex, then we denote by $\mathcal{C}^\infty(M, Z)$ the G-algebra of smooth sections of the algebra functions which vanish over Z, in particular $\mathcal{C}^\infty(M) = \mathcal{C}^\infty(M, \emptyset)$. Recall that a (complex) differential k-form ω on M is a G-basic form if it is G-invariant and valued in the normal bundle to the orbits, i.e. if

$$g^*\omega = \omega \text{ and } i_{\tilde{X}}\omega = 0, \quad \forall g \in G \text{ and } X \in \mathfrak{g},$$

where $\tilde{X}$ is the vector field generated by a fixed vector X from the Lie algebra $\mathfrak{g}$ of G. The de Rham subcomplex of G-basic differential forms on M is denoted $(\Omega^*_{G-bas}(M), d)$. With the extra subspace Z, we denote by $(\Omega^*_{G-bas}(M, Z), d)$ the subcomplex of G-basic forms which vanish of infinite order on the points of Z. The G-basic cohomology spaces are denoted $H^*_{G-bas}(M)$ and $H^*_{G-bas}(M, Z)$. Recall that $\mathcal{C}^\infty_\infty(M, Z)^G$ is the closed ideal of G-invariant smooth functions on M which vanish of infinite order on Z. We shall prove the following

WithoutV **Proposition B.1.** *Let $\mathcal{I}$ be a closed ideal in $\mathcal{C}^\infty_0(M, Z)$ which contains $\mathcal{C}^\infty_\infty(M, Z)$ and such that $G\mathcal{I} = \mathcal{I}$. Then $\mathcal{I}^G$ is a Connes algebra whose periodic cyclic homology is isomorphic to the G-basic cohomology of the G-pair (M, Z), i.e.*

$$\mathrm{HP}^{\mathrm{top}}_i\left(\mathcal{I}^G\right) \simeq \oplus_{k \geq 0} \mathrm{H}^{i+2k}_{G-bas}(M, Z), \quad i = 0, 1.$$

Moreover, this latter is also isomorphic to the relative Čech cohomology of the pair $(M/G, Z/G)$.

Proof. The algebra $\mathcal{C}_0^\infty(M,Z)^G$ is dense in the C^*-algebra $C_0(M\smallsetminus Z)^G$ and stable under holomorphic functional calculus, therefore

$$RK_*\left(\mathcal{C}_0^\infty(M,Z)^G\right)\simeq K_*\left(C_0(M\smallsetminus Z)^G\right)\simeq K^*(M/G,Z/G).$$

Since the quotient Fréchet m-algebra $\mathcal{C}_0^\infty(M,Z)^G/\mathcal{I}^G$ is topologically nilpotent, we deduce from excision and Proposition **??** that

$$RK_*\left(\mathcal{I}^G\right)\simeq RK_*\left(\mathcal{C}_0^\infty(M,Z)^G\right)\simeq K_*\left(C_0(M\smallsetminus Z)^G\right)\ \text{ and }\ \mathrm{HP}_*^{\mathrm{top}}\left(\mathcal{I}^G\right)\simeq \mathrm{HP}_*^{\mathrm{top}}\left(\mathcal{C}_0^\infty(M,Z)^G\right).$$

Since this is as well true when $\mathcal{I}=\mathcal{C}_\infty^\infty(M,Z)$, we only need to identify the periodic cyclic homology and the Chern-Connes-Karoubi character for $\mathcal{C}_\infty^\infty(M,Z)^G$.

On the other hand, the topological Chern character is known to be a rational isomorphism, for all finite CW-complexes and hence also for finite CW-pairs like $(M/G,Z/G)$, i.e. it yields an isomorphism, for $i\in\mathbb{Z}_2$, from $K^i(M/G,Z/G)\otimes\mathbb{C}$ to the Čech cohomology $\check{H}^{[i]}(M/G,Z/G)=\oplus_{k\geq 0}\check{H}^{i+2k}(M/G,Z/G)$ with complex coefficients. Now, by using a good open cover of M/G Verona [?][Corollary 3.8], or rather its pull-back in M, we can form as usually a Čech-de Rham bicomplex but built out of the G-basic forms. This proves by the standard argument that each G-basic cohomology space $H^k_{G-bas}(M,Z)$ is isomorphic to the corresponding Čech cohomology space $\check{H}^k(M/G,Z/G)$, both with complex coefficients. Now, the HKR map yields a well defined chain map χ between the periodic cyclic complex of the algebra $\mathcal{C}_\infty^\infty(M,Z)^G$ and the G-basic complex of the compact G-pair (M,Z). We hence end up with a linear map

$$\chi_*:\mathrm{HP}_i\left(\mathcal{C}_\infty^\infty(M,Z)^G\right)\longrightarrow \oplus_k H^{i+2k}_{G-bas}(M,Z)\simeq\oplus_{k\geq 0}\check{H}^{i+2k}(M/G,Z/G),$$

such that, the following diagram, whose horizontal maps are the Chern-Connes-Karoubi character and the topological Chern character, commutes

$$\begin{tikzcd}
K_i\left(\mathcal{C}_\infty^\infty(M,Z)^G\right)\otimes\mathbb{C} \arrow[r] \arrow[d] & \mathrm{HP}_i^{\mathrm{top}}\left(\mathcal{C}_\infty^\infty(M,Z)^G\right) \arrow[d] \\
K^i(M/G,Z/G)\otimes\mathbb{C} \arrow[r] & \oplus_k\check{H}^{i+2k}(M/G,Z/G)\simeq\oplus_k H^{i+2k}_{G-bas}(M,Z)
\end{tikzcd}$$

This is a counterpart of the Chern-Weil realization theorem in our setting. It thus only remains to check that χ_* is an isomorphism, a standard fact which can be obtained for instance by comparing the Čech-de Rham bicomplex alluded to above with the similar Čech-$\mathrm{HP}^{\mathrm{top}}$ bicomplex, and the fact that the HKR map gives a chain morphism between them which becomes in an obvious way an isomorphism for one of the filtrations. See for instance BlockGetzler [?] and also MathaiStevenson [?] where this approach is used with the $\mathbb{Z}$-graded version of the periodic cyclic complex. We conclude that $\mathcal{I}^G$ is a Connes algebra which is spectrally invariant in its C^*-completion $\mathcal{C}_0(M\smallsetminus Z)^G$ and that the periodic cyclic homology of $\mathcal{I}^G$ is isomorphic to the G-basic relative de Rham cohomology of the pair (M,Z) (forms vanishing of infinite order at Z). The proof is now complete. □

Corollary B.2. *The quotient algebra $\mathcal{C}_\infty^\infty(M)^G/\mathcal{C}_\infty^\infty(M,F)^G$ of G-invariant Whitney functions on $Z\subset M$, is a Connes algebra.*

Appendix C. Survey on spectral invariance

M2ALL: NOT REALLY NEEDED, MAYBE REMOVE EVENTUALLY?

Given a unital algebra morphism $\phi : C \to D$, we shall say that C is a ϕ-full in D if $\phi^{-1}(G(D)) = G(C)$ where $G(\bullet)$ denotes the group of invertibles of a given unital algebra $\bullet$. In other words, C is ϕ-full in D if any $c \in C$ whose image $\phi(c)$ is invertible, is already invertible in C. If ϕ is injective, then C is ϕ-full in D if the subalgebra $\phi(C)$ if a full subalgebra of D, in this case we shall sometimes identify C with its image in D and simply call C a full subalgebra of D when ϕ is clear from the context. All ideals will be two sided.

We assume that the (unital) topological algebra $\mathcal{A}$ belongs to the category $\mathfrak{B}$ and hence has a Banach algebra completion $A = \overline{\mathcal{A}}$. The following is a convenient criterion.

Lemma C.1. [?] *$\mathcal{A}$ is a full subalgebra of A if and only if there exists $\epsilon > 0$ such that the following property holds for any $a \in \mathcal{A}$:*

$$||a|| < \epsilon \Longrightarrow (1 + a) \in G(\mathcal{A}).$$

Proof. If $\mathcal{A}$ is a full subagebra of its Banach algebra completion A, then one just takes $\epsilon = 1$. Assume conversely that such $\epsilon > 0$ exists, and let x be an element in $\mathcal{A} \cap G(A)$. Denote by L_x left multiplication by x and let $W \subset G(A)$ by the inverse image by L_x of the open ϵ-ball centered at $1 \in \mathcal{A}$. We then choose an element $y \in W \cap \mathcal{A}$. Then we have

$$||1 - xy|| < \epsilon \text{ and } xy \in \mathcal{A} \cap G(A).$$

Therefore and by assumption, we deduce that there exists $z \in \mathcal{A}$ such that $(xy)^{-1} = 1 + z$. We conclude that $x^{-1} = y(xy)^{-1}$ also belongs to $\mathcal{A}$. □

Let now $\mathcal{I}$ be a closed ideal in $\mathcal{A}$ and denote by I its completion in A, so that I is a closed two-sided ideal in A. Denote by $\varphi : \mathcal{B} := \mathcal{A}/\mathcal{I} \longrightarrow B := A/I$ the induced map. The following lemma is proved in [?]:

Lemma C.2. *$\mathcal{A}$ is full in A if and only if $\mathcal{I}$ is full in I and $\mathcal{B}$ is φ-full in B.*

Proof. Assume that $\mathcal{A}$ is full in A and let $x \in \mathcal{I}$ be such that $||x|| < 1$. Then $(1 + x)^{-1} \in \mathcal{A}$ and

$$(1 + x)^{-1} - 1 = -(1 + x)^{-1}x \in \mathcal{I}.$$

Assume now that $x \in \mathcal{A}$ is such that there exist $y \in A$ and $(z_1, z_2) \in I^2$ with

$$y j_A(x) - z_1 = 1 \text{ and } j_A(x)y - z_2 = 1.$$

A density argument shows that there exist $y' \in \mathcal{A}$ and $(z'_1, z'_2) \in \mathcal{I}^2$ such that

$$||j_A(xy' - \iota(z'_1)) - 1|| < 1 \text{ and } ||j_A(y'x - \iota(z'_2)) - 1|| < 1.$$

Hence, $j_A(xy' - \iota(z'_1))$ and $j_A(y'x - \iota(z'_2))$ are invertible in A and thus in $\mathcal{A}$. In other words, $\exists (v, v') \in \mathcal{A}^2$ such that

$$(xy' - \iota(z'_1))v = 1 \text{ and } v'(y'x - \iota(z'_2)) = 1.$$

Applying π we conclude.

For the converse, we reproduce the argument from [?] LMN. Choose $\eta > 0$ such that for any $a \in A$ we have

$$||\hat{\pi}(a) - 1|| < \eta \Longrightarrow ||\hat{\pi}(a)^{-1} - 1|| < \frac{1}{8}.$$

Let then $x \in \mathcal{A}$ be such that $||j_A(x) - 1|| < \eta$. Then there exists $y \in \mathcal{A}$ such that

$$z_1 := xy - 1 \in \mathcal{I} \text{ and } ||\hat{\pi}(j_A(y)) - 1|| < \frac{1}{8}.$$

By density of $\mathcal{I}$ in I, there exists $z_2 \in \mathcal{I}$ such that $||y - 1 + z_2|| < \frac{1}{4}$. But then $z_1 + xz_2 \in \mathcal{I}$ and we have

$$\begin{aligned} ||z_1 + xz_2|| &= ||xy - 1 + x(z_2 + y - 1) - x(y-1)|| = ||x - 1 + x(z_2 + y - 1)|| \\ &\leq ||x - 1|| + ||x|| \times ||z_2 + y - 1|| < \frac{1}{2} + (1 + (1/2))\frac{1}{4} < 1. \end{aligned}$$

Therefore, $1 + z_1 + xz_2$ is invertible in $1 + I$ and hence in $1 + \mathcal{I}$. It is now clear that $(y + z_2)(1 + z_1 + xz_2)^{-1}$ belongs to $\mathcal{A}$ and that

$$x\left[(y + z_2)(1 + z_1 + xz_2)^{-1}\right] = 1.$$

□

M2M: ADD RELATION WITH HFC AND ALSO WITH K-THEORY.

Institut fur Analysis, Welfengarten 1, 30167 Hannover, Germany
Email address: alexandre.baldare@math.uni-hannover.de

IMAG, Univ. Montpellier, CNRS, 34090 Montpellier, France
Email address: moulay.benameur@umontpellier.fr

IECL, Université Lorraine, CNRS, 57000 Metz, France
Email address: nistor@univ-lorraine.fr
URL: http://www.iecl.univ-lorraine.fr/~Victor.Nistor