

22050 Signals and linear systems in continuous time

Kaj-Åge Henneberg

L01

Course introduction

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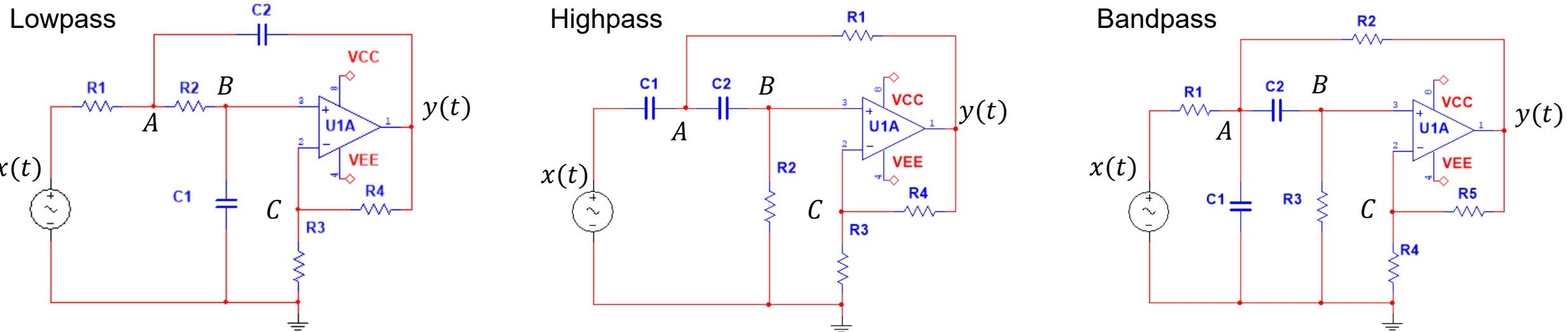
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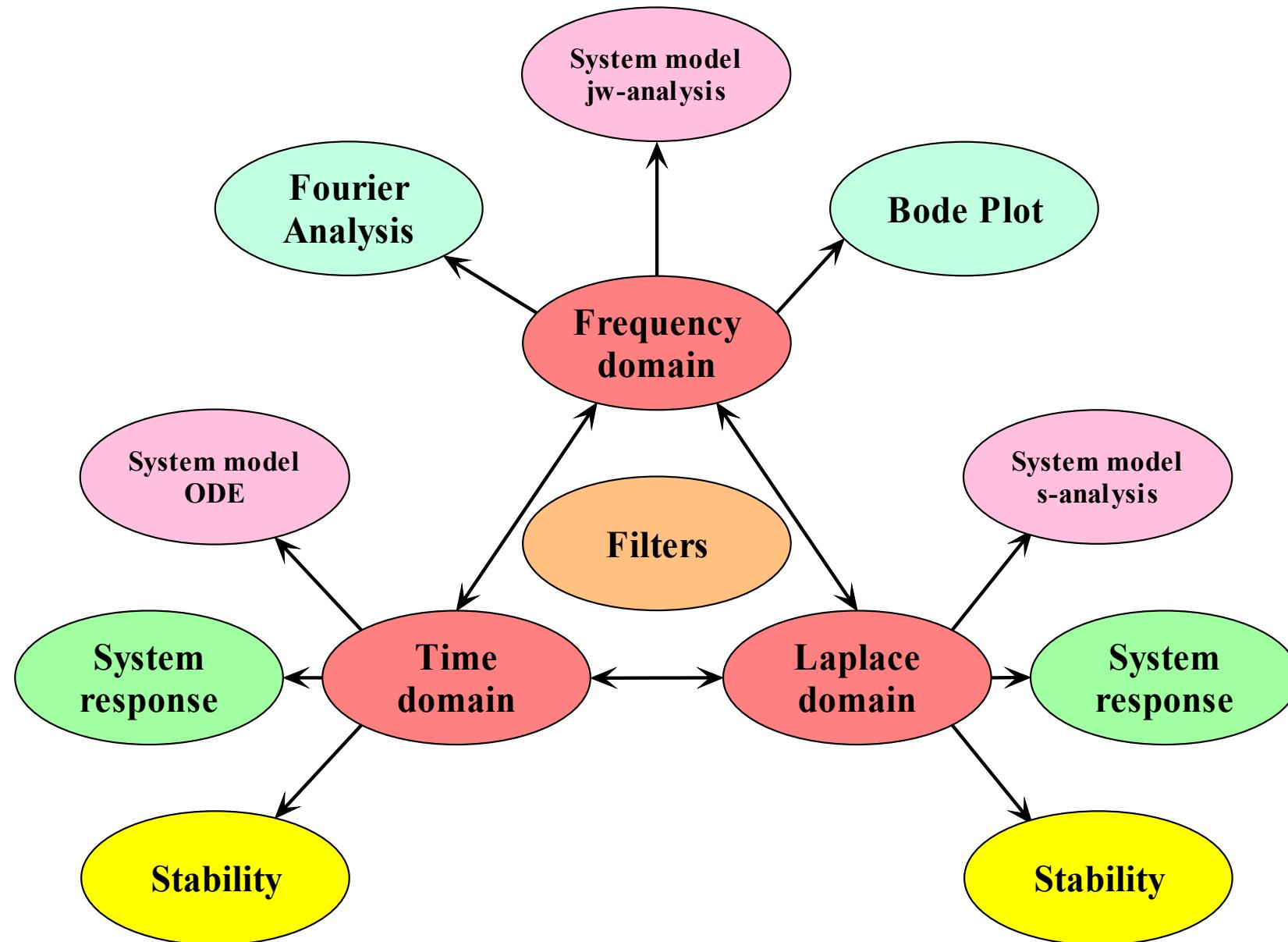
Analysis of systems performance in analogue signal processing circuits

1. Time domain analysis of system response to test functions.
2. Frequency domain properties of filter circuits.
3. Design of filter circuits with robust performance.

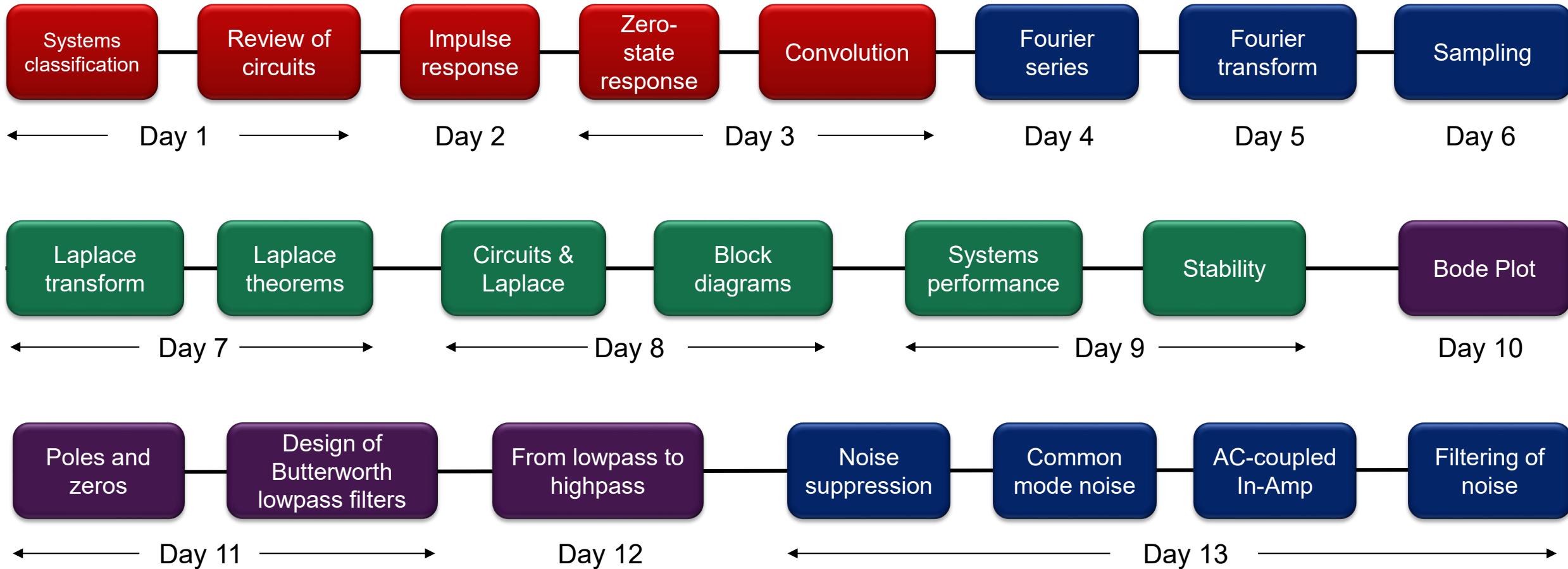


Course topics

- System
 - Model
 - Response
 - Performance
- Domains
 - Time
 - Frequency
 - s-domain



Roadmap of the course



Overview day-by-day

- **Time domain** – day 1 to 3
 - **Day 1**
 - Classification of signals and systems into various classes.
 - Setting up differential equations for active filter circuits.
 - Practice the use of Maple and Multisim
 - **Day 2**
 - **Zero-input response:** Calculation of output signals due to initial conditions within a circuit.
 - **Impulse response:** Calculation of output signal due to an impulse input signal.
 - **Day 3**
 - **Zero-state response:** Calculation of output signals due to input signals, when initial conditions are zero.
 - **Convolution:** Any input signal can be thought of as an infinite number of points on a curve. Each point has a time delay from zero and an amplitude. We can calculate the output response for each such point and add up all the responses to obtain the complete response. This procedure is called “convolution”. Convolution allows us to calculate output signals from input signals represented as sampled data. When we solve differential equations, we need a function to describe the input signal. With convolution, we can use data points.

Overview day-by-day

- **Fourier domain** – day 4 to 6

- Day 4

- **Trigonometric Fourier series:** Signals can be represented as a sum of sinusoidal signals. The mathematical procedure of the Fourier series calculates the amplitude and time delay of each frequency component making up the signal.
 - **Complex exponential Fourier series:** It is mathematically convenient to express signals as a sum of complex exponential functions because differentiation of exponential functions returns an exponential function. Hence differentiation of a basis function returns the same basis function.

- Day 5

- **Fourier transform:** Fourier series cannot represent aperiodic signals with infinite duration. The Fourier transform is an extension of the Fourier series expansion which handles aperiodic signals.
 - **Fourier theorems:** The Fourier transformation creates frequency domain operators equivalent to integration, differentiation, time shift, frequency shift, convolutions and more.

- Day 6

- **Analog to digital conversion:** Sampling analog signals with an analog-to-digital converter (ADC) measures a signal at equidistant time intervals (sampling) and rounds off the amplitude (quantization) depending on the number of bits used by the ADC. A lowpass filter must precede the ADC. The parameters of the ADC define the performance criteria for the lowpass filter.

Overview day-by-day

- **Laplace domain** – day 7 to 9

- Day 7

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 - **Laplace theorems:** The Laplace transformation creates Laplace domain operators equivalent to integration, differentiation, time shift, frequency shift, convolutions and more.

- Day 8

- **Solving dif. equations:** We will use the concepts of Laplace transformation to set up circuit equations directly in the Laplace domain. We will then solve the equation to obtain the step response.

- Day 9

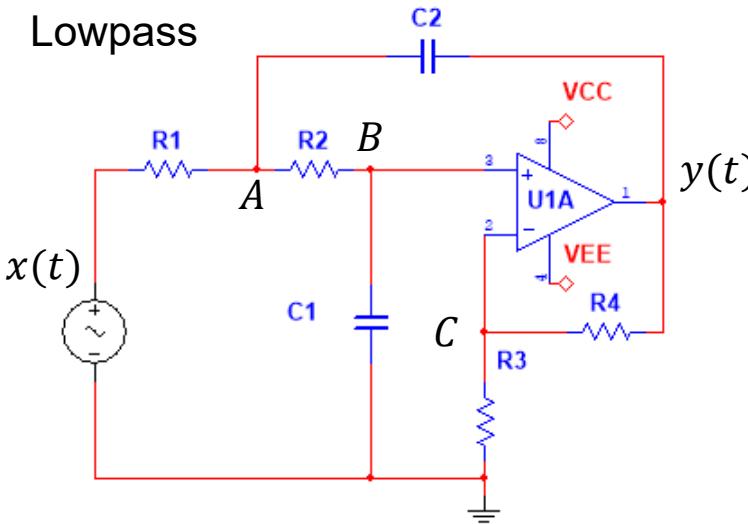
- **Systems performance:** Laplace domain analysis is used to analyze the properties of overdamped, critically damped and underdamped systems in both the time domain and the frequency domain.
 - **Stability:** Laplace domain analysis is used to study the stability of second order systems and to elucidate the conditions for stability.

Overview day-by-day

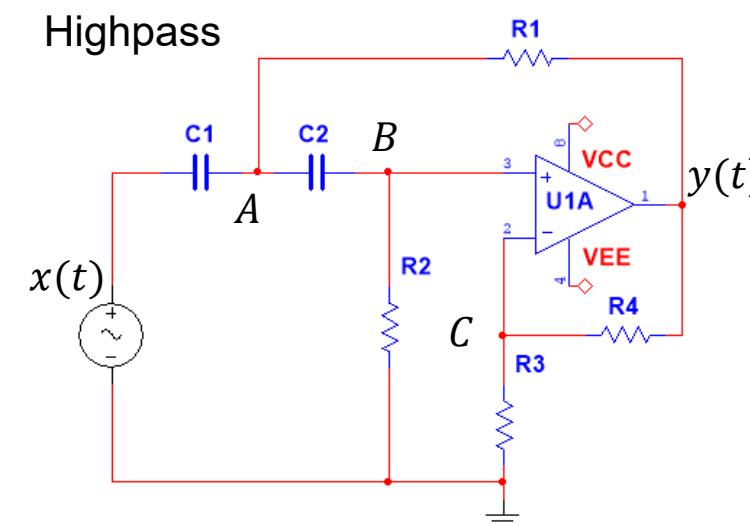
- **Filter design** – day 10 to 13
 - Day 10
 - **Bode plot:** Filter design requires standardized procedures for plotting amplitude and phase spectra. The Bode plot procedure defines this standard and enable us to identify which part of a system that is responsible for different features in the amplitude and phase spectra. The Bode plot procedure thus tells us if there is a problem with system performance, and also what part of the system is responsible for the undesirable performance.
 - Day 11
 - **Designing active filters:** The performance function for a filter of the Butterworth class is derived and the resistors and capacitors in a filter of the Sallen-Key circuit topology are calculated.
 - Day 12
 - Design of highpass filters by frequency transformation of lowpass filters.
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 - **Noise suppression:** Application of frequency domain concepts to evaluate suppression of common mode and differential mode noise using differential amplifiers and filters.

Course topics

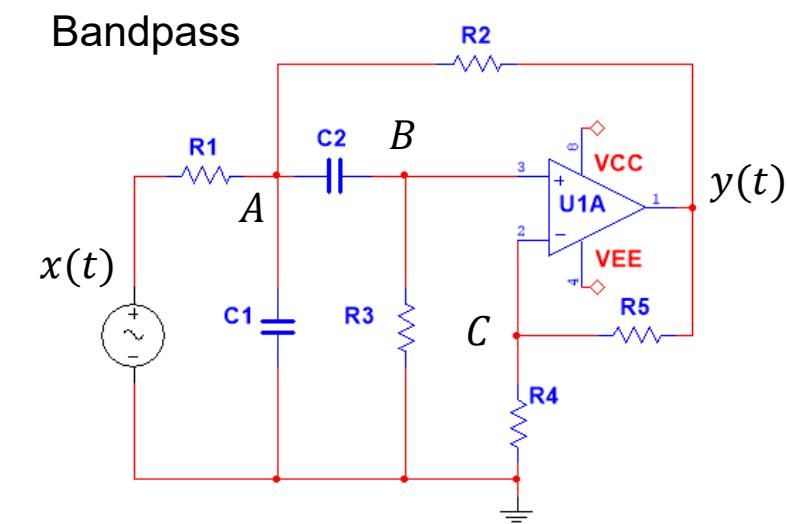
Lowpass



Highpass



Bandpass



All three circuits can be modelled by this differential equation: $\ddot{y} + a_1\dot{y} + a_0y = b_2\ddot{x} + b_1\dot{x} + b_0x$

The solution of the differential equation depends on the roots of the characteristic equation:

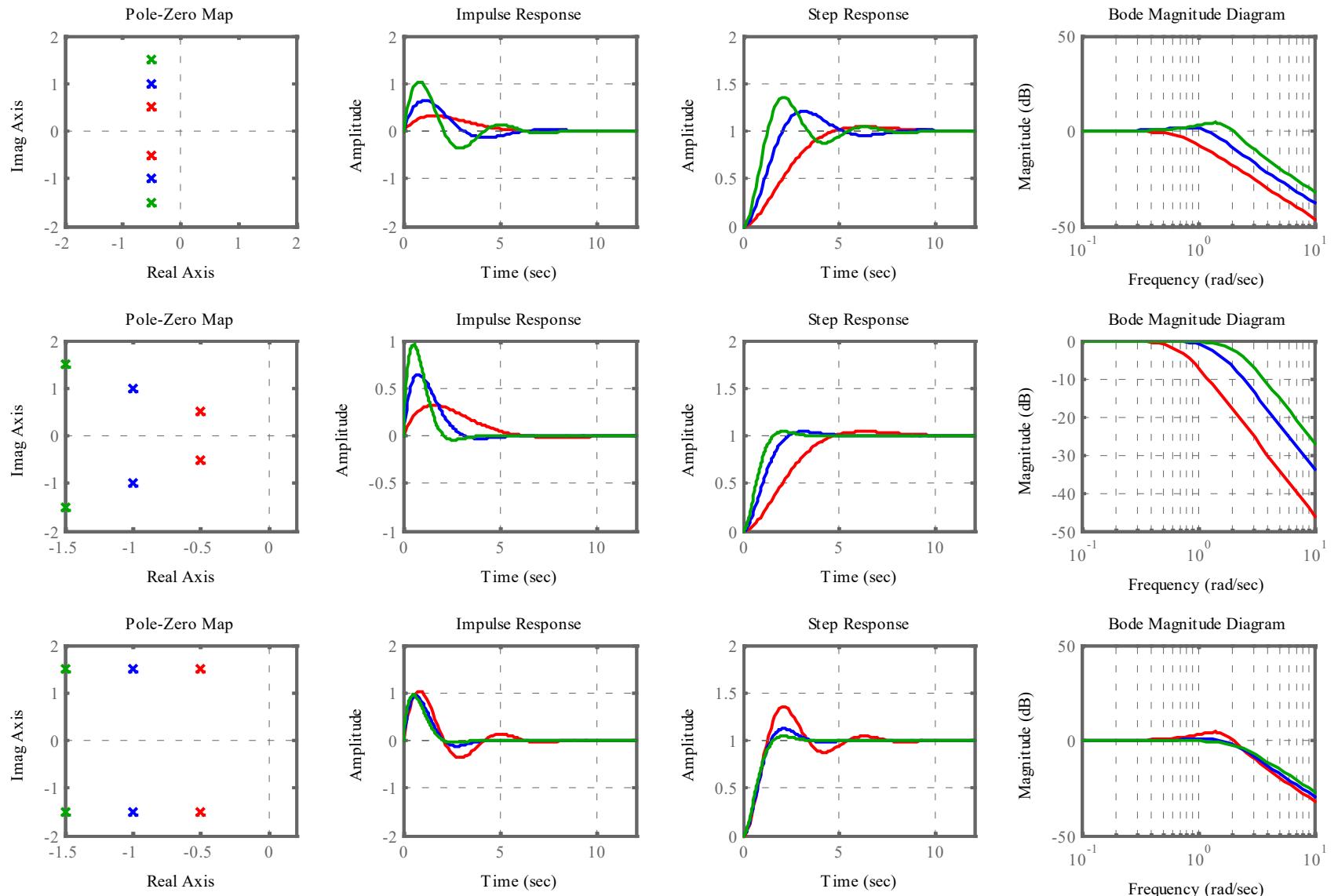
1. Two real and distinct roots.
2. A double (repeated) root
3. Complex conjugate roots.

The position of the roots in the complex plane dictates system performance.

Course topics

Here two complex conjugate roots are moved around the in the complex plane, and variations in the system performance is observed.

At the end of the course, we should be able to tell which root position produces which performance.



Course Introduction

- **Prerequisite competencies**

- Analog circuit theory. AC circuits will be used extensively in this course.
- Maple. This tool will be used extensively.
- Mathematics 1 (01005).
- Mathematics 2 (01035) is recommended.

- **Tools**

- KiCad. We will use KiCad to simulate circuits. Simulations will have the purpose of confirming analysis done in Maple.
 - If you prefer another circuit simulation program that is fine. Teaching assistants only have solutions in KiCad.
 - You will need it on the first day. Have it ready.
 - A zip file with KiCad user files (spice models and circuit templates) will be available on DTU Learn.

- **Materials for lecture preparation:**

- The book chapter for the day. See the reading list on DTU Learn.
- Slide set highlighting some, but not all, of the textbook material.
- Prerecorded video-lecture.

- **Lectures:**

- The lecture time will be used to highlight selected topics. There is not time to cover all topics in the book chapter.
- It is assumed that you have studied the book chapter and the slide set before the lecture.
- Occasionally, video lectures on textbook topics can replace auditorium lectures. Typically, because auditorium time is used to demonstrate difficult solution techniques.
- Videos are inadequate as sole preparatory material. They are meant as a tool for topic reviews.

- **Questions and Answers (Q&A):**

- On DTU Learn there is a general Q&A and a curriculum Q&A. Here you can post questions about general issues or topic specific issues. For topic specific issues make sure you post your question in that Q&A.
- Read the already posted questions before posting your own. The answer might already be there.
- Answers will come as soon as I have time, but do not expect answers in the evenings or weekends.

- **Exercise work:**
 - Organize yourselves into informal teams and collaborate on solving the problems.
 - The solutions to the problems are included in the problem set.
 - Warning: If there are errors in the slides and solutions, and these errors mislead you at the exam, this will be your responsibility.
 - The instructor and TAs will rotate between the tutorial rooms.
- **Quizzes:**
 - Multiple Choice Quizzes (MCQ) is available on DTU Learn for some but not all weeks.
 - Each quiz is a collection of old exam questions focusing on a specific topic. Use these quizzes to test your week-by-week learning.
 - Quizzes are assessment quizzes, not learning quizzes. The score is your feedback on your level of understanding. Their purpose is not to teach you how to solve the problem, only to indicate if more reading is required.
 - You may answer the MCQ as many times you want. Gradebook is disabled on weekly quizzes. There is no track recording of your quiz results.
 - All quizzes have a solution key included.

Course Introduction

- **Textbook:** B.P. Lathi. Signal processing and linear systems. Oxford University Press. E-book is sold at the Polytechnic Bookstore.
- **Exam:**
 - 4 hours, with all aids, closed internet.
 - Multiple Choice type. Calculations needed to answer the questions. Calculations are not included in the submission.
 - You are likely to use both Maple and KiCad/Spice to answer the exam.
- **Feedback:**
 - As there will be no hand-ins, you have to self-monitor your progress and learning curve by self-assessing how readily you can approach the problems.
 - You can answer a series of quizzes to measure your learning.
- **DTU Learn**
 - All messages are broadcast from DTU Learn
 - There is a node for each week, and within each week there are sub-nodes for lecture materials, Q&A, MCQ, and other items.

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Classification of signals and systems

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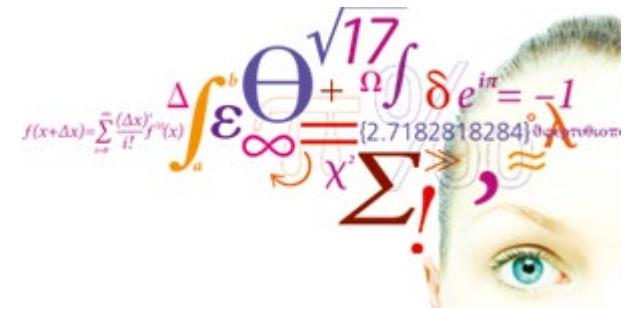
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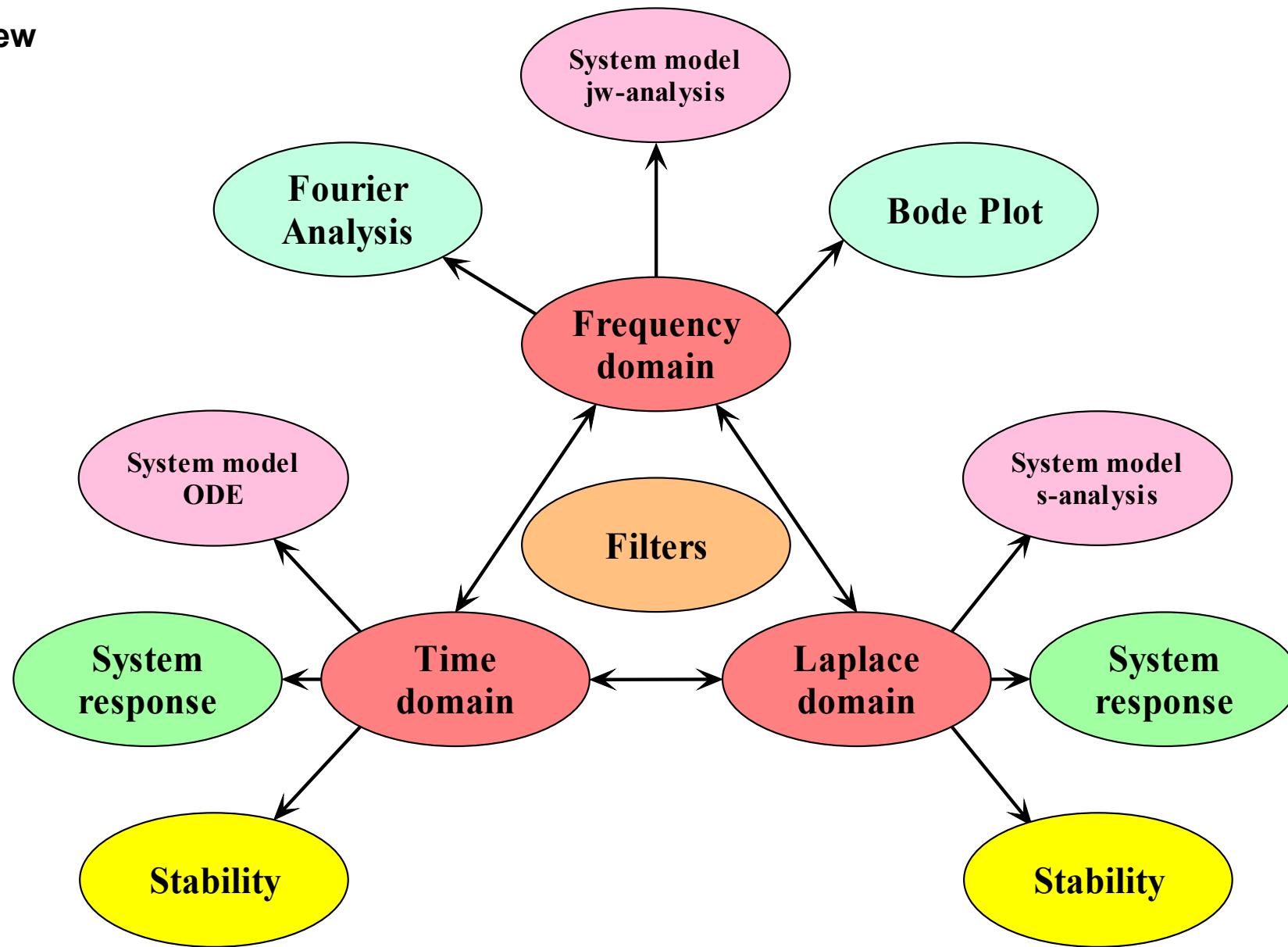
Content of slide set

- Lecture
 - Overview of "Signals and Systems part"
 - Classification of signals
 - Operations on signals
 - Special signals
 - Classification of systems
 - Overview of filter problems
- Exercises

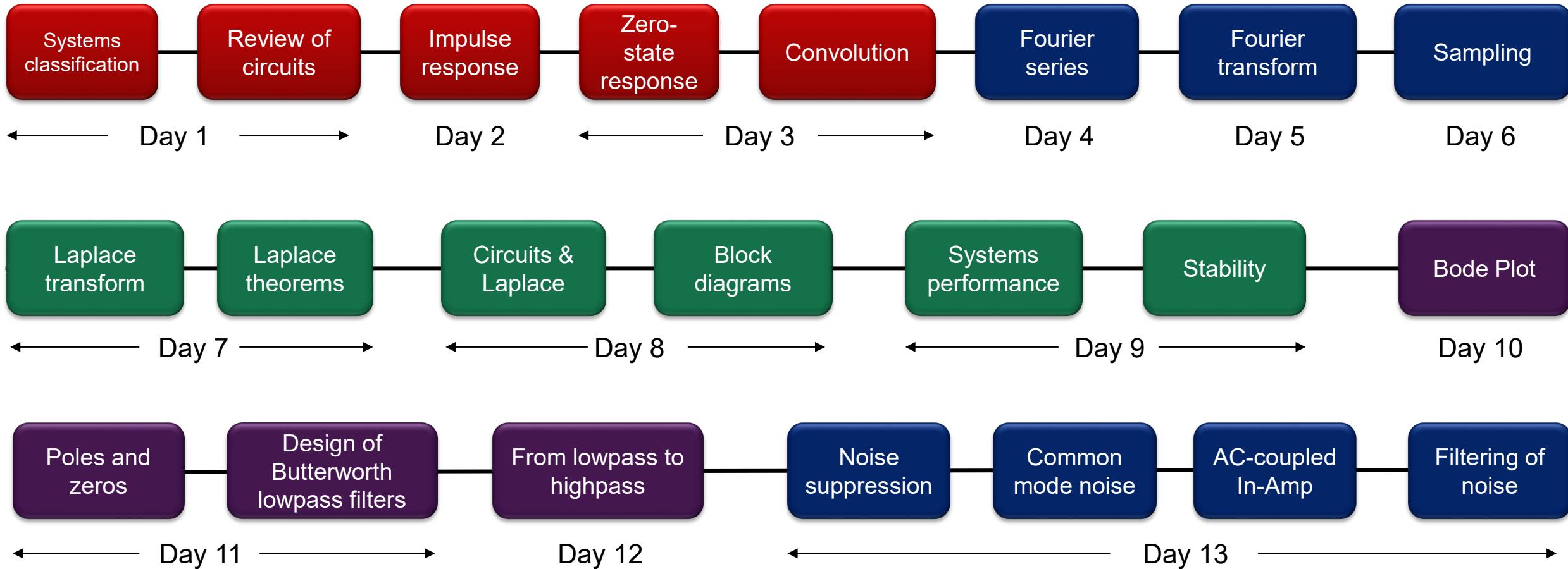


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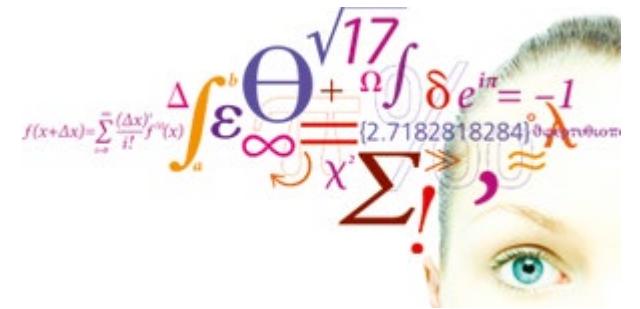
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Video 2

Deterministic versus stochastic signals

Deterministic signal:

$$x(t) = \begin{cases} 0 & t < 0 \\ t^4 e^{-t^2/3} & t \geq 0 \end{cases}$$

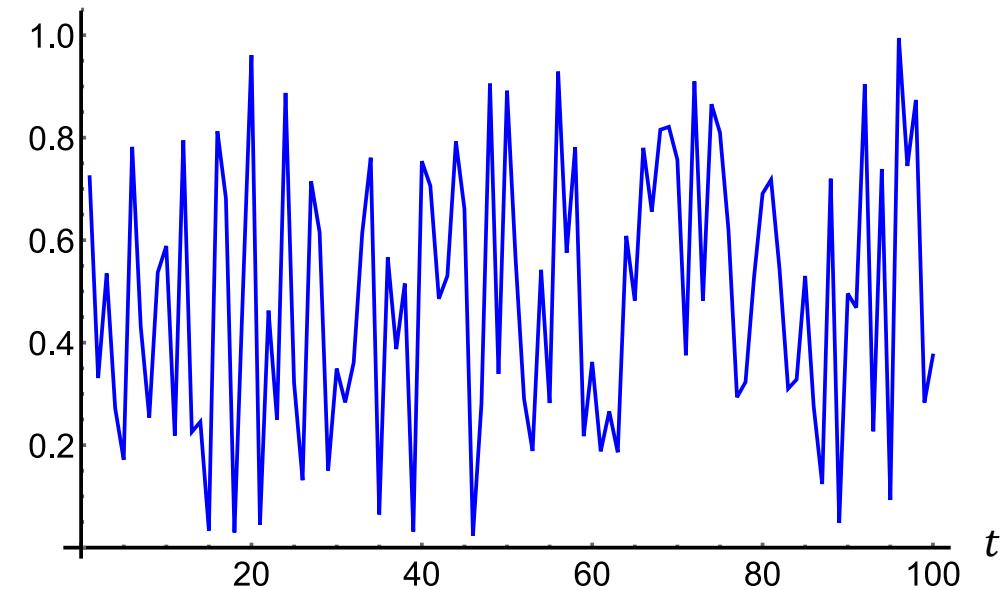
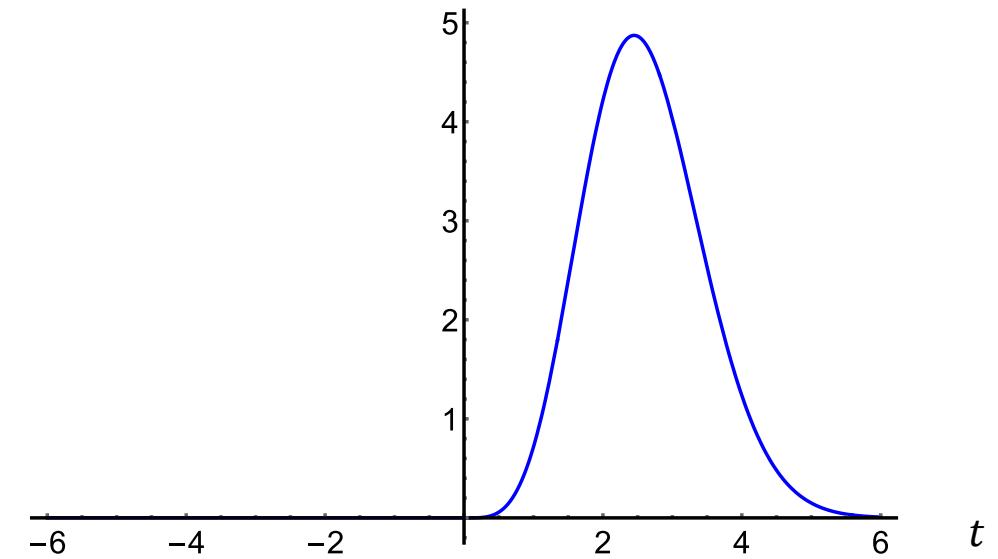
Can be expressed in mathematical equations.

Stochastic signals:

The process generating the signal is random.

No equations can be used to express the amplitude at any given time.

A deterministic signal has a known functional dependency of the independent variable, which often is time, but also could be space or some other quantity. Thus, for a deterministic time signal we can always tell at time t_0 what the signal amplitude will be at time t_1 even when $t_1 > t_0$.

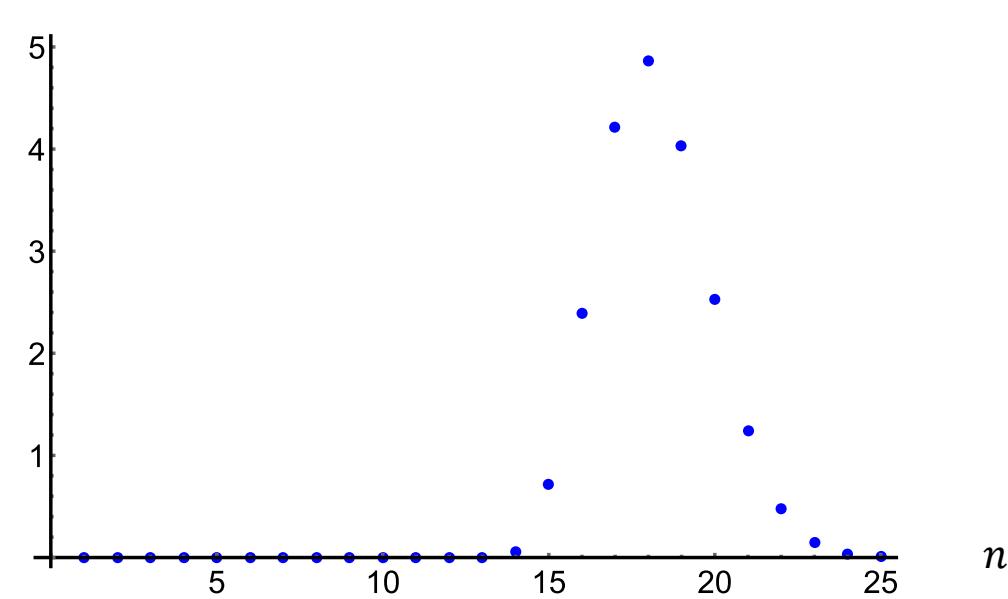
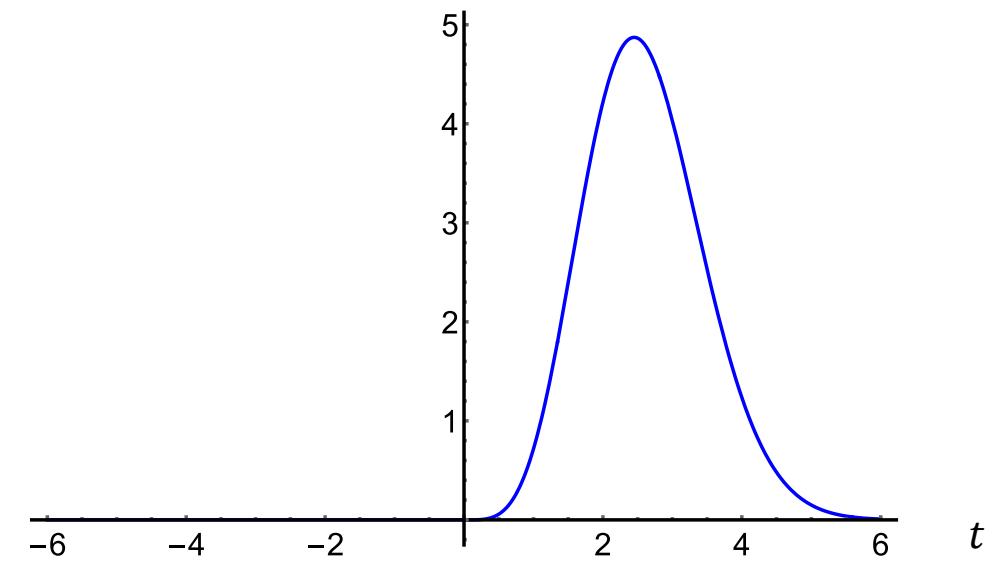


Continuous time versus discrete time signals

Continuous time signal: $x(t) = \begin{cases} 0 & t < 0 \\ t^4 e^{-t^2/3} & t \geq 0 \end{cases}$

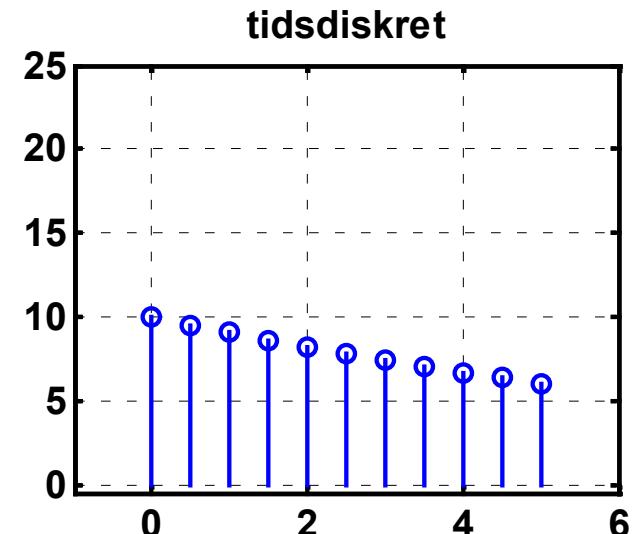
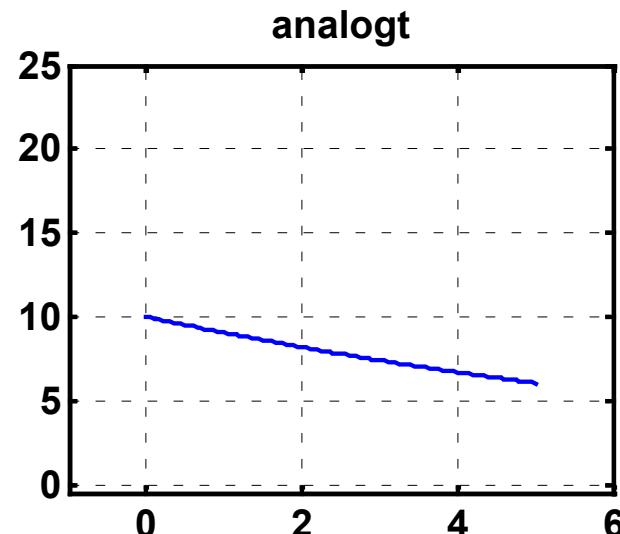
Discrete time signals: $x(n\Delta t) = \begin{cases} 0 & n < 0 \\ (n\Delta t)^4 e^{-(n\Delta t)^2/3} & n \geq 0 \end{cases}$

Analog signals are continuous both in amplitude and time. Analog signals are also often called continuous-time (CT) signals. Discrete-time (DT) signals are defined only at certain (often constant) time intervals. The samples of DT signals are assumed to have continuous amplitude values. A **digital signal** is discrete both in time and in amplitude, i.e., the amplitude of each sample is represented by a finite number of bits.



Analogue versus digital signal

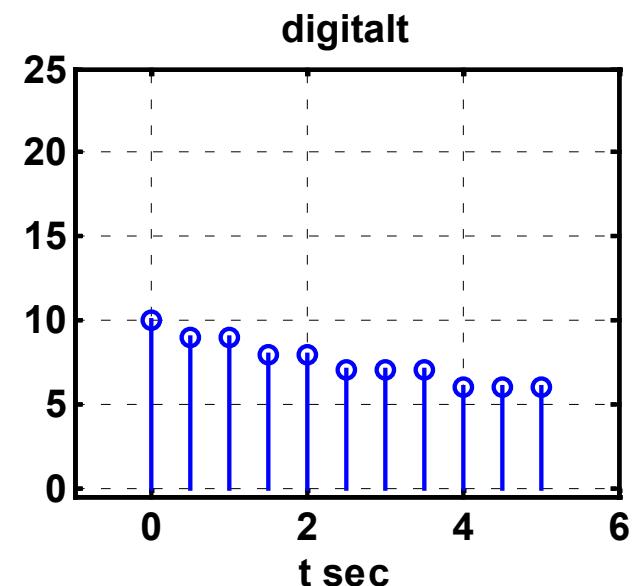
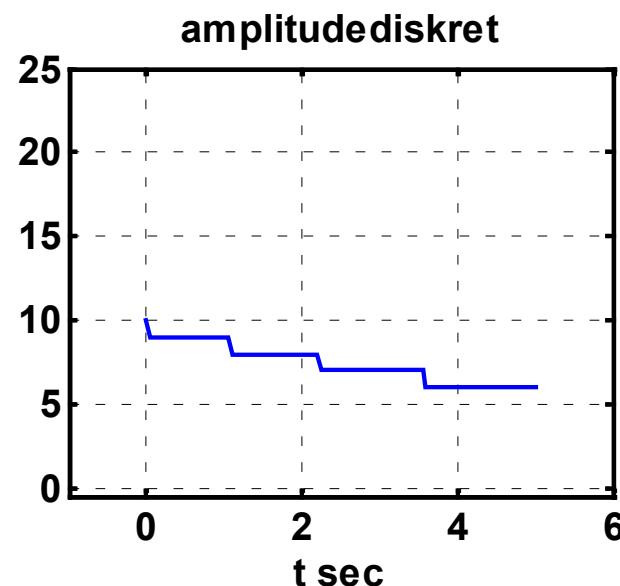
Analogue signals are continuous in both time and amplitude.



Digital signals are discrete in both time and amplitude.

Digital samples are stored in binary form with a finite number of bits.

$$\Delta V_{LSB} = \frac{ADC\ Voltage\ range}{2^N}$$



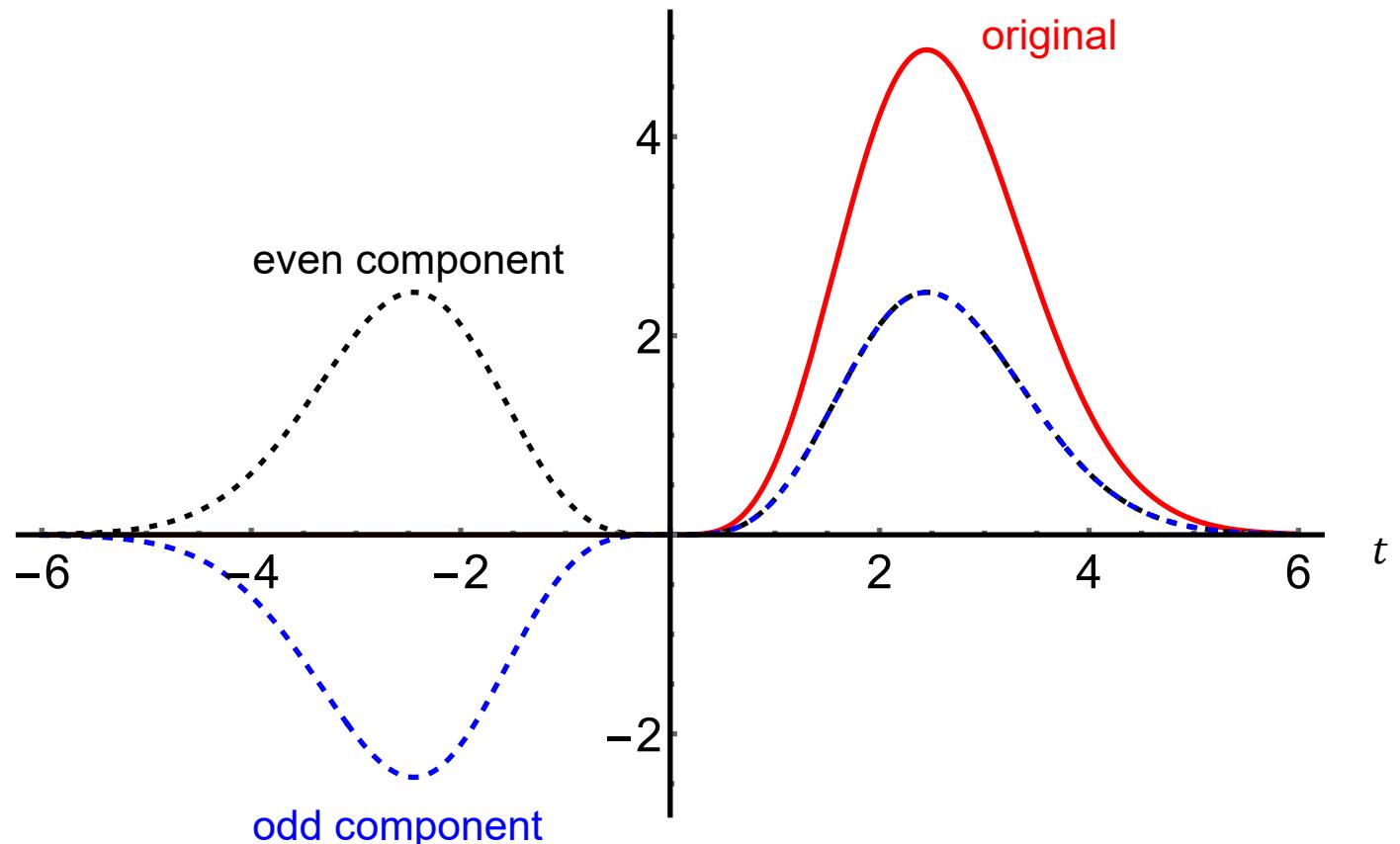
Even and odd signals

Even signal:

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

Odd signal:

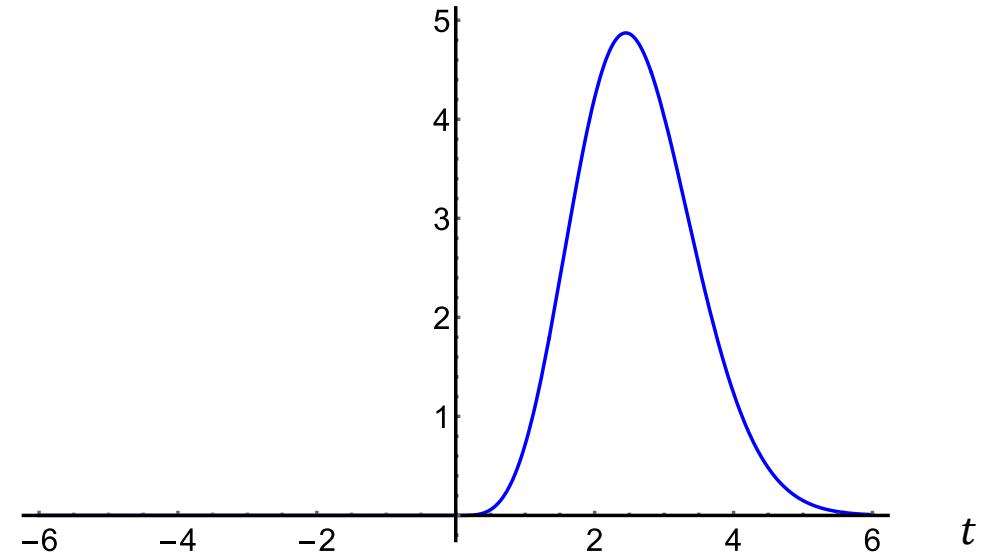
$$x_o(t) = \frac{x(t) - x(-t)}{2}$$



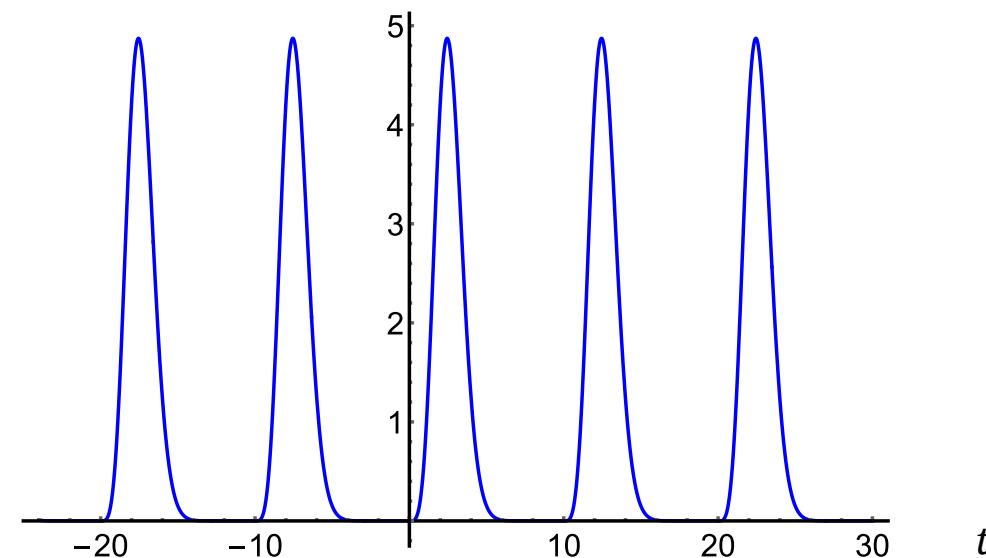
- A signal that is neither even nor odd still has an even component and an odd component.
- A signal that is even has zero odd component.
- A signal that is odd has zero even component.

Aperiodic and periodic signals

Aperiodic signal: $x(t) = \begin{cases} 0 & t < 0 \\ t^4 e^{-t^2/3} & t \geq 0 \end{cases}$

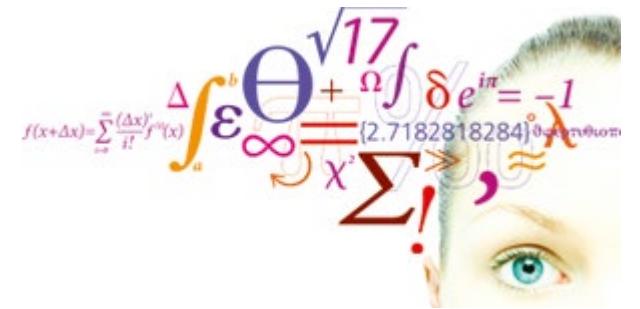


Periodic signal: $y(t) = \sum_{i=-3}^3 x(t - i \cdot T_0) \quad T_0 = 10$



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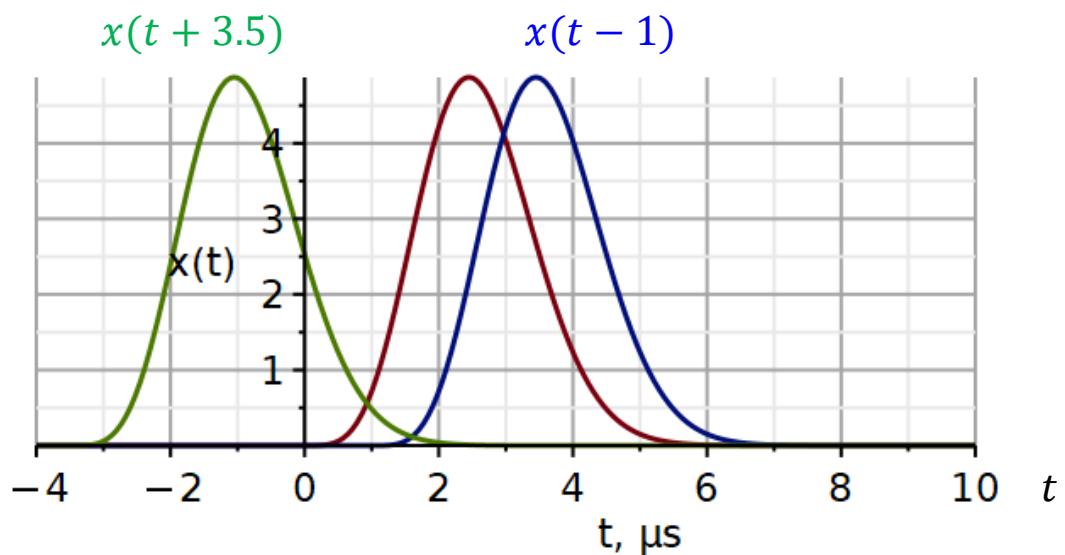
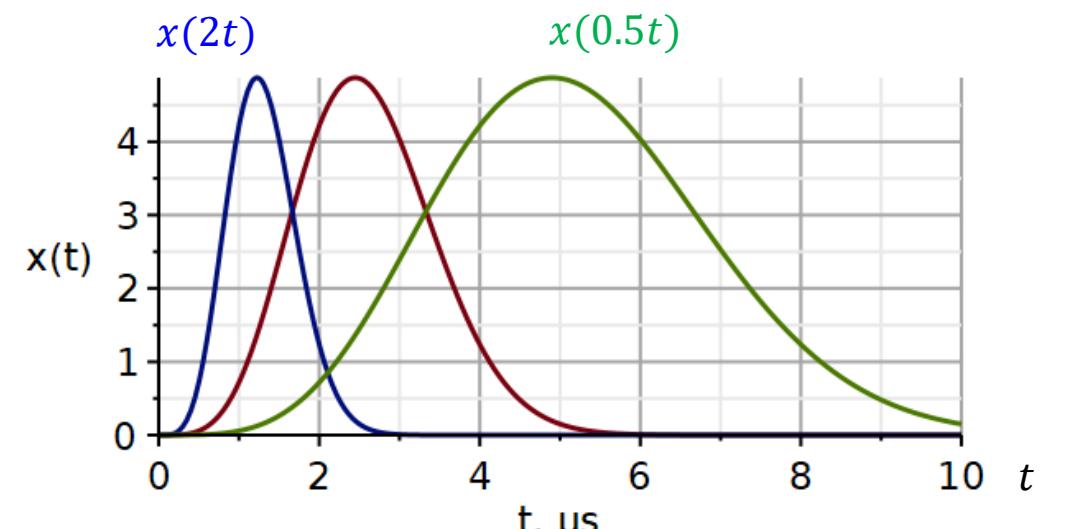


Time scaling and time shift

Aperiodic signal: $x(t) = \begin{cases} 0 & t < 0 \\ t^4 e^{-t^2/3} & t \geq 0 \end{cases}$

Time scaling: $y(t) = x(a \cdot t)$

Time shift: $y(t) = x(t - t_0)$



1.3.4 Combined operations. Order of operations is important

Scaling → Shift

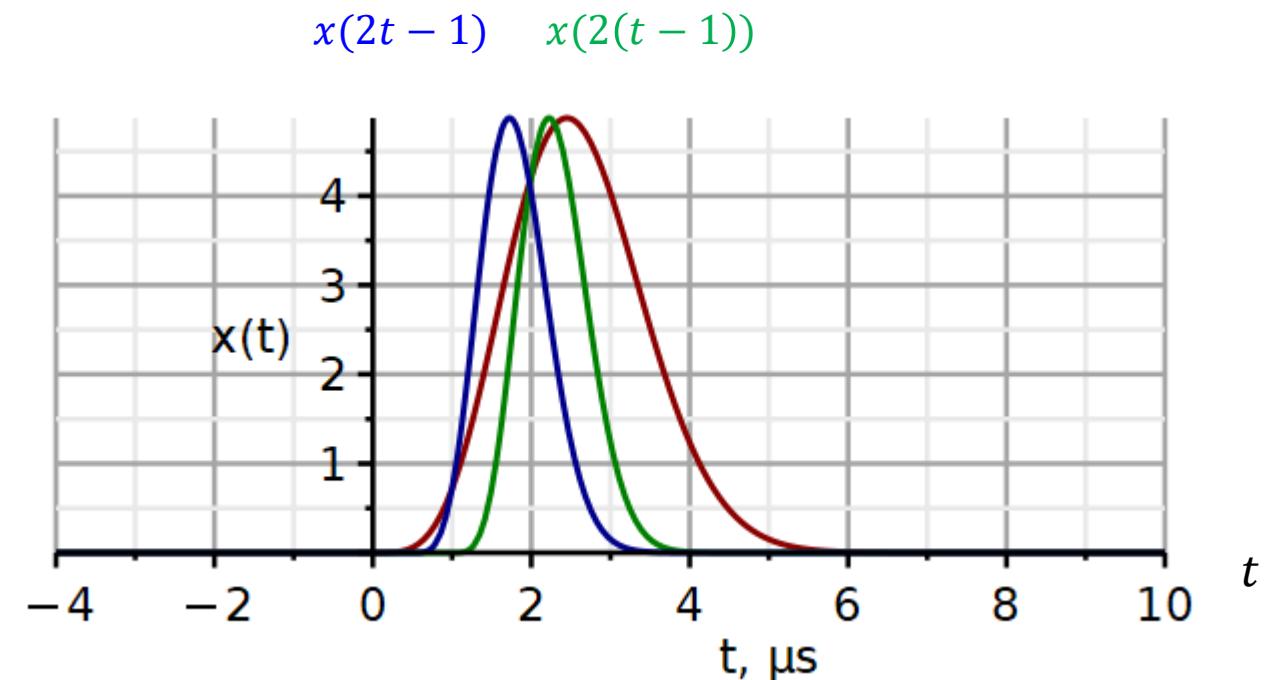
$$s(t) = x(at)$$

$$y(t) = s(t - t_0) = x(a(t - t_0))$$

Shift → Scaling

$$s(t) = x(t - t_0)$$

$$y(t) = s(at) = x(at - t_0)$$

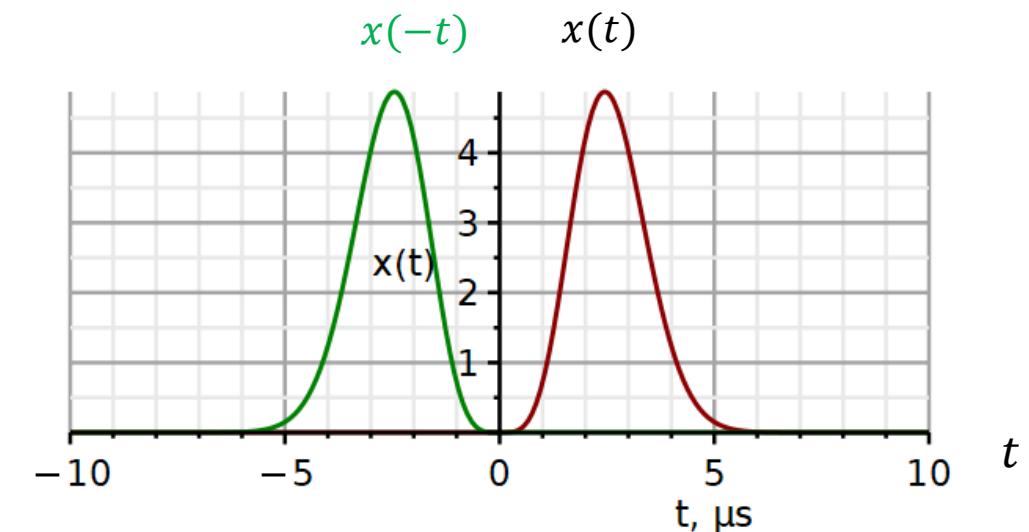


Time reflection**Aperiodic signal:**

$$x(t) = \begin{cases} 0 & t < 0 \\ t^4 e^{-t^2/3} & t \geq 0 \end{cases}$$

Time reflection:

$$y(t) = x(-t)$$

*Shift → Reflect*

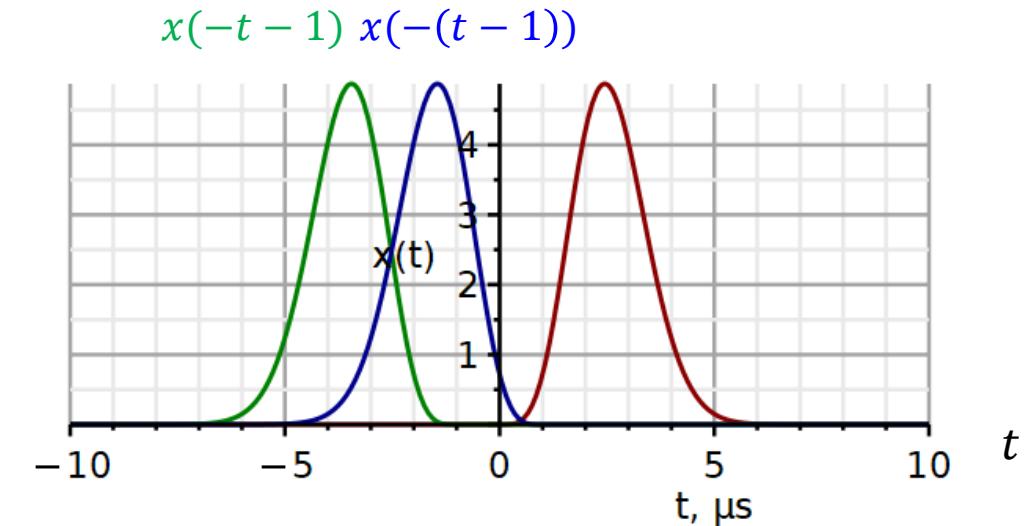
$$s(t) = x(t - t_0)$$

$$y(t) = s(-t) = x(-t - t_0)$$

Reflect → Shift

$$s(t) = x(-t)$$

$$y(t) = s(t - t_0) = x(-(t - t_0))$$

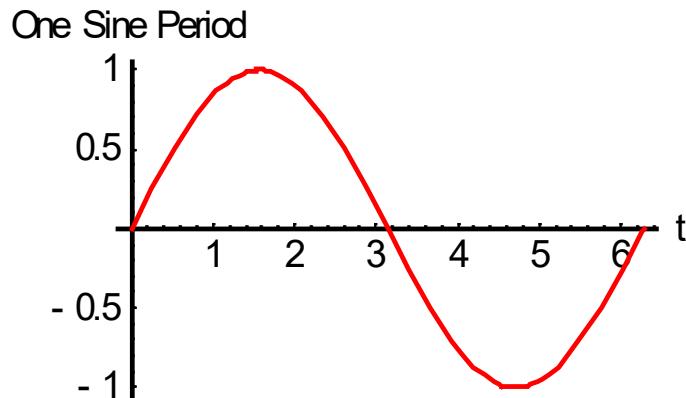


Signal energy

Signal energy:

$$E_x \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Example:



$$x(t) = \begin{cases} 0 & , t < 0 \\ A \sin(\omega t) & , 0 \leq t \leq 2\pi/\omega \\ 0 & , t > 2\pi/\omega \end{cases}$$

$$\begin{aligned} E_x &= \int_0^{2\pi/\omega} A^2 \sin(\omega t)^2 dt = \frac{A^2}{2} \int_0^{2\pi/\omega} [\cos(\omega t - \omega t) - \cos(\omega t + \omega t)] dt \\ &= \frac{A^2}{2} \int_0^{2\pi/\omega} [1 - \cos(2\omega t)] dt = A^2 \frac{\pi}{\omega} \end{aligned}$$

Signal Power

Signal power:

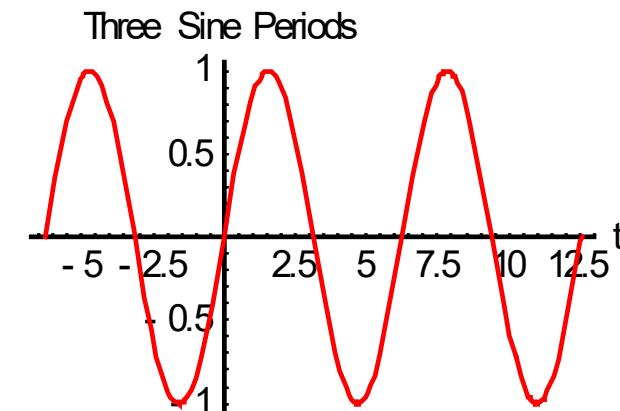
For a periodic signal, the energy is finite in every period. Thus, the total energy in all periods is infinite. Instead, we can define its power as the energy in one period divided by the period duration.

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$P_x = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} |x(t)|^2 dt$$

$$P_x \stackrel{\text{def}}{=} \lim_{T_0 \rightarrow \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 dt$$

$$x(t) = \sin(t)$$



$$P_x = \frac{E_{x,1p}}{T_0} = \frac{A^2\pi/\omega}{2\pi/\omega} = \frac{A^2}{2}$$

$$x_{rms} = \sqrt{P_x} = \frac{A}{\sqrt{2}} \approx 0.7071 \cdot A$$

Energy and power signals

- Energy signal: A signal with finite energy
- Power signal: A signal with finite power
- A periodic signal with finite energy in one period will have infinite energy in all periods
- Thus a periodic signal is never an energy signal.
- Signals with infinite power is neither an energy nor a power signal

Measure of deviation:

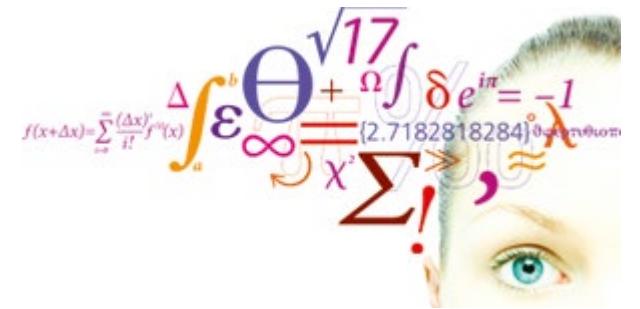
Mean Square Error

$$MSE \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} |x(t) - \tilde{x}(t)|^2 dt$$

Energy of error signal

Content of slide set

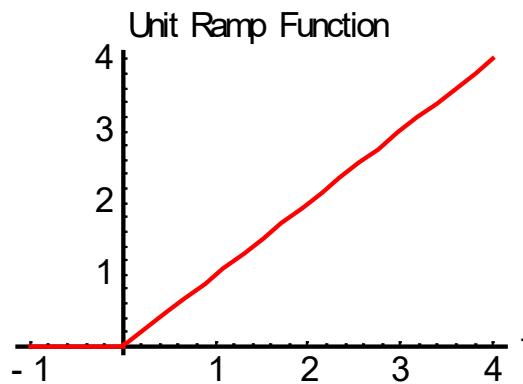
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Special signals

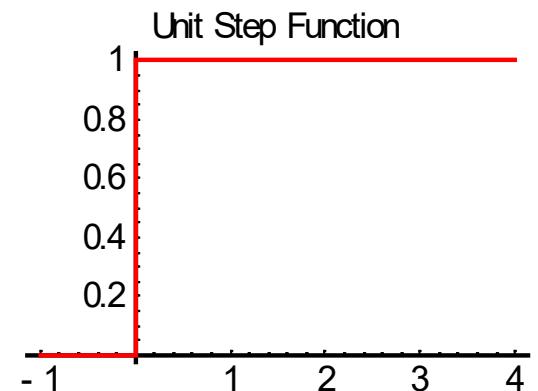
Unit ramp:

$$r(t) \stackrel{\text{def}}{=} \begin{cases} 0 & , t < 0 \\ t & , t \geq 0 \end{cases}$$



Unit step:

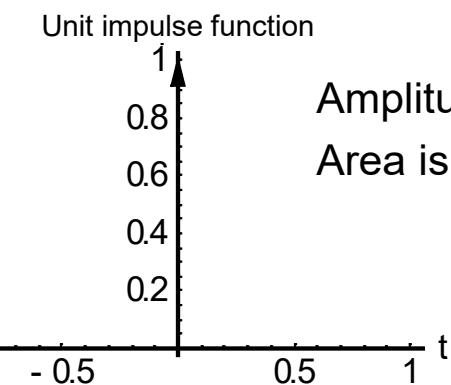
$$u(t) \stackrel{\text{def}}{=} \begin{cases} 0 & , t < 0 \\ 1 & , t \geq 0 \end{cases}$$



Unit impulse:

$$\delta(t) = 0, t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$



$$u(t) = \frac{dr}{dt}, r(t) = \int_{-\infty}^t u(\tau) d\tau$$

$$\delta(t) = \frac{du}{dt}, u(t) = \int_{-\infty}^t \delta(\tau) d\tau$$

Properties of Impulse Function

Property 1: $\delta(t) = \delta(-t)$

Even function

Property 2:
$$\int_{0_-}^{0_+} A\delta(t) dt = A$$

The area under the impulse function equals its strength

Property 3: $A\delta(t - t_0) = 0, t \neq t_0$

The impulse function is zero except at $t = t_0$

Property 4: $A\delta(t - t_0) + B\delta(t - t_0) = (A + B)\delta(t - t_0)$

Superimposed impulse functions add their strengths

Property 5: $y(t)\delta(t - t_0) = y(t_0)\delta(t - t_0)$

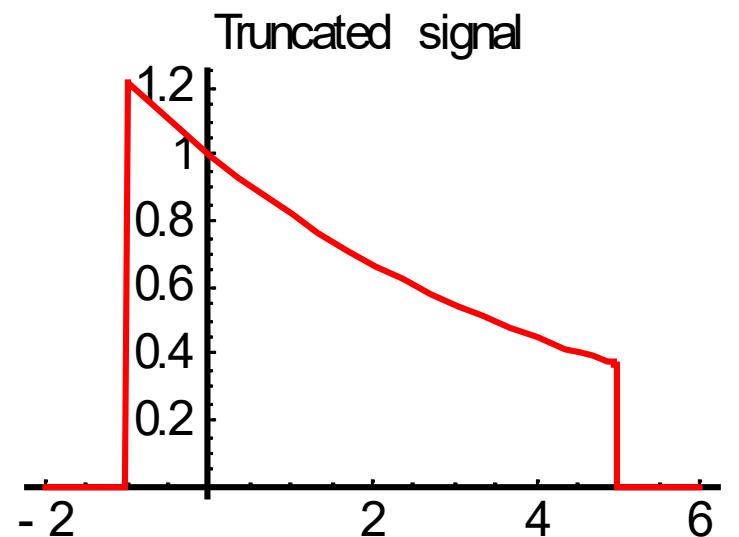
Multiplication of a function by an impulse function yields an impulse function with strength equal the function amplitude at the impulse location

Application of step function

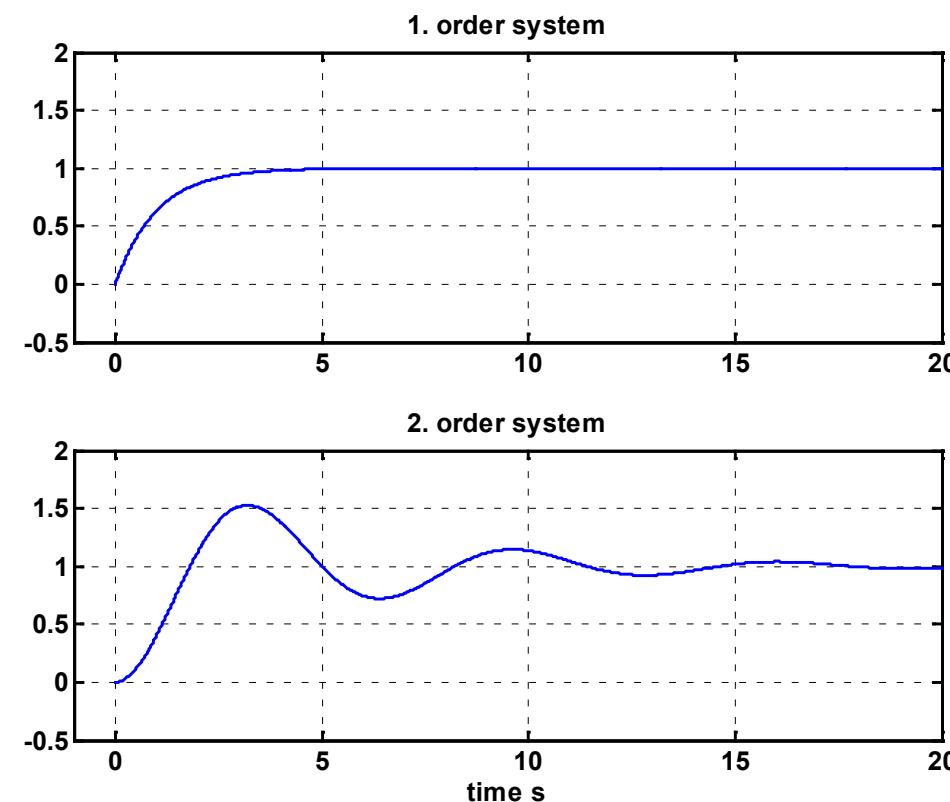
1: As a test signal to analyze the "Step Response" of a system



2: Truncation of signals



$$x(t) = e^{-t/5}[u(t + 1) - u(t - 5)]$$

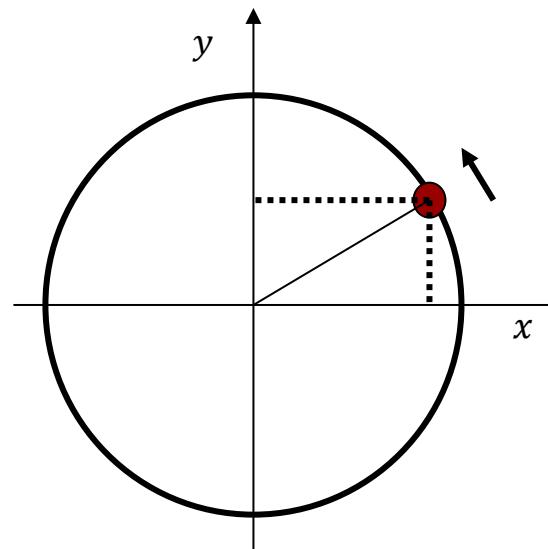


Sinusoidal signals

$$x(t) = A \cdot \cos[\theta(t)]$$

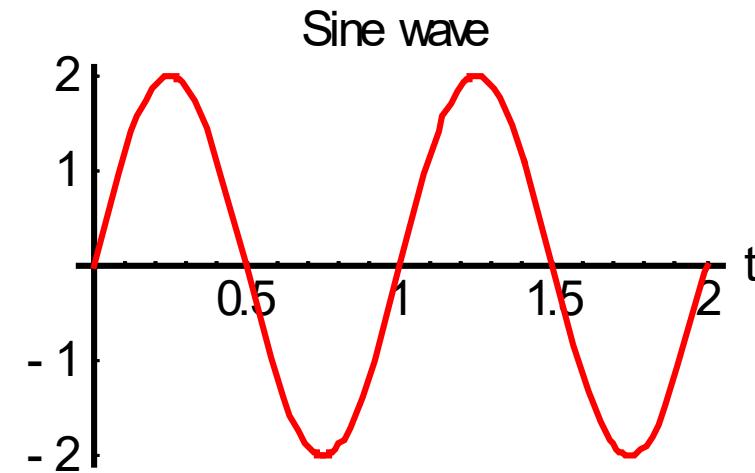
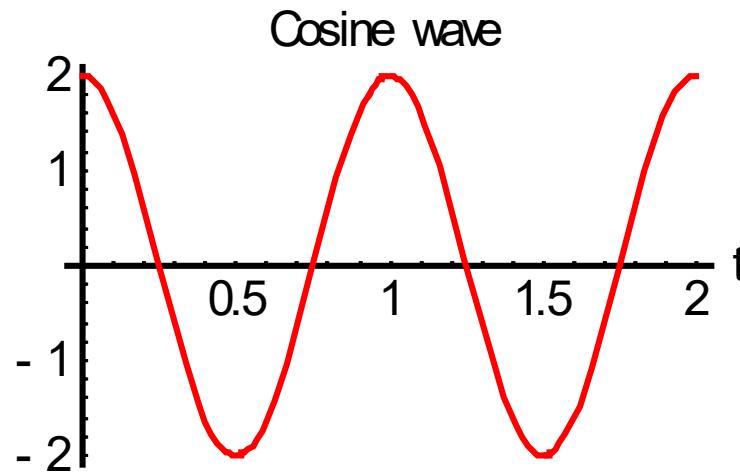
$$x(t) = A \cdot \cos[2\pi \cdot f \cdot t + \theta]$$

$$y(t) = A \cdot \sin[2\pi \cdot f \cdot t + \theta]$$

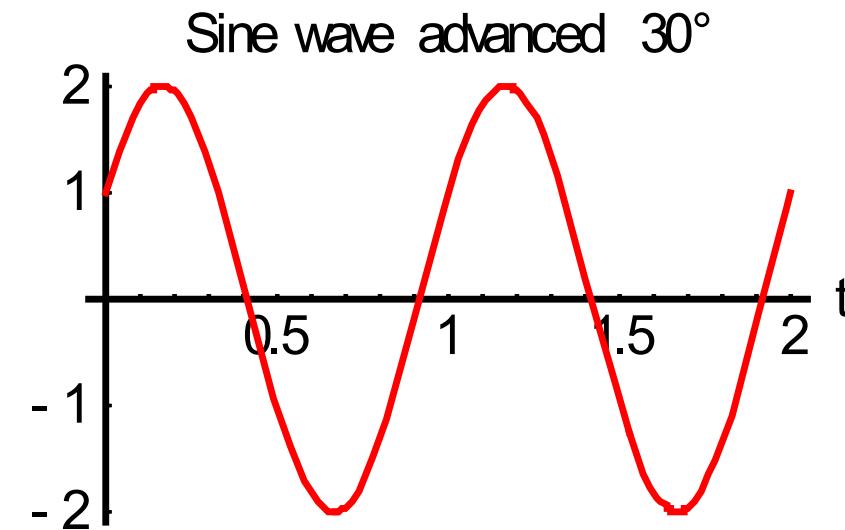
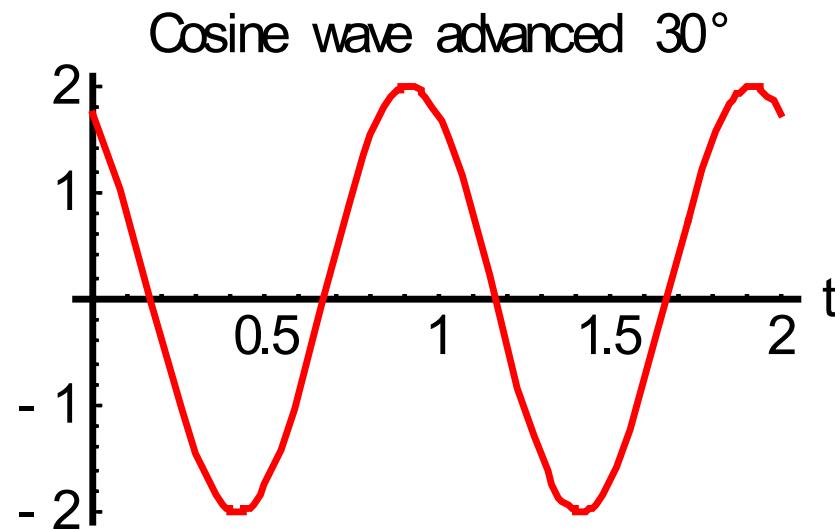
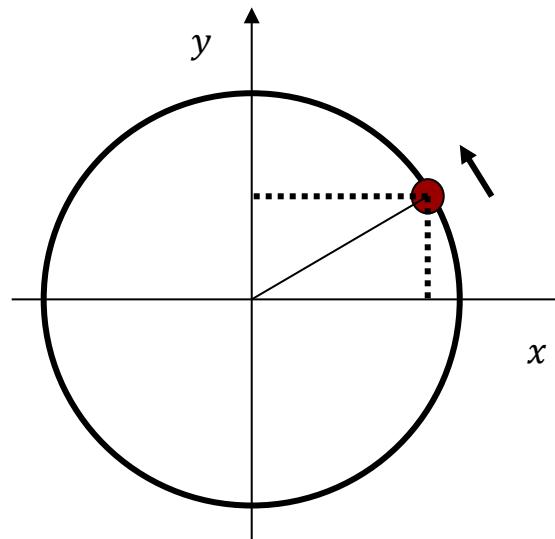


$$\theta(t) = \frac{d\theta}{dt} \cdot t = \omega \cdot t$$

$$\omega = \frac{2\pi}{T_0} = 2\pi \cdot f$$



Phase angle

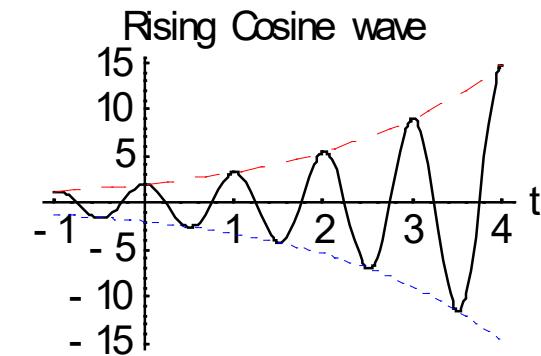
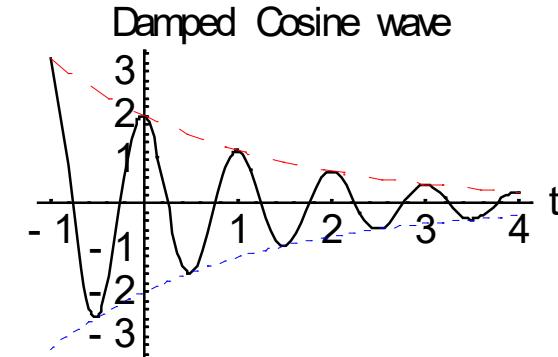
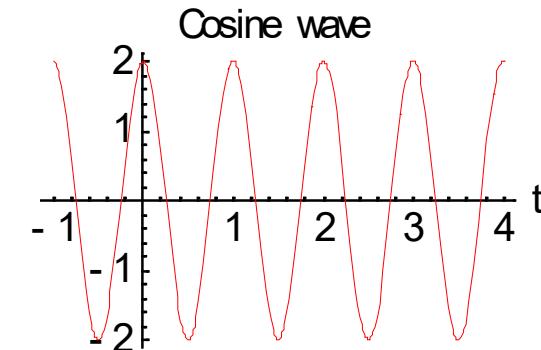


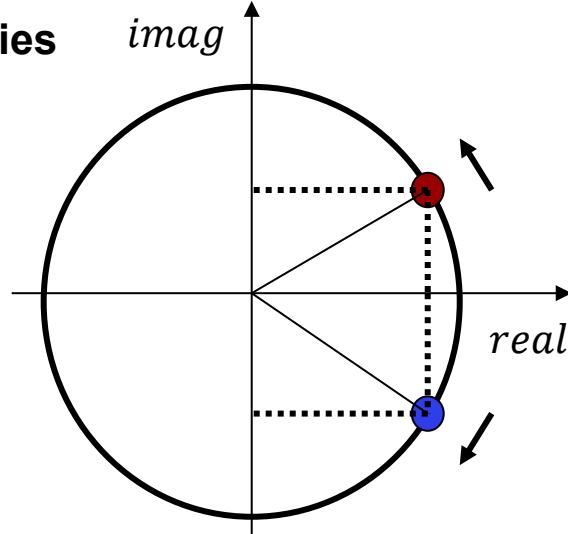
Exponentially varying sinusoids

$$x_1(t) = A \cdot \cos[2\pi \cdot f \cdot t]$$

$$x_2(t) = \underbrace{A \cdot e^{-t/\tau}}_{\text{envelope}} \cdot \cos[2\pi \cdot f \cdot t]$$

$$x_3(t) = \underbrace{A \cdot e^{t/\tau}}_{\text{envelope}} \cdot \cos[2\pi \cdot f \cdot t]$$



Euler's identities

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$
$$e^{-j\omega t} = \cos(\omega t) - j \sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2 \cos(\omega t)$$
$$e^{j\omega t} - e^{-j\omega t} = j2 \sin(\omega t)$$

As a mathematical abstraction, we can represent any sinusoidal oscillation as the sum of two rotations, one with a positive frequency, and one with an identical negative frequency.

From a physical perspective, oscillations have only positive frequencies.

Why would we ever use complex exponentials rather than cos and sin?

If we insert a cos function into a differential equation, we obtain a sum of cos and sine functions, which are hard to add up. If we insert a complex exponential function, we obtain a sum of only complex exponential functions.

Complex exponential signals

Complex exponential:

$$\begin{aligned} z(t) &= e^{\sigma t} [\cos(\omega t) + j \sin(\omega t)] \\ &= e^{\sigma t} e^{j\omega t} = e^{(\sigma+j\omega)t} \\ &= e^{st}, s \stackrel{\text{def}}{=} \sigma + j\omega \end{aligned}$$

A complex exponential has one parameter s .

$\operatorname{Re}(s)$ describes the exponential change in amplitude.

$\operatorname{Im}(s)$ describes the frequency of oscillation.

Complex conjugated:

$$\begin{aligned} z^*(t) &= e^{\sigma t} [\cos(\omega t) - j \sin(\omega t)] \\ &= e^{\sigma t} e^{-j\omega t} = e^{(\sigma-j\omega)t} \\ &= e^{s^*t}, s^* \stackrel{\text{def}}{=} \sigma - j\omega \end{aligned}$$

$$z(t) = x(t) + jy(t) \quad z^*(t) = x(t) - jy(t)$$

Even function:

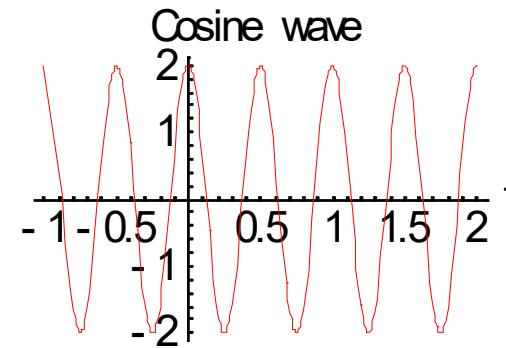
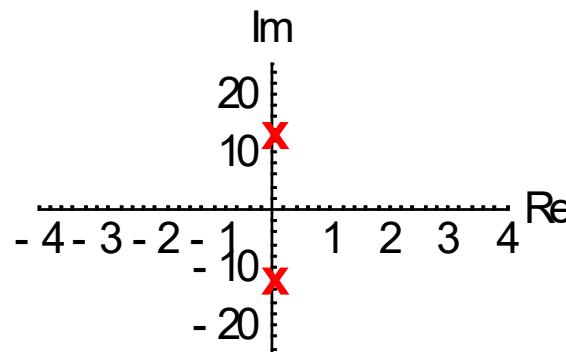
$$x(t) = \frac{1}{2} (z(t) + z^*(t)) = \frac{1}{2} (e^{st} + e^{s^*t}) = e^{\sigma t} \cos(\omega t)$$

Odd function

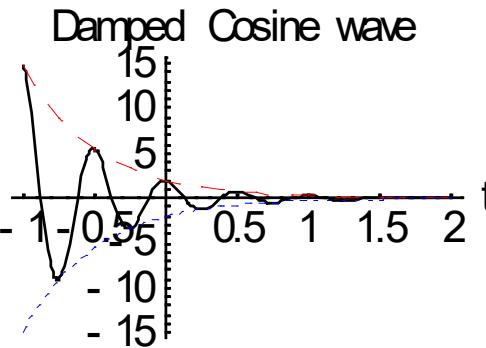
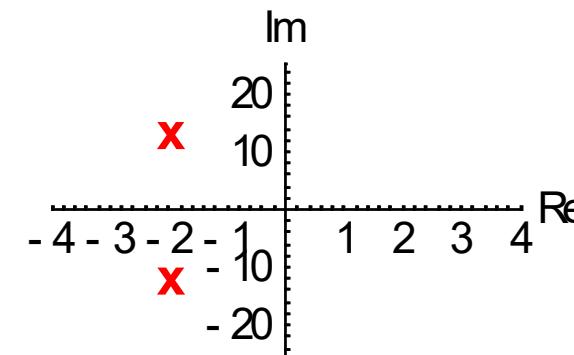
$$y(t) = \frac{1}{2} (z(t) - z^*(t)) = \frac{1}{2} (e^{st} - e^{s^*t}) = e^{\sigma t} \sin(\omega t)$$

Complex plane

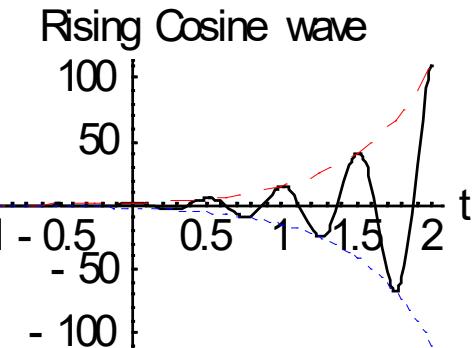
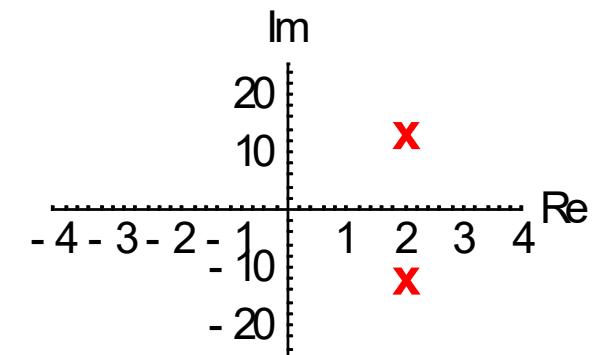
$$s = j2\pi \cdot 2$$



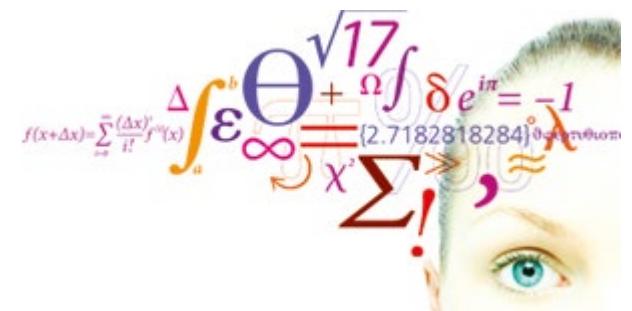
$$s = -2 + j2\pi \cdot 2$$



$$s = 2 + j2\pi \cdot 2$$



- Lecture
 - Overview of "Signals and Systems part"
 - Classification of signals
 - Operations on signals
 - Special signals
 - **Classification of systems**
 - Overview of filter problems
- Exercises

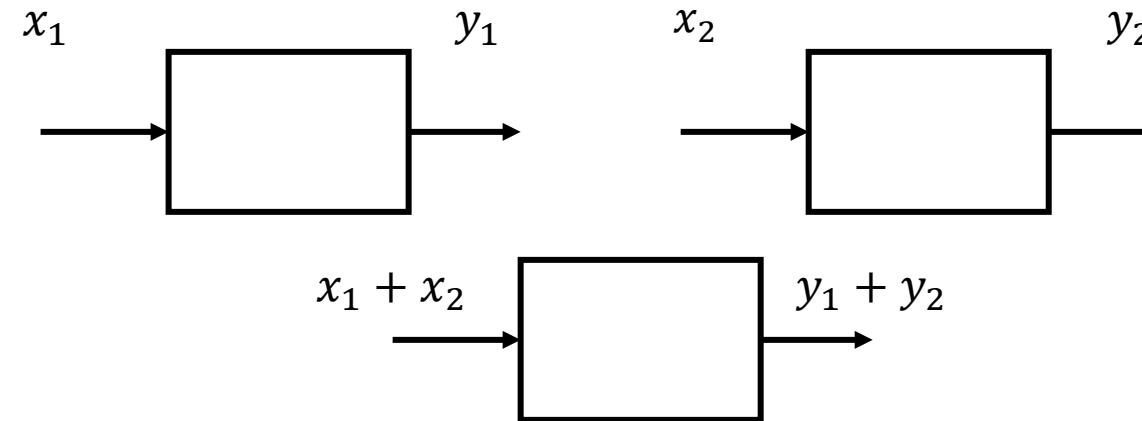


Video 3

Linear systems

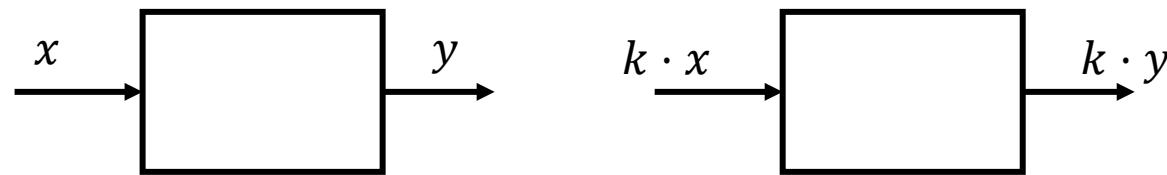
Additive property:

+

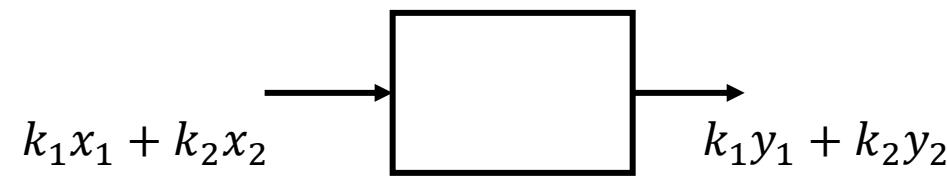


Homogeneity property:

=



Superposition property:



A system is linear if and only if the superposition property holds.



$$x(t) = k_1 x_1(t) + k_2 x_2(t)$$

$$k_1 \ddot{y}_1(t) + a_1 k_1 \dot{y}_1(t) + a_0 k_1 y_1(t) = b_0 k_1 x_1(t)$$

$$y(t) = k_1 y_1(t) + k_2 y_2(t)$$

$$k_2 \ddot{y}_2(t) + a_1 k_2 \dot{y}_2(t) + a_0 k_2 y_2(t) = b_0 k_2 x_2(t)$$

$$\dot{y}(t) = k_1 \dot{y}_1(t) + k_2 \dot{y}_2(t)$$

$$\ddot{y}(t) = k_1 \ddot{y}_1(t) + k_2 \ddot{y}_2(t)$$

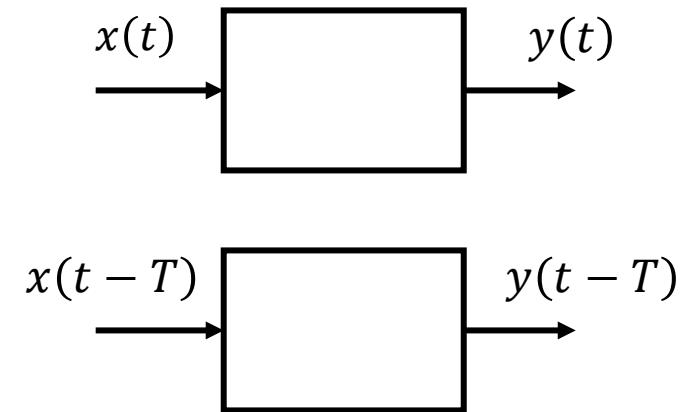
$$\underbrace{k_1 \ddot{y}_1(t) + k_2 \ddot{y}_2(t)}_{\ddot{y}(t)} + a_1 \left(\underbrace{k_1 \dot{y}_1(t) + k_2 \dot{y}_2(t)}_{\dot{y}(t)} \right) + a_0 \left(\underbrace{k_1 y_1(t) + k_2 y_2(t)}_{y(t)} \right) = b_0 \left(\underbrace{k_1 x(t) + k_2 x_2(t)}_{x(t)} \right)$$

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_0 x(t)$$

If a system has a linear differential equation, it is a linear system.

Time-Invariant systems

A system is time-invariant if a time-shifted input yields a correspondingly time-shifted output.



A system is time-invariant if the system itself doesn't vary with time. This holds if all system parameters are constants.

The differential equation for a time-invariant system must have **constant coefficients**.

Time-invariance refers ONLY to the system, NOT the signals. For a time-invariant system, the input and output signals can vary as functions of time.

LTI System = Linear Time-Invariant System

A system is causal if its output depends only on present and past input.

A noncausal system can have output that depend on future input. It must predict future values of the input signal. A noncausal system can have an output before the input is applied, i.e., without "cause".

Physical systems are always causal.

Noncausal systems are only of theoretical interest.

Which system is causal?

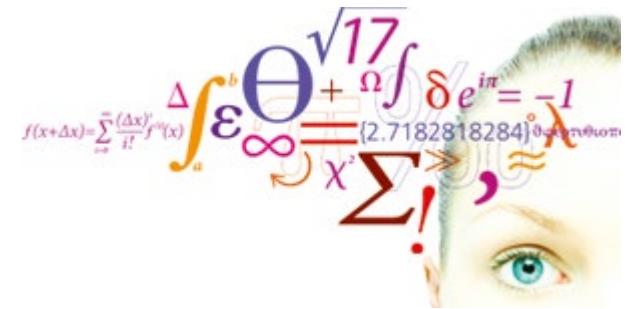
$$A: y(t) = \int_{t-5}^{t+5} x(\tau) d\tau$$

$$B: y(t) = \int_{t-5}^t x(\tau) d\tau$$

LTIC System = Linear Time-Invariant Causal System

Content of slide set

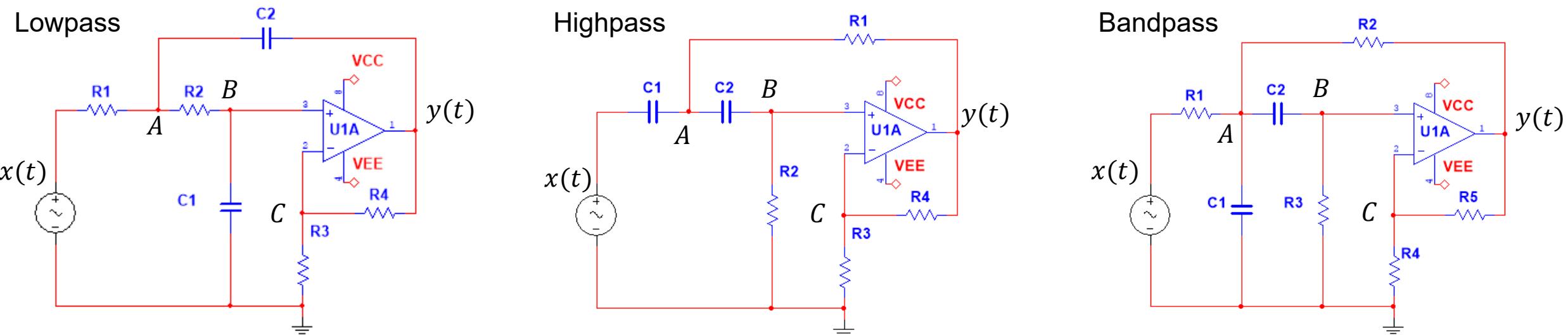
- Lecture
 - Overview of "Signals and Systems part"
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 - Special signals
 - Classification of systems
 - **Overview of filter problems**
- Exercises



Filter circuits used in this course.

Being able to analyze the performance of filter circuits and to design new filter circuits meeting given design specifications is a very valuable competency for students in both Electrotechnology and in Biomedical engineering.

This course will use Sallen-Key filter circuits as a common platform for application of theories and methods.



All three circuits can be modelled by this differential equation: $\ddot{y} + a_1\dot{y} + a_0y = b_2\ddot{x} + b_1\dot{x} + b_0x$

The differential equation for a lowpass filter

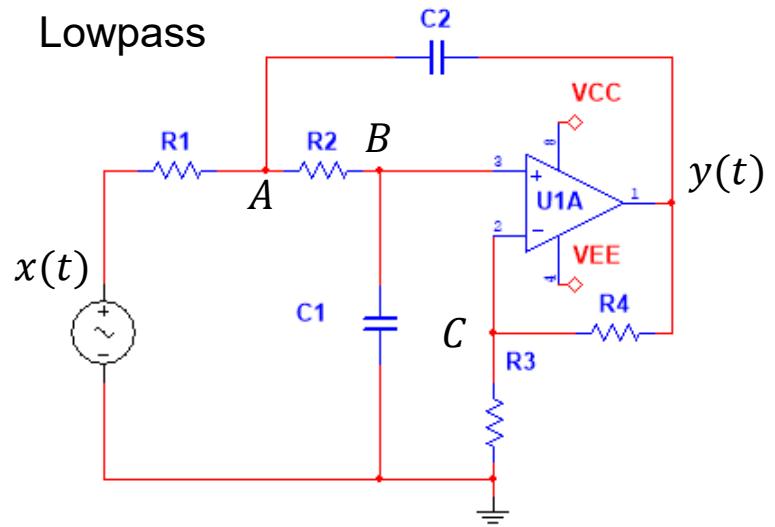
Why is this a lowpass filter?

At high frequencies, C_1 will become a short circuit. The voltage V_B will approach 0V and $y(t) = KV_B(t)$ will also approach zero.

$$K = 1 + \frac{R_4}{R_3}$$

$$a_1 = \frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} + \frac{1 - K}{R_2 C_1} \quad a_0 = \frac{1}{R_1 R_2 C_1 C_2}$$

$$b_2 = 0 \quad b_1 = 0 \quad b_0 = \frac{K}{R_1 R_2 C_1 C_2}$$



$$\ddot{y} + a_1 \dot{y} + a_0 y = b_0 x$$

The differential equation for a highpass filter

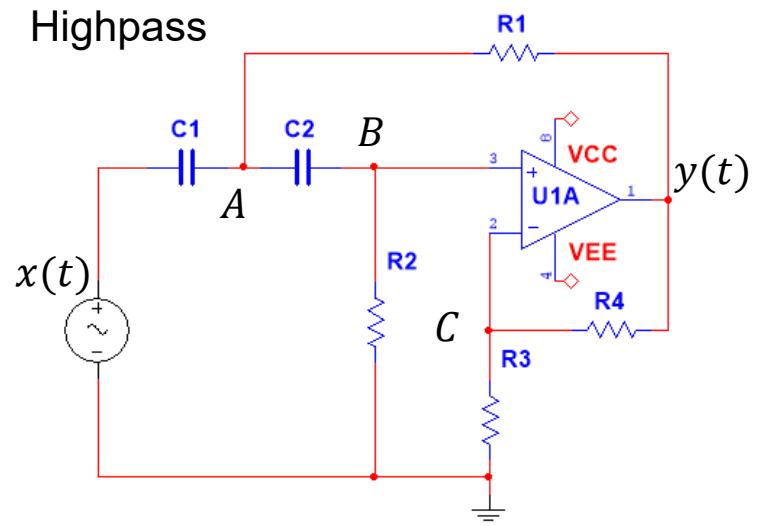
Why is this a highpass filter?

At low frequencies, C_1 and C_2 will block currents. As the current through R_2 approaches zero, the voltage V_B will approach 0V and $y(t) = KV_B(t)$ will also approach 0V.

$$K = 1 + \frac{R_4}{R_3}$$

$$a_1 = \frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} + \frac{1 - K}{R_2 C_1} \quad a_0 = \frac{1}{R_1 R_2 C_1 C_2}$$

$$b_2 = K \quad b_1 = 0 \quad b_0 = 0$$



$$\ddot{y} + a_1\dot{y} + a_0y = b_2\ddot{x}$$

The differential equation for a bandpass filter

Why is this a bandpass filter?

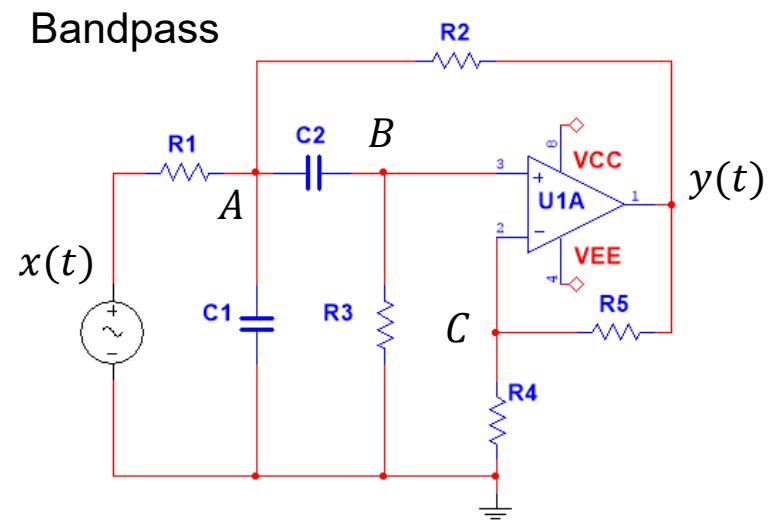
At low frequencies, C_2 will block currents. The voltage V_B will approach 0V and $y(t) = KV_B(t)$ will also approach zero.

At high frequencies, C_1 will short circuit. The voltage V_B will approach 0V and $y(t) = KV_B(t)$ will also approach zero.

$$K = 1 + \frac{R_4}{R_3}$$

$$a_1 = \frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} + \frac{1 - K}{R_2 C_1} \quad a_0 = \frac{1}{R_1 R_2 C_1 C_2}$$

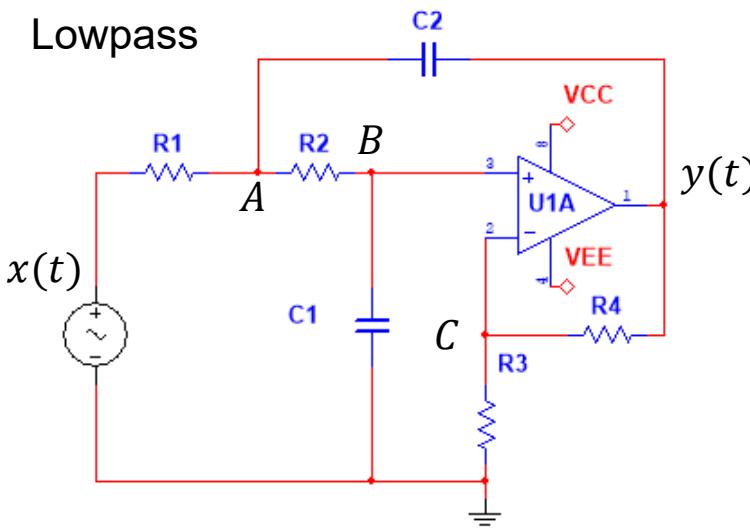
$$b_2 = 0 \quad b_1 = K \frac{D}{R_1 C_1} \quad b_0 = 0$$



$$\ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{x}$$

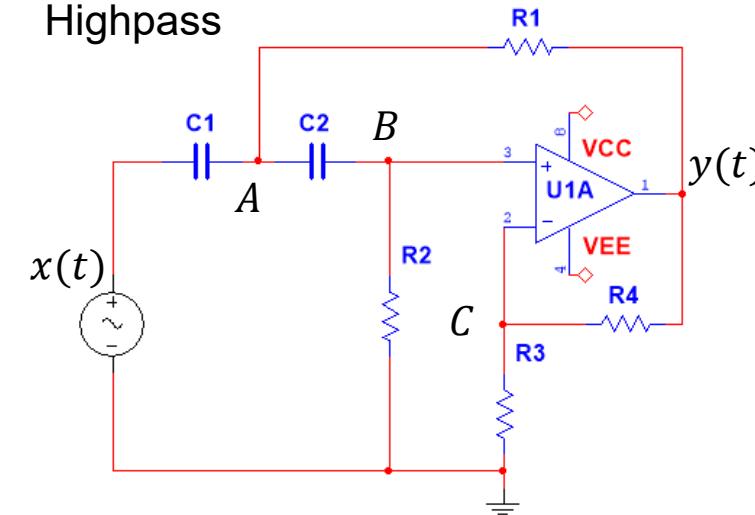
Filter circuits used in this course.

Lowpass



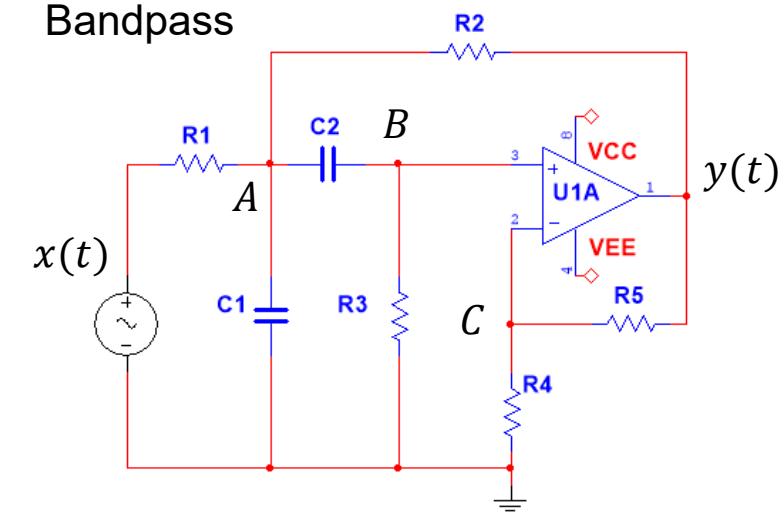
$$\ddot{y} + a_1\dot{y} + a_0y = b_0x$$

Highpass



$$\ddot{y} + a_1\dot{y} + a_0y = b_2\ddot{x}$$

Bandpass



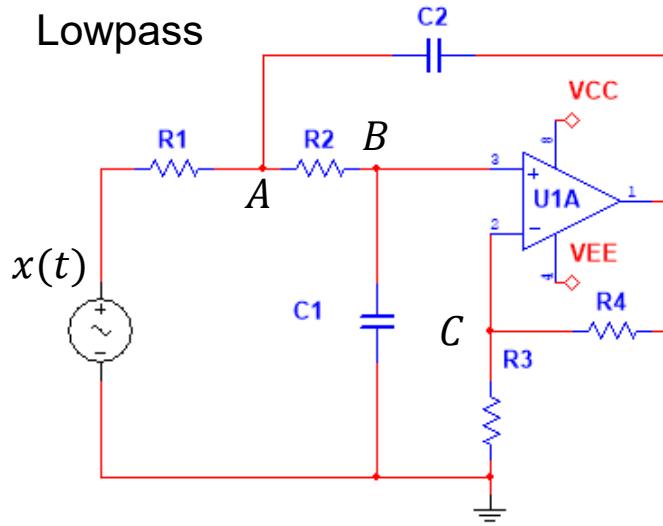
$$\ddot{y} + a_1\dot{y} + a_0y = b_1\dot{x}$$

Observation:

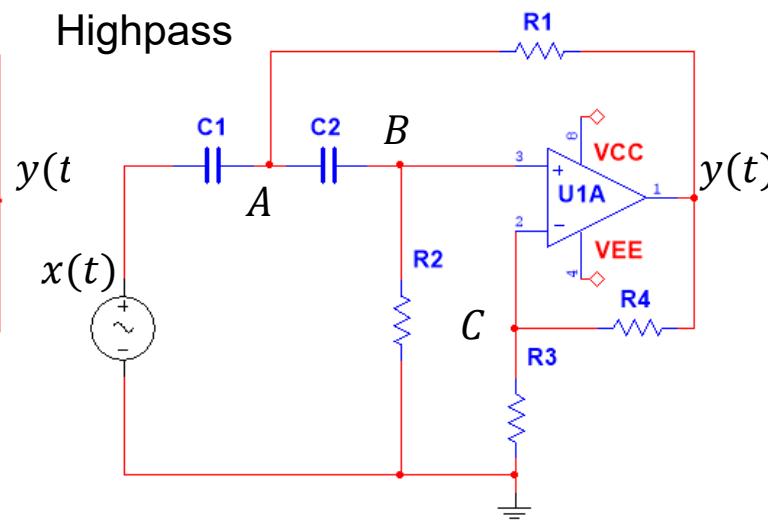
- Same left-hand side. Different right-hand side, hence different particular solution.
- The characteristic equation can have one of three different solutions: two real roots, one double root, a complex conjugated pair.
- The three filters above have component values that will cover all of the three possibilities. Although the filter problems look identical, the techniques used to solve them differ because of the difference in the characteristic equations.

Sallen-Key filters used in this course

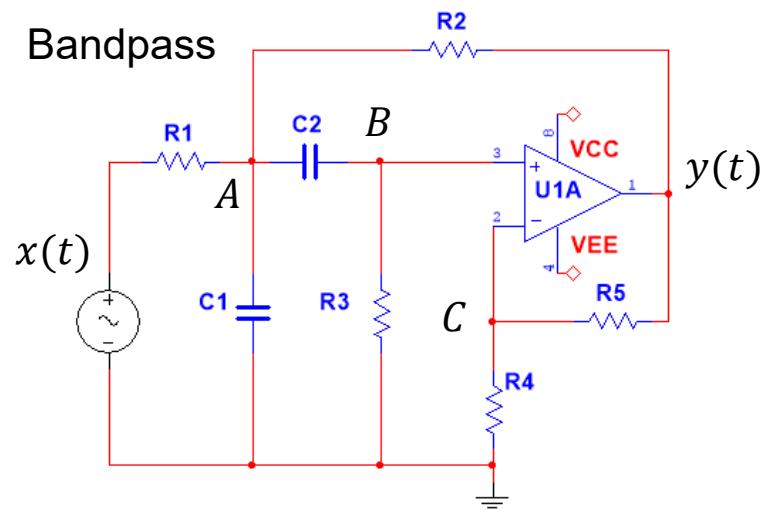
Lowpass



Highpass



Bandpass



Filter table

		Lowpass	Highpass	Bandpass
R_1	$k\Omega$	3.9894	1784.1	56.419
R_2	$k\Omega$	0.8865	892.06	34.117
R_3	$k\Omega$	1.0	1.0	149.72
R_4	$k\Omega$	1.0	1.0	1.0
R_5	$k\Omega$	—	—	1.0
C_1	nF	1795.2	1784.1	56.419
C_2	nF	398.94	3568.2	56.419
a_1		$2.83 \cdot 10^3$	$6.283 \cdot 10^{-1}$	$3.141 \cdot 10^1$
a_0		$3.95 \cdot 10^5$	$9.87 \cdot 10^{-2}$	$9.87 \cdot 10^4$
b_2		0	2	0
b_1		0	0	$6.283 \cdot 10^2$
b_0		$7.89 \cdot 10^5$	0	0

22050 Signals and linear systems in continuous time

Kaj-Åge Henneberg

L02

Time-domain Impulse response

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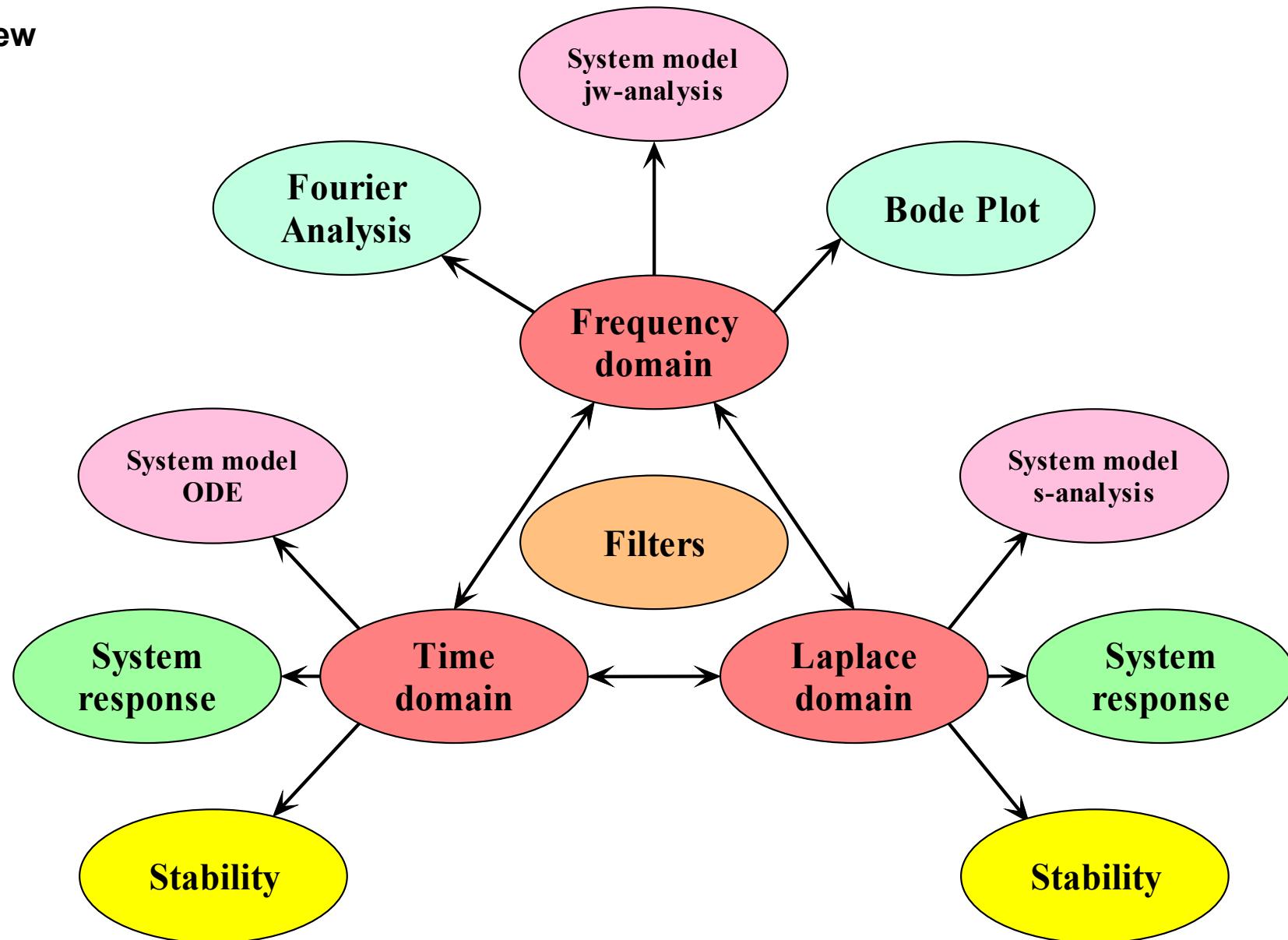
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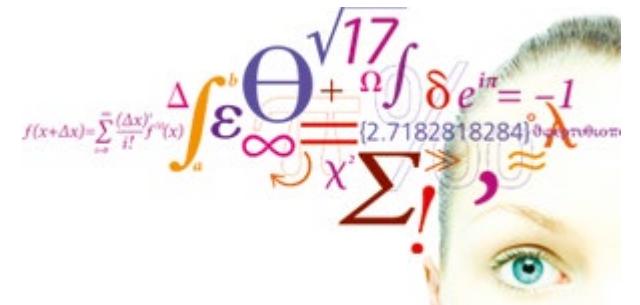
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Course overview

- System
 - Model
 - Response
 - Performance
- Domains
 - Time
 - Frequency
 - s-domain

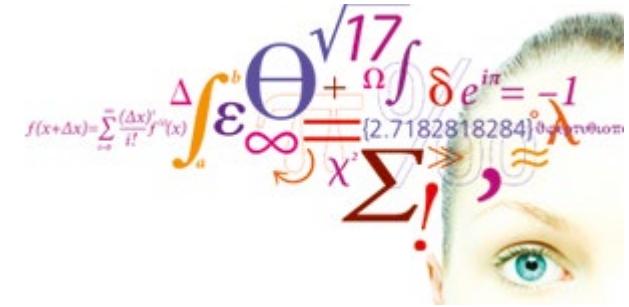


- System equation for mechanical system
- Decomposition property
- Zero-input response
- Unit impulse response
- Examples



Lathi: 2.1 – 2.3 + 2.8

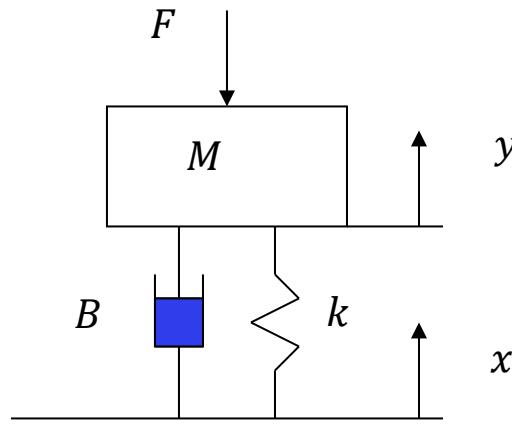
Next topic



Decomposition property

Video 1

Wheel suspension model



Mass M could be a car, k its springs, and B its shock absorbers.

Ground level is $x(t)$ and the level of a marked point on the car is $y(t)$.

ΔL : Spring compression due to gravity

$$\underbrace{M\ddot{y}}_{\text{Acceleration}} = - \underbrace{\frac{F}{M}}_{\text{Force}} - \underbrace{\frac{Mg}{M}}_{\text{Gravity}} - \underbrace{k(y - x - \Delta L)}_{\text{Spring}} - \underbrace{B(\dot{y} - \dot{x})}_{\text{Damper}}$$

$$M\ddot{y} + B\dot{y} + ky = -F - Mg + B\dot{x} + kx + k\Delta L$$

At rest: $y = \dot{y} = \ddot{y} = x = \dot{x} = F = 0 \Rightarrow Mg = k\Delta L$

$$M\ddot{y} + B\dot{y} + ky = -F + B\dot{x} + kx$$

Zero force: $\ddot{y} + \frac{B}{M}\dot{y} + \frac{k}{M}y = \frac{B}{M}\dot{x} + \frac{k}{M}x$ Standard form

Operator format

Standard form:
n'th order

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y(t) = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \dots + b_1 \frac{dx}{dt} + b_0 x(t)$$

Operator form:

$$(D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0)y(t) = (b_m D^m + b_{m-1} D^{m-1} + \dots + b_1 D + b_0)x(t)$$

Generic form:

$$Q(D)y(t) = P(D)x(t)$$

To avoid amplifying noise in the input signal, the order of the polynomial P should not be higher than Q , i.e. $m \leq n$.

Example:

$$Q(D) = D^2 + \frac{1}{\tau_1 + \tau_2}D + \frac{k}{\tau_1 \tau_2}$$

$n = 2$

$$P(D) = \frac{1}{\tau_1 + \tau_2}D + \frac{k}{\tau_1 \tau_2}$$

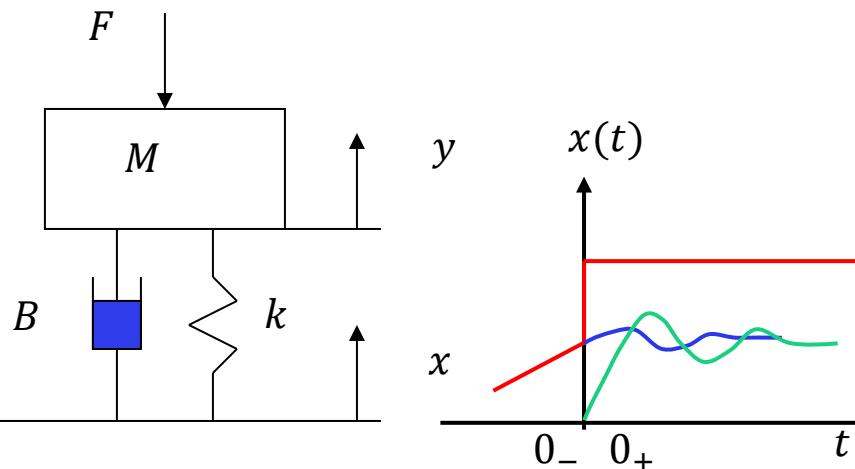
$m = 1$

Initial conditions

There are different ways to define the initial conditions. We need to understand the difference and their pros and cons.

Let us think of the suspension of a car passing a sudden increase in elevation. We will assume this happens at $t = 0$.

We want to describe the elevation $y(t)$ of the car after passing the sudden elevation.



$$\ddot{y} + \frac{B}{M}\dot{y} + \frac{k}{M}y = \frac{B}{M}\dot{x} + \frac{k}{M}x$$

Classical solution of differential equations: $y(0_+) = y_0, \quad \dot{y}(0_+) = \dot{y}_0$

The shortcoming of this method is that we don't know how much the jump in ground elevation (input $x(t)$) contributed to this new state and how much the state of the car suspension before the jump ($y(0_-)$, $\dot{y}(0_-)$) contributes to the new state. We cannot separate out the energy in the new state into what energy was in the system before the jump and how much came from the jump.

New approach:

$$y(0_-) = y_0, \quad \dot{y}(0_-) = \dot{y}_0$$

In this approach, we specify the state of the system, before the jump. This state is the result of all previous ($t < 0$) changes in ground level elevation but is completely independent of what happens for $t \geq 0$. The state (energy) of the system at $t = 0_-$ will dissipate and settle at its equilibrium elevation (blue curve). Independent of this, the car will react to the jump in elevation (green curve). The advantage of this approach is that we can develop methods to compute the response to any arbitrary changes in input measured by sensors, not just inputs stated as mathematical equations.

Decomposition property

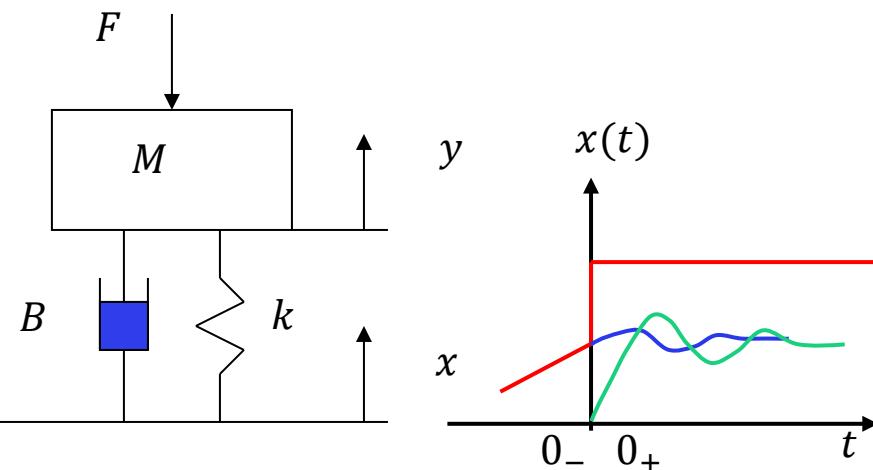
This mechanical system will change its state at $t > 0$ due to three cause:

Zero-input response

1. The state is allowed to dissipate to its resting equilibrium at $t > 0$ after an initial displacement from rest. This produces a transient response. **Blue curve**.

Zero-state response

2. The ground level x is displaced. This produces an **input-driven response**. **Green curve**.
3. A combination of 1 and 2.



$$\text{Total response} = \begin{array}{ll} \text{zero-input response} & \text{(internal energy)} \\ + \text{zero-state response} & \text{(external energy)} \end{array}$$

Decomposition is based on the principle of superposition; thus, **decomposition is only possible for linear systems**.

Decomposition property

Decomposition is based on the principle of **superposition**.

Thus, decomposition is only possible for linear systems, i.e., where $Q(D)$ and $P(D)$ are linear operator polynomials.

In terms of techniques, we still need to be able to solve differential equations, and the methods are almost identical. But the new approach also allows us to apply new methods.

$$\begin{aligned} \text{Total response} &= \text{zero-input response} && (\text{internal energy}) \\ &+ \text{zero-state response} && (\text{external energy}) \end{aligned}$$

$$y(t) = \underbrace{y_{zi}(t)}_{\substack{\text{zero} \\ \text{input}}} + \underbrace{y_{zs}(t)}_{\substack{\text{zero} \\ \text{state}}}$$

$$Q(D)y(t) = P(D)x(t)$$

$$Q(D)(y_{zi}(t) + y_{zs}(t)) = P(D)x(t)$$

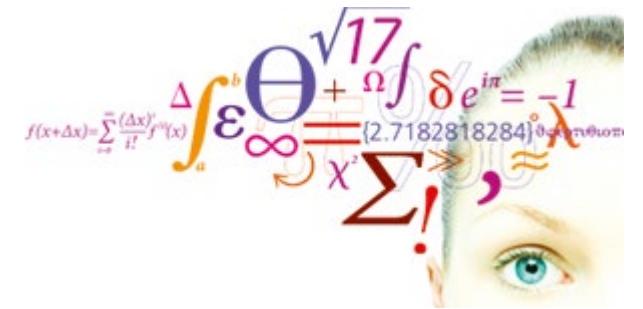
$$Q(D)y_{zi}(t) + Q(D)y_{zs}(t) = P(D)x(t)$$

$$Q(D)y_{zi}(t) = 0 \quad \text{Homogeneous differential equation}$$

$$Q(D)y_{zs}(t) = P(D)x(t) \quad \text{Inhomogeneous differential equation with zero initial conditions}$$

Zero-input response

Video 2



Zero-input response

In order that the left-hand side can equal zero for all times t , the solution to the homogeneous equation, $y_0(t)$, and all its derivatives must have the same form, that is:

Inserting in (1) yields:

Because $e^{\lambda t}$ is outside the parenthesis, the equation can equal zero for all times only if the content inside the parenthesis is zero.

Characteristic equation and its roots:

$$(1): (D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0)y(t) = 0$$

$$y_0(t) = Ce^{\lambda t}$$

$$C(\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0)e^{\lambda t} = 0$$

$$\begin{aligned} Q(\lambda) &= \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0 \\ &= (\lambda - \lambda_n)(\lambda - \lambda_{n-1}) \dots (\lambda - \lambda_1) = 0 \end{aligned}$$

Natural modes:

$$C_1e^{\lambda_1 t}, C_2e^{\lambda_2 t}, \dots, C_ne^{\lambda_n t}$$

Natural response types

1. If n different real roots:

$$y_0(t) = \sum_{j=1}^n C_j e^{\lambda_j t}, t \geq 0$$

$\lambda_j < 0$, exponentially decreasing response

$\lambda_j > 0$, exponentially increasing response

2. If r repeated real roots:

$$y_0(t) = (C_1 + C_2 t + \dots + C_r t^{r-1}) e^{\lambda_1 t} + C_{r+1} e^{\lambda_{r+1} t} + \dots + C_n e^{\lambda_n t}$$

3. If complex conjugated roots:

For a polynomial $Q(\lambda)$ with real coefficients, complex roots will always appear in conjugated pairs.

$$\begin{aligned} y_0(t) &= C_1 e^{(\sigma+j\omega)t} + C_2 e^{(\sigma-j\omega)t} \\ &= C_a e^{\sigma t} \cos(\omega t) + C_b e^{\sigma t} \sin(\omega t) \\ &= C_c e^{\sigma t} \cos(\omega t + \theta) \end{aligned}$$

Three typical forms

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = 0;$$

$$\ddot{y}(t) + 2\sigma \dot{y}(t) + \sigma^2 y(t) + \omega^2 y(t) = 0;$$

$$\sigma = \frac{a_1}{2} \quad \omega^2 = a_0 - \sigma^2$$

Addition of sinusoidal functions

$$C \cos(\omega t + \theta) = C \cos(\theta) \cos(\omega t) - C \sin(\theta) \sin(\omega t) = a \cos(\omega t) + b \sin(\omega t)$$

$$a = C \cos(\theta), b = -C \sin(\theta)$$

$$C = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1} \left(-\frac{b}{a} \right)$$

Determination of coefficients

n unknown coefficients require n initial conditions

Example:

System with two real distinct roots

We are here only concerned with homogeneous diff. equations. Hence there are no particular solution here.

$$\begin{aligned}y_0(t) &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ \dot{y}_0(t) &= C_1 \lambda_1 e^{\lambda_1 t} + C_2 \lambda_2 e^{\lambda_2 t}\end{aligned}$$

$$C_1 + C_2 = y_0(0_-)$$

$$C_1 \lambda_1 + C_2 \lambda_2 = \dot{y}_0(0_-)$$

$$C_2 = \frac{\lambda_1 y_0(0_-) - \dot{y}_0(0_-)}{\lambda_1 - \lambda_2}$$

$$C_1 = y_0(0_-) - C_2 = -\frac{\lambda_2 y_0(0_-) - \dot{y}_0(0_-)}{\lambda_1 - \lambda_2}$$

Example 1: Two real roots

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 0; y(0_-) = 2, \dot{y}(0_-) = 1$$

$$Q(\lambda) = \lambda^2 + 3\lambda + 2 = 0 \quad \frac{3^2}{2} > 4 \Leftrightarrow \text{real roots}$$

$$Q(\lambda) = (\lambda + 2)(\lambda + 1) = 0 \\ \Leftrightarrow \lambda_1 = -2 \wedge \lambda_2 = -1$$

$$y_0(t) = C_1 e^{-2t} + C_2 e^{-t}; t \geq 0$$

$$C_2 = \frac{\lambda_1 y_0(0_-) - \dot{y}_0(0_-)}{\lambda_1 - \lambda_2} = \frac{-2 \cdot 2 - 1}{-2 - (-1)} = 5$$

$$C_1 = y_0(0_-) - C_2 = 2 - 5 = -3$$

$$y_0(t) = 5e^{-t} - 3e^{-2t}; t \geq 0$$

Initial conditions

$$ics := y(0) = 2, D(y)(0) = 1$$

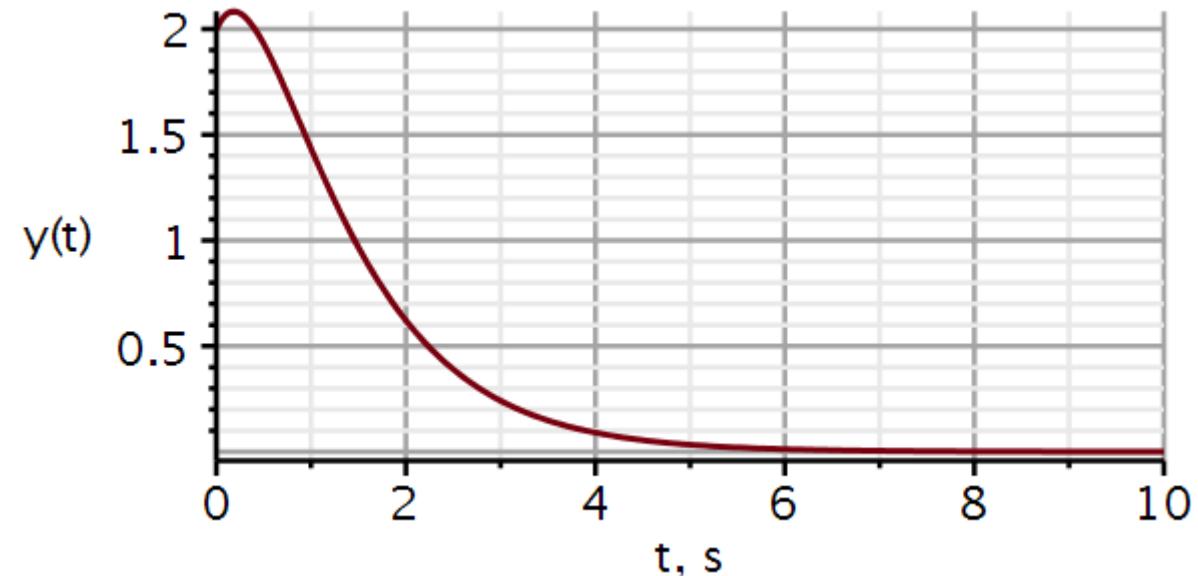
$$ics := y(0) = 2, D(y)(0) = 1$$

Differential equation

$$ode := diff(y(t), t, t) + 3 \cdot diff(y(t), t) + 2 \cdot y(t) = 0;$$

$$solution := evalf(dsolve(\{ode, ics\}, y(t)))$$

$$solution := y(t) = 5 \cdot e^{-1 \cdot t} - 3 \cdot e^{-2 \cdot t}$$



Example 2: A double root

$$\ddot{y}(t) + 6\dot{y}(t) + 9y(t) = 0; y(0_-) = 2, \dot{y}(0_-) = 1$$

$$Q(\lambda) = \lambda^2 + 6\lambda + 9 = 0 \quad \frac{6^2}{9} = 4 \Leftrightarrow \text{double roots}$$

$$Q(\lambda) = (\lambda + 3)(\lambda + 3) = 0 \Leftrightarrow \lambda = -3$$

$$y_0(t) = (C_1 + C_2 t)e^{-3t}; t \geq 0$$

$$y_0(0_-) = C_1 = 2$$

$$\dot{y}_0(t) = (C_1\lambda + C_2 + C_2\lambda t)e^{\lambda t}$$

$$\dot{y}_0(0_-) = C_1\lambda + C_2 \Rightarrow C_2 = \dot{y}_0(0_-) - C_1\lambda$$

$$C_2 = 1 - 2(-3) = 7$$

$$y_0(t) = (2 + 7t)e^{-3t}; t \geq 0$$

Initial conditions

$$ics := y(0) = 2, D(y)(0) = 1$$

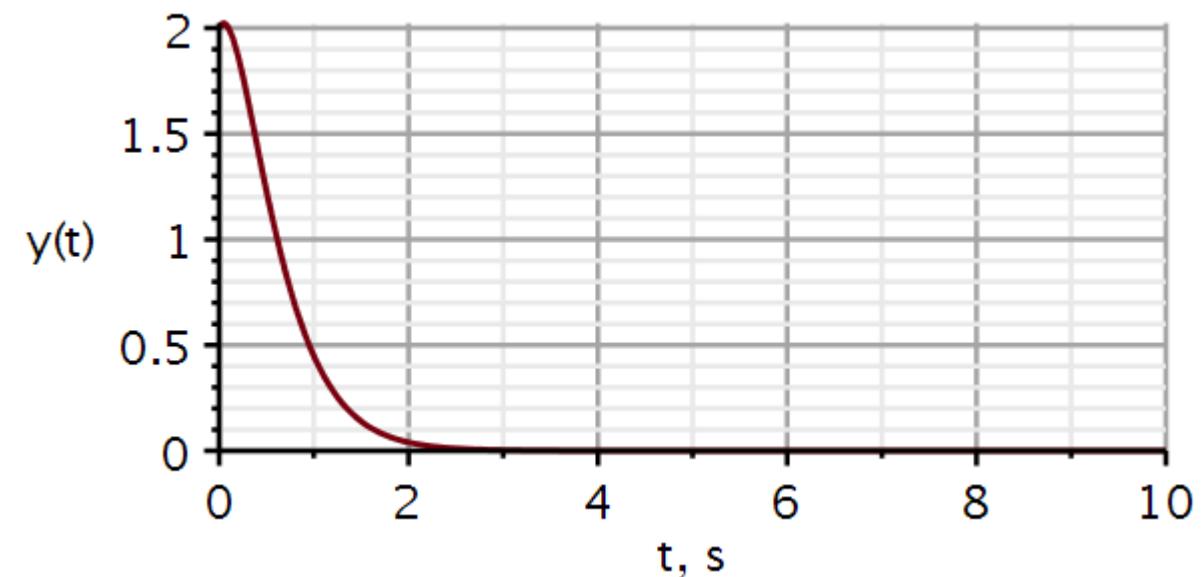
$$ics := y(0) = 2, D(y)(0) = 1$$

Differential equation

$$ode := \text{diff}(y(t), t, t) + 6 \cdot \text{diff}(y(t), t) + 9 \cdot y(t) = 0 :$$

$$solution := \text{evalf}(\text{dsolve}(\{ode, ics\}, y(t)))$$

$$solution := y(t) = 2 \cdot e^{-3 \cdot t} + 7 \cdot e^{-3 \cdot t} t$$



Example 3: Complex conjugated roots

$$\ddot{y}(t) + 4\dot{y}(t) + 13y(t) = 0; y(0_-) = 2, \dot{y}(0_-) = 1$$

$$Q(\lambda) = \lambda^2 + 4\lambda + 13 = 0 \quad \frac{4^2}{13} < 4 \Leftrightarrow \text{comp conj roots}$$

$$Q(\lambda) = (\lambda + 2 + j3)(\lambda + 2 - j3) = 0 \Leftrightarrow \lambda = -2 \pm j3$$

$$y_0(t) = C_1 e^{-2t} \cos(3t) + C_2 e^{-2t} \sin(3t); t \geq 0$$

$$y_0(0_-) = C_1 = 2$$

$$\begin{aligned}\dot{y}_0(t) = & C_1 \sigma e^{\sigma t} \cos(\omega t) - C_1 \omega \sin(\omega t) \\ & + C_2 \sigma e^{\sigma t} \sin(\omega t) + C_2 \omega e^{\sigma t} \cos(\omega t)\end{aligned}$$

$$\dot{y}_0(0_-) = C_1 \sigma + C_2 \omega = 1 \Rightarrow C_2 = \frac{\dot{y}_0(0_-) - C_1 \sigma}{\omega}$$

$$\Rightarrow C_2 = \frac{1 - 2(-2)}{3} = \frac{5}{3}$$

$$y_0(t) = 2e^{-2t} \cos(3t) + \frac{5}{3}e^{-2t} \sin(3t); t \geq 0$$

Initial conditions

$$ics := y(0) = 2, D(y)(0) = 1$$

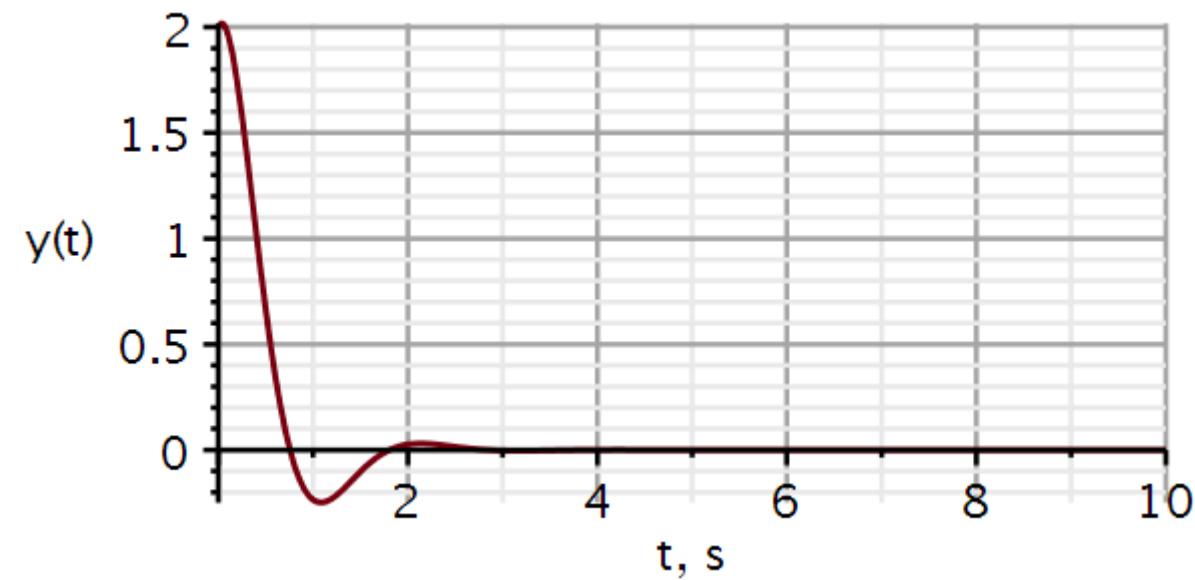
$$ics := y(0) = 2, D(y)(0) = 1$$

Differential equation

$$ode := diff(y(t), t, t) + 4 \cdot diff(y(t), t) + 13 \cdot y(t) = 0 :$$

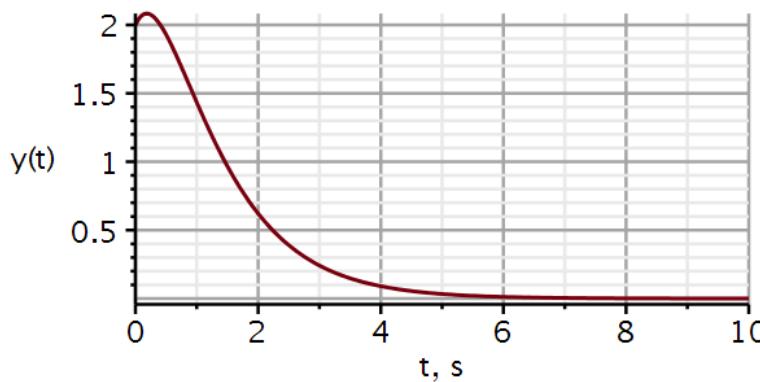
$$solution := evalf(dsolve(\{ode, ics\}, y(t)))$$

$$solution := y(t) = 1.666666667 e^{-2 \cdot t} \sin(3 \cdot t) + 2 \cdot e^{-2 \cdot t} \cos(3 \cdot t)$$

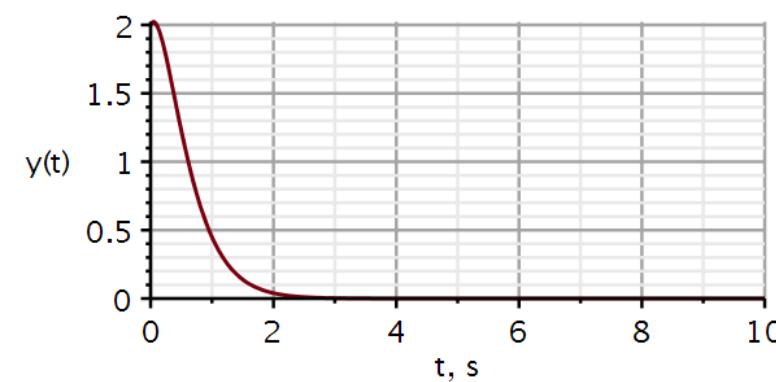


Variation in dynamics

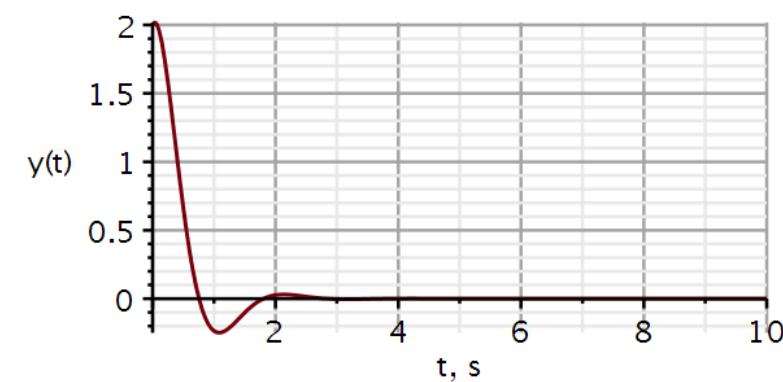
Two real roots



A double root



Complex conjugated roots



Slow return to resting state

Largest time constant slows down decay

Fast return to resting state

Fastest dissipation but
overshoots and
oscillates

The roots of the characteristic equation define the nature of the system.

Maple package: DynamicSystems

We will use Maple throughout the course and use some relevant tools. One package of high relevance is **DynamicSystems**:

u: input signal

x: system state

y: output signal

t: time

s: complex frequency: $\sigma + j\omega$

> *with(DynamicSystems)* :

Return a sequence of all options and their values.

> *SystemOptions()*

inputvariable = u, statevariable = x, discretefreqvar = z, relativeerror = 0.001, colors = ["#78000E", "#000E78", "#4A7800", "#3E578A", "#780072", "#00786A", "#604191", "#004A78", "#784C00", "#91414A", "#3E738A", "#78003B", "#00783F", "#914186", "#510078", "#777800"], duration = 10.0, radians = false, discrete = false, linearsolvemethod = none, cancellation = false, conjugate = false, decibels = true, discretetimevar = q, samplecount = 10, outputvariable = y, continuoustimevar = t, parameters = Ø, complexfreqvar = s, hertz = false, sampletime = 1.

Normally we use $u(t)$ to denote the step function. When using DynamicSystems, $u(t)$ is reserved to denote input. In this situation I use $v(t)$ to denote the step function in Maple scripts.

Maple: Defining systems

There are several ways to define a system.

In the example shown here, a function named **Coefficients** are used. It takes as input:

$$\frac{P(s)}{Q(s)}$$

In Maple, “s” denotes the differential operator.

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$$

$$s^2y(t) + 3sy(t) + 2y(t) = x(t) \Rightarrow y(t) = \frac{1}{s^2 + 3s + 2}x(t)$$

$$\frac{P(s)}{Q(s)} = \frac{1}{s^2 + 3s + 2}$$

Dynamic systems

```
restart
with(plots):
with(DynamicSystems):
sys1 := Coefficients( $\frac{1}{s^2 + 3 \cdot s + 2}$ );
PrintSystem(sys1)
```

Coefficients

continuous

1 output(s); 1 input(s)

inputvariable = [$u1(s)$]

outputvariable = [$y1(s)$]

$num_{1,1} = [1]$

$den_{1,1} = [1, 3, 2]$

Maple: Plotting roots of numerator and denominator operators

$$Q(D)y(t) = P(D)x(t)$$

$$y(t) = \frac{P(D)}{Q(D)}x(t)$$

The roots of $P(D) = 0$ are called “**ZEROS**”

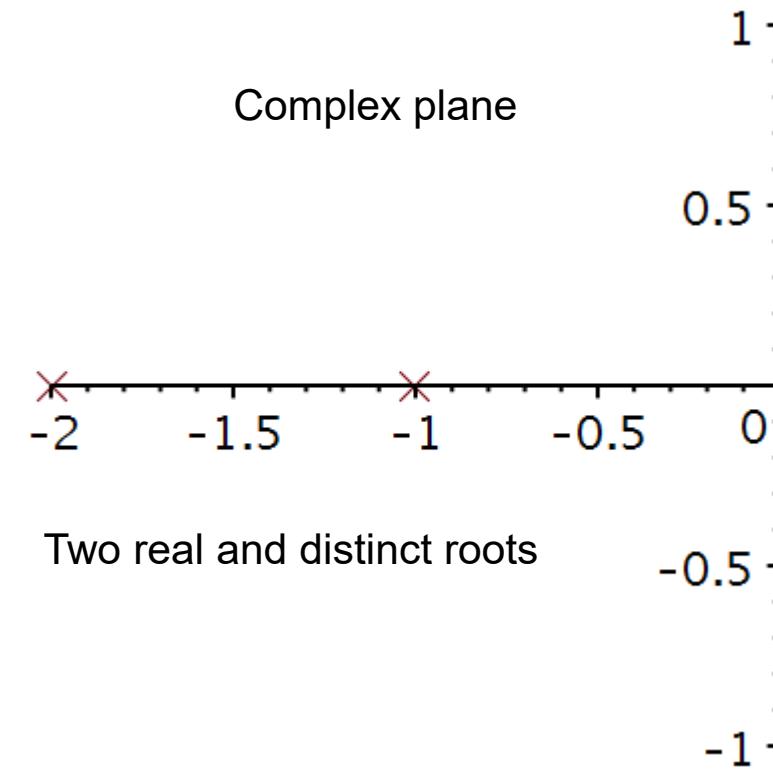
The roots of $Q(D) = 0$ are called “**POLES**”

The function **ZeroPolePlot** will plot the zeros and poles for a predefined system, here named *sys1*.

Observation

The poles are also the roots of the characteristic equation.

```
ZeroPolePlot[sys1, symbolsize = 30, font = [Helvetica, roman, 18],  
axis[2] = [thickness = 2.0], axis[1] = [mode = linear, thickness  
= 2.0], size = [400, 400])
```

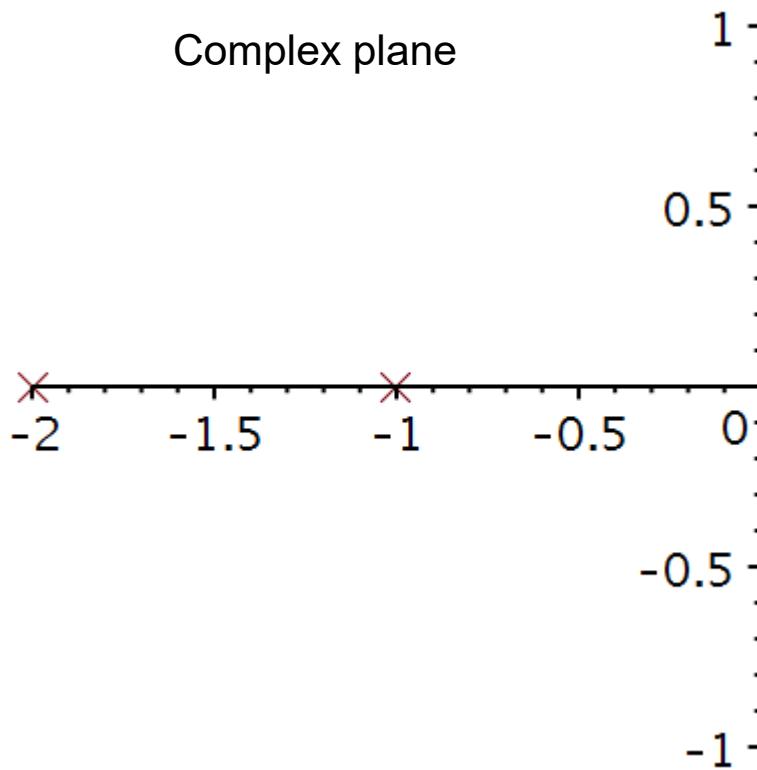


Roots of characteristic equations

Two real poles

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$$

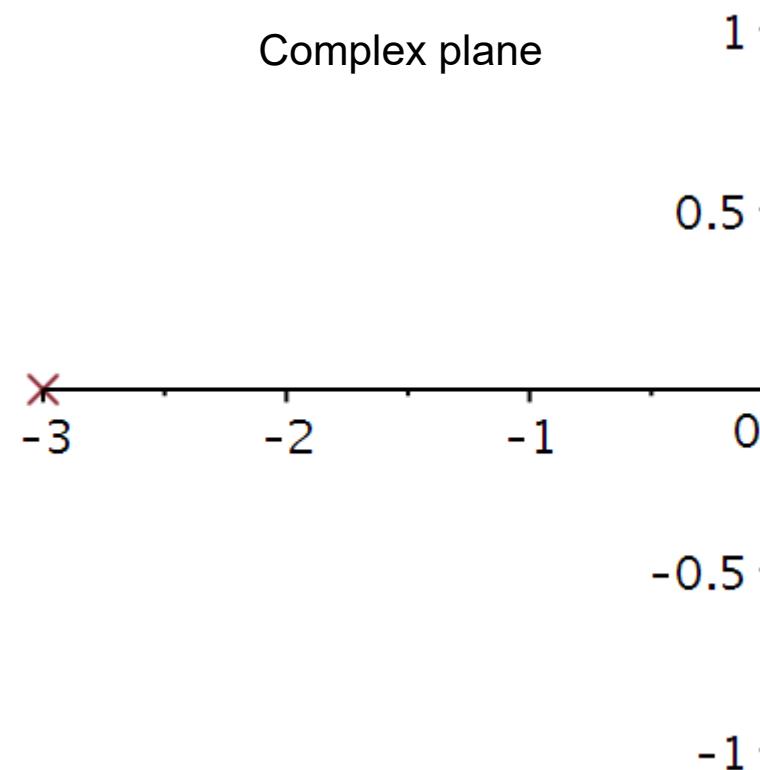
Complex plane



A double pole

$$\ddot{y}(t) + 6\dot{y}(t) + 9y(t) = x(t)$$

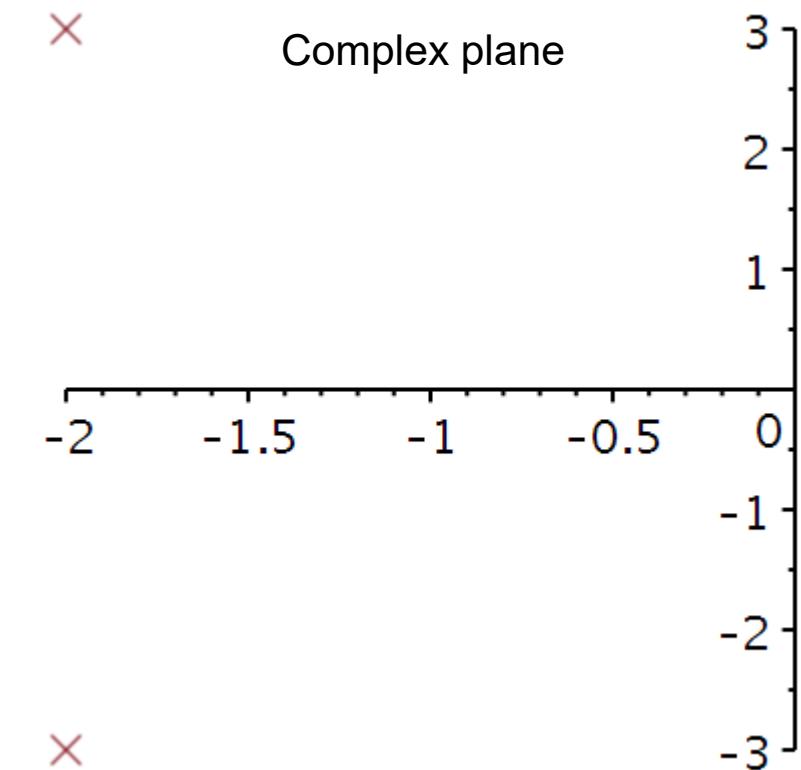
Complex plane



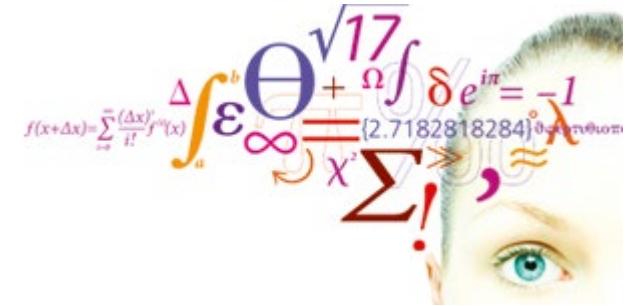
Complex conjugated poles

$$\ddot{y}(t) + 4\dot{y}(t) + 13y(t) = x(t)$$

Complex plane



Break 10 min.

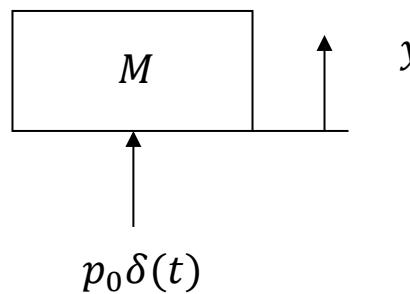


Impulse response

Video 3

Effect of an impulse disturbance

Imagine that we hit the mass M with a hammer from beneath. The hammer will act with an impulse p_0 on the mass and the momentum of the mass will increase by an amount p_0 :



Observation:

The effect of an impulse disturbance is an instantaneous change in velocity \dot{y} .

Because the change in velocity is finite, there is not enough time for a change in position y .

$$p(t) = p_0 u(t) = Mv(t) = M\dot{y}(t)$$

$$\frac{dp}{dt} = M\dot{v} = M\ddot{y} = p_0 \dot{u}(t) = p_0 \delta(t)$$

$$F = M\ddot{y} = p_0 \delta(t) \quad \wedge \quad \dot{y}(0_-) = 0 \quad \wedge \quad y(0_-) = 0$$

$$M\dot{y}(0_+) - M\dot{y}(0_-) = \int_{0_-}^{0_+} M\ddot{y} dt = p_0 \underbrace{\int_{0_-}^{0_+} \delta(t) dt}_{=1} = p_0$$

$$\underbrace{\dot{y}(0_+) - \dot{y}(0_-)}_{\text{velocity change}} = \frac{p_0}{M}$$

$$\int_{0_-}^{0_+} \dot{y} dt = \underbrace{y(0_+) - y(0_-)}_{\text{position change}} = \frac{p_0}{M} (0_+ - 0_-) = 0$$

Response to sudden change on input

The differential equation and its initial conditions shown here describes a second order system, which is at rest for $t < 0$.

$$\ddot{y} + a_1 \dot{y} + a_0 y = x(t), \quad \dot{y}(0_-) = 0, y(0_-) = 0$$

If the input $x(t)$ changes instantaneously, the highest derivative must be able to follow this instantaneous change.

Fx, we cannot have a change in position without a velocity, and we cannot have a change in velocity without an acceleration.

An instantaneous change in velocity requires an infinitely large acceleration.

An instantaneous change in position requires an infinitely large velocity.

Response to sudden change on input

Here the input is an impulse function.

At $t = 0$, acceleration becomes an impulse, velocity a step function and position becomes a ramp function.

Here the input is the derivative of the impulse function.

At $t = 0$, acceleration becomes the derivative of the impulse, velocity an impulse function and position becomes a step function.

Here the input is the 2nd derivative of the impulse function.

At $t = 0$, acceleration becomes the 2nd derivative of the impulse, velocity is the 1st derivative of the impulse function and position becomes an impulse function.

$$\ddot{y} + a_1\dot{y} + a_0y = \delta(t), \quad \dot{y}(0_-) = 0, y(0_-) = 0$$

$$@ t = 0: \quad \ddot{y} \propto \delta(t) \quad \dot{y} \propto u(t) \quad y \propto r(t)$$

$$\ddot{y} + a_1\dot{y} + a_0y = \dot{\delta}(t), \quad \dot{y}(0_-) = 0, y(0_-) = 0$$

$$@ t = 0: \quad \ddot{y} \propto \dot{\delta}(t) \quad \dot{y} \propto \delta(t) \quad y \propto u(t)$$

$$\ddot{y} + a_1\dot{y} + a_0y = \ddot{\delta}(t), \quad \dot{y}(0_-) = 0, y(0_-) = 0$$

$$@ t = 0: \quad \ddot{y} \propto \ddot{\delta}(t) \quad \dot{y} \propto \dot{\delta}(t) \quad y \propto \delta(t)$$

Effect at $t = 0$ of an impulse disturbance

At $t = 0$, acceleration behaves like an impulse:

At $t = 0$:

The acceleration is an impulse, the velocity changes as a step function, and the position changes as a ramp function.

We observe that the effect of a unit impulse is an instantaneous unit jump in velocity:

$$\ddot{y} + a_1 \dot{y} + a_0 y = \delta(t), \quad \dot{y}(0_-) = 0, y(0_-) = 0$$

$$\ddot{y}(t) = \delta(t)$$

$$\int_{0_-}^t \ddot{y}(s) ds = \dot{y}(t) - \dot{y}(0_-) = \int_{0_-}^t \delta(s) ds = u(t)$$

$$\int_{0_-}^t \dot{y}(s) ds = y(t) - y(0_-) = \int_{0_-}^t u(s) ds = r(t)$$

$$\dot{y}(0_+) - \dot{y}(0_-) = 1$$

$$y(0_+) - y(0_-) = r(0_+) = 0$$

Effect at $t = 0$ of an impulse disturbance

At $t = 0$:

Acceleration is the derivative of the impulse function, velocity \dot{y} is an impulse function, and position is a step function.

$$\ddot{y} + a_1 \dot{y} + a_0 y = \delta(t), \quad \dot{y}(0_-) = 0, y(0_-) = 0$$

$$\int_{0_-}^t \ddot{y}(s) ds = \dot{y}(t) - \dot{y}(0_-) = \int_{0_-}^t \delta(s) ds$$

$$\dot{y}(t) - \dot{y}(0_-) = \delta(t)$$

$$\int_{0_-}^t \dot{y}(s) ds = y(t) - y(0_-) = \int_{0_-}^t \delta(s) ds = u(t)$$

We observe that the effect of a unit impulse change in velocity creates an instantaneous unit jump in position:

$$y(0_+) - y(0_-) = 1$$

Effect at $t = 0$ of an impulse disturbance

At $t = 0$:

acceleration is the 2nd derivative of the impulse function, velocity \dot{y} is the 1st derivative of the impulse function, and position is an impulse function.

$$\ddot{y} + a_1 \dot{y} + a_0 y = \ddot{\delta}(t), \quad \dot{y}(0_-) = 0, y(0_-) = 0$$

$$\int_{0_-}^t \ddot{y}(s) ds = \dot{y}(t) - \dot{y}(0_-) = \int_{0_-}^t \ddot{\delta}(s) ds$$

$$\dot{y}(t) - \dot{y}(0_-) = \dot{\delta}(t)$$

$$\int_{0_-}^t \dot{y}(s) ds = y(t) - y(0_-) = \int_{0_-}^t \dot{\delta}(s) ds = \delta(t)$$

We observe that if a system gets a 2nd derivative of the impulse as input, an impulse will occur on the output.

$$y(t) - y(0_-) = \delta(t)$$

Initial conditions for response to normalized impulse perturbation

An impulse perturbation produces an instant change in system state.

Thus, for a system at rest, we can represent the impulse perturbation by specifying the initial conditions:

$$\begin{aligned}\ddot{y} + a_1\dot{y} + a_0y &= \delta(t), & \dot{y}(0_-) = 0, y(0_-) = 0 \\ \Rightarrow \dot{y}(0_+) - \dot{y}(0_-) &= 1 \\ \Rightarrow y(0_+) - y(0_-) &= 0\end{aligned}$$

$$\ddot{y} + a_1\dot{y} + a_0y = 0 \quad \dot{y}(0_+) = 1, y(0_+) = 0$$

For an n'th order system:

$$y^{(n-1)}(0_+) = 1 \quad y^{(n-2)}(0_+) = y^{(n-3)}(0_+) = \dots = y(0_+) = 0$$

Observation:

- If we have a differential equation where the forcing function $x(t) = \delta(t)$, then we have in principle an inhomogeneous differential equation.
- We have seen that for a 2nd order system the effect of a unit impulse is a sudden increase in $\dot{y}(t)$ by 1. If $\dot{y}(0_-) = 0$, then $\dot{y}(0_+) = 1$.
- Instead of solving an inhomogeneous differential equation, we will solve the corresponding homogeneous equation, using the initial conditions caused by the unit impulse.

Solving for impulse response

When seeking the response to an impulse perturbation, we replace the input function by a unit impulse function. We can solve this equation by assigning appropriate initial conditions to the homogeneous equation. Subscript n refers to the *natural* response.

$$\begin{aligned}\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) &= x(t) \\ \ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) &= \delta(t) \\ \ddot{y}_n(t) + a_1\dot{y}_n(t) + a_0y_n(t) &= 0 \quad y(0_+) = 0, \dot{y}(0_+) = 1\end{aligned}$$

The impulse response is the solution to a homogeneous differential equation:

It plays a vital role and is by convention denoted h .

$$h(t) \stackrel{\text{def}}{=} y_n(t)u(t) = \sum_{j=1}^n C_j e^{\lambda_j t}; t \geq 0_+, m < n$$

We have considered only the case when $b_0 = 1$. We need to generalize the result to a general operator polynomial $P(D)x(t)$

Solving for impulse response

Unscaled input:

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) = x(t) \quad y(0_+) = 0, \dot{y}(0_+) = 1$$

$$h(t) = y_n(t)u(t)$$

If we scale the input by a factor b_0 ,
we must scale the output by that
same factor b_0 :

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) = b_0x(t)$$

$$h(t) = b_0y_n(t)u(t)$$

Solving for impulse response

Unscaled input:

If we differentiate on the rhs, we must differentiate on the lhs:

$$\ddot{y}_1(t) + a_1\dot{y}_1(t) + a_0y_1(t) = x(t) \quad h_1(t) = y_1(t)u(t)$$

$$\ddot{y}_1(t) + a_1\ddot{y}_1(t) + a_0\dot{y}_1(t) = \dot{x}(t)$$

$$\ddot{y}_2(t) + a_1\dot{y}_2(t) + a_0y_2(t) = \dot{x}(t) \quad h_2(t) = y_2(t)u(t)$$

$$\ddot{y}_2(t) + a_1\ddot{y}_2(t) + a_0\dot{y}_2(t) = b_1\dot{x}(t) \quad h_2(t) = b_1y_2(t)u(t)$$

$$y_2(t) = \dot{y}_1(t) \quad h_2(t) = b_1 \frac{d}{dt} (y_1(t)u(t))$$

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) = b_1\dot{x}(t) + b_0x(t) = P(D)x(t)$$

$$h(t) = b_1 \frac{d}{dt} (y_n(t)u(t)) + b_0y_n(t)u(t) = P(D)(y_n(t)u(t))$$

If the input is a linear combination, then the output will be a linear combination:

Solving for impulse response

If we have a differential equation with a derivative on the right-hand side, we will never attempt to solve this.

We will solve for the simpler case, and post-manipulate the solution to fit the equation at hand:

$$\ddot{y}_2(t) + a_1\dot{y}_2(t) + a_0y_2(t) = b_1\dot{x}(t)$$

$$\ddot{y}_2(t) + a_1\dot{y}_2(t) + a_0y_2(t) = b_1\delta(t)$$

$$\ddot{y}_1(t) + a_1\dot{y}_1(t) + a_0y_1(t) = \delta(t)$$

$$\ddot{y}_1(t) + a_1\dot{y}_1(t) + a_0y_1(t) = 0 \quad y_1(0_+) = 0, \dot{y}_1(0_+) = 1$$

$$h_2(t) = b_1 \frac{d}{dt} (y_1(t)u(t))$$

Solving for impulse response

We can split the differential operation in two terms:

$$\begin{aligned} h(t) = P(D)(y_n(t)u(t)) &= (b_m D^m + \dots + b_1 D + b_0)(y_n(t)u(t)) \\ &= b_m D^m(y_n(t)u(t)) + b_{m-1} D^{m-1}(y_n(t)u(t)) \dots + b_0(y_n(t)u(t)) \end{aligned}$$

$$\begin{aligned} (b_m D^m + \dots + b_1 D + b_0)(y_n(t)u(t)) &= b_0 y_n(t)u(t) + b_m y_n(t)D^m(u(t)) + b_m u(t)D^m(y_n(t)) + \\ &\quad \dots + b_1 y_n(t)D(u(t)) + b_1 u(t)D(y_n(t)) \end{aligned}$$

The blue terms are zero.

$$t > 0: u(t) = 1, \quad y_n(t)D^m(u(t)) = 0$$

$$t = 0: y_n(0) = 0, \quad y_n(0)D^m(u(t)) = 0$$

$$h(t) = P(D)(y_n(t)u(t)) = u(t)P(D)(y_n(t))$$

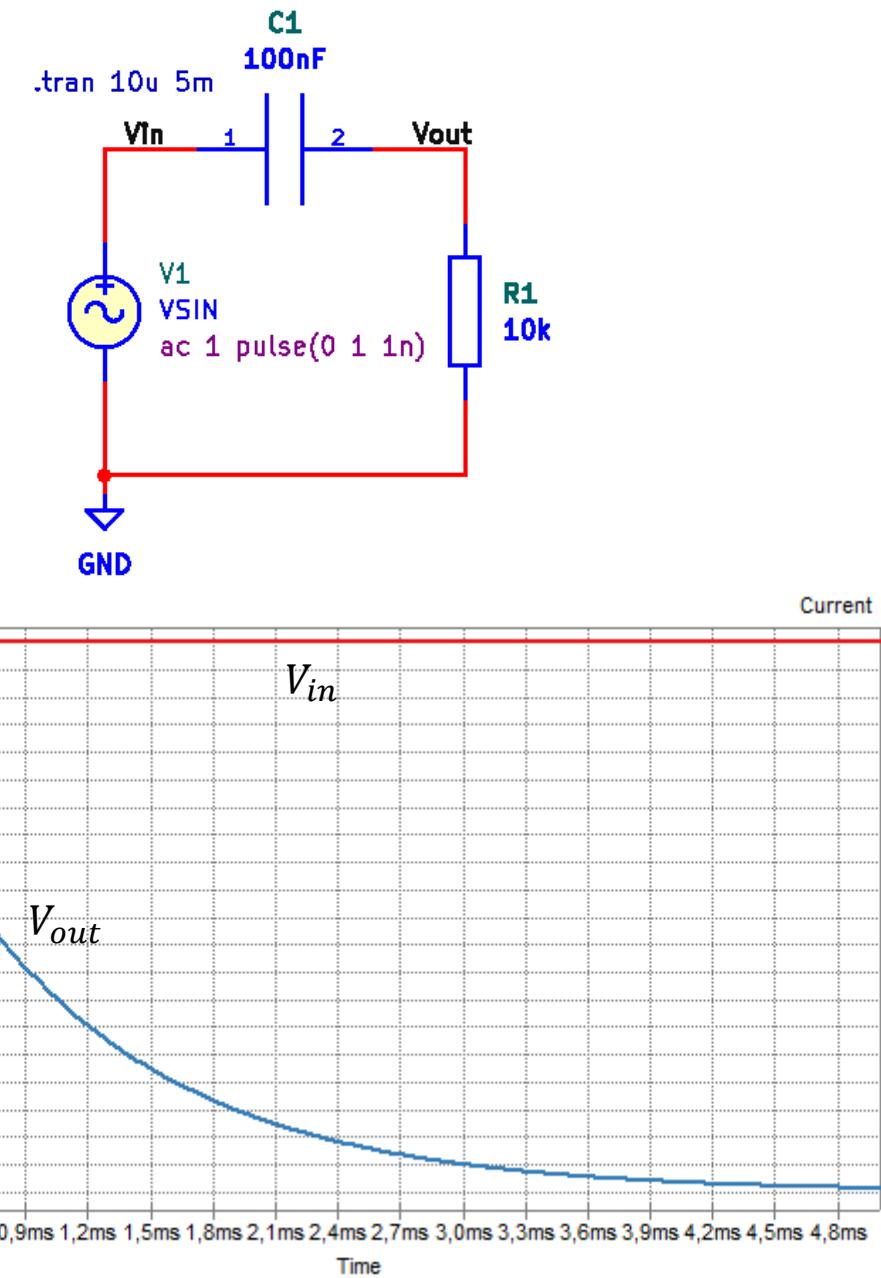
Solving for impulse response

The present circuit will respond with a sudden jump on the output if a sudden jump is applied at the input.

We recall that the voltage drop across a capacitor cannot change instantaneously. The unit step response seen on the output of this RC circuit clearly shows that the output response includes a unit step from zero to one at $t = 0$.

Because the unit impulse function is the derivative of the unit step function, the unit impulse response will also be the derivative of the unit step response.

If we are to differentiate the blue curve at $t = 0$, we need a unit impulse function.



Solving for impulse response

There are systems, like the circuit on the previous slide, where a sudden change of input results in a synchronous sudden change on the output.

In the differential equation shown here, the right-hand side contains the n'th derivative of the impulse function.

This requires the impulse response to also include an impulse function, such that the n'th derivative of the impulse function is present on both sides.

Matching the terms with the highest order of derivative of $\delta(t)$ we obtain:

When the order m of the $P(D)$ operator polynomial equals the order n of the $Q(D)$ operator polynomial, an impulse function must be included in the impulse response.

$$\begin{aligned} m = n \quad & (D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0)h(t) \\ & = (b_nD^n + b_{n-1}D^{n-1} + \dots + b_1D + b_0)\delta(t) \end{aligned}$$

$$h(t) = A_0\delta(t) + (P(D)y_n(t))u(t)$$

$$D^n(A_0\delta(t)) = b_nD^n\delta(t) \Rightarrow A_0 = b_n$$

$$h(t) = b_n\delta(t) + (P(D)y_n(t))u(t)$$

Examples

Let us find the natural response:

$$\ddot{y}_n + 4\dot{y}_n + 4y_n = 0, \quad y_n(0_+) = 0, \dot{y}_n(0_+) = 1$$

We solve the homogeneous equation
with special initial conditions.

$$\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0 \Rightarrow \lambda = -2$$

What type of roots will we see?

- A. Two real roots
- B. A double root
- C. Complex conjugated roots

$$y_n(t) = (C_1 + C_2 t)e^{-2t}$$

$$\dot{y}_n(t) = (-2C_1 + C_2(1 - 2t))e^{-2t}$$

$$y_n(0_+) = 0 \Rightarrow C_1 = 0 \wedge \dot{y}(0_+) = 1 \Rightarrow C_2 = 1$$

$$y_n(t) = te^{-2t}; t \geq 0$$

$$h(t) = b_n \delta(t) + (P(D)y_n(t))u(t), m \leq n$$

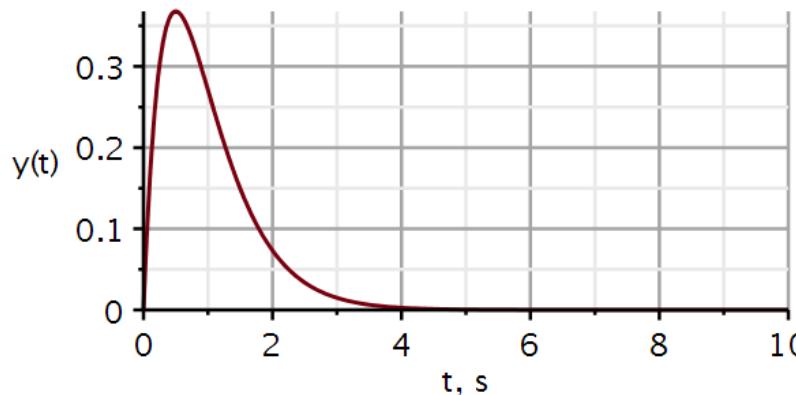
Without knowing the $P(D)$ polynomial, we cannot write the final form of the impulse response.

Examples

$$\ddot{y} + 4\dot{y} + 4y = 2\dot{x}$$

$m < n$

$$\begin{aligned} h(t) &= [P(D)te^{-2t}]u(t) \\ &= [2te^{-2t}]u(t) \end{aligned}$$

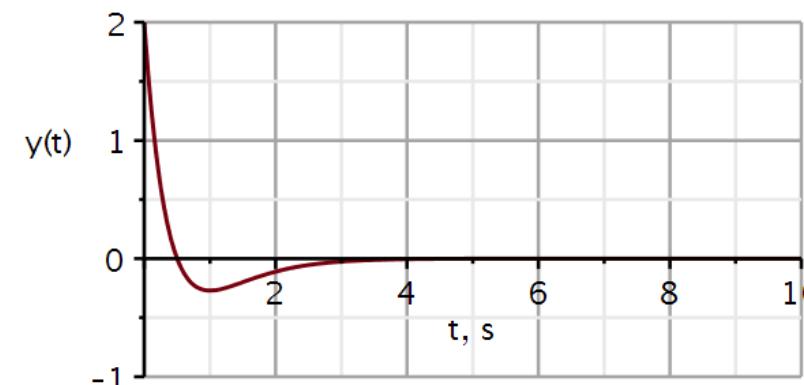


@ $t = 0$: $y(t) \propto r(t)$

$$\ddot{y} + 4\dot{y} + 4y = 2\dot{x}$$

$m < n$

$$\begin{aligned} h(t) &= [P(D)te^{-2t}]u(t) \\ &= \left[2 \frac{d}{dt}(te^{-2t}) \right] u(t) \\ &= 2[1 - 2t]e^{-2t}u(t) \end{aligned}$$

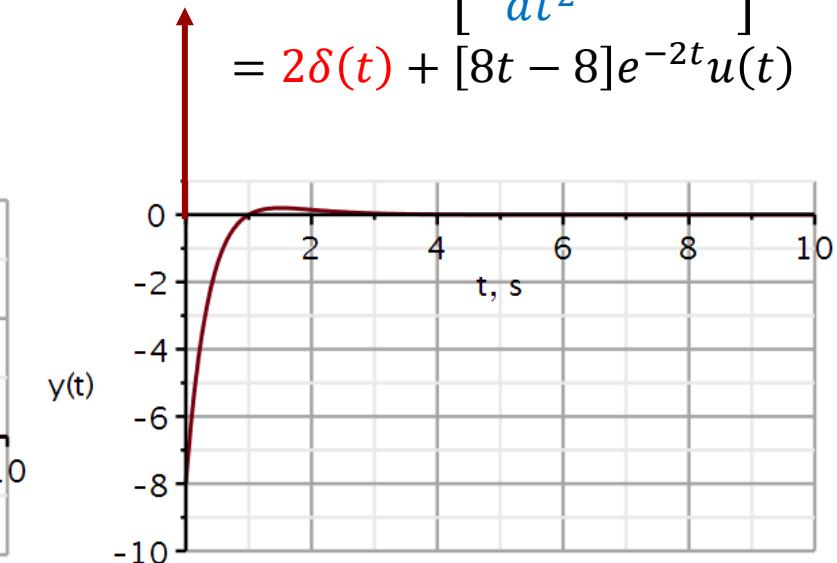


@ $t = 0$: $y(t) \propto u(t)$

$$\ddot{y} + 4\dot{y} + 4y = 2\ddot{x}$$

$m = n$

$$\begin{aligned} h(t) &= 2\delta(t) + [P(D)te^{-2t}]u(t) \\ &= 2\delta(t) + \left[2 \frac{d^2}{dt^2}(te^{-2t}) \right] u(t) \\ &= 2\delta(t) + [8t - 8]e^{-2t}u(t) \end{aligned}$$



@ $t = 0$: $y(t) \propto \delta(t)$

Impulse response – Two real roots

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$$

Analytic impulse response

$$\ddot{y}_n(t) + 3\dot{y}_n(t) + 2y_n(t) = 0, \dot{y}_n(0_+) = 1, y_n(0_+) = 0$$

Initial conditions

$$Q(\lambda) = \lambda^2 + 3\lambda + 2 = 0$$

$$ics := y(0) = 0, D(y)(0) = 1$$

$$Q(\lambda) = (\lambda + 2)(\lambda + 1) = 0$$

$$ics := y(0) = 0, D(y)(0) = 1$$

$$\Leftrightarrow \lambda_1 = -2 \wedge \lambda_2 = -1$$

$$y_n(t) = C_1 e^{-2t} + C_2 e^{-t}; t \geq 0$$

Differential equation

$$C_2 = \frac{\lambda_1 y_n(0_+) - \dot{y}_n(0_+)}{\lambda_1 - \lambda_2} = \frac{-2 \cdot 0 - 1}{-2 - (-1)} = 1$$

$$ode := diff(y(t), t, t) + 3 \cdot diff(y(t), t) + 2 \cdot y(t) = 0 :$$

$$C_1 = y_n(0_+) - C_2 = 0 - 1 = -1$$

$$solution := evalf(dsolve(\{ode, ics\}, y(t)))$$

$$y_n(t) = (e^{-t} - e^{-2t})$$

$$solution := y(t) = -1. \cdot e^{-2 \cdot t} + e^{-1 \cdot t}$$

$$h(t) = (e^{-t} - e^{-2t})u(t)$$

$$h1 := t \mapsto (e^{-1 \cdot t} - e^{-2 \cdot t}) \cdot u(t) :$$

Impulse response – One double root

$$\ddot{y}(t) + 6\dot{y}(t) + 9y(t) = 0; y(0) = 0, \dot{y}(0) = 1$$

$$Q(\lambda) = \lambda^2 + 6\lambda + 9 = 0$$

$$Q(\lambda) = (\lambda + 3)(\lambda + 3) = 0 \Leftrightarrow \lambda = -3$$

$$y_n(t) = (C_1 + C_2 t)e^{-3t}; t \geq 0$$

$$y_n(0) = C_1 = 0$$

$$\dot{y}_n(t) = (C_1\lambda + C_2 + C_2\lambda t)e^{\lambda t}$$

$$\dot{y}_n(0) = C_1\lambda + C_2 \Rightarrow C_2 = \dot{y}_n(0) - C_1\lambda$$

$$C_2 = 1 - 0(-3) = 1$$

$$y_n(t) = te^{-3t}$$

$$h(t) = te^{-3t}u(t)$$

Analytic impulse response

$u := t \rightarrow \text{Heaviside}(t) :$

|

Initial conditions

$ics := y(0) = 0, D(y)(0) = 1$

$ics := y(0) = 0, D(y)(0) = 1$

Differential equation

$ode := diff(y(t), t, t) + 6 \cdot diff(y(t), t) + 9 \cdot y(t) = 0 :$

$solution := evalf(dsolve(\{ode, ics\}, y(t)))$

$solution := y(t) = e^{-3 \cdot t} t$

$h2 := t \rightarrow t \cdot e^{-3 \cdot t} u(t) :$

Impulse response – Complex conjugated roots

$$\ddot{y}(t) + 4\dot{y}(t) + 13y(t) = 0; y(0_+) = 0, \dot{y}(0_+) = 1$$

$$Q(\lambda) = \lambda^2 + 4\lambda + 13 = 0$$

$$Q(\lambda) = (\lambda + 2 + j3)(\lambda + 2 - j3) = 0 \Leftrightarrow \lambda = -2 \pm j3$$

$$y_n(t) = C_1 e^{-2t} \cos(3t) + C_2 e^{-2t} \sin(3t); t \geq 0$$

$$y_n(0) = C_1 = 0$$

$$\begin{aligned}\dot{y}_n(t) &= C_1 \sigma e^{\sigma t} \cos(\omega t) - C_1 \omega \sin(\omega t) \\ &\quad + C_2 \sigma e^{\sigma t} \sin(\omega t) + C_2 \omega e^{\sigma t} \cos(\omega t)\end{aligned}$$

$$\begin{aligned}\dot{y}_n(0_+) &= C_1 \sigma + C_2 \omega = 1 \Rightarrow C_2 = \frac{\dot{y}_n(0_+) - C_1 \sigma}{\omega} \\ \Rightarrow C_2 &= \frac{1 - 0(-2)}{3} = \frac{1}{3}\end{aligned}$$

$$y_n(t) = \frac{1}{3} e^{-2t} \sin(3t)$$

$$h(t) = \frac{1}{3} e^{-2t} \sin(3t) u(t)$$

Analytic impulse response

Initial conditions

$$ics := y(0) = 0, D(y)(0) = 1$$

$$ics := y(0) = 0, D(y)(0) = 1$$

Differential equation

$$ode := diff(y(t), t, t) + 4 \cdot diff(y(t), t) + 13 \cdot y(t) = 0 :$$

$$solution := evalf(dsolve(\{ode, ics\}, y(t)))$$

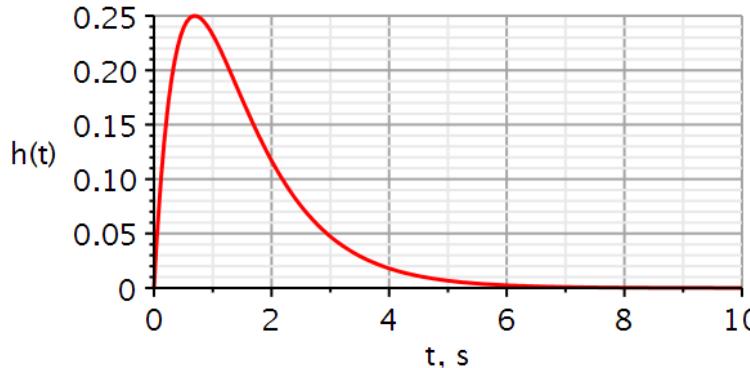
$$solution := y(t) = 0.3333333333 e^{-2 \cdot t} \sin(3 \cdot t)$$

$$h3 := t \rightarrow 0.3333333333 e^{-2 \cdot t} \sin(3 \cdot t) \cdot u(t)$$

Impulse responses

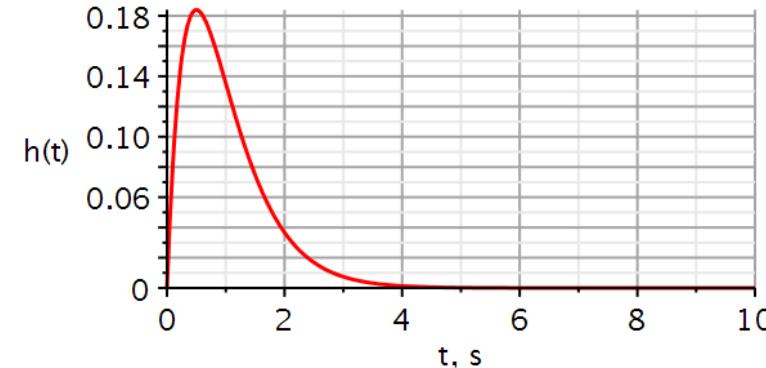
Two real poles

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \delta(t)$$



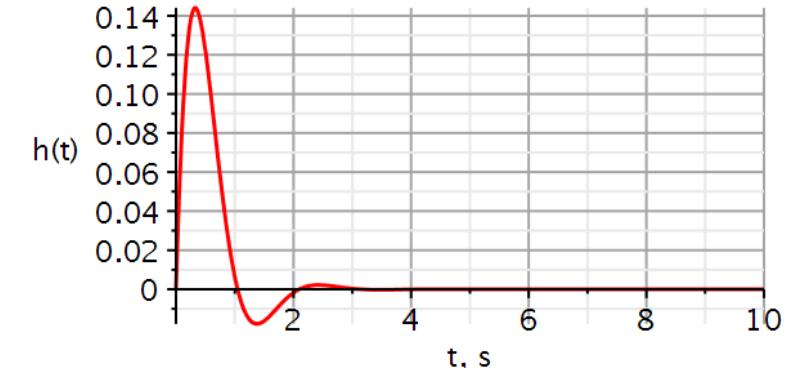
A double pole

$$\ddot{y}(t) + 6\dot{y}(t) + 9y(t) = \delta(t)$$



Complex conjugated poles

$$\ddot{y}(t) + 4\dot{y}(t) + 13y(t) = \delta(t)$$



`with(DynamicSystems)`



```
→ ImpulseResponsePlot(sys1, 10, color = [red], thickness = 3, axesfont
= [ "Helvetica", "ROMAN", 18 ], axis[2] = [ thickness = 2.5 ],
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labelfont = [ "HELVETICA", 18 ], gridlines, size = [ 600, 300 ])
```

Relationship between impulse response and step response

We obtain the step response $y_u(t)$ by applying the step function $u(t)$ as input.

If we differentiate the input signal, we must differentiate the output. Since the unit impulse function is the derivative of the unit step function, **the impulse response is the derivative of the step response.**

Hence, if we already have the step response, we can obtain the impulse response simply by differentiating the step response.

This is mostly **important in experimental determination** of the impulse response. It is almost impossible to create a perfect impulse. It is easier to create a good step function, (e.g., flipping a switch), measure the step response, fit that to an equation, and differentiate that to obtain the impulse response.

$$Q(D)y(t) = P(D)x(t)$$

$$Q(D)y_u(t) = P(D)u(t) \quad \text{step}$$

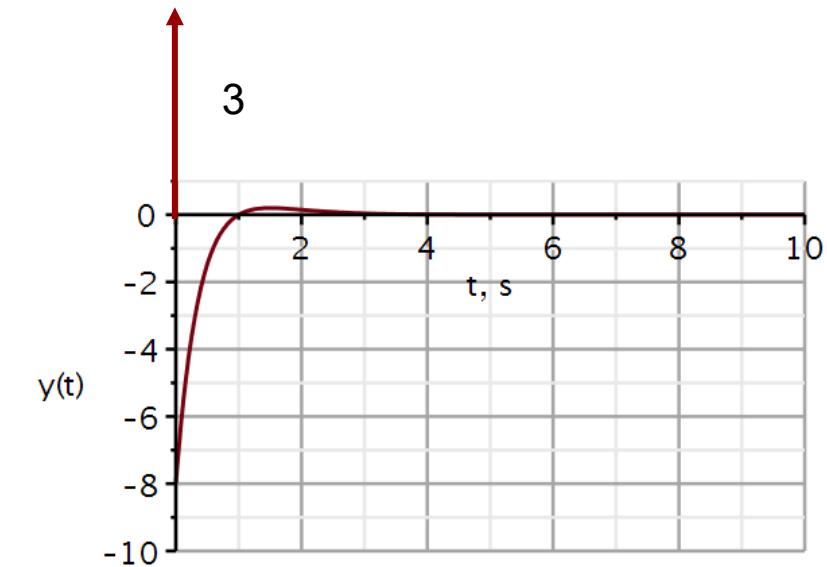
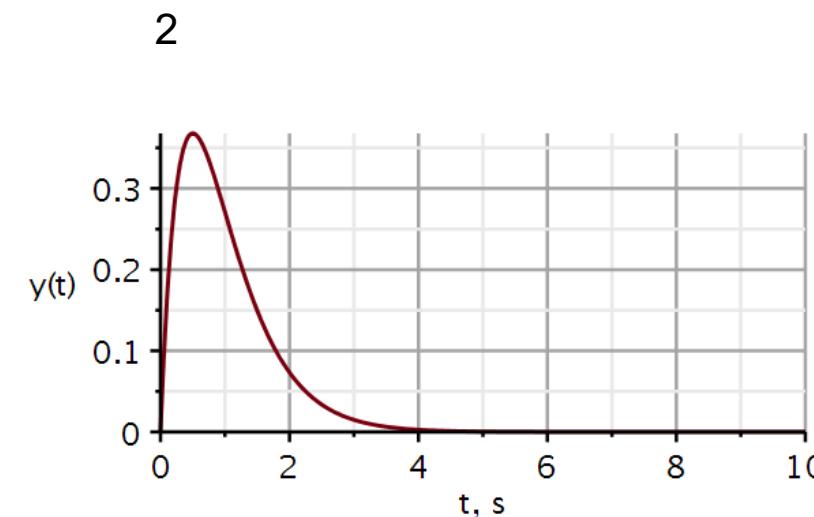
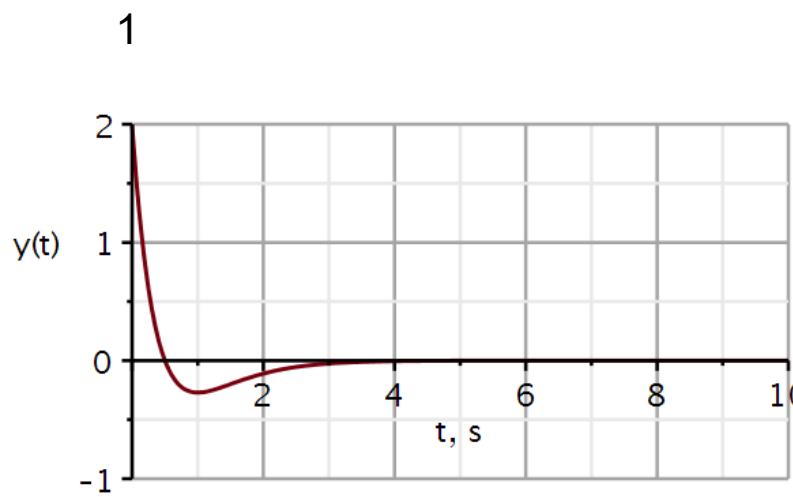
$$Q(D)h(t) = P(D)\delta(t) \quad \text{impulse}$$

$$\delta(t) = \frac{du}{dt} \Rightarrow h(t) = \frac{dy_u(t)}{dt}$$

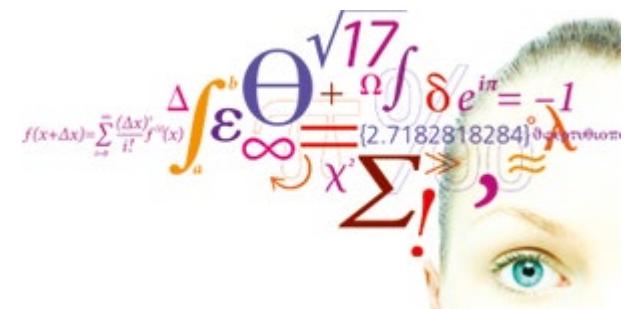
Quick quiz

Which impulse response belongs to which filter?

- A. 2nd order Lowpass filter
- B. 2nd order Highpass filter
- C. 2nd order Bandpass filter

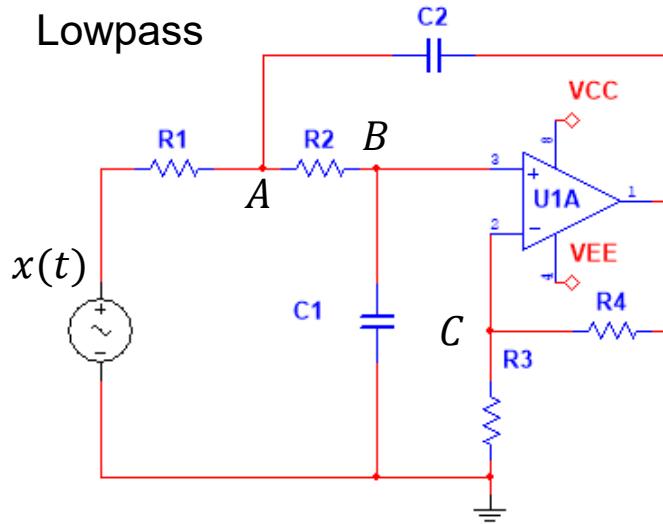


Problems

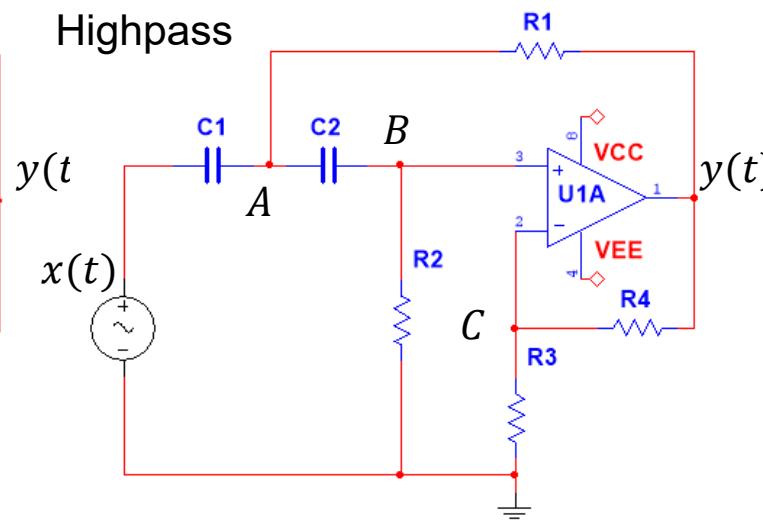


Sallen-Key filters used in this course

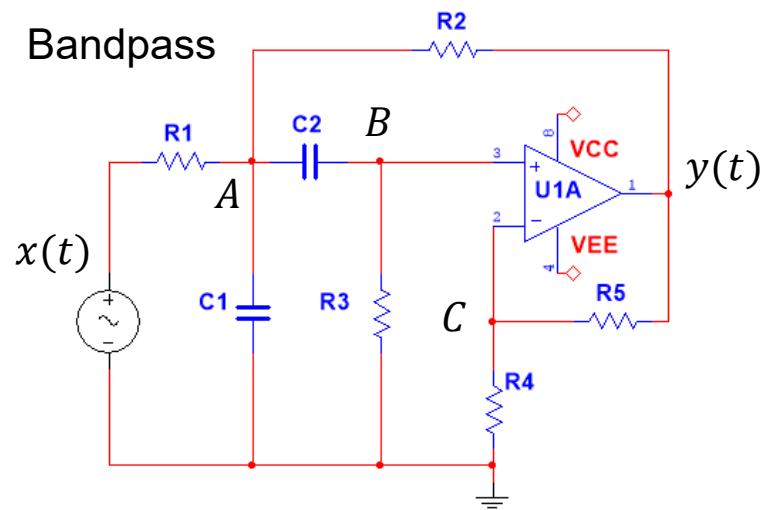
Lowpass



Highpass



Bandpass



Filter table

		Lowpass	Highpass	Bandpass
R_1	$k\Omega$	3.9894	1784.1	56.419
R_2	$k\Omega$	0.8865	892.06	34.117
R_3	$k\Omega$	1.0	1.0	149.72
R_4	$k\Omega$	1.0	1.0	1.0
R_5	$k\Omega$	—	—	1.0
C_1	nF	1795.2	1784.1	56.419
C_2	nF	398.94	3568.2	56.419
a_1		$2.83 \cdot 10^3$	$6.283 \cdot 10^{-1}$	$3.141 \cdot 10^1$
a_0		$3.95 \cdot 10^5$	$9.87 \cdot 10^{-2}$	$9.87 \cdot 10^4$
b_2		0	2	0
b_1		0	0	$6.283 \cdot 10^2$
b_0		$7.89 \cdot 10^5$	0	0

Filter 4: Lowpass filter – impulse response

1. Classify the system as linear/nonlinear, time-invariant/time-variant, causal/noncausal.
2. Draw the roots of the characteristic equation in the complex plane.
3. Based on the roots of the characteristic equation, classify the system as underdamped, critically damped, or overdamped. If the system is close to being critically damped (but not exactly), you can assume that is what was intended.
4. Check if the decomposition property holds for the system. If it does, the solution to the homogeneous differential equation is also the system's zero-input response
5. Using the initial conditions derived in Filter Problem 1, show that the homogeneous differential equation has the solution: $y_0(t) = (2.116e^{-147.31t} - 0.116 e^{-2680.2t})u(t)$
6. Define the initial conditions appropriate for obtaining the impulse response.
7. Show that the impulse response is: $h(t) = 311.8(e^{-147.31t} - e^{-2680.2t})u(t)$ and plot it.

Maple file available at DTU Learn.

Filter 5: Highpass filter – impulse response

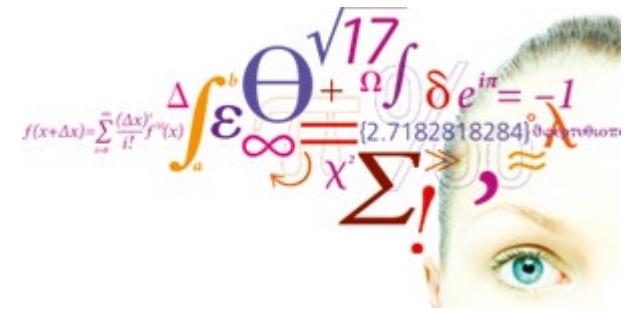
1. Classify the system as linear/nonlinear, time-invariant/time-variant, causal/noncausal.
2. Draw the roots of the characteristic equation in the complex plane.
3. Based on the roots of the characteristic equation, classify the system as underdamped, critically damped, or overdamped. If the system is close to being critically damped (but not exactly), you can assume that is what was intended.
4. Check if the decomposition property holds for the system. If it does, the solution to the homogeneous differential equation is also the system's zero-input response
5. Using the initial conditions derived in Filter Problem 2, show that the homogeneous differential equation has the solution: $y_0(t) = 0$
6. Define the initial conditions appropriate for obtaining the impulse response.
7. Show that the impulse response is: $h(t) = 2\delta(t) - 0.6283(2 - 0.31415 t)e^{-0.31415 t}u(t)$ and plot it.

Maple file NOT available at DTU Learn. Adapt from Filter 4 problem.

Filter 6: Bandpass filter – impulse response

1. Classify the system as linear/nonlinear, time-invariant/time-variant, causal/noncausal.
2. Draw the roots of the characteristic equation in the complex plane.
3. Based on the roots of the characteristic equation, classify the system as underdamped, critically damped, or overdamped. If the system is close to being critically damped (but not exactly), you can assume that is what was intended.
4. Check if the decomposition property holds for the system. If it does, the solution to the homogeneous differential equation is also the system's zero-input response
5. Using the initial conditions derived in Filter Problem 3, show that the homogeneous differential equation has the solution: $y_0(t) = 0$
6. Define the initial conditions appropriate for obtaining the impulse response.
7. Show that the impulse response is: $h(t) = 628.3 e^{-15.7 t} \cos(314 t)u(t) - 31.4e^{-15.7 t} \sin(314 t) u(t)$ or equivalently $h(t) = 629.1 e^{-15.7 t} \cos(314 t + 0.05) u(t)$ and plot it.

Maple file NOT available at DTU Learn. Adapt from Filter 4 problem.



Solutions

Filter 4: Lowpass filter – impulse response

1. Classify the system as linear/nonlinear, time-invariant/time-variant, causal/noncausal.
2. Draw the roots of the characteristic equation in the complex plane.
3. Based on the roots of the characteristic equation, classify the system as underdamped, critically damped, or overdamped. If the system is close to being critically damped (but not exactly), you can assume that is what was intended.
4. Check if the decomposition property holds for the system. If it does, the solution to the homogeneous differential equation is also the system's zero-input response
5. Using the initial conditions derived in Filter Problem 1, show that the homogeneous differential equation has the solution: $y_0(t) = (2.116e^{-147.33 t} - 0.116 e^{-2680.2 t})u(t)$
6. Define the initial conditions appropriate for obtaining the impulse response.
7. Show that the impulse response is: $h(t) = 311.8(e^{-147.31 t} - e^{-2680.2 t})u(t)$ and plot it.

Filter 4: Lowpass filter – impulse response (sol)

1. Classify the system as linear/nonlinear, time-invariant/time-variant, causal/noncausal.

We know from problem Filter 1 that the system has a linear differential equation with constant coefficients. It is therefore linear and time-invariant.

The system does not require future input values and is therefore causal.

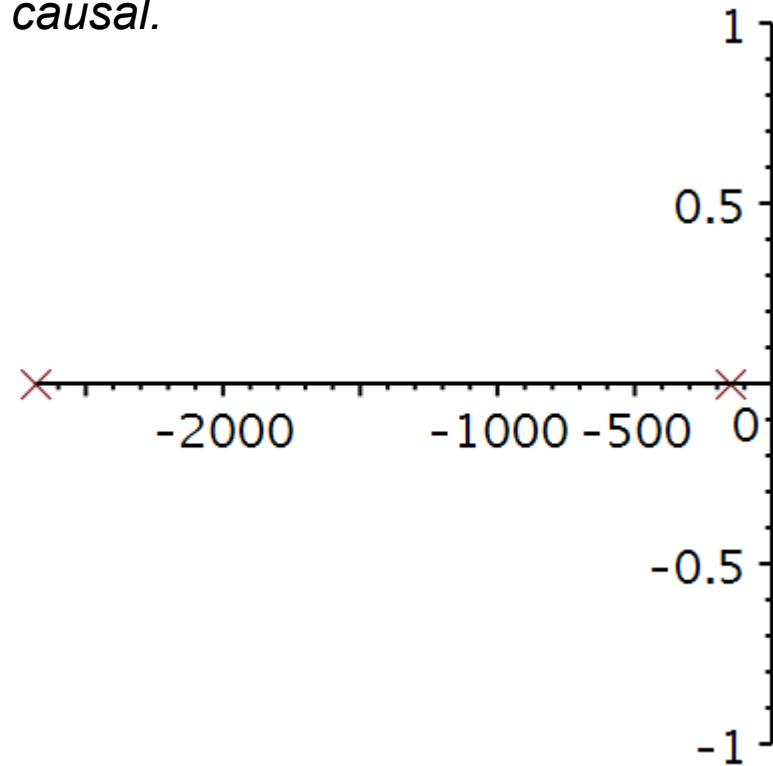
2. Draw the roots of the characteristic equation in the complex plane.

Roots of characteristic equation

$$\text{solve}(\lambda^2 + a_1 \cdot \lambda + a_0 = 0, \lambda)$$
$$-147.3061597, -2680.231742$$

3. Two real roots, hence an overdamped system.

4. Decomposition property? *The system is linear and time-invariant. Superposition holds and decomposition is possible.*



Filter 4: Lowpass filter – impulse response (sol)

5. Using the initial conditions derived in Filter Problem 1, show that the homogeneous differential equation has the solution: $y_0(t) = (2.116e^{-147.33 t} - 0.116 e^{-2680.2 t})u(t)$

Initial conditions $y(0_-) = 2, \dot{y}(0_-) = 0$

$ics := y(0) = 2, D(y)(0) = 0$

$\textcolor{blue}{ics := y(0) = 2, D(y)(0) = 0}$

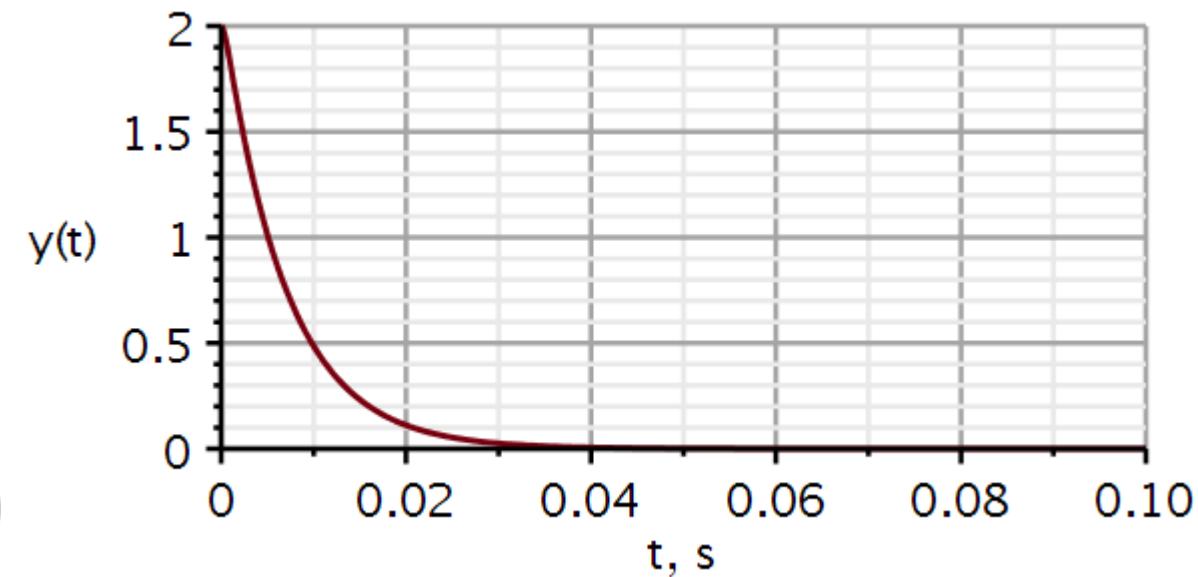
Differential equation

$ode := \text{diff}(y(t), t, t) + a1 \cdot \text{diff}(y(t), t) + a0 \cdot y(t) = 0 :$

$solution := \text{evalf}(\text{dsolve}(\{ode, ics\}, y(t)), 5)$

$\textcolor{blue}{solution := y(t) = 2.1163 e^{-147.33 t} - 0.1163 e^{-2680.2 t}}$

$y1 := t \rightarrow (2.1163 e^{-147.33 t} - 0.1163 e^{-2680.2 t}) \cdot \text{Heaviside}(t)$



Filter 4: Lowpass filter – impulse response (sol)

6. Define the initial conditions appropriate for obtaining the impulse response.

$$y(0_+) = 0, \dot{y}(0_+) = 1$$

7. Show that the impulse response is: $h(t) = 311.8(e^{-147.31t} - e^{-2680.2t})u(t)$ and plot it.

Analytic impulse response

Initial conditions

$$ics := y(0) = 0, D(y)(0) = 1$$

$$ics := y(0) = 0, D(y)(0) = 1$$

Differential equation

$$ode := \text{diff}(y(t), t, t) + a1 \cdot \text{diff}(y(t), t) + a0 \cdot y(t) = 0 :$$

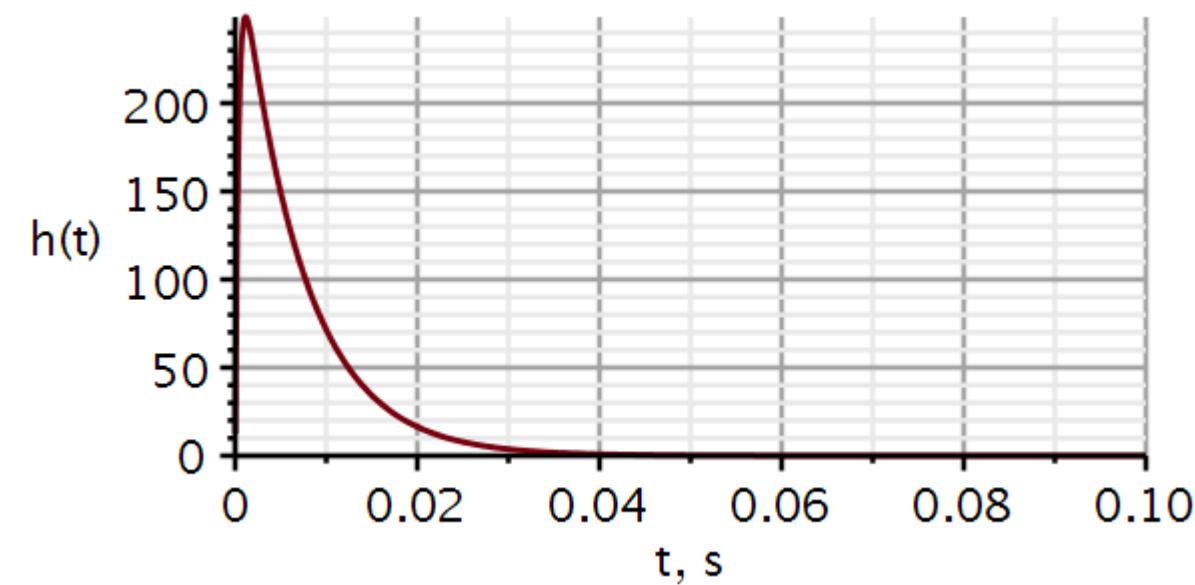
$$solutionh := \text{evalf}(\text{dsolve}(\{ode, ics\}, y(t)))$$

$$\begin{aligned} solutionh := y(t) &= 0.0003948003868 e^{-147.3061599 t} \\ &\quad - 0.0003948003868 e^{-2680.231742 t} \end{aligned}$$

`assign(solutionh)`

$$\text{evalf}(b0 \cdot y(t), 5) \cdot \text{Heaviside}(t)$$

$$(311.75 e^{-147.31 t} - 311.75 e^{-2680.2 t}) \text{Heaviside}(t)$$



Filter 5: Highpass filter – impulse response

1. Classify the system as linear/nonlinear, time-invariant/time-variant, causal/noncausal.
2. Draw the roots of the characteristic equation in the complex plane.
3. Based on the roots of the characteristic equation, classify the system as underdamped, critically damped, or overdamped. If the system is close to being critically damped (but not exactly), you can assume that is what was intended.
4. Check if the decomposition property holds for the system. If it does, the solution to the homogeneous differential equation is also the system's zero-input response
5. Using the initial conditions derived in Filter Problem 2, show that the homogeneous differential equation has the solution: $y_0(t) = 0$
6. Define the initial conditions appropriate for obtaining the impulse response.
7. Show that the impulse response is: $h(t) = 2\delta(t) - 0.6283(2 - 0.31415 t)e^{-0.31415 t}u(t)$ and plot it.

Filter 5: Highpass filter – impulse response (sol)

1. Classify the system as linear/nonlinear, time-invariant/time-variant, causal/noncausal.

We know from problem Filter 2 that the system has a linear differential equation with constant coefficients. It is therefore linear and time-invariant.

The system can be built and is therefore causal.

2. Draw the roots of the characteristic equation in the complex plane.

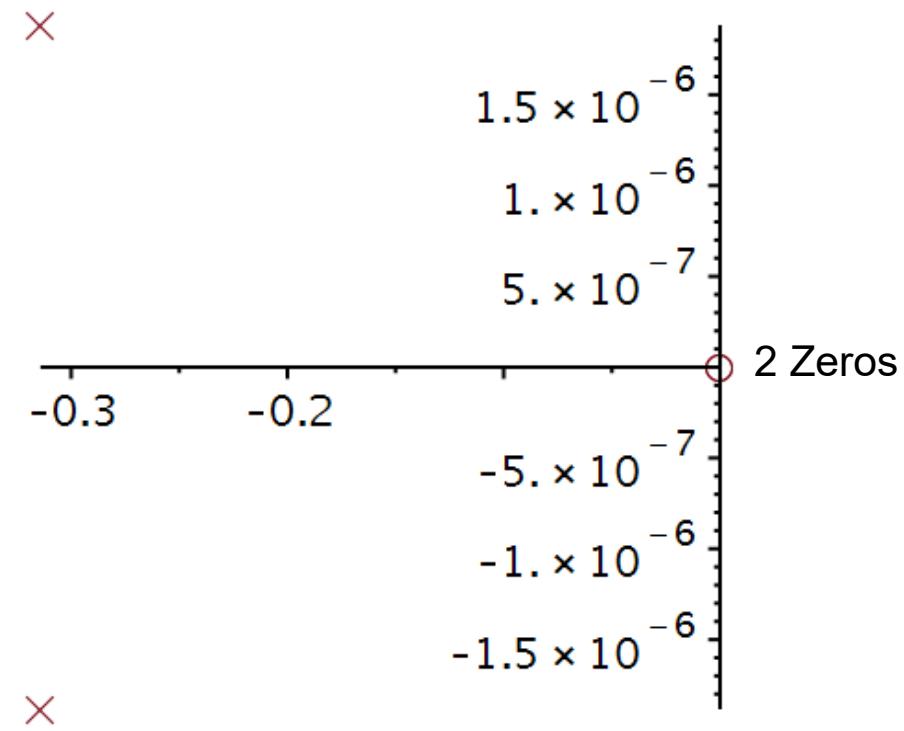
Roots of characteristic equation

```

HProots := solve(λ² + a1·λ + a0 = 0, λ)
HProots := -0.3141624760 + 0.001487536067 I,
           -0.3141624760 - 0.001487536067 I
p := -Re(HProots[1])
      p := 0.3141624760
a11 := 2·p
      a11 := 0.6283249520
a00 := p²
      a00 := 0.09869806133
  
```

With decimal coefficients, it is impossible to get repeated poles, even if we ignore the imaginary part.

$$p = \frac{\pi}{10}$$



Filter 5: Highpass filter – impulse response (sol)

3. A double root, hence, a critically damped system.
4. The system is linear and time-invariant. Superposition holds and decomposition is possible.
5. Using the initial conditions derived in Filter Problem 2, show that the homogeneous differential equation has the solution: $y_0(t) = 0$
6. Define the initial conditions appropriate for obtaining the impulse response.

$$y(0_-) = 0, \dot{y}(0_-) = 0 \Rightarrow y_0(t) = 0$$

$$y(0_+) = 0, \dot{y}(0_+) = 1$$

Filter 5: Highpass filter – impulse response (sol)

7. Show that the impulse response is: $h(t) = 2\delta(t) - 0.6283(2 - 0.31415t)e^{-0.31415t}u(t)$ and plot it.

$$\ddot{y}(t) + 0.6248\dot{y}(t) + 0.09871y(t) = 0; y(0) = 0, \dot{y}(0) = 1$$

$$Q(\lambda) = \lambda^2 + 0.6248\lambda + 0.09871 = 0$$

$$Q(\lambda) \approx (\lambda + 0.3141624760)(\lambda + 0.3141624760) = 0 \Leftrightarrow \lambda = -\frac{\pi}{10}$$

$$y_n(t) = (C_1 + C_2 t)e^{-\lambda t}; t \geq 0$$

$$y_n(0_+) = C_1 = 0$$

$$\dot{y}_n(t) = (C_1\lambda + C_2 + C_2\lambda t)e^{\lambda t}$$

$$\dot{y}_n(0_+) = C_1\lambda + C_2 \Rightarrow C_2 = \dot{y}_n(0_+) - C_1\lambda$$

$$C_2 = 1 - 0(-3) = 1$$

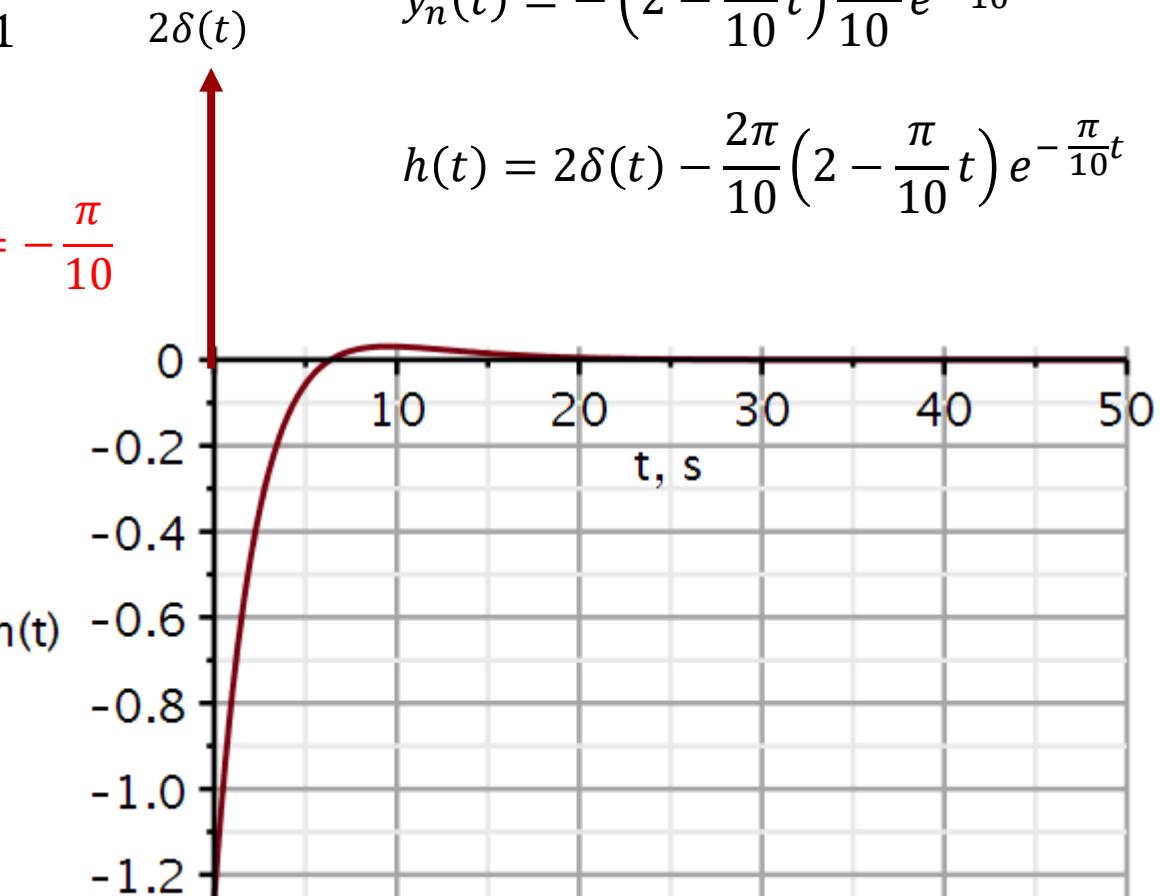
$$y_n(t) = te^{-\frac{\pi}{10}t}u(t)$$

$$h(t) = b_2\delta(t) + b_2\dot{y}_n(t)$$

$$\dot{y}_n(t) = \left(1 - \frac{\pi}{10}t\right)e^{-\frac{\pi}{10}t}$$

$$\dot{y}_n(t) = -\left(2 - \frac{\pi}{10}t\right)\frac{\pi}{10}e^{-\frac{\pi}{10}t}$$

$$h(t) = 2\delta(t) - \frac{2\pi}{10}\left(2 - \frac{\pi}{10}t\right)e^{-\frac{\pi}{10}t}$$



Filter 6: Bandpass filter – impulse response

1. Classify the system as linear/nonlinear, time-invariant/time-variant, causal/noncausal.
2. Draw the roots of the characteristic equation in the complex plane.
3. Based on the roots of the characteristic equation, classify the system as underdamped, critically damped, or overdamped. If the system is close to being critically damped (but not exactly), you can assume that is what was intended.
4. Check if the decomposition property holds for the system. If it does, the solution to the homogeneous differential equation is also the system's zero-input response
5. Using the initial conditions derived in Filter Problem 3, show that the homogeneous differential equation has the solution: $y_0(t) = 0$
6. Define the initial conditions appropriate for obtaining the impulse response.
7. Show that the impulse response is: $h(t) = 628.3 e^{-15.7 t} \cos(314 t)u(t) - 31.4e^{-15.7 t} \sin(314 t) u(t)$ or equivalently $h(t) = 629.1 e^{-15.7 t} \cos(314 t + 0.05) u(t)$ and plot it.

Filter 6: Bandpass filter – impulse response (sol)

1. Classify the system as linear/nonlinear, time-invariant/time-variant, causal/noncausal.

We know from problem Filter 3 that the system has a linear differential equation with constant coefficients. It is therefore linear and time-invariant.

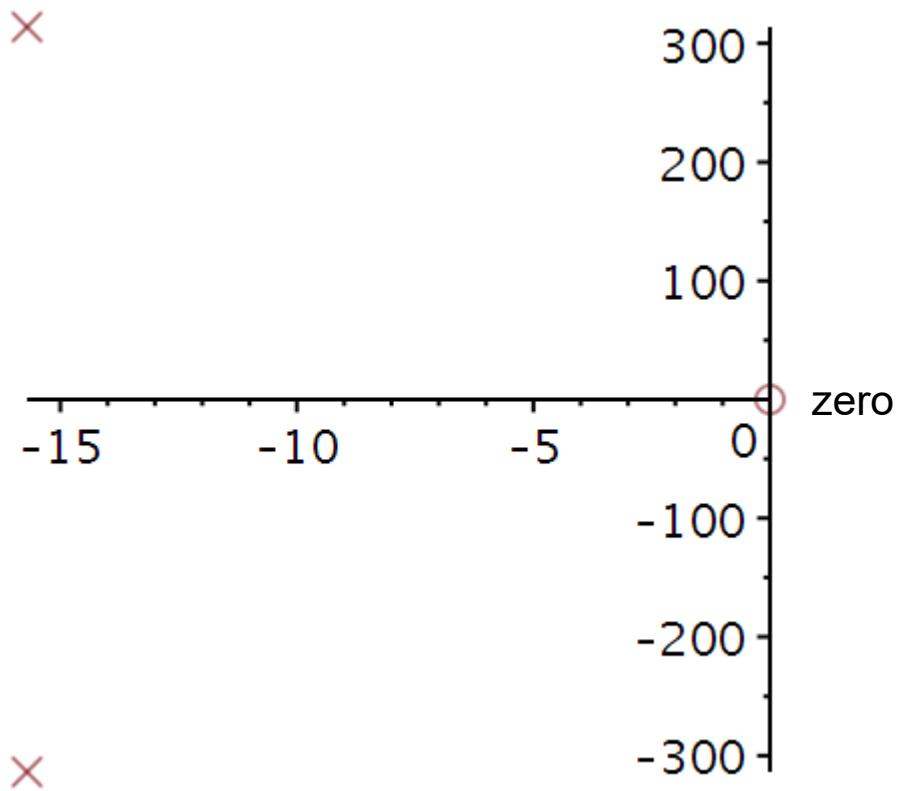
The system can be built and is therefore causal.

2. Draw the roots of the characteristic equation in the complex plane.

Roots of characteristic equation

$$\text{solve}(\lambda^2 + a1 \cdot \lambda + a0 = 0, \lambda)$$

$$-15.70298545 + 313.7646688 I, -15.70298545 - 313.7646688 I$$



Filter 6: Bandpass filter – impulse response (sol)

3. Complex conjugated roots, hence, an underdamped system.
4. The system is linear and time-invariant. Superposition holds and decomposition is possible.
5. Using the initial conditions derived in Filter Problem 3, show that the homogeneous differential equation has the solution: $y_0(t) = 0$
6. Define the initial conditions appropriate for obtaining the impulse response.

$$y(0_-) = 0, \dot{y}(0_-) = 0 \Rightarrow y_0(t) = 0$$

$$y(0_+) = 0, \dot{y}(0_+) = 1$$

Filter 6: Bandpass filter – impulse response (sol)

7. Show that the impulse response is: $h(t) = 628.3 e^{-15.7t} \cos(314t)u(t) - 31.4e^{-15.7t} \sin(314t) u(t)$ or equivalently $h(t) = 629.1 e^{-15.7t} \cos(314t + 0.05) u(t)$ and plot it.

$$C = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(-\frac{b}{a}\right)$$

Initial conditions

$$ics := y(0) = 0, D(y)(0) = 1$$

$$ics := y(0) = 0, D(y)(0) = 1$$

Differential equation

$$ode := \text{diff}(y(t), t, t) + a1 \cdot \text{diff}(y(t), t) + a0 \cdot y(t) = 0 :$$

$$solutionh := \text{evalf}(\text{dsolve}(\{ode, ics\}, y(t)))$$

$$solutionh := y(t) = 0.003187101990 e^{-15.70298545 t} \sin(313.7646688 t)$$

`assign(solutionh)`

$$\text{evalf}(b1 \cdot \dot{y}(t), 5) \cdot \text{Heaviside}(t)$$

$$(-31.447 e^{-15.703 t} \sin(313.76 t))$$

$$+ 628.33 e^{-15.703 t} \cos(313.76 t) \text{ Heaviside}(t)$$

$$A := 628.33 :$$

$$B := -31.447 :$$

$$\omega := 313.76 :$$

$$C := \sqrt{A^2 + B^2}$$

$$C := 629.1164461$$

$$\theta := \arctan(-B, A)$$

$$\theta := 0.05000681593$$

$$h3a := t \rightarrow C \cdot e^{-15.703 t} \cdot \cos(\omega \cdot t + \theta)$$

$$h3a := t \mapsto C \cdot e^{(-1) \cdot 15.703 \cdot t} \cdot \cos(\omega \cdot t + \theta)$$

$$h3a(0.02)$$

$$459.1482785$$

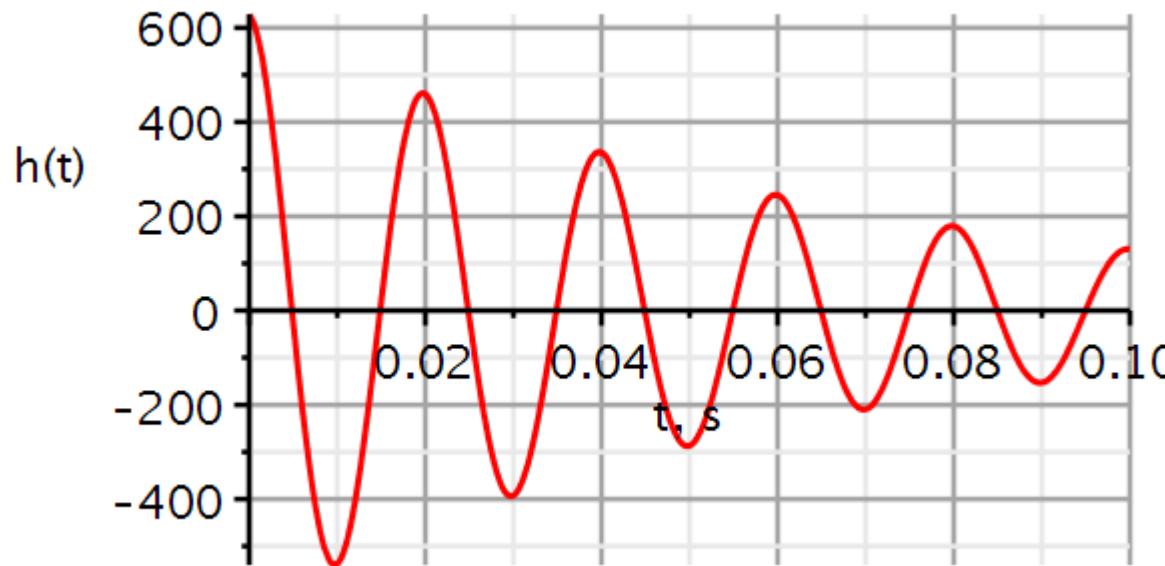
$$h3b := t \rightarrow (-31.447 e^{-15.703 t} \cdot \sin(313.76 \cdot t) + 628.33 e^{-15.703 t} \cdot \cos(313.76 \cdot t))$$

$$h3b(0.02)$$

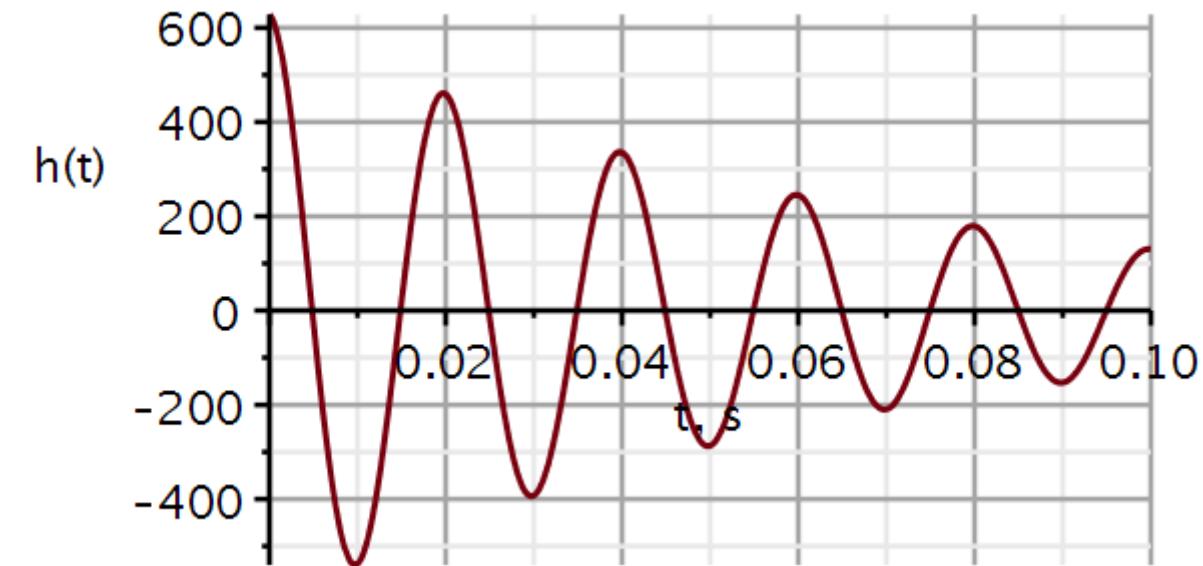
$$459.1482785$$

Filter 6: Bandpass filter – impulse response (sol)

```
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= linear, thickness = 2.5], labels = ["t, s", "h(t)", labelfont = ["HELVETICA",  
18], gridlines, size = [600, 300]])
```



Analytical



22050 Signals and linear systems in continuous time

Kaj-Åge Henneberg

L03

**Convolution
Zero-state response
Classical solutions
Stability**

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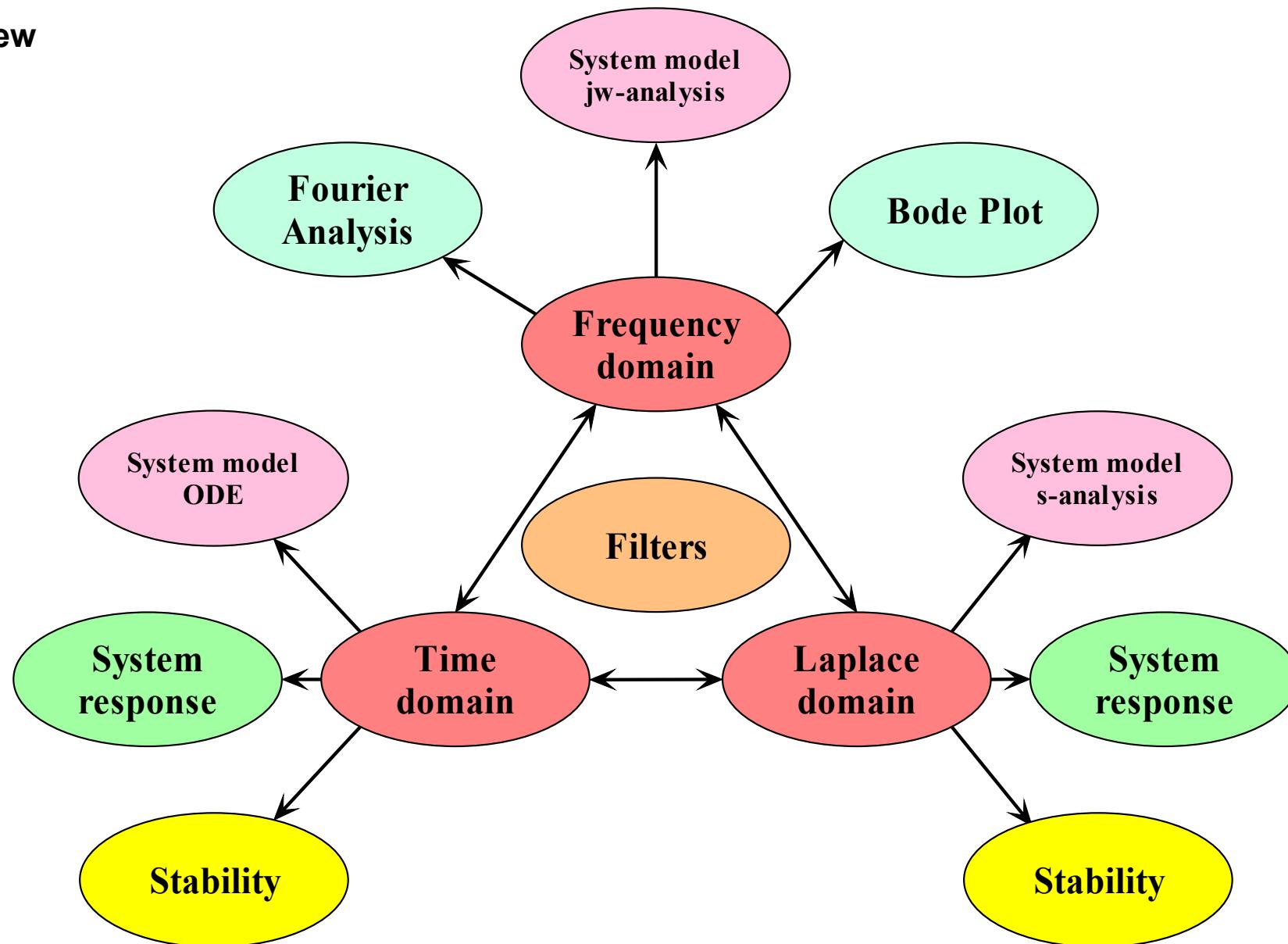
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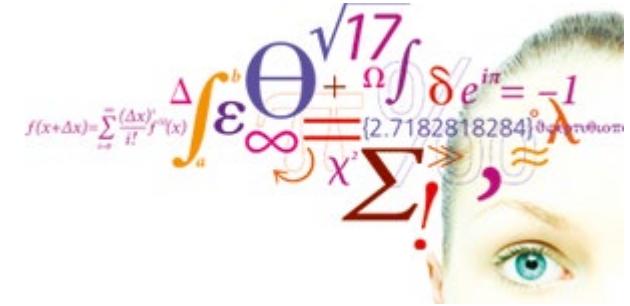
Course overview

- System
 - Model
 - Response
 - Performance
- Domains
 - Time
 - Frequency
 - s-domain



Break 10 min

- Review video
 - Decomposition property
 - Zero input response
 - Unit impulse response
- Zero-state response
 - Superposition
 - Convolution integral
- Classical method
- Stability
- Problems



Review Video 1

After lectures 1 - 3 you should be able to

After lectures 1 - 3 you should be able to

- Classify signals and systems
- Perform operations on signals
- Calculate zero-input responses
- Define the impulse response $h(t)$
- Define initial values and solve for impulse responses
- Calculate convolution integrals for zero-state responses
- Use KiCad/Spice for simulation of electrical circuits
- Use Maple or equivalent tool to perform calculations on signals and linear systems and plot signals

Properties of linear systems

Unscaled input:

$$\ddot{y}_1(t) + a_1\dot{y}_1(t) + a_0y_1(t) = x(t)$$

$$h_1(t) = y_1(t)u(t)$$

If we differentiate on the rhs,
we must differentiate on the
lhs:

$$\ddot{y}_1(t) + a_1\ddot{y}_1(t) + a_0\dot{y}_1(t) = \dot{x}(t)$$

$$y_2(t) \stackrel{\text{def}}{=} \dot{y}_1(t)$$

Defining new unknown:

$$\ddot{y}_2(t) + a_1\dot{y}_2(t) + a_0y_2(t) = \dot{x}(t)$$

$$h_2(t) = y_2(t)u(t) = \dot{y}_1(t)u(t)$$

$$b_1\ddot{y}_2(t) + a_1b_1\dot{y}_2(t) + a_0b_1y_2(t) = b_1\dot{x}(t)$$

$$y_3(t) \stackrel{\text{def}}{=} b_1y_2(t)$$

$$\ddot{y}_3(t) + a_1\dot{y}_3(t) + a_0y_3(t) = b_1\dot{x}(t)$$

$$\begin{aligned} h_3(t) &= y_3(t)u(t) \\ &= b_1y_2u(t) \\ &= b_1\dot{y}_1(t)u(t) \end{aligned}$$

$$\ddot{y}_3(t) + a_1\dot{y}_3(t) + a_0y_3(t) = P(D)x(t)$$

$$h_3(t) = P(D)(y_1(t)u(t))$$

Response to sudden change on input

Here the input is an impulse function.

At $t = 0$, acceleration becomes an impulse, velocity a step function and position becomes a ramp function.

Here the input is the derivative of the impulse function.

At $t = 0$, acceleration becomes the derivative of the impulse, velocity an impulse function and position becomes a step function.

Here the input is the 2nd derivative of the impulse function.

At $t = 0$, acceleration becomes the 2nd derivative of the impulse, velocity is the 1st derivative of the impulse function and position becomes an impulse function.

$$\ddot{y} + a_1\dot{y} + a_0y = \delta(t), \quad \dot{y}(0_-) = 0, y(0_-) = 0$$

$$@ t = 0: \quad \ddot{y} \propto \delta(t) \quad \dot{y} \propto u(t) \quad y \propto r(t)$$

$$\ddot{y} + a_1\dot{y} + a_0y = \dot{\delta}(t), \quad \dot{y}(0_-) = 0, y(0_-) = 0$$

$$@ t = 0: \quad \ddot{y} \propto \dot{\delta}(t) \quad \dot{y} \propto \delta(t) \quad y \propto u(t)$$

$$\ddot{y} + a_1\dot{y} + a_0y = \ddot{\delta}(t), \quad \dot{y}(0_-) = 0, y(0_-) = 0$$

$$@ t = 0: \quad \ddot{y} \propto \ddot{\delta}(t) \quad \dot{y} \propto \dot{\delta}(t) \quad y \propto \delta(t)$$

Review: Solving for impulse response

The last two examples on the previous slide show that there are systems where a sudden change of input produce a synchronous sudden change on the output. In the last case the impulse response must include an impulse:

$$m = n$$

$$h(t) = A_0 \delta(t) + (P(D)y_n(t))u(t)$$

$$(D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0)h(t) = (b_n D^n + b_{n-1} D^{n-1} + \dots + b_1 D + b_0)\delta(t)$$

Inserting in the differential equation we see that the coefficient of the highest order derivative of $\delta(t)$ must be the same on both sides. Hence $A_0 = b_n$.

$$h(t) = b_n \delta(t) + (P(D)y_n(t))u(t), m \leq n$$

$$h(t) = P(D)(y_n(t)u(t)), m > n$$

Initial conditions:

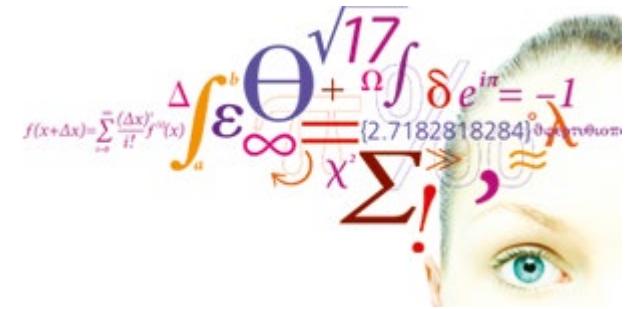
$$\begin{aligned} y_n(0_+) &= y_n^{(1)}(0_+) = \dots = y_n^{(n-2)}(0_+) = 0 \\ y_n^{(n-1)}(0_+) &= 1 \end{aligned}$$

Break 10 min

- Review
 - Decomposition property
 - Zero input response
 - Unit impulse response
- **Zero-state response**
 - Superposition
 - Convolution integral
- Classical method
- Stability
- Problems

Zero-state response

Video 2

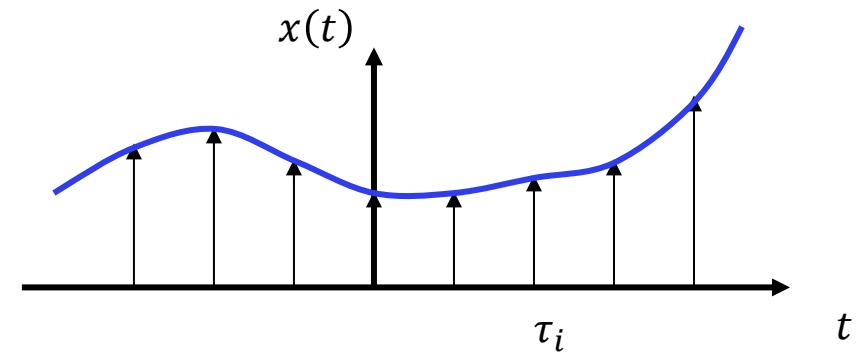


Impulse response method

If we can consider any arbitrary input signal to be a sum of the same very simple signal, then we can use the superposition principle to construct the system response.

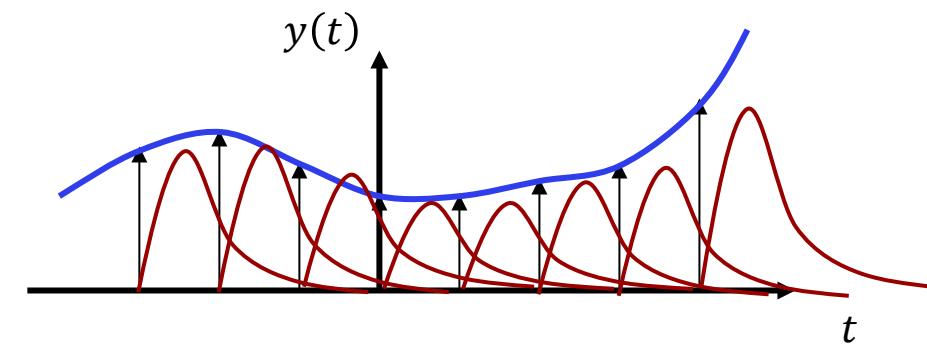
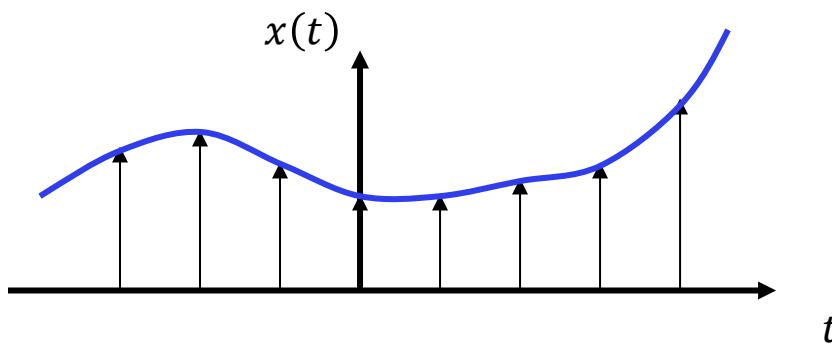
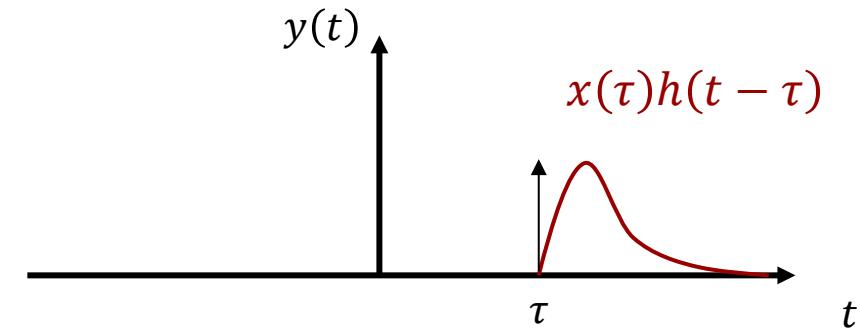
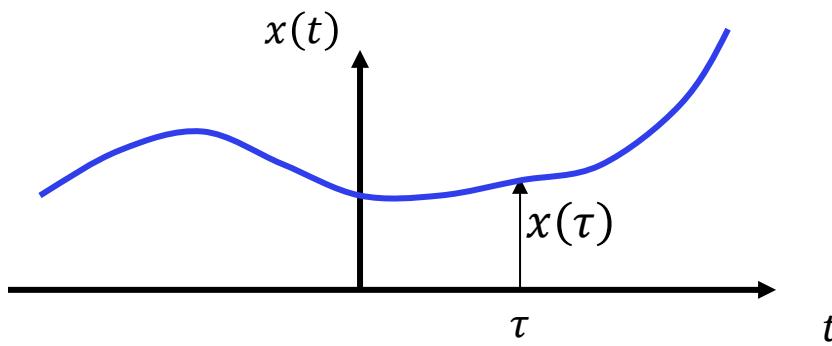
The ideal candidate is the impulse function:

In the limit as the impulse functions are placed closer and closer together, the signal $x(t)$ becomes a superposition of an infinite number of impulse functions time-shifted along the time axis.



$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$$

Superposition of impulse responses



$$\sum_{i=1}^N x(\tau_i) \delta(t - \tau_i)$$

$$\sum_{i=1}^N x(\tau_i) h(t - \tau_i)$$

Superposition integral

Input → output

Impulse response:

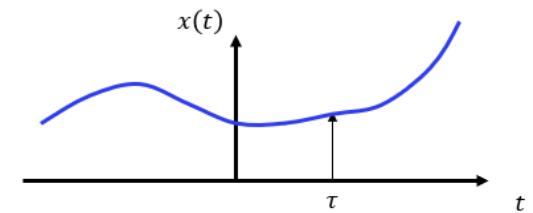
$$\delta(t) \rightarrow h(t)$$

Time invariance:

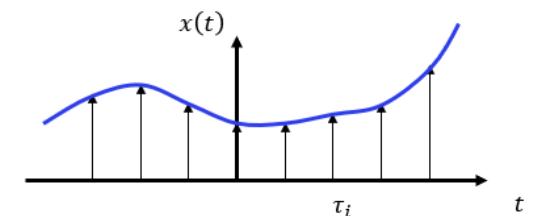
$$\delta(t - \tau) \rightarrow h(t - \tau)$$

Homogeneity:

$$x(\tau)\delta(t - \tau) \rightarrow x(\tau)h(t - \tau)$$

**Superposition integral:**

$$\underbrace{\int_{-\infty}^{\infty} x(\tau)\delta(t - \tau) d\tau}_{x(t)} \rightarrow \underbrace{\int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau}_{y_{zs}(t)}$$

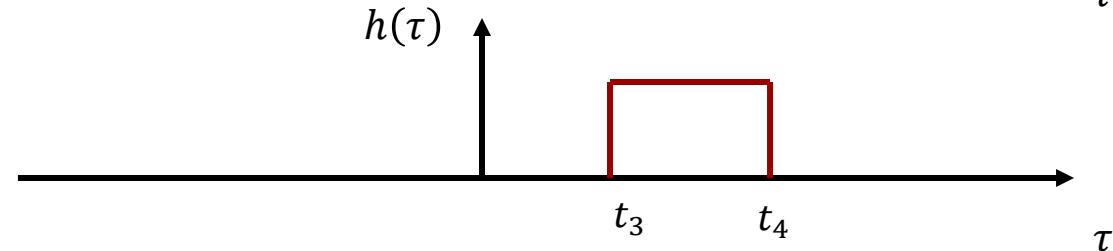
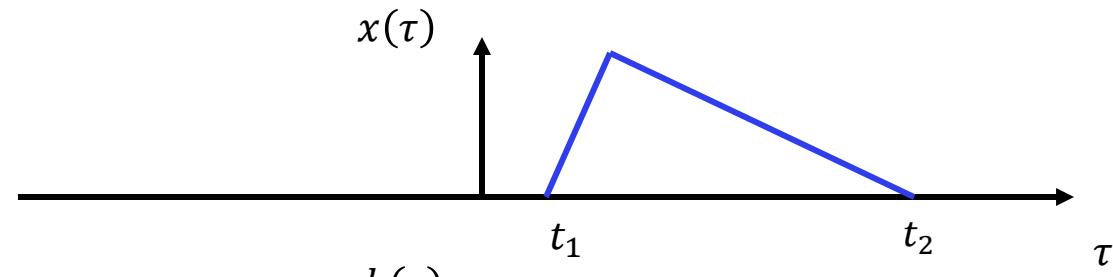
**Convolution integral:**

$$y_{zs}(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau$$

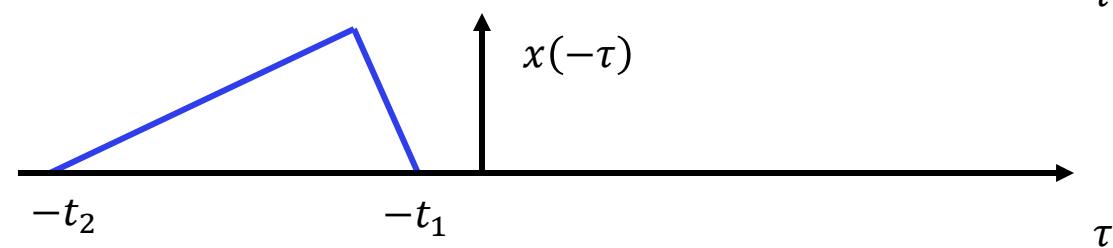
In order to calculate the zero-state response with the impulse response method, we must first determine the impulse response function $h(t)$.

Convolution

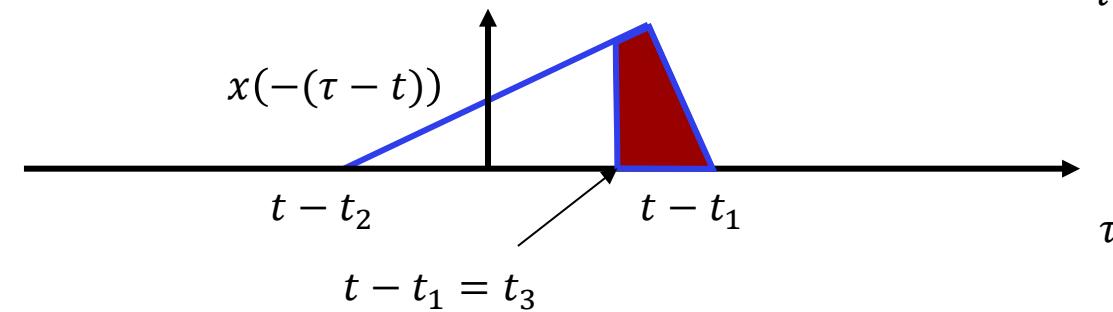
$$y_{zs}(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau = \int_{-\infty}^{\infty} h(\tau) \underbrace{x(-(\tau - t))}_{\text{reflected and shifted toward the right}} d\tau$$



Time inversion:



Time shift:



Convolution

If we have causal *signals*:

$$y_{zs}(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$

$$x(t) = 0, \quad t < 0_-$$

$$y_{zs}(t) = h(t) * x(t) = \int_{0_-}^{\infty} x(\tau) h(t - \tau) d\tau$$

If we have causal *systems*:

$$h(t) = 0, \quad t < 0 \quad h(t - \tau) = 0, \quad \tau > t$$

$$y_{zs}(t) = h(t) * x(t) = \int_{-\infty}^{t} x(\tau) h(t - \tau) d\tau$$

If we have both:

$$y_{zs}(t) = h(t) * x(t) = \int_{0_-}^{t} x(\tau) h(t - \tau) d\tau$$

We will almost always have both causal signals and systems.

Convolution Properties

1. Commutative: $x_1(t) * x_2(t) = x_2(t) * x_1(t)$

2. Distributive: $x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$

3. Associative: $x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$

4. Time shift: $x_1(t) * x_2(t) = c(t)$

$$x_1(t - T_1) * x_2(t) = c(t - T_1)$$

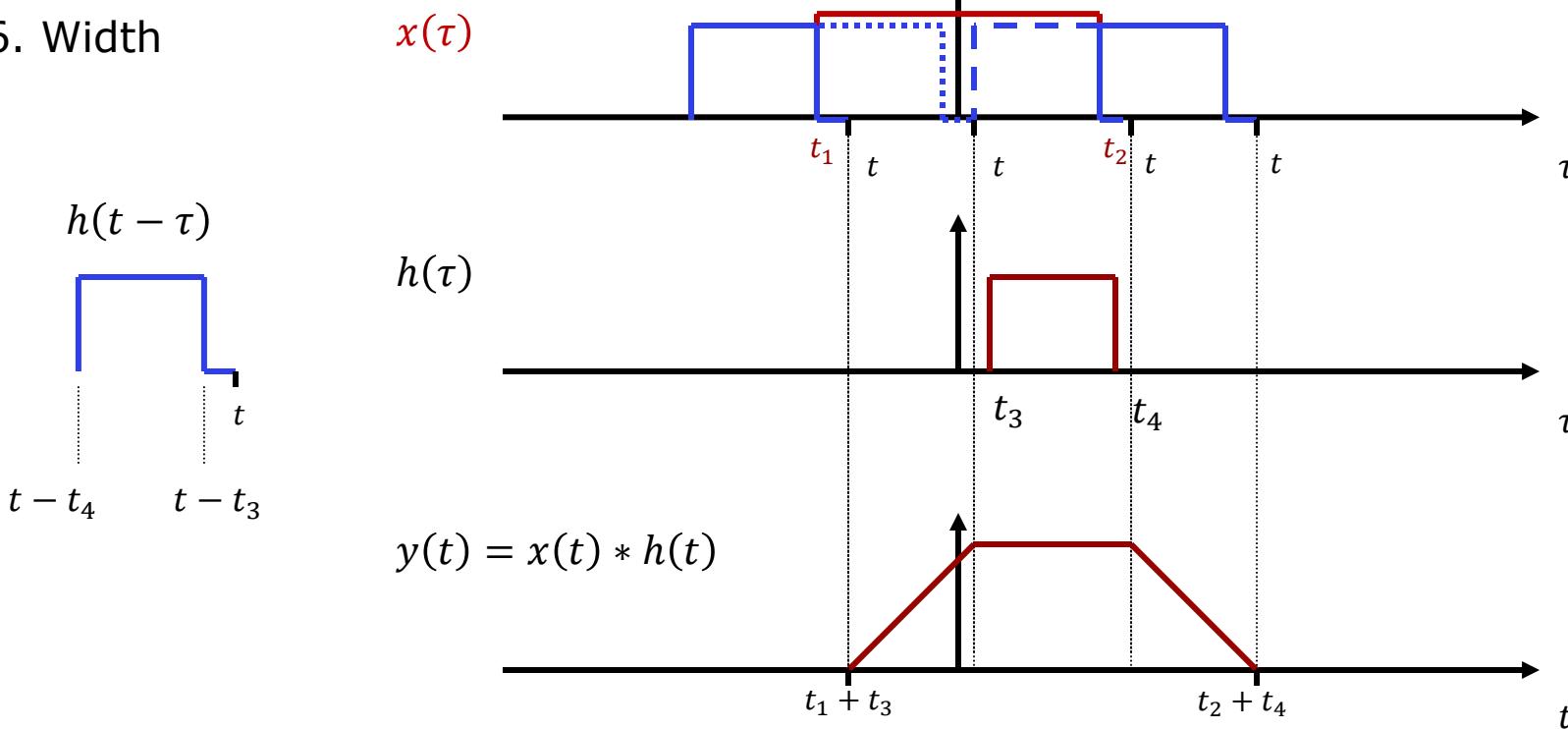
$$x_1(t) * x_2(t - T_2) = c(t - T_2)$$

$$x_1(t - T_1) * f_2(t - T_2) = c(t - T_1 - T_2)$$

5. Identity: $x(t) * \delta(t) = x(t)$

Convolution Properties

6. Width

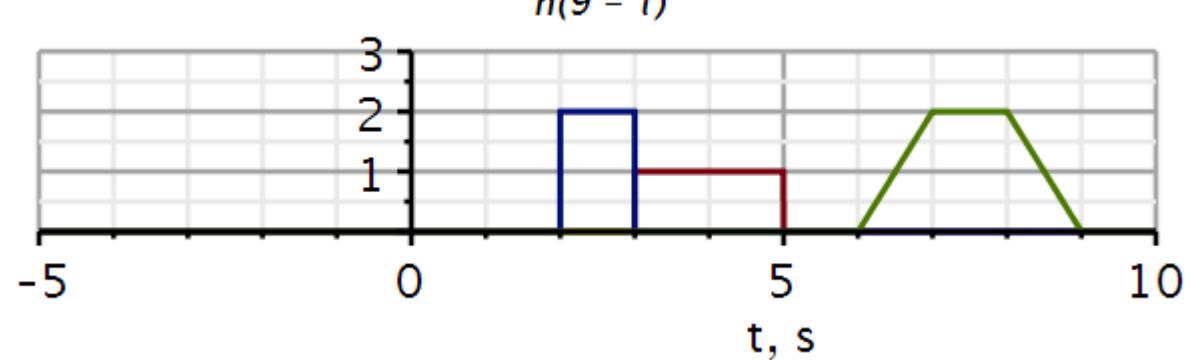
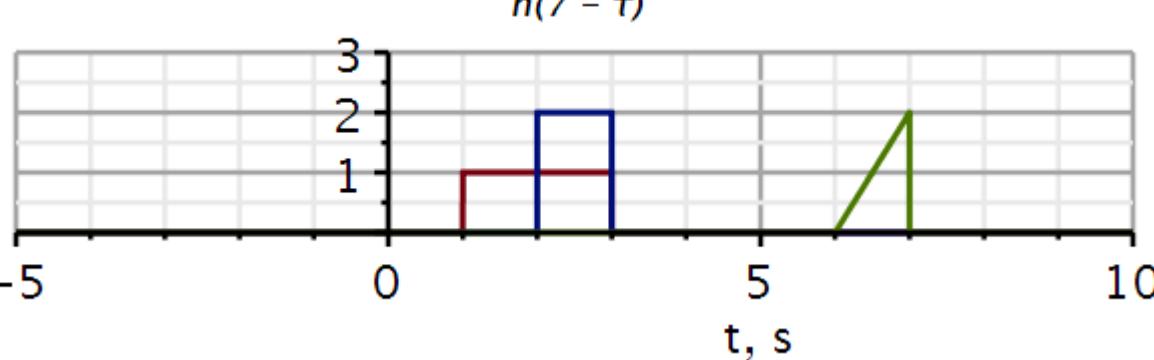
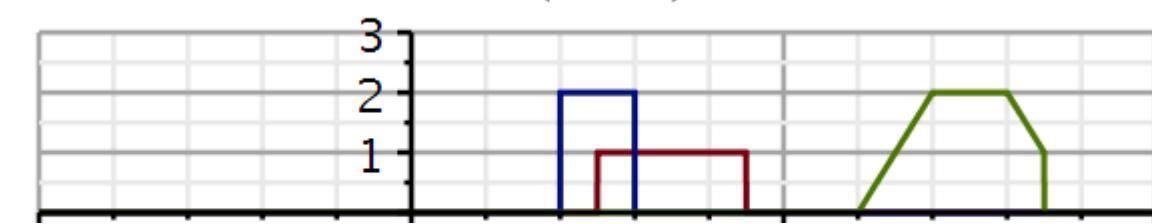
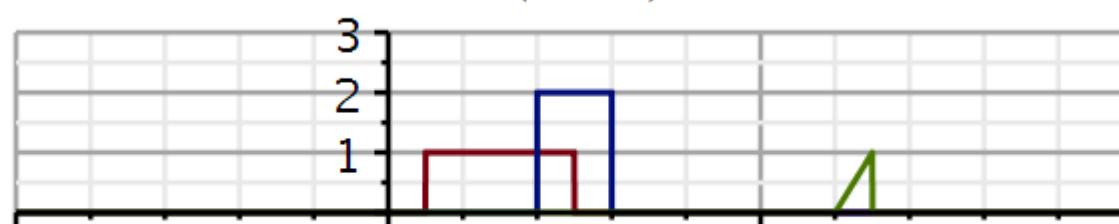
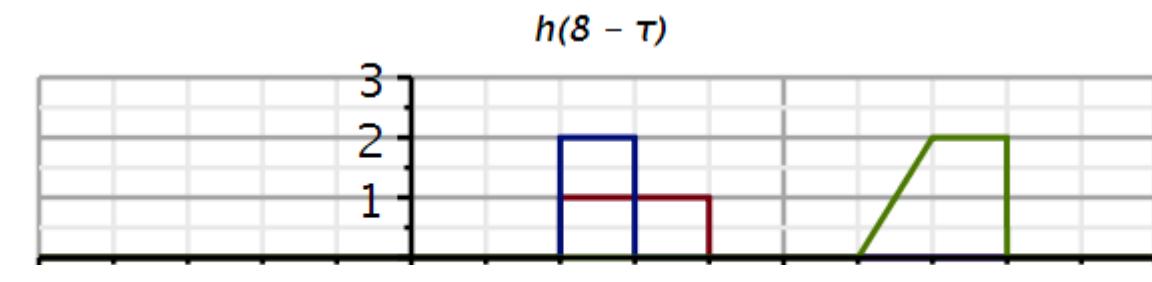
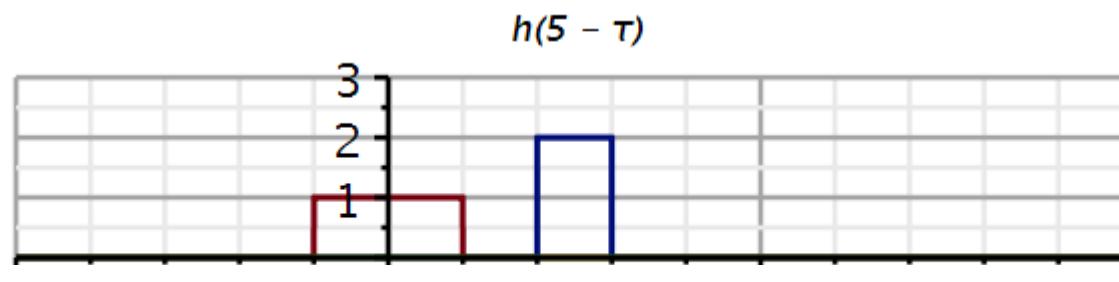
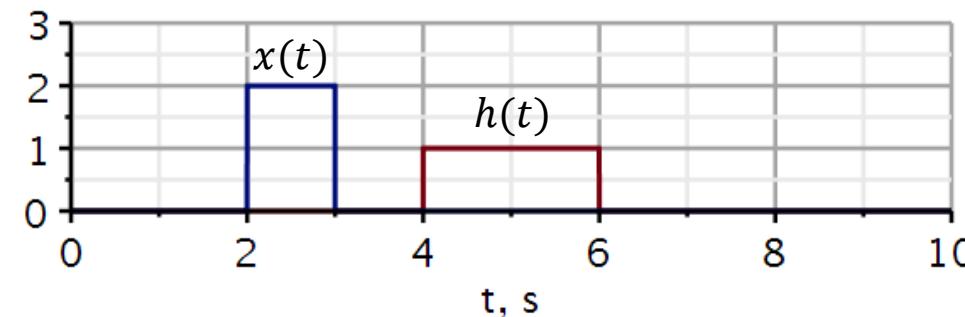


Useful check of results:

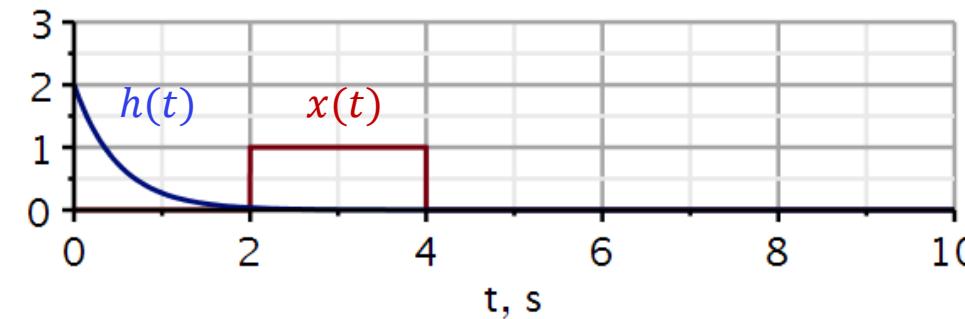
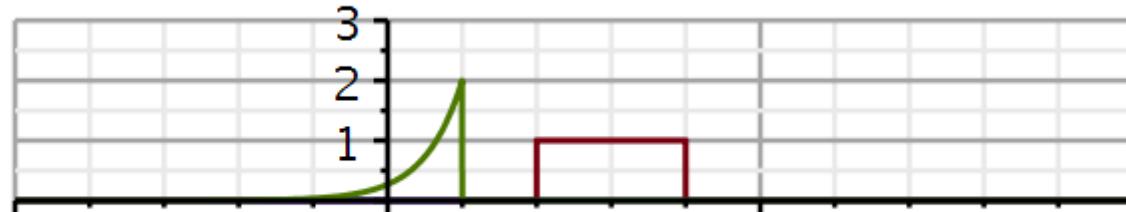
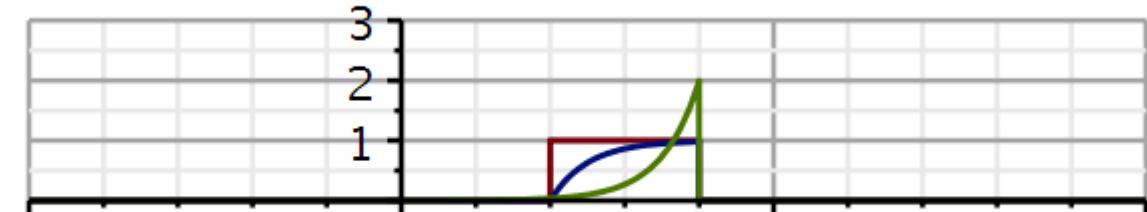
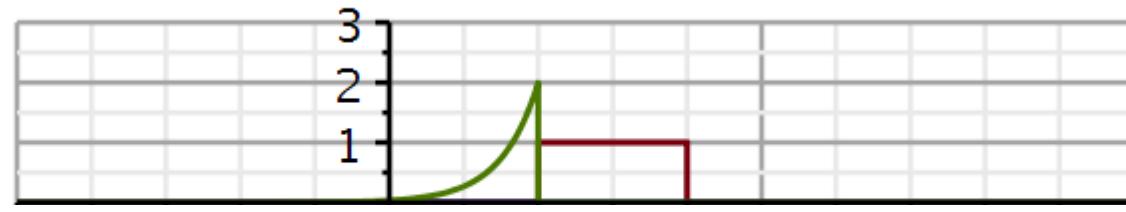
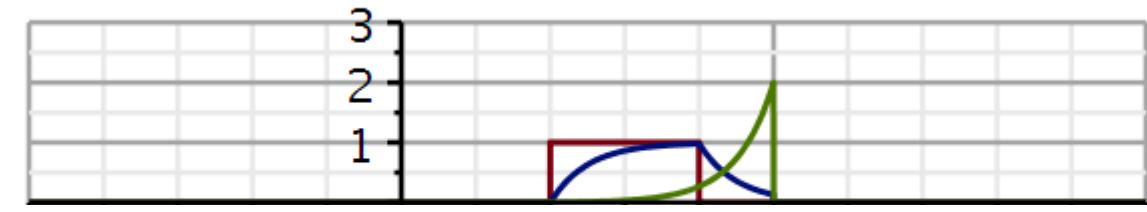
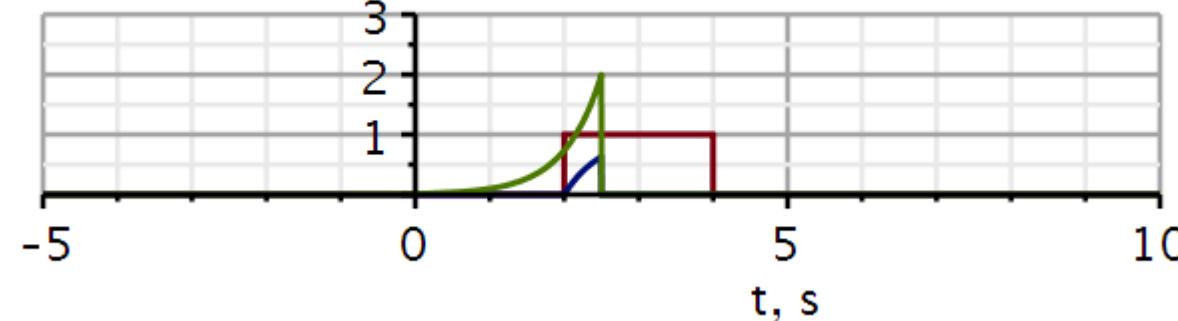
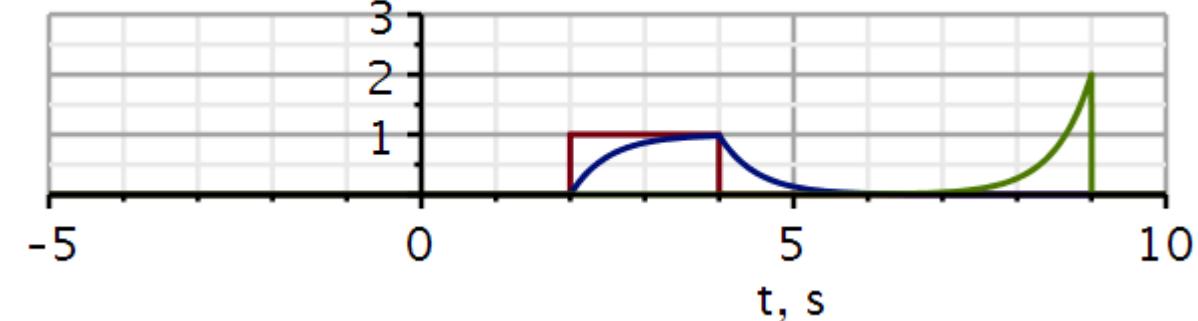
$$T_y = (t_2 + t_4) - (t_1 + t_3) = (\textcolor{red}{t}_2 - \textcolor{red}{t}_1) + (\textcolor{blue}{t}_4 - \textcolor{blue}{t}_3) = \textcolor{red}{T}_x + \textcolor{blue}{T}_h$$

- The *starting point* of the result is the sum of the starting points of the two signals.
- The *end point* of the result is the sum of the end points of the two signals.
- The *duration* of the result is the sum of the durations of the two signals.

Convolution examples



Convolution examples

 $h(1 - \tau)$  $h(4 - \tau)$  $h(2 - \tau)$  $h(5 - \tau)$  $h(2.5 - \tau)$  $h(9 - \tau)$ 

Tools in Maple

```

restart
with(plots):
with(DynamicSystems):
v := t→Heaviside(t):
conv := (x, h) → t → int(x(τ) * h(t - τ), τ = 0 .. t);
conv := (x, h) ↪ t ↪ ∫₀^t x(τ) · h(t - τ) dτ

```

Example signals

```

x1 := t→2·(v(t - 2) - v(t - 3)) :          # input signal
h1 := t→(v(t - 4) - v(t - 6)) :            # impulse response
y1 := t→conv(t→x1(t), t→h1(t))(t)          # output signal
y1 := t ↪ conv(t ↪ x1(t), t ↪ h1(t))(t)

h1r := (t, T) → h1(T - t) :                 # impulse response reversed and shifted T
y11 := (t, T) → piecewise(0 < T - t, y1(t)) # truncate output at T
y11 := (t, T) ↪ { y1(t)   0 < T - t

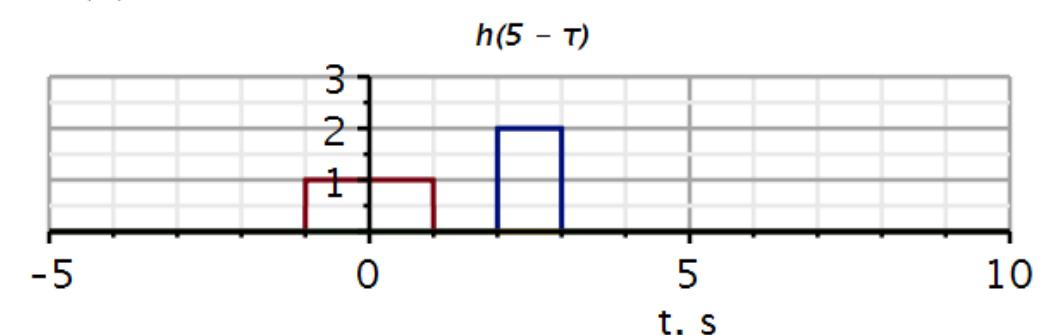
```

```

convT := T → plot( {x1(t), h1r(t, T), y11(t, T)}, t = -5 .. 10, 0 .. 3, thickness = 3,
axesfont = [ "Helvetica", "ROMAN", 18], axis[2] = [ thickness = 2.5], axis[1] =
[ mode = linear, thickness = 2.5], labels = [ "t, s", " "], labelfont
= [ "HELVETICA", 18], gridlines, size = [ 600, 200], titlefont = [ "Helvetica",
"ITALIC", 14], title = cat("h(", convert(T, string), " - τ)")) :
# a function to plot signals at T

```

convT(5)



Example signals

```

h2 := t->piecewise(t < 5, 2·e-2·t) :          # impulse response
x2 := t->(v(t - 2) - v(t - 4)) :               # input signal
y2 := t->conv(t->x2(t), t->h2(t))(t)           # output signal
y2 := t-> conv(t->x2(t), t->h2(t))(t)
h2r := (t, T)->piecewise(0 < T - t, h2(T - t)) :
# impulse response reversed and shifted T
y2I := (t, T)->piecewise(0 < T - t, y2(t))      # truncate output at T
y2I := (t, T) -> { y2(t)   0 < T - t

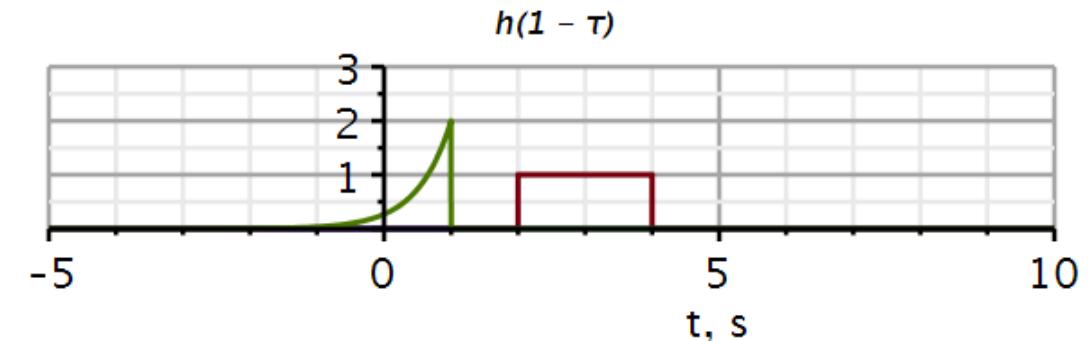
```

```

convT2 := T->plot( {x2(t), h2r(t, T), y2I(t, T)}, t=-5..10, 0..3, thickness = 3,
axesfont = [ "Helvetica", "ROMAN", 18], axis[ 2 ] = [ thickness = 2.5 ], axis[ 1 ]
= [ mode = linear, thickness = 2.5 ], labels = [ "t, s" , " " ], labelfont
= [ "HELVETICA", 18 ], gridlines, size = [ 600, 200 ], titlefont = [ "Helvetica",
"ITALIC", 14 ], title = cat("h(", convert(T, string), " - \u03c4)") ) :
# a function to plot signals at T

```

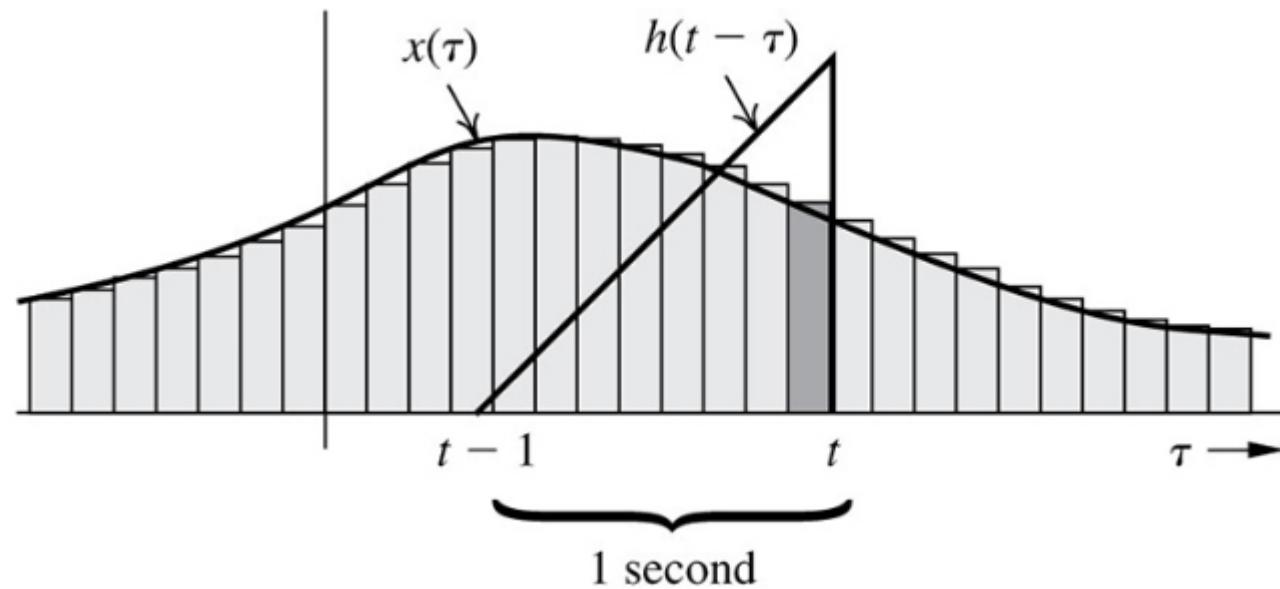
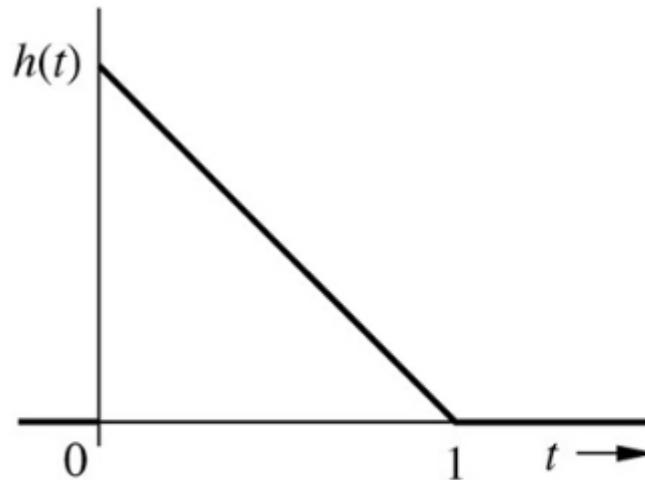
`convT2(1)`



Notice that the impulse response is truncated at $t = 5$.

The reversed impulse response is truncated at T .

Impulse response weighs previous inputs



The output of the convolution can be thought of as a weighted sum (integral) of previous inputs. The most recent input contributes with the highest weight, older inputs contribute to the output with smaller and smaller weights. This system has a finite memory span of 1s.

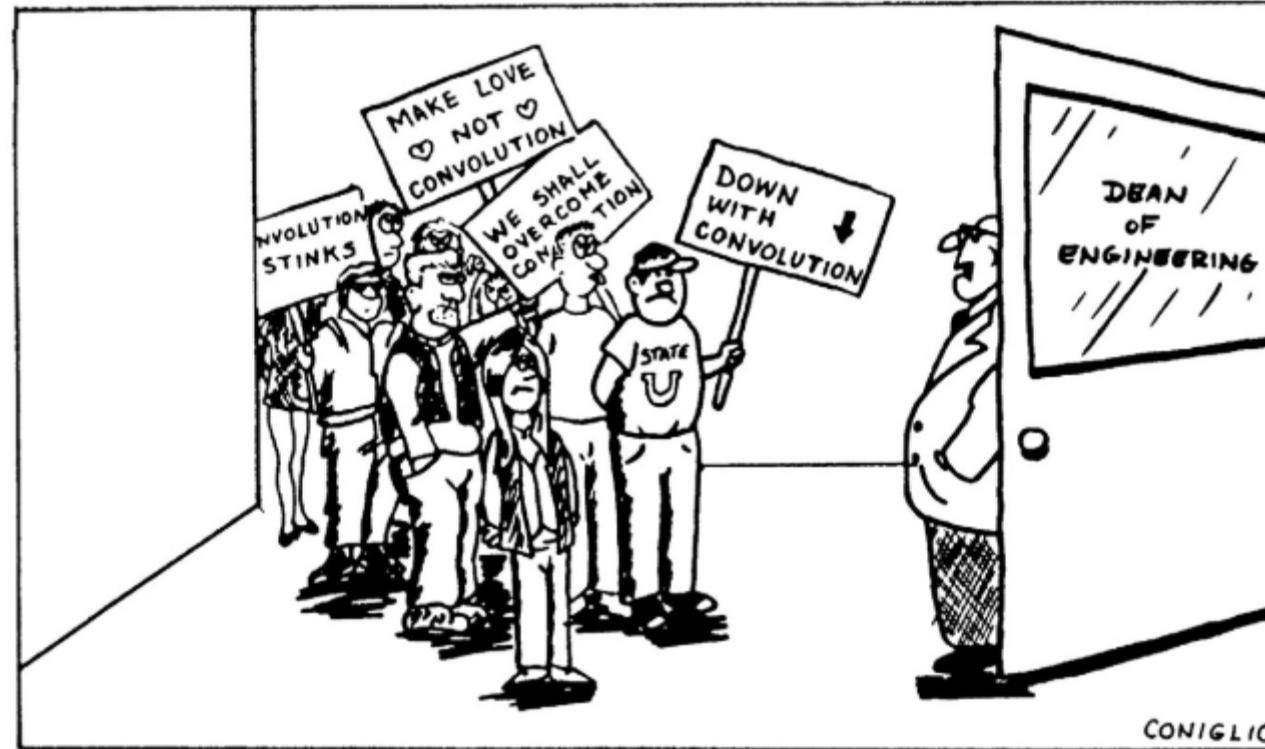
Convolution table

Do not forget the table of convolutions.

Convolution of $x(t)$ with a step function $u(t)$ is equivalent to integration of $x(t)$.

No.	$x_1(t)$	$x_2(t)$	$x_1(t) * x_2(t) = x_2(t) * x_1(t)$
1	$x(t)$	$\delta(t - T)$	$x(t - T)$
2	$e^{\lambda t} u(t)$	$u(t)$	$\frac{1 - e^{\lambda t}}{-\lambda} u(t)$
3	$u(t)$	$u(t)$	$t u(t)$
4	$e^{\lambda_1 t} u(t)$	$e^{\lambda_2 t} u(t)$	$\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} u(t) \quad \lambda_1 \neq \lambda_2$
5	$e^{\lambda t} u(t)$	$e^{\lambda t} u(t)$	$t e^{\lambda t} u(t)$
6	$t e^{\lambda t} u(t)$	$e^{\lambda t} u(t)$	$\frac{1}{2} t^2 e^{\lambda t} u(t)$
7	$t^N u(t)$	$e^{\lambda t} u(t)$	$\frac{N! e^{\lambda t}}{\lambda^{N+1}} u(t) - \sum_{k=0}^N \frac{N! t^{N-k}}{\lambda^{k+1} (N-k)!} u(t)$
8	$t^M u(t)$	$t^N u(t)$	$\frac{M! N!}{(M+N+1)!} t^{M+N+1} u(t)$
9	$t e^{\lambda_1 t} u(t)$	$e^{\lambda_2 t} u(t)$	$\frac{e^{\lambda_2 t} - e^{\lambda_1 t} + (\lambda_1 - \lambda_2) t e^{\lambda_1 t}}{(\lambda_1 - \lambda_2)^2} u(t)$

Convolution

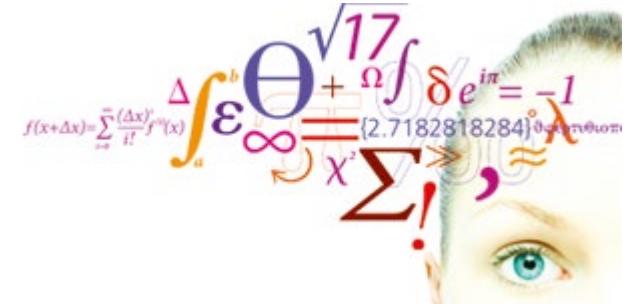


Break 10 min

- Review
 - Decomposition property
 - Zero input response
 - Unit impulse response
- Zero-state response
 - Superposition
 - Convolution integral
- Classical method
- Stability
- Problems

Classical methods

Video 3



Classical solution method for differential equations

The main difference between the classical method and the method used here is that in the classical approach, the initial conditions are defined at $t = 0_+$.

Students have been introduced to the classical method in Mat 1 and Mat 2. It is assumed that students master this method, and it will only be reviewed here with a few examples.

Classical solution techniques for ordinary differential equations (ODEs)

1. Solving for *particular* solution
2. Add *particular* solution and *homogeneous* solution
3. Solve for unknown coefficients using initial conditions at $t = 0_+$

$$Q(D)y(t) = P(D)x(t)$$

$$Q(D)[y_0(t) + y_\varphi(t)] = P(D)x(t)$$

Natural response: $Q(D)y_0(t) = 0$

Forced response: $Q(D)y_\varphi(t) = P(D)x(t)$

The right-hand side is a weighted sum of $x(t)$ and its derivatives. Thus, $y_\varphi(t)$ must be chosen so that the left-hand side can match every term on the right-hand side.

Example 1: Polynomial input

$$(D^2 + 5D + 6)y(t) = (D + 1)x(t)$$
$$x(t) = 6t^2, y(0_+) = 25/18, \dot{y}(0_+) = -2/3$$

Guess:

$$y_\varphi(t) = \beta_2 t^2 + \beta_1 t + \beta_0, t > 0$$

$y_\varphi(t)$ inserted:

$$2\beta_2 + 5(2\beta_2 t + \beta_1) + 6(\beta_2 t^2 + \beta_1 t + \beta_0) = 6t^2 + 12t$$

Matching coefficients
for equal power terms

$$\begin{aligned} t^2: 6\beta_2 t^2 &= 6t^2 \Rightarrow \beta_2 = 1 \\ t^1: 10\beta_2 t + 6\beta_1 &= 12t \Rightarrow \beta_1 = 1/3 \\ t^0: 2 + 5/3 + 6\beta_0 &= 0 \Rightarrow \beta_0 = -11/18 \end{aligned}$$

Particular solution:

$$y_\varphi(t) = \textcolor{red}{t^2 + \frac{1}{3}t - \frac{11}{18}}, t > 0$$

Example 1: Polynomial input

Homogeneous equation:

$$(D^2 + 5D + 6)y_0(t) = 0; y(0_+) = 25/18, \dot{y}(0_+) = -2/3$$

Characteristic equation:

$$\lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) = 0$$

Generic solution form:

$$y_0(t) = C_1 e^{-2t} + C_2 e^{-3t}; t \geq 0$$

Solving for unknown coefficients:

$$y(t) = C_1 e^{-2t} + C_2 e^{-3t} + t^2 + \frac{1}{3}t - \frac{11}{18}$$

$$\dot{y}(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t} + 2t + \frac{1}{3}$$

$$\begin{cases} y(0_+) = C_1 + C_2 - \frac{11}{18} = \frac{25}{18} \\ \dot{y}(0_+) = -2C_1 - 3C_2 + \frac{1}{3} = -\frac{2}{3} \end{cases} \Rightarrow \begin{cases} C_1 = 5 \\ C_2 = -3 \end{cases}$$

$$y(t) = 5e^{-2t} - 3e^{-3t} + t^2 + \frac{1}{3}t - \frac{11}{18}; t \geq 0$$

Response to exponential inputs

While the convolution method allow us to compute the response to a large number of input signals, the method does not provide clear insight into what action the system performs on the input signal.

To get some insight into this at this point in the course, we will use the classical method.

When the input signal is exponential, the differential operator polynomial produces an ordinary polynomial. Hence after all the differentiations, we end up with an algebraic equation.

$$Q(D)y(t) = P(D)x(t) \quad D^i y(0_+) = \alpha_i, i = 0, \dots, n-1$$

$$x(t) = e^{\gamma t}$$

$$y_\phi(t) = \beta e^{\gamma t}$$

$$De^{\gamma t} = \gamma e^{\gamma t}$$

$$D^2e^{\gamma t} = \gamma^2 e^{\gamma t}$$

$$D^r e^{\gamma t} = \gamma^r e^{\gamma t}$$

$$Q(D)\beta e^{\gamma t} = Q(\gamma)\beta e^{\gamma t} \quad P(D)e^{\gamma t} = P(\gamma)e^{\gamma t}$$

$$Q(D)y_\phi(t) = Q(\gamma)\beta e^{\gamma t} = P(\gamma)e^{\gamma t} = P(D)x(t)$$

$$\beta = \frac{P(\gamma)}{Q(\gamma)}$$

Response to exponential inputs

Example:

Exponential input:

$$x(t) = e^{\gamma t}$$

Exponential output:

$$y_\phi(t) = \beta e^{\gamma t}, \quad \beta = \text{gain}$$

Differential equation:

$$\underbrace{(D^2 + a_1 D + a_0)}_{Q(D)} \beta e^{\gamma t} = \underbrace{(b_1 D + b_0)}_{P(D)} e^{\gamma t}$$

Algebraic equation:

$$\underbrace{(\gamma^2 + a_1\gamma + a_0)}_{Q(\gamma)} \beta e^{\gamma t} = \underbrace{(b_1\gamma + b_0)}_{P(\gamma)} e^{\gamma t}$$

$$\underbrace{(\gamma^2 + a_1\gamma + a_0)}_{Q(\gamma)} \beta = \underbrace{(b_1\gamma + b_0)}_{P(\gamma)}$$

We observe that the gain β is a function of γ .

$$\beta(\gamma) = \frac{(b_1\gamma + b_0)}{(\gamma^2 + a_1\gamma + a_0)} = \frac{P(\gamma)}{Q(\gamma)}$$

$$y_\phi(t) = \underbrace{\beta(\gamma)}_{\text{gain}} e^{\gamma t} = \frac{P(\gamma)}{Q(\gamma)} x(t)$$

Response to exponential inputs

We call this gain the **Transfer function** because it shows how the input is transferred to the output.

The forced response (particular solution) is:

The complete solution is the sum of the natural response and the particular solution:

$$H(\gamma) \stackrel{\text{def}}{=} \frac{P(\gamma)}{Q(\gamma)}$$

$$y_\phi(t) = \underbrace{H(\gamma)}_{\text{gain}} e^{\gamma t}$$

$$\begin{aligned} y(t) &= \underbrace{y_0(t)}_{n} + \underbrace{y_\phi(t)}_{\text{forced response}} \\ &= \underbrace{\sum_{j=1}^n K_j e^{\lambda_j t}}_{\text{natural response}} + \underbrace{H(\gamma) e^{\gamma t}}_{\text{forced response}}, \quad \gamma \neq \lambda_j \end{aligned}$$

Response to sinusoidal inputs

Let us investigate how **sinusoidal signals** of different frequencies are transferred through the system.

$$e^{\gamma t} = e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

$$e^{-\gamma t} = e^{-j\omega t} = \cos(\omega t) - j \sin(\omega t)$$

Adding two complex conjugated numbers yields twice the real part:

$$y_\phi(t) = H(\gamma)e^{\gamma t}$$

$$H(\gamma) \stackrel{\text{def}}{=} \frac{P(\gamma)}{Q(\gamma)}$$

$$y_{\phi,1}(t) = H(j\omega)e^{j\omega t}$$

$$y_{\phi,2}(t) = H(-j\omega)e^{-j\omega t}$$

$$y_\phi(t) = \frac{1}{2}H(j\omega)e^{j\omega t} + \frac{1}{2}H(-j\omega)e^{-j\omega t} = \operatorname{Re}\{H(j\omega)e^{j\omega t}\}$$

$$y_\phi(t) = \operatorname{Re}\{|H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t}\}$$

$$y_\phi(t) = \operatorname{Re}\{|H(j\omega)|e^{j(\omega t + \angle H(j\omega))}\}$$

$$y_\phi(t) = |H(j\omega)| \operatorname{Re}\{e^{j(\omega t + \angle H(j\omega))}\}$$

$$y_\phi(t) = \underbrace{|H(j\omega)|}_{\text{gain}} \cos \left(\omega t + \underbrace{\angle H(j\omega)}_{\text{phase}} \right)$$

We see that the modulus of the transfer function defines the system gain as a function of frequency:

Response to sinusoidal inputs

$$H(\gamma) \stackrel{\text{def}}{=} \frac{P(\gamma)}{Q(\gamma)} \quad H(j\omega) \stackrel{\text{def}}{=} \frac{P(j\omega)}{Q(j\omega)}$$

Transfer functions

Two real roots: $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 2x(t)$

$$H(j\omega) \stackrel{\text{def}}{=} \frac{P(j\omega)}{Q(j\omega)} = \frac{2}{(j\omega)^2 + 3(j\omega) + 2}$$

$$|H(j\omega)| = \frac{2}{\sqrt{(2 - \omega^2)^2 + (3\omega)^2}}$$

A double root: $\ddot{y}(t) + 6\dot{y}(t) + 9y(t) = 10\ddot{x}(t)$

$$H(j\omega) \stackrel{\text{def}}{=} \frac{P(j\omega)}{Q(j\omega)} = \frac{10(j\omega)^2}{(j\omega)^2 + 6(j\omega) + 9}$$

$$|H(j\omega)| = \frac{10\omega^2}{\sqrt{(9 - \omega^2)^2 + (6\omega)^2}}$$

Complex conjugated roots:

$$\ddot{y}(t) + 4\dot{y}(t) + 13y(t) = 4\dot{x}(t)$$

$$H(j\omega) \stackrel{\text{def}}{=} \frac{P(j\omega)}{Q(j\omega)} = \frac{4(j\omega)}{(j\omega)^2 + 4(j\omega) + 13}$$

$$|H(j\omega)| = \frac{4\omega}{\sqrt{(13 - \omega^2)^2 + (4\omega)^2}}$$

Response to sinusoidal inputs

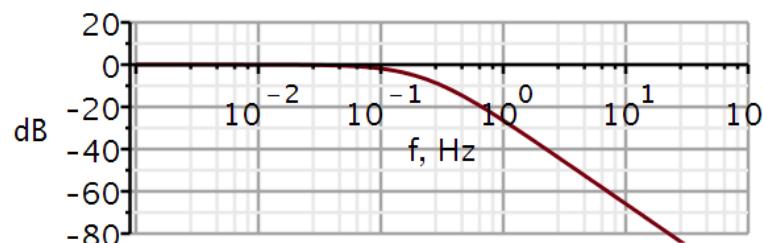
$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 2\dot{x}(t)$$

$$H(j\omega) = \frac{2}{(j\omega)^2 + 3(j\omega) + 2}$$

$$|H(j\omega)| = \frac{2}{\sqrt{(2 - \omega^2)^2 + (3\omega)^2}}$$

$$|H(j\omega)|_{\omega \rightarrow 0} = \frac{2}{2} = 1$$

$$|H(j\omega)|_{\omega \rightarrow \infty} = \frac{2}{\infty} = 0$$



Low pass filter

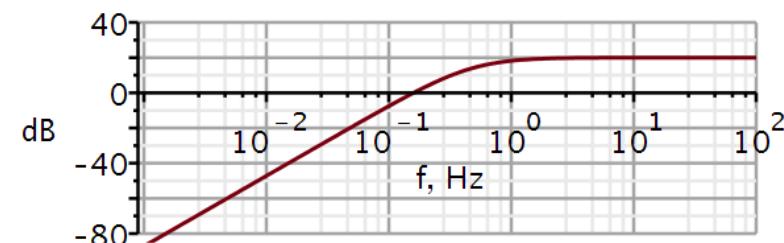
$$\ddot{y}(t) + 6\dot{y}(t) + 9y(t) = 10\ddot{x}(t)$$

$$H(j\omega) = \frac{10(j\omega)^2}{(j\omega)^2 + 6(j\omega) + 9}$$

$$|H(j\omega)| = \frac{10\omega^2}{\sqrt{(9 - \omega^2)^2 + (6\omega)^2}}$$

$$|H(j\omega)|_{\omega \rightarrow 0} = \frac{0}{9} = 0$$

$$|H(j\omega)|_{\omega \rightarrow \infty} = \frac{10\omega^2}{\omega^2} = 10$$



High pass filter

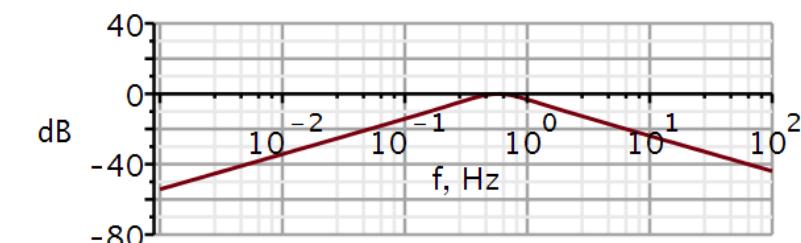
$$\ddot{y}(t) + 4\dot{y}(t) + 13y(t) = 4\dot{x}(t)$$

$$H(j\omega) = \frac{4(j\omega)}{(j\omega)^2 + 4(j\omega) + 13}$$

$$|H(j\omega)| = \frac{4\omega}{\sqrt{(13 - \omega^2)^2 + (4\omega)^2}}$$

$$|H(j\omega)|_{\omega \rightarrow 0} = \frac{0}{13} = 0$$

$$|H(j\omega)|_{\omega \rightarrow \infty} = \frac{4}{\omega} = 0$$



Band pass filter

Maple tools for plotting transfer function

```

restart
with(plots):
dB := ω→ 20·log10(|H(ω)|): # converts to decibel
angle := ω→argument(H(ω))· $\frac{180}{\pi}$ : # converts from radians to degrees

```

Transfer function

$$H := \omega \rightarrow \frac{b2 \cdot (j \cdot \omega)^2 + b1 \cdot (j \cdot \omega) + b0}{(j \cdot \omega)^2 + a1 \cdot (j \cdot \omega) + a0} : \quad \# \text{2nd order transfer function}$$

Example system

$$\begin{aligned} b2 &:= 0 \;;\; b1 := 0 \;;\; b0 := 2 : & \# \text{numerator coefficients} \\ a1 &:= 3 \;;\; a0 := 2 : & \# \text{denominator coefficients} \end{aligned}$$

Plotting solution

```

semilogplot(dB(2·π·f), f= 0.001 ..1E2, -80 ..20, thickness = 3, font = ["Helvetica",
"roman", 18], axis[2] = [thickness = 2.5], axis[1] = [thickness = 2.5], labels
= ["f, Hz", "dB"], labelfont = ["HELVETICA", 18], numpoints = 100, gridlines,
size = [600, 200])

```

```

semilogplot(angle(2·π·f), f= 0.001 ...1E3, -200 ..200, thickness = 3, font
= [Helvetica, roman, 18], axis[2] = [thickness = 2.5], axis[1] = [thickness
= 2.5], labels = ["f, Hz", " "], labelfont = ["HELVETICA", 18], numpoints
= 100, gridlines, size = [600, 200])

```

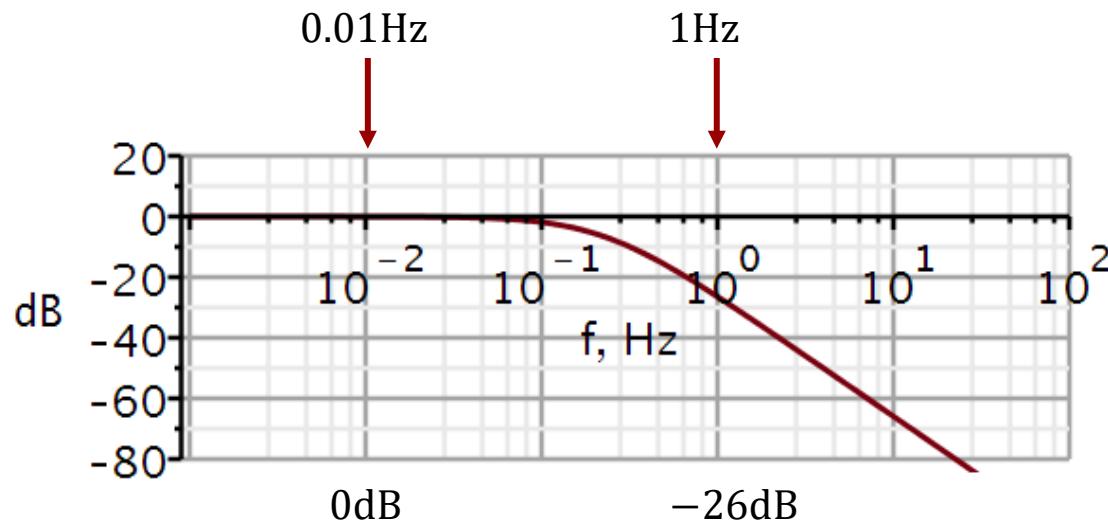
Calculating systems output using convolution

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 2x(t)$$

```

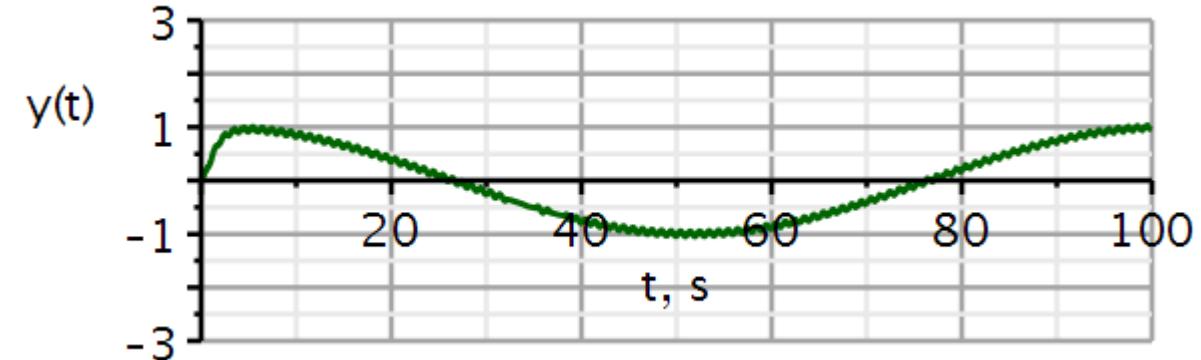
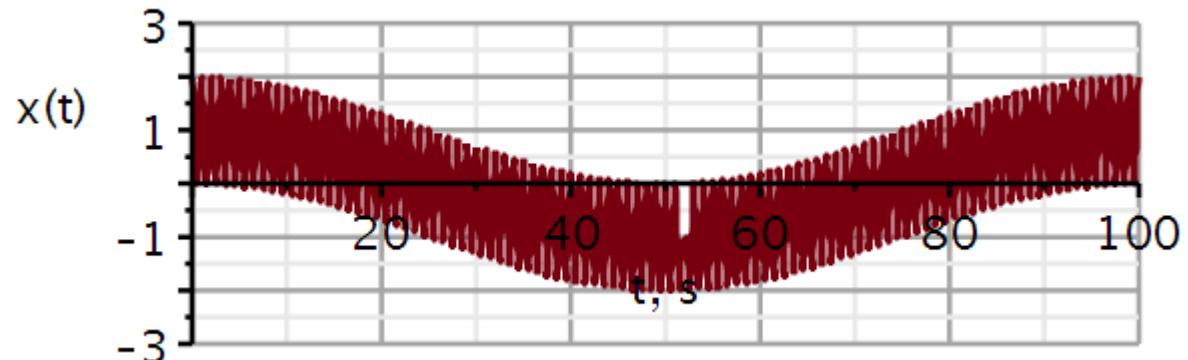
x1 := t→cos(2·π·0.01·t) + cos(2·π·t) :      # input signal
h1 := t→2·(e-t-e-2·t) :                      # impulse response
y1 := t→conv(t→x1(t), t→h1(t))(t)           # output signal
yI := t↦ conv(t↦ x1(t), t↦ h1(t))(t)

```



$$-26\text{dB} = -20\text{dB} - 6\text{dB} \leftrightarrow 10^{-1} \times 2^{-1} = 20^{-1}$$

Input: 0.01 Hz + 1 Hz



1 Hz component removed (almost)
0.01 Hz component passed through

Calculating systems output using convolution

$$\ddot{y}(t) + 6\dot{y}(t) + 9y(t) = 10\ddot{x}(t) \quad h(t) = 10\delta(t) + 10 \frac{d^2}{dt^2}(t e^{-3t})$$

$x1 := t \rightarrow \cos(2\pi \cdot 0.01 \cdot t) + \cos(2\pi \cdot t)$: # input signal

$$10 \cdot \frac{d^2}{dt^2}(t e^{-3t})$$

$$-60e^{-3t} + 90te^{-3t}$$

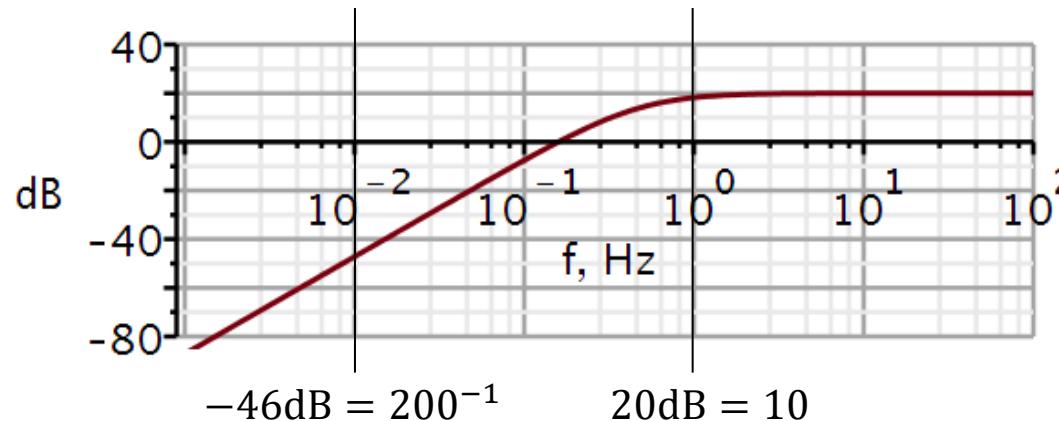
$h2 := t \rightarrow -60e^{-3t} + 90te^{-3t}$: # impulse response

$y1 := t \rightarrow conv(t \rightarrow x1(t), t \rightarrow h2(t))(t)$ # output signal

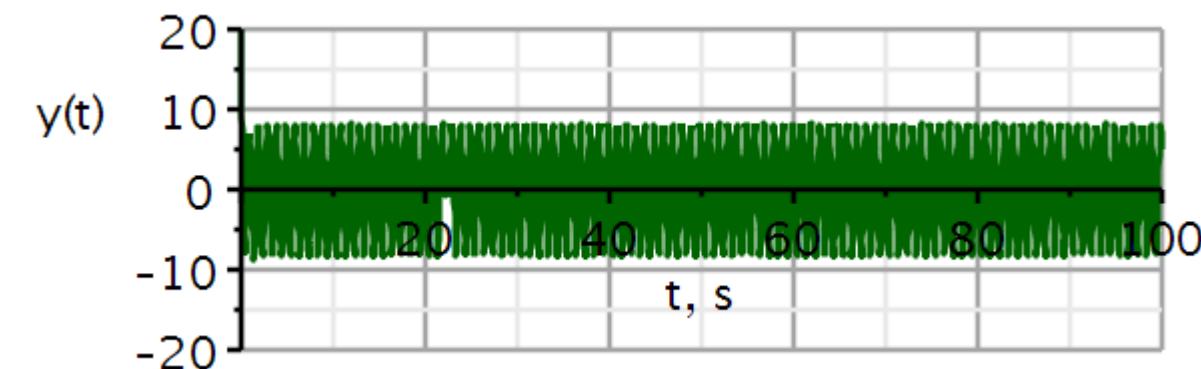
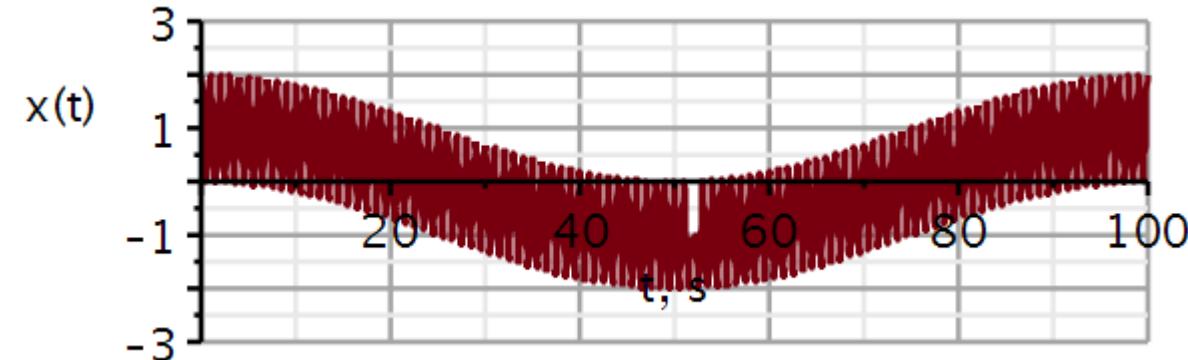
$$y1 := t \mapsto conv(t \mapsto x1(t), t \mapsto h2(t))(t)$$

$y2 := t \rightarrow y1(t) + 10 \cdot x1(t)$:

?



Input: 0.01 Hz + 1 Hz



0.01 Hz component removed
1 Hz component passed through

Example 2: Natural mode input

This is an exponential input. Let us try and use the transfer function to obtain the forced response $y_\varphi(t)$:

By finding the roots of the characteristic equation, or as here the poles of H , we see that we are heading for trouble.

$$(D^2 + 5D + 6)y(t) = (D + 1)x(t)$$
$$x(t) = 5e^{-2t}, y(0_+) = 1, \dot{y}(0_+) = -1$$

$$H(\gamma) = \frac{P(\gamma)}{Q(\gamma)} = \frac{\gamma + 1}{\gamma^2 + 5\gamma + 6} = \frac{\gamma + 1}{(\gamma + 2)(\gamma + 3)}$$

$$y_\varphi(t) = H(-2)x(t) = \frac{\gamma + 1}{(\gamma + 2)(\gamma + 3)} \Big|_{\gamma=-2} 5e^{-2t}$$
$$= 5 \frac{-2 + 1}{(-2 + 2)(-2 + 3)} e^{-2t} = \frac{-5}{0} e^{-2t}$$

If the input is a scaled copy of one of the natural modes, we must use a different approach.

Example 2: Natural mode input

Particular solution assuming double root:

$$D(\beta te^{\gamma t}) = \beta e^{\gamma t} + \beta t \gamma e^{\gamma t}$$

$$D^2(\beta te^{\gamma t}) = 2\beta \gamma e^{\gamma t} + \beta t \gamma^2 e^{\gamma t}$$

$$y_\varphi(t) = \beta te^{\gamma t}; t > 0$$

$$(D^2 + 5D + 6)(\beta te^{\gamma t}) = (D + 1)(5e^{\gamma t})$$

$$2\beta \gamma e^{\gamma t} + \beta t \gamma^2 e^{\gamma t} + 5 \cdot (\beta e^{\gamma t} + \beta t \gamma e^{\gamma t}) + 6 \cdot \beta t e^{\gamma t} = 5 \cdot (\gamma e^{\gamma t} + e^{\gamma t})$$

$$2\beta \gamma + \beta t \gamma^2 + 5 \cdot (\beta + \beta t \gamma) + 6 \cdot \beta t = 5 \cdot (\gamma + 1)$$

Matching left-side and right-side terms with t :

$$\gamma = -2$$

$$\beta t \gamma^2 + 5 \cdot (\beta t \gamma) + 6 \cdot \beta t = 0$$

$$\beta t 4 - 5 \cdot (\beta t 2) + 6 \cdot \beta t = (10 - 10)\beta t = 0$$

These terms turn out to be independent of β

Matching left-side and right-side terms without t :

$$2\beta(-2) + 5 \cdot (\beta) = 5 \cdot (-2 + 1) \Rightarrow \beta = -\frac{5}{5 - 4} = -5$$

$$y_\varphi(t) = -5te^{-2t}; t > 0$$

Example 2: Natural mode input

Homogeneous equation:

$$(D^2 + 5D + 6)y_n(t) = 0$$

$$e^{\lambda t} \quad \lambda^2 + 5\lambda + 6 = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = -3$$

$$y_n(t) = K_1 e^{-2t} + K_2 e^{-3t}$$

$$y(t) = K_1 e^{-2t} + K_2 e^{-3t} - 5te^{-2t}$$

$$y(0_+) = 1, \dot{y}(0_+) = -1$$

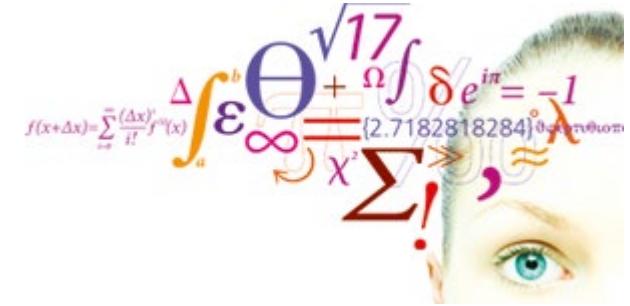
$$\dot{y}(t) = -2K_1 e^{-2t} - 3K_2 e^{-3t} - 5e^{-2t} + 10te^{-2t}$$

$$\begin{aligned} y(0_+) &= K_1 + K_2 = 1 \\ \dot{y}(0_+) &= -2K_1 - 3K_2 - 5 = -1 \end{aligned} \quad \Rightarrow K_1 = 7, K_2 = -6$$

$$y(t) = ((7 - 5t)e^{-2t} - 6e^{-3t})u(t)$$

Break 10 min

- Review
 - Decomposition property
 - Zero input response
 - Unit impulse response
- Zero-state response
 - Superposition
 - Convolution integral
- Classical method
- **Stability**
- Problems



$$Q(D)y(t) = P(D)x(t)$$

Characteristic equation:

$$Q(\lambda) = 0$$

For unrepeated roots:

Natural response:

$$y_0(t) = \sum_{j=1}^n C_j e^{\lambda_j t}, t \geq 0$$

Asymptotically stable:

$Re\{\lambda_j\} < 0$, exponentially decreasing response

Unstable system:

$Re\{\lambda_j\} > 0$, exponentially increasing response

Marginally stable:

$Re\{\lambda_j\} = 0$, oscillating with constant amplitude

For repeated roots:

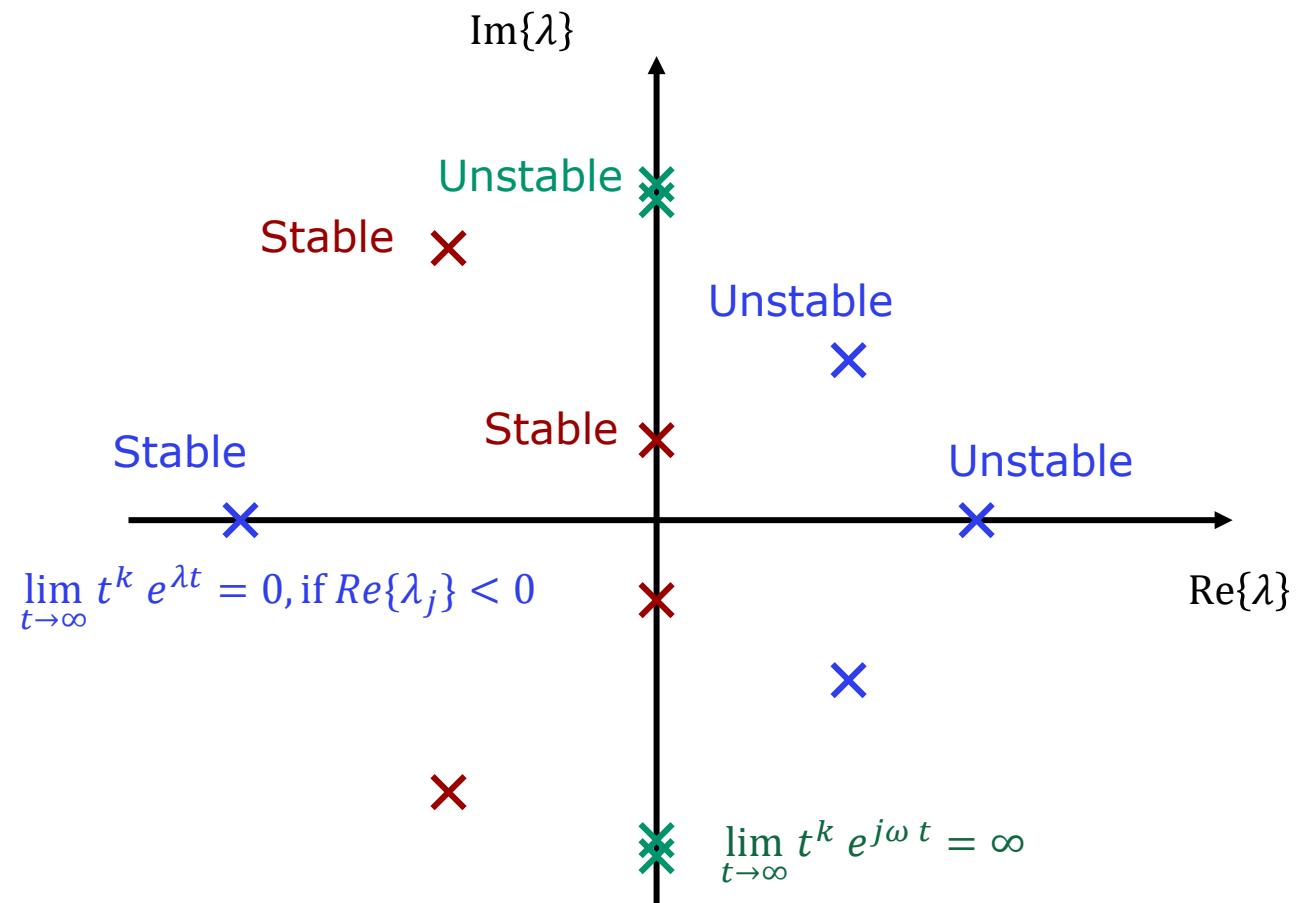
$$\lim_{t \rightarrow \infty} t^k e^{\lambda t} = 0, \quad \text{if } Re\{\lambda_j\} < 0$$

$$\lim_{t \rightarrow \infty} t^k e^{j\omega t} = \infty, \quad \text{if } Re\{\lambda_j\} = 0$$

For systems with repeated roots, none of these can be on the imaginary axis.

Stability

An unconditionally stable system has all its roots in the:
LEFT HALF PLANE



Bounded Input- Bounded Output (BIBO) stability

If we do not know the natural modes of the system, we need an alternative means of determining if the system is stable.

In an experiment, we can provide an input and measure the output. We may also set up some relations to give us insight in what to expect of a stable system.

For an asymptotically stable system :
 $Re\{\lambda_j\} < 0$

- An asymptotically stable system is always BIBO stable.
- BIBO stability ensures that the zero-state response is always amplitude bounded.
- BIBO stability only concerns what can be observed on the output (external).

$$y_{zs}(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

$$|y_{zs}(t)| \leq \int_{-\infty}^{\infty} |h(\tau)||x(t - \tau)|d\tau$$

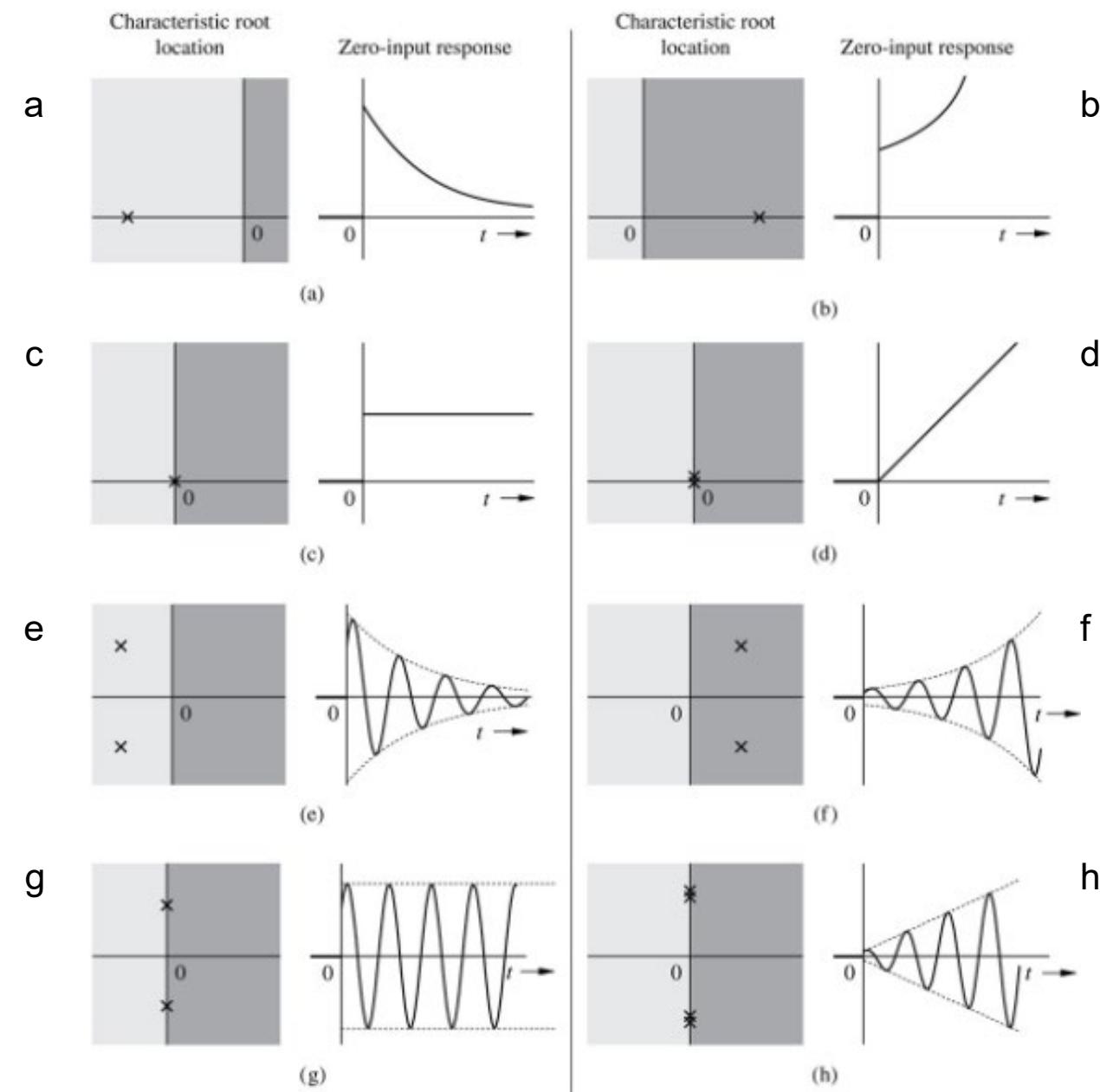
$|x(t - \tau)| \leq K_1$ Bounded input:

$$|y_{zs}(t)| \leq K_1 \int_{-\infty}^{\infty} |h(\tau)|d\tau$$

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau \leq K_2$$

$|y_{zs}(t)| \leq K_1 K_2$ Bounded output

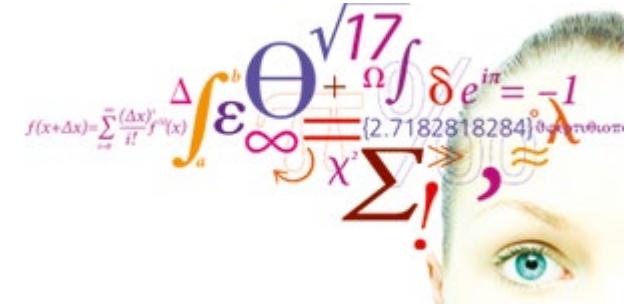
Quick quiz



Break 10 min

Learning priorities from Problem Solving:

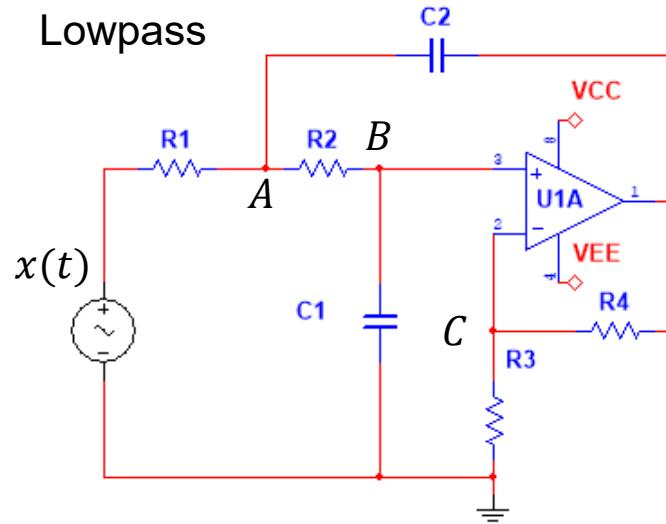
1. Why this method?
2. When is this method valid?
3. How is this method used?
4. What is the result?
5. How do I validate the result?



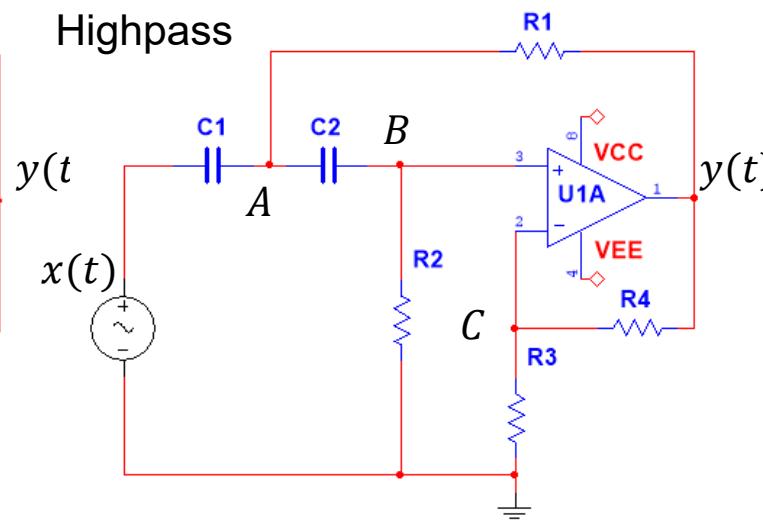
Problems

Sallen-Key filters used in this course

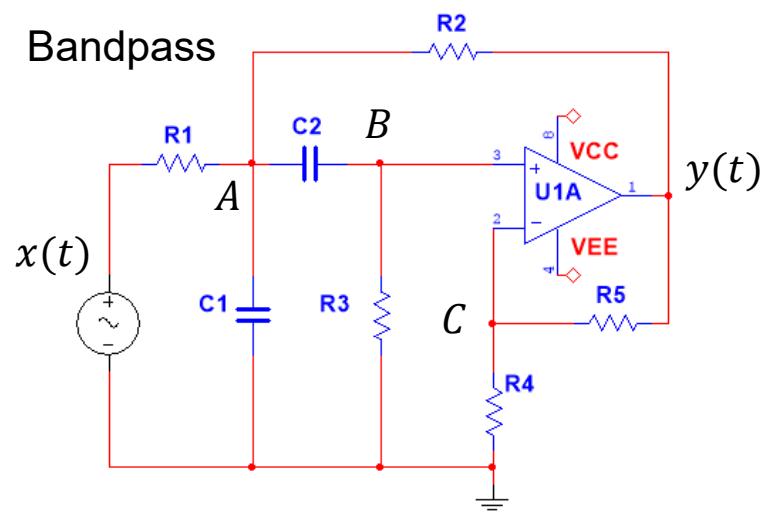
Lowpass



Highpass



Bandpass



Filter table

		Lowpass	Highpass	Bandpass
R_1	$k\Omega$	3.9894	1784.1	56.419
R_2	$k\Omega$	0.8865	892.06	34.117
R_3	$k\Omega$	1.0	1.0	149.72
R_4	$k\Omega$	1.0	1.0	1.0
R_5	$k\Omega$	—	—	1.0
C_1	nF	1795.2	1784.1	56.419
C_2	nF	398.94	3568.2	56.419
a_1		$2.83 \cdot 10^3$	$6.283 \cdot 10^{-1}$	$3.141 \cdot 10^1$
a_0		$3.95 \cdot 10^5$	$9.87 \cdot 10^{-2}$	$9.87 \cdot 10^4$
b_2		0	2	0
b_1		0	0	$6.283 \cdot 10^2$
b_0		$7.89 \cdot 10^5$	0	0

Filter 7: Lowpass filter - Convolution

To solve this problem, you will need the solutions to problems solved in weeks 1 and 2.

$$h(t) = 311.8(e^{-147.31t} - e^{-2680.2t})u(t)$$

1. Determine if the lowpass filter is asymptotically stable, marginally stable or unstable.
2. Plot $|H(j\omega)|$ in decibels.
3. Using Maple, apply the convolution integral to calculate the zero-state response to the input $x(t) = 2u(t)$.
4. Use the Maple package *DynamicSystems* to verify the result in (3).
5. As another validation, use KiCad/Spice to plot the step response.

Short answer to (3): $y_{zs}(t) = (4 + 0.23 e^{-2680.2t} - 4.23 e^{-147.25t})u(t)$

Filter 8: Highpass filter - Convolution

To solve this problem, you will need the solutions to problems solved in weeks 1 and 2.

$$h(t) = 2\delta(t) - 0.6283(2 - 0.31415 t)e^{-0.31415 t}u(t)$$

1. Determine if the highpass filter is asymptotically stable, marginally stable or unstable.
2. Plot $|H(j\omega)|$ in decibels.
3. Using Maple, apply the convolution integral to calculate the zero-state response to the input $x(t) = 2u(t)$.
4. Use the Maple package *DynamicSystems* to verify the result in (3).
5. As another validation, use KiCad/Spice to plot the step response.

Short answer to (3): $y_{zs}(t) = 4(1 - 0.31415 t)e^{-0.31415 t}u(t)$

Filter 9: Bandpass filter - Convolution

To solve this problem, you will need the solutions to problems solved in weeks 1 and 2.

$$h(t) = 628.3 e^{-15.7 t} \cos(314 t) u(t) - 31.4 e^{-15.7 t} \sin(314 t) u(t)$$

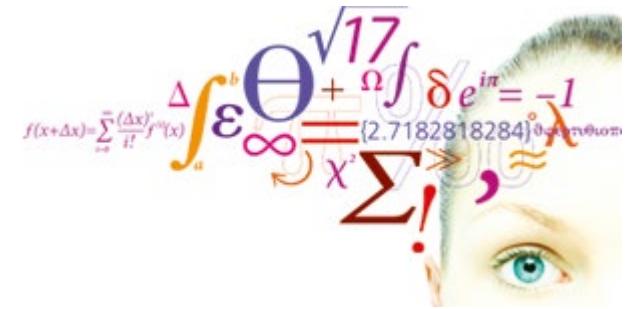
1. Determine if the bandpass filter is asymptotically stable, marginally stable or unstable.
2. Plot $|H(j\omega)|$ in decibels.
3. Using Maple, apply the convolution integral to calculate the zero-state response to the input $x(t) = 2u(t)$.
4. Use the Maple package *DynamicSystems* to verify the result in (3).
5. As another validation, use KiCad/Spice to plot the step response.

Short answer to (3): $y_{zs}(t) = 4e^{-15.7 t} \sin(313.76 t) u(t) = 4e^{-15.7 t} \cos\left(313.76 t + \frac{\pi}{2}\right) u(t)$

Problem set (solutions)

Learning priorities from Problem Solving:

1. Why this method?
2. When is this method valid?
3. How is this method used?
4. What is the result?
5. How do I validate the result?



Solutions

Filter 7: Lowpass filter – Convolution (sol)

To solve this problem, you will need the solutions to problems solved in weeks 1 and 2.

$$h(t) = 311.8(e^{-147.31t} - e^{-2680.2t})u(t)$$

1. Determine if the lowpass filter is asymptotically stable, marginally stable or unstable.
2. Plot $|H(j\omega)|$ in decibels.
3. Using Maple, apply the convolution integral to calculate the zero-state response to the input $x(t) = 2u(t)$.
4. Use the Maple package *DynamicSystems* to verify the result in (3).
5. As another validation, use KiCad/Spice to plot the step response.

Short answer to (3): $y_{zs}(t) = (4 + 0.23 e^{-2680.2t} - 4.23 e^{-147.25t})u(t)$

Filter 7: Lowpass filter – Convolution (sol)

1. The poles are all in the left-side plane. Hence the filter is asymptotically stable.

2. Amplitude characteristic:

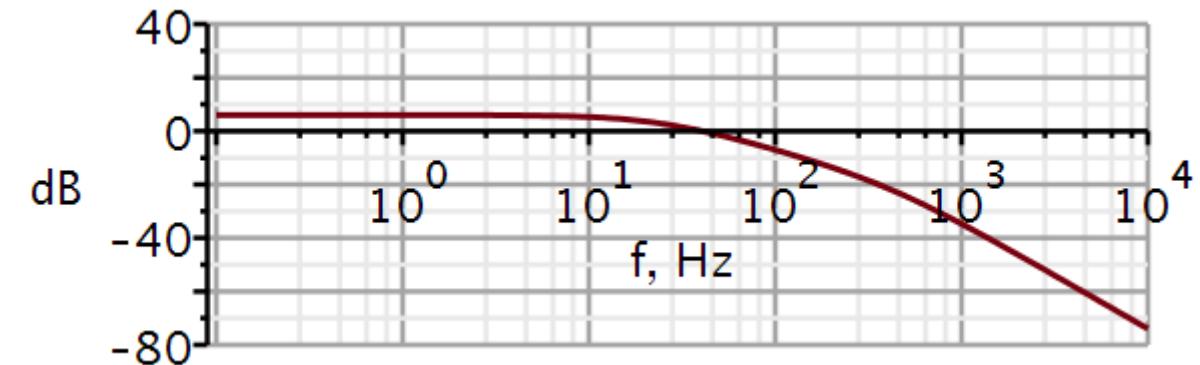
Transfer function

$$H := \omega \rightarrow \frac{b2 \cdot (j \cdot \omega)^2 + b1 \cdot (j \cdot \omega) + b0}{(j \cdot \omega)^2 + a1 \cdot (j \cdot \omega) + a0} : \quad \# \text{ 2nd order transfer function}$$

simplify(evalf(H(\omega)))

$$\frac{7.896292901 \cdot 10^9}{-10000 \cdot \omega^2 + 2.827537902 \cdot 10^7 \cdot j \omega + 3.948146451 \cdot 10^9}$$

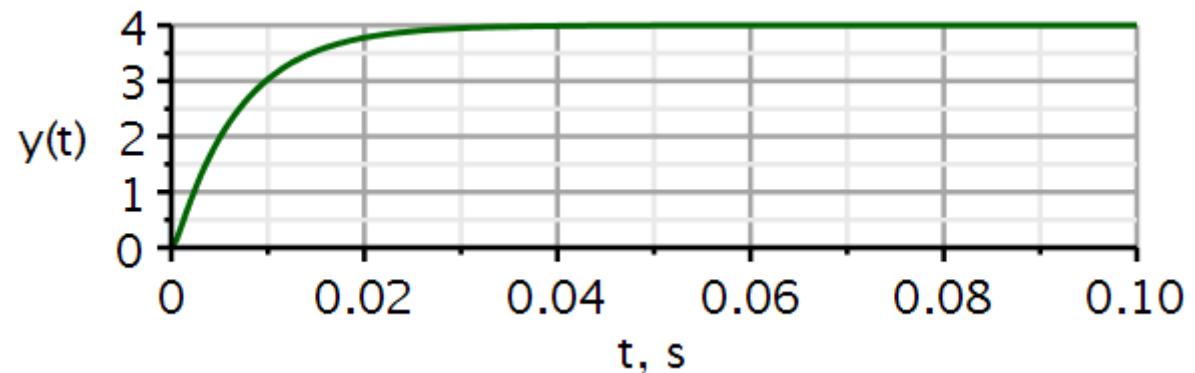
$$dB := \omega \rightarrow 20 \cdot \log10(|H(\omega)|) : \quad \# \text{ converts to decibel}$$



```
semilogplot(dB(2 * pi * f), f = 0.1 .. 1E4, -80 .. 40, thickness = 3, axesfont = ["Helvetica", "roman", 18], axis[2] = [thickness = 2.5], axis[1] = [thickness = 2.5], labels = ["f, Hz", "dB"], labelfont = ["HELVETICA", 18], numpoints = 100, gridlines, size = [600, 200])
```

Filter 7: Lowpass filter – Convolution (sol)

3. Zero-state response using convolution:



$$xI := t \rightarrow 2 \cdot v(t) :$$

input signal

$$hI := t \rightarrow (311.75 e^{-147.31 t} - 311.75 e^{-2680.2 t}) \text{ Heaviside}(t) : \quad \# \text{ impulse response}$$

$$\text{evalf}\left(\int_0^t xI(\tau) \cdot hI(t - \tau) d\tau, 5\right)$$

$$3.9999 - 4.2326 e^{-147.31 t} + 0.23263 e^{-2680.2 t}$$

Short answer to (3): $y_{zs}(t) = (4 + 0.23 e^{-2680.2 t} - 4.23 e^{-147.25 t}) u(t)$

Filter 7: Lowpass filter – Convolution (sol)

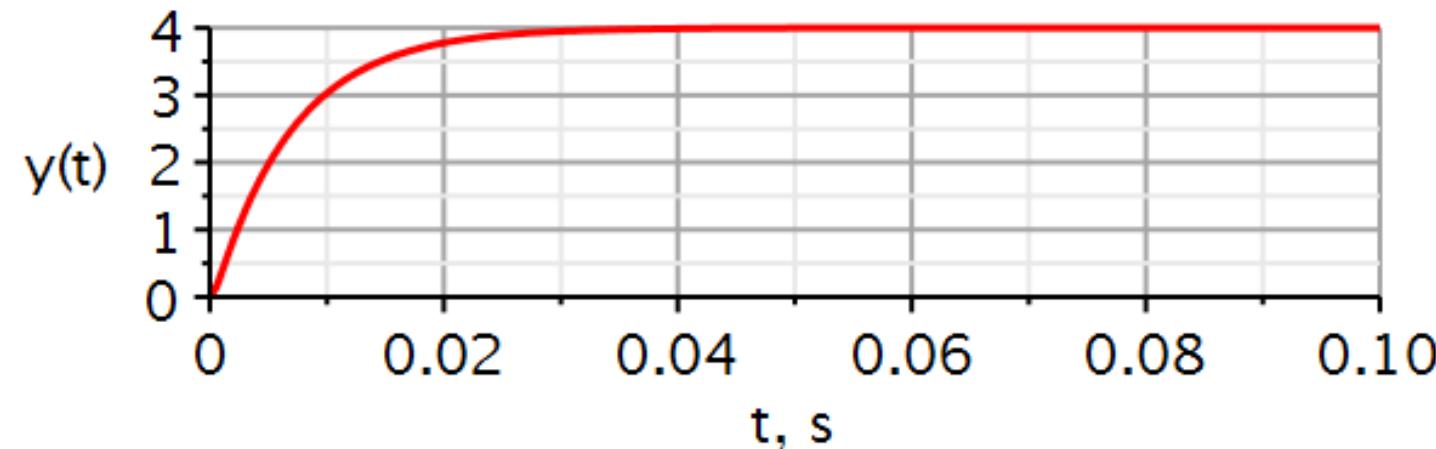
4. Use the Maple package *DynamicSystems* to verify the result.

$$sysLP := \text{Coefficients}\left(\frac{b0}{s^2 + al \cdot s + a0}\right);$$

vin := *Step*(2, 0, 0)

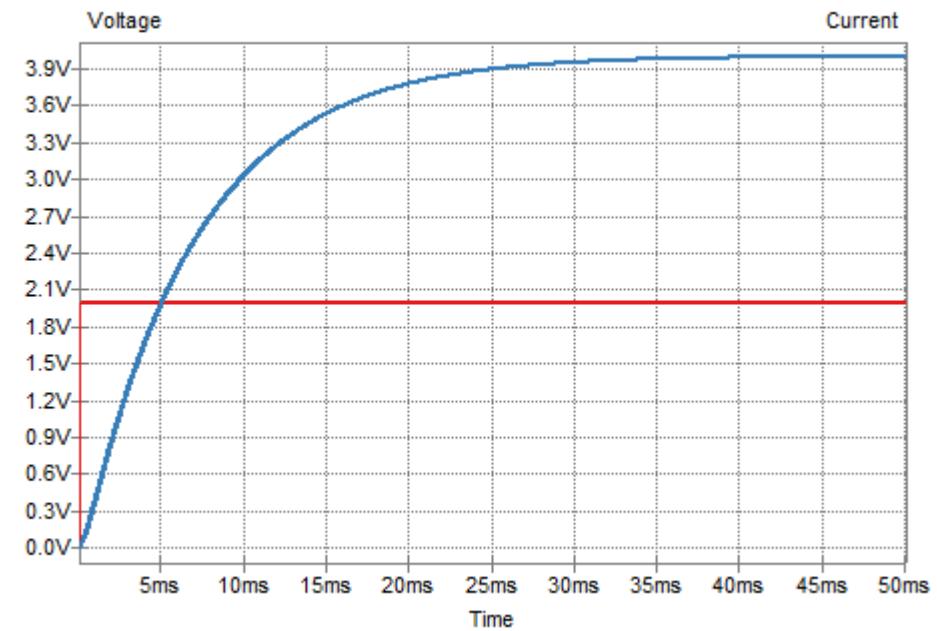
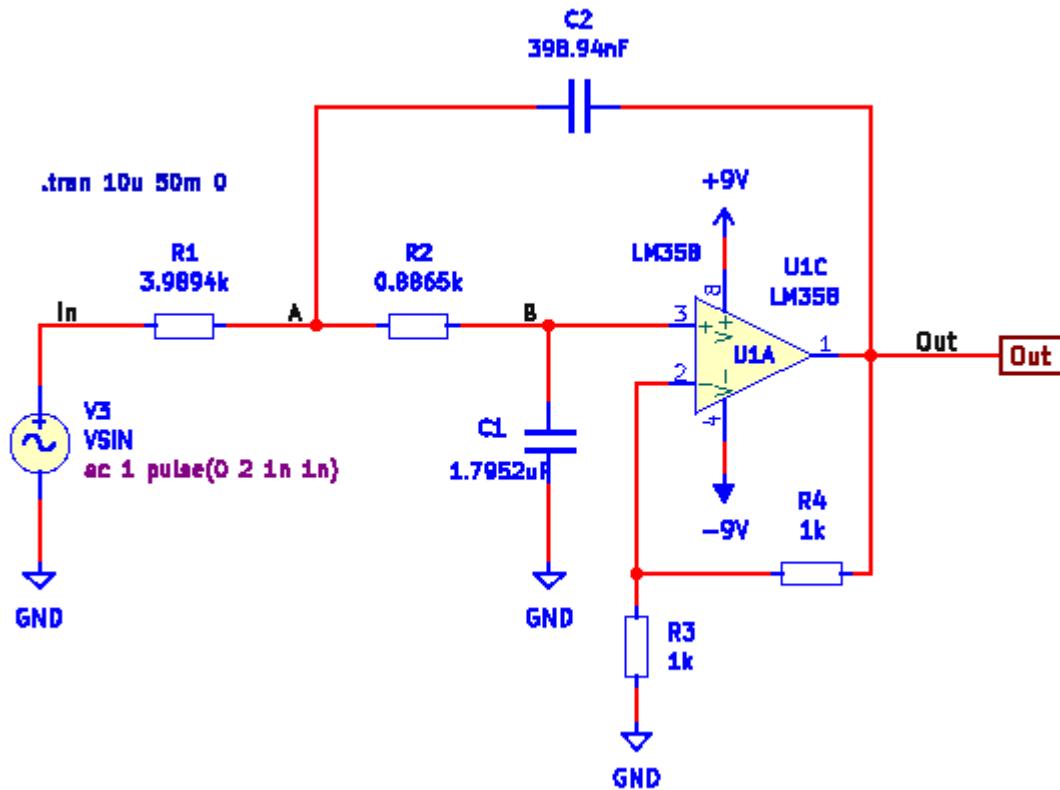
vin := 2 Heaviside(*t*)

```
ResponsePlot(sysLP, vin, duration = 0.1, color = [red], thickness = 3, axesfont = ["Helvetica", "ROMAN", 18], axis[2] = [thickness = 2.5],  
axis[1] = [mode = linear, thickness = 2.5], labels = ["t, s", "y(t)"], labelfont = ["HELVETICA", 18], gridlines, size = [600, 200])
```



Filter 7: Lowpass filter – Convolution (sol)

- As another validation, use KiCad/Spice to plot the step response.



Filter 8: Highpass filter – Convolution (sol)

To solve this problem, you will need the solutions to problems solved in weeks 1 and 2.

$$h(t) = 2\delta(t) - 0.6283(2 - 0.31415 t)e^{-0.31415 t}u(t)$$

1. Determine if the highpass filter is asymptotically stable, marginally stable or unstable.
2. Plot $|H(j\omega)|$ in decibels.
3. Using Maple, apply the convolution integral to calculate the zero-state response to the input $x(t) = 2u(t)$.
4. Use the Maple package *DynamicSystems* to verify the result in (3).
5. As another validation, use KiCad/Spice to plot the step response.

Short answer to (3): $y_{zs}(t) = 4(1 - 0.31415 t)e^{-0.31415 t}u(t)$

Filter 8: Highpass filter – Convolution (sol)

1. The poles are all in the left-side plane. Hence the filter is asymptotically stable.
2. Amplitude characteristic:

Transfer function

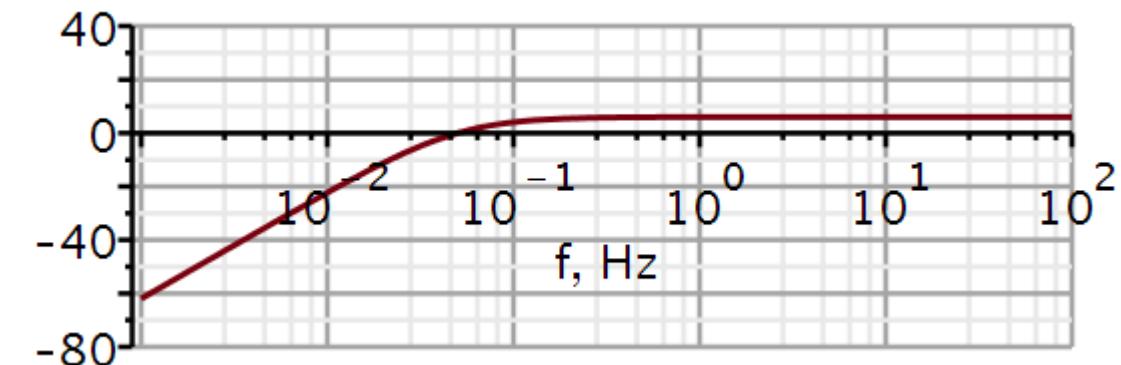
$$H := \omega \rightarrow \frac{b2 \cdot (\text{j} \cdot \omega)^2 + b1 \cdot (\text{j} \cdot \omega) + b0}{(\text{j} \cdot \omega)^2 + a1 \cdot (\text{j} \cdot \omega) + a0} : \quad \# \text{ 2nd order transfer function}$$

simplify(evalf(H(ω)))

$$= \frac{2 \cdot \omega^2}{-\omega^2 + 0.6283249520 \text{j} \omega + 0.09870027409}$$

$$dB := \omega \rightarrow 20 \cdot \log10(|H(\omega)|) : \quad \# \text{ converts to decibel}$$

```
semilogplot(dB(2 · π · f), f = 0.001 .. 1E2, -80 .. 40, thickness = 3, axesfont = ["Helvetica", "roman", 18], axis[2] = [thickness = 2.5], axis[1] = [thickness = 2.5], labels = ["f, Hz", "dB"], labelfont = ["HELVETICA", 18], numpoints = 100, gridlines, size = [600, 200])
```



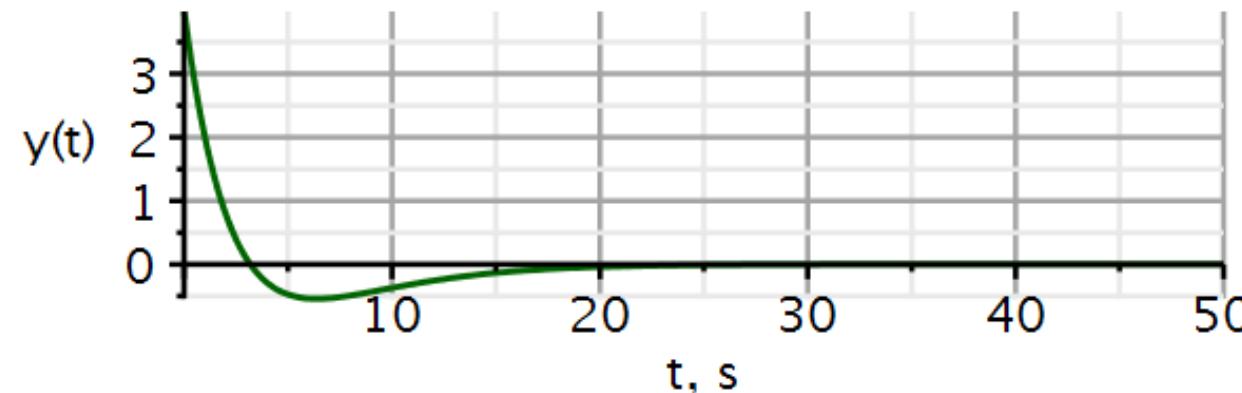
Filter 8: Highpass filter – Convolution (sol)

3. Zero-state response using convolution:

```

xl := t→2·v(t) : # input signal
h1 := t→(0.19739 e-0.31416 t t - 1.2566 e-0.31416 t) Heaviside(t) : # impulse response
2·xl(t) + evalf(∫0t xl(τ) · h1(t - τ) dτ, 5)
4 Heaviside(t) - 3.9998 - 1.2566 e-0.31416 t t + 3.9998 e-0.31416 t
y1 := t→2·xl(t) + conv(t→xl(t), t→h1(t))(t) # output signal
y1 := t ↦ 2·xl(t) + conv(t ↦ xl(t), t ↦ h1(t))(t)
plot({y1(t)}, t=0..50, thickness = 3, color = "DarkGreen", axesfont = ["Helvetica", "ROMAN", 18], axis[2] = [thickness = 2.5], axis[1] = [mode = linear, thickness = 2.5], labels = ["t, s", "y(t)"], labelfont = ["HELVETICA", 18], gridlines, size = [600, 200])

```



When $b_n \neq 0$, remember to add $b_n \delta(t)$ to the impulse response.

Short answer to (3): $y_{zs}(t) = 4(1 - 0.31415 t)e^{-0.31415 t}u(t)$

Filter 8: Highpass filter – Convolution (sol)

4. Use the Maple package *DynamicSystems* to verify the result.

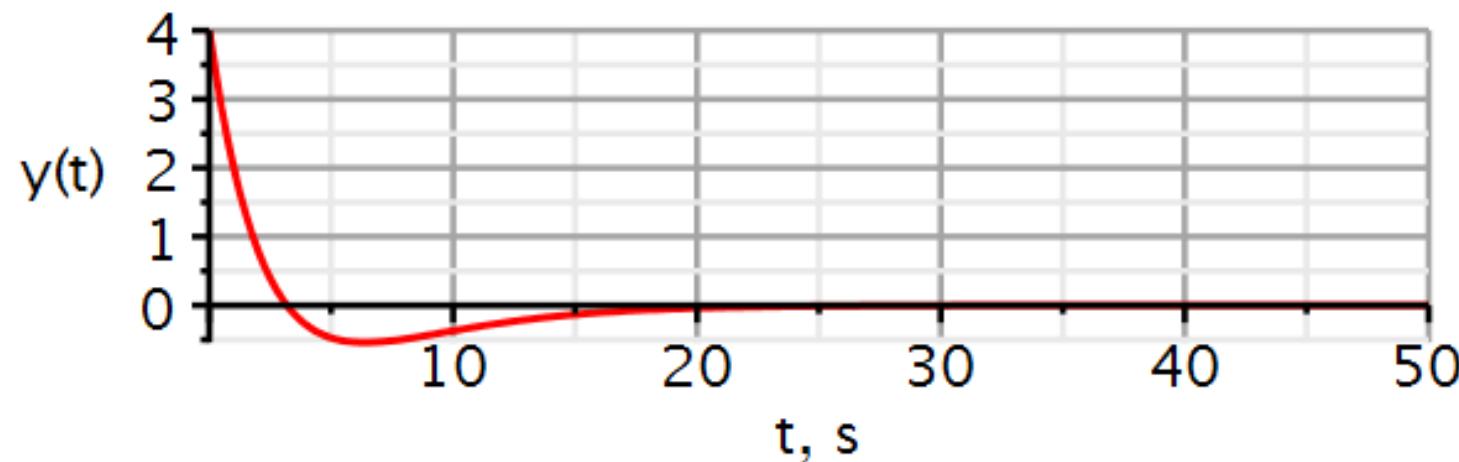
Dynamic systems

$$\text{sysHP} := \text{Coefficients}\left(\frac{b2 \cdot s^2}{s^2 + a1 \cdot s + a0}\right);$$

$$\text{vin} := \text{Step}(2, 0, 0)$$

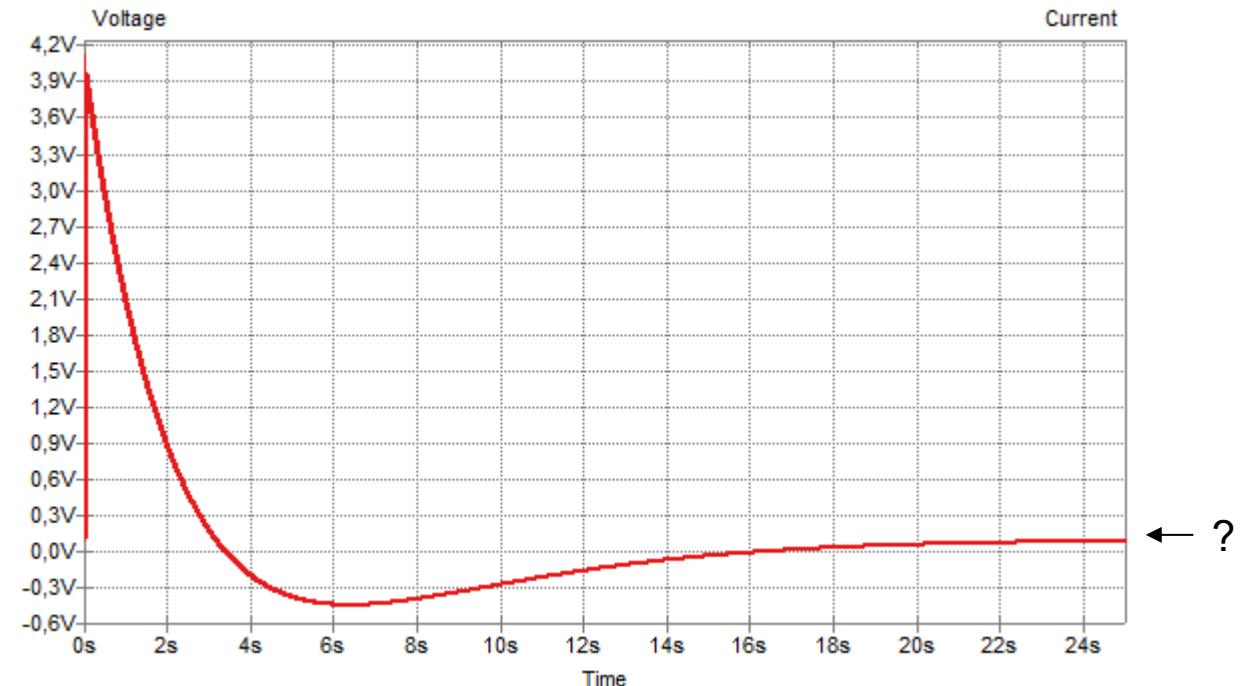
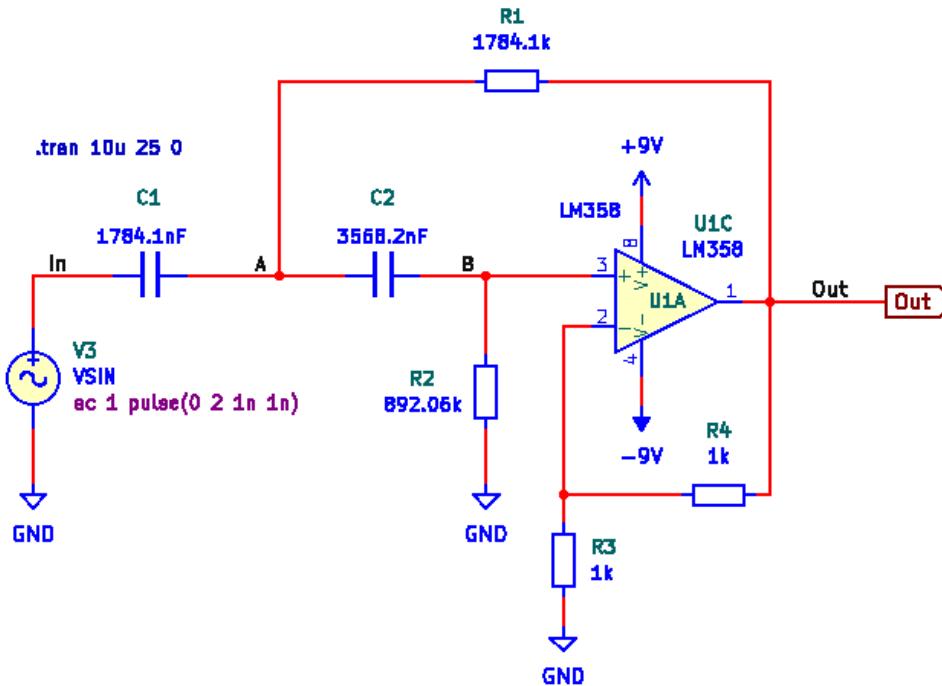
$$\text{vin} := 2 \text{ Heaviside}(t)$$

```
ResponsePlot(sysHP, vin, duration = 50, color = [red], thickness = 3, axesfont = ["Helvetica", "ROMAN", 18], axis[2] = [thickness = 2.5],  
axis[1] = [mode = linear, thickness = 2.5], labels = ["t, s", "y(t)", labelfont = ["HELVETICA", 18], gridlines, size = [600, 200]])
```



Filter 8: Highpass filter – Convolution (sol)

- As another validation, use KiCad/Spice to plot the step response.



Filter 9: Bandpass filter – Convolution (sol)

To solve this problem, you will need the solutions to problems solved in weeks 1 and 2.

$$h(t) = 628.3 e^{-15.7 t} \cos(314 t) u(t) - 31.4 e^{-15.7 t} \sin(314 t) u(t)$$

1. Determine if the bandpass filter is asymptotically stable, marginally stable or unstable.
2. Plot $|H(j\omega)|$ in decibels.
3. Using Maple, apply the convolution integral to calculate the zero-state response to the input $x(t) = 2u(t)$.
4. Use the Maple package *DynamicSystems* to verify the result in (3).
5. As another validation, use KiCad/Spice to plot the step response.

Short answer to (3): $y_{zs}(t) = 4e^{-15.7 t} \sin(313.76 t) u(t) = 4e^{-15.7 t} \cos\left(313.76 t + \frac{\pi}{2}\right) u(t)$

Filter 9: Bandpass filter – Convolution (sol)

1. The poles are all in the left-side plane. Hence the filter is asymptotically stable.
2. Amplitude characteristic:

Transfer function

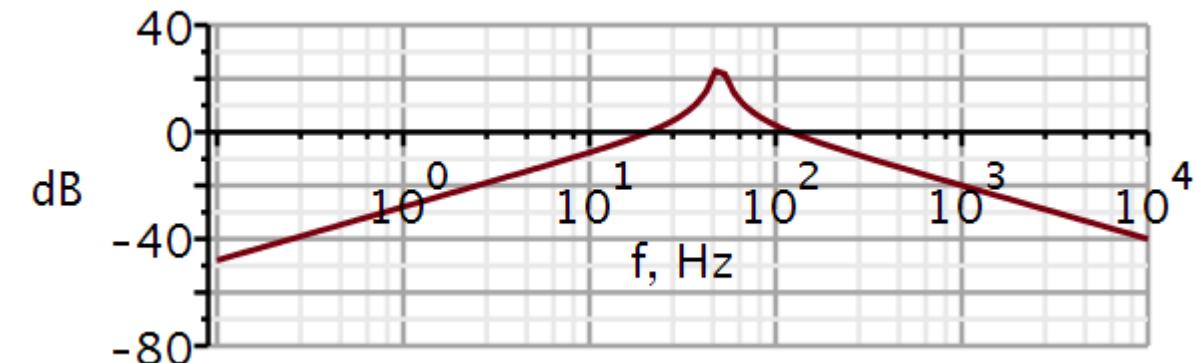
$$H := \omega \rightarrow \frac{b_2 \cdot (j\omega)^2 + b_1 \cdot (j\omega) + b_0}{(j\omega)^2 + a_1 \cdot (j\omega) + a_0} : \quad \# \text{ 2nd order transfer function}$$

$$\text{simplify}(\text{evalf}(H(\omega)))$$

$$\frac{1.96349251 \cdot 10^8 I\omega}{-312500. \omega^2 + 9.814365906 \cdot 10^6 I\omega + 3.084214098 \cdot 10^{10}}$$

$$dB := \omega \rightarrow 20 \cdot \log10(|H(\omega)|) : \quad \# \text{ converts to decibel}$$

```
semilogplot(dB(2 * pi * f), f = 0.1 .. 1E4, -80 .. 40, thickness = 3, axesfont = ["Helvetica", "roman", 18], axis[2] = [thickness = 2.5], axis[1] = [thickness = 2.5], labels = ["f, Hz", "dB"], labelfont = ["HELVETICA", 18], numpoints = 100, gridlines, size = [600, 200])
```



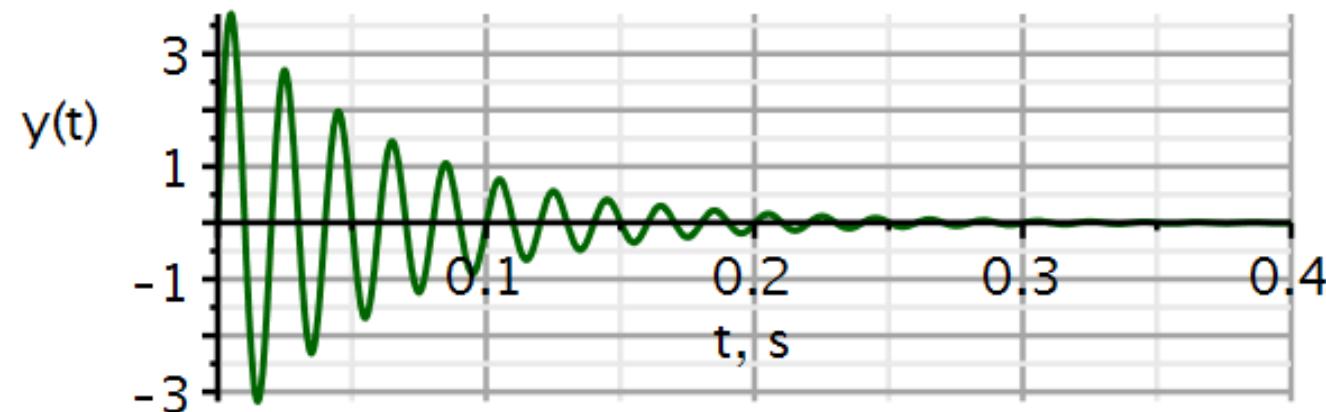
Filter 9: Bandpass filter – Convolution (sol)

3. Zero-state response using convolution:

```

xl := t→2·v(t) : # input signal
hl := t→(-31.447 e-15.703 t sin(313.76 t) + 628.33 e-15.703 t cos(313.76 t)) Heaviside(t) : # impulse response
evalf∫0t xl(τ) · hl(t - τ) dτ, 5
-2.9330 10-6 + 2.9330 10-6 e-15.703 t cos(313.76 t) + 4.0052 e-15.703 t sin(313.76 t)
yI := t→ conv(t→xl(t), t→hl(t))(t) # output signal
yI := t ↦ conv(t ↦ xl(t), t ↦ hl(t))(t)
plot({yI(t)}, t=0..0.4, thickness = 3, color = "DarkGreen", axesfont = ["Helvetica", "ROMAN", 18], axis[2] = [thickness = 2.5], axis[1] = [mode = linear, thickness = 2.5], labels = ["t, s", "y(t)"], labelfont = ["HELVETICA", 18], gridlines, size = [600, 200])

```



Short answer to (3): $y_{zs}(t) = 4e^{-15.7 t} \sin(313.76 t) u(t) = 4e^{-15.7 t} \cos\left(313.76 t + \frac{\pi}{2}\right) u(t)$

Filter 9: Bandpass filter – Convolution (sol)

4. Use the Maple package *DynamicSystems* to verify the result.

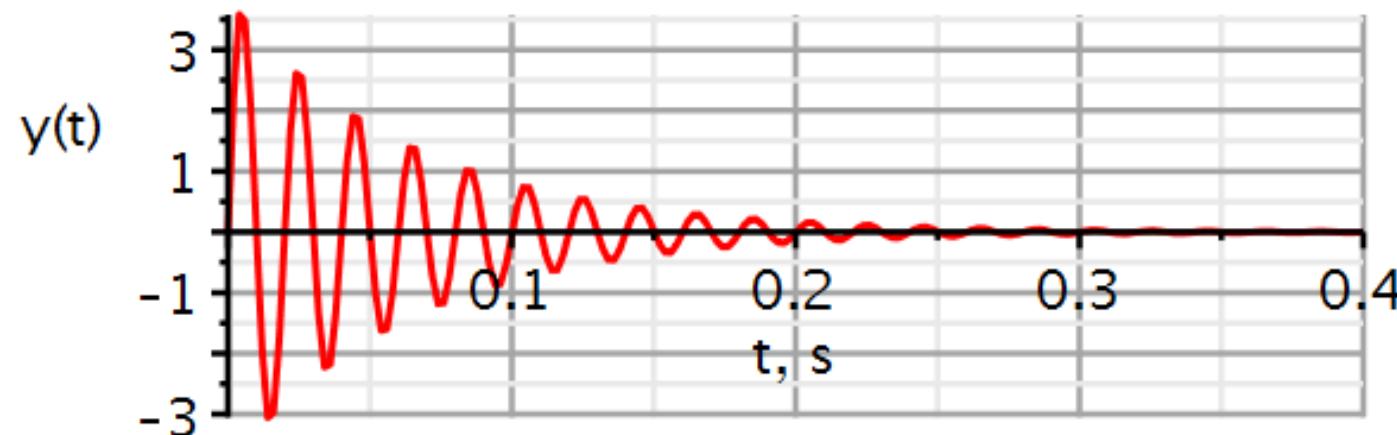
Dynamic systems

$$\text{sysBP} := \text{Coefficients}\left(\frac{b1 \cdot s}{s^2 + a1 \cdot s + a0}\right);$$

$$\text{vin} := \text{Step}(2, 0, 0)$$

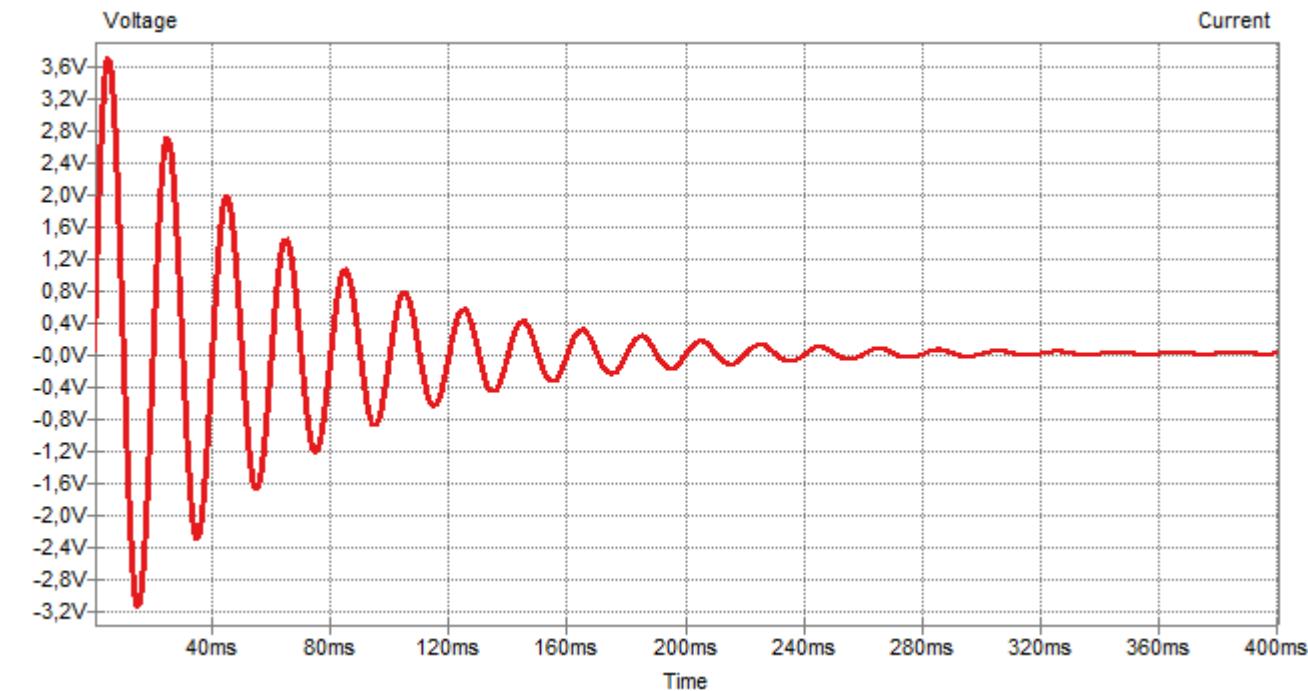
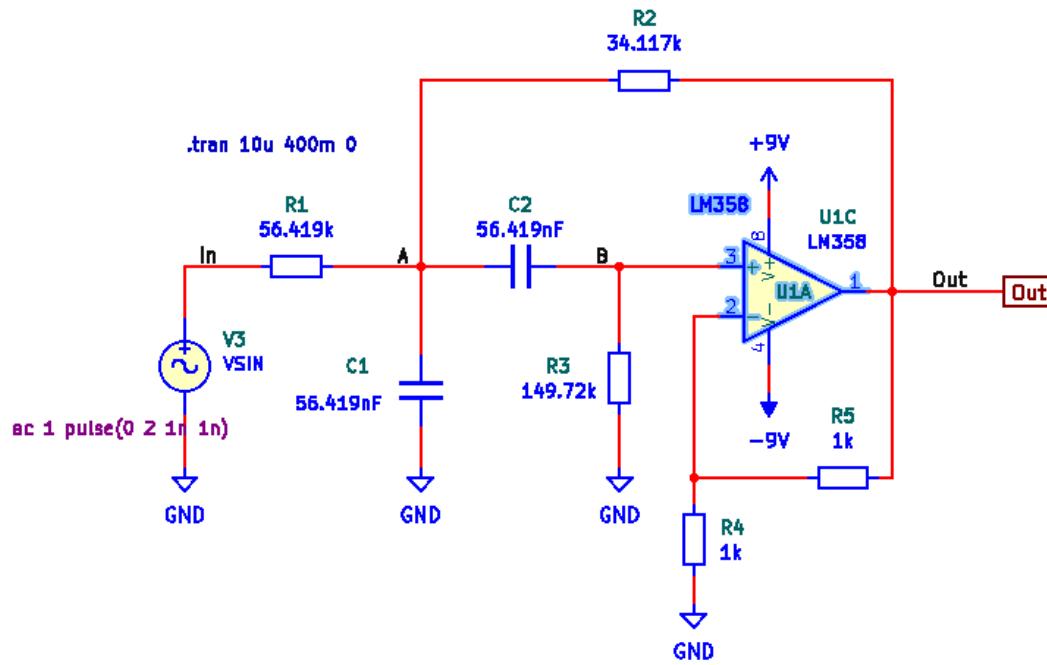
$$\text{vin} := 2 \text{ Heaviside}(t)$$

```
ResponsePlot(sysBP, vin, duration = 0.4, color = [red], thickness = 3, axesfont = ["Helvetica", "ROMAN", 18], axis[2] = [thickness = 2.5],  
axis[1] = [mode = linear, thickness = 2.5], labels = ["t, s" , "y(t)" ], labelfont = ["HELVETICA", 18], gridlines, size = [600, 200])
```



Filter 9: Bandpass filter – Convolution (sol)

- As another validation, use KiCad/Spice to plot the step response.



22050 Continuous-Time Signals and Linear Systems

Kaj-Åge Henneberg

L04

Expansion of signals using orthogonal functions.
Expansion and reconstruction of signals using
Fourier series.

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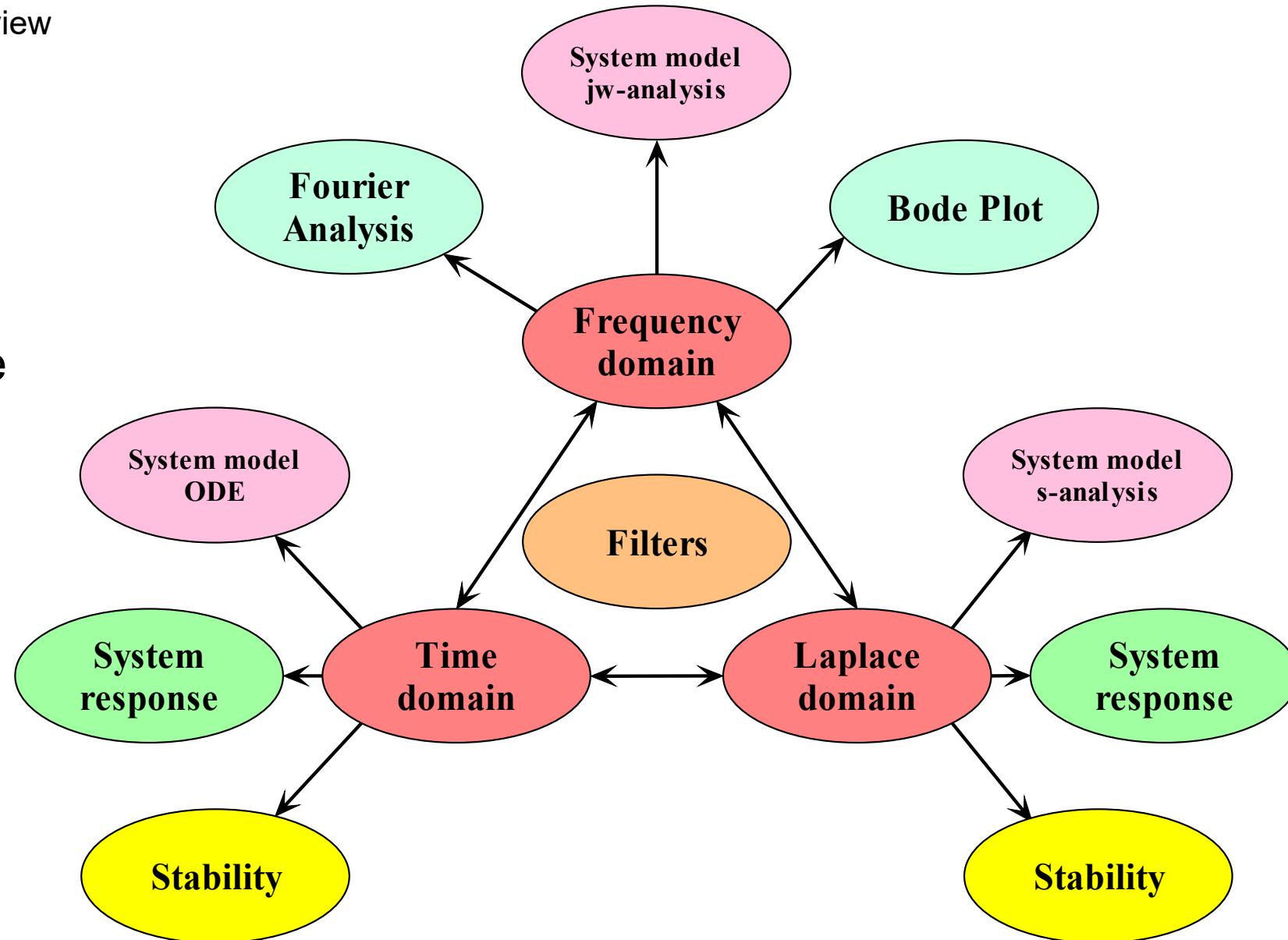
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Course overview

- System
 - Model
 - Response
 - Performance
- Domains
 - Time
 - Frequency
 - s-domain



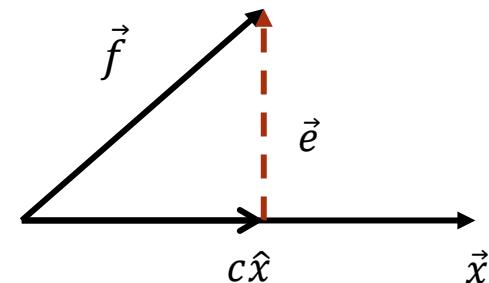
- **Signal representation 3.1 – 3.3**
 - Components of signals
 - Signal correlation and orthogonality
 - Orthogonal signal space
- Trigonometric Fourier Series 3.4 – 3.5
 - Orthogonality of trigonometric basis functions
 - Trigonometric Fourier Series.
 - Compact trigonometric Fourier Series.
 - Conditions for existence.
 - Symmetry in even and odd signals
- Complex exponential Fourier Series 3.6
 - Orthogonality of complex exponential basis functions.
 - Determining coefficients
 - Symmetry properties
 - Negative frequencies
 - Power spectrum

Lathi: Ch. 3.1 – 3.3 incl.

Similarity Video 1

Components of a vector and signals

We approximate a vector \vec{f} by another vector $c\hat{x}$, where $|\hat{x}| = 1$.



$$\vec{f} \approx c\hat{x}$$

$$\vec{e} = \vec{f} - c\hat{x}$$

$$|\vec{e}|^2 = \vec{e} \cdot \vec{e} = (\vec{f} - c\hat{x}) \cdot (\vec{f} - c\hat{x}) \\ (\vec{f} \cdot \vec{f}) + c^2(\hat{x} \cdot \hat{x}) - 2c(\vec{f} \cdot \hat{x})$$

$$\frac{d(|\vec{e}|^2)}{dc} = 0 + 2c|\hat{x}|^2 - 2(\vec{f} \cdot \hat{x}) = 0 \Rightarrow$$

$$c = \frac{\vec{f} \cdot \hat{x}}{|\hat{x}|^2} = \vec{f} \cdot \hat{x}$$

The difference defines the error:

The length of the error vector indicates the magnitude of the error.

We want to choose c so that the error is the smallest possible:

Signal space

$$f(t) \approx cx(t), \quad t_1 \leq t \leq t_2$$

$$e(t) = \begin{cases} f(t) - cx(t) & t_1 \leq t \leq t_2 \\ 0 & \text{otherwise} \end{cases}$$

We optimize the approximation by minimizing the energy of the error signal:

$$E_e = \int_{t_1}^{t_2} e^2(t) dt = \int_{t_1}^{t_2} [f(t) - cx(t)]^2 dt$$

$$\frac{dE_e}{dc} = 0 \Rightarrow$$

$$c = \frac{1}{E_x} \int_{t_1}^{t_2} f(t)x^*(t)dt$$

Derivation on next two slide not lectured. Read for your self.

We will elaborate on this briefly.

$$E_e = \int_{t_1}^{t_2} e^2(t) dt = \int_{t_1}^{t_2} [f(t) - cx(t)]^2 dt$$

Assuming functions to be complex-valued:

$$|u + v|^2 = (u + v)(u^* + v^*) = |u|^2 + |v|^2 + u^*v + uv^*$$

$$u(t) \stackrel{\text{def}}{=} f(t) \quad v(t) \stackrel{\text{def}}{=} -c x(t)$$

$$|f - cx|^2 = (f - cx)(f^* - cx^*) = |f|^2 + c^2|x|^2 - f^*cx - fcx^*$$

We aim to split the energy of the error into several terms, some of which will include the scaling factor c :

We want to determine the value of c which minimizes the energy of the error signal.

$$\begin{aligned} E_e &= \int_{t_1}^{t_2} (|f|^2 + c^2|x|^2 - f^*cx - fcx^*) dt \\ &= \int_{t_1}^{t_2} |f|^2 dt + c^2 \int_{t_1}^{t_2} |x|^2 dt - c \int_{t_1}^{t_2} f^*x dt - c \int_{t_1}^{t_2} fx^* dt \\ &= E_f + c^2 E_x - c \int_{t_1}^{t_2} f^*x dt - c \int_{t_1}^{t_2} fx^* dt \end{aligned}$$

$$E_e = E_f + c^2 E_x - c \int_{t_1}^{t_2} f^* x \, dt - c \int_{t_1}^{t_2} f x^* \, dt + \frac{1}{E_x} \int_{t_1}^{t_2} f x^* \, dt \int_{t_1}^{t_2} f^* x \, dt - \frac{1}{E_x} \int_{t_1}^{t_2} f x^* \, dt \int_{t_1}^{t_2} f^* x \, dt$$

add and subtract

$$E_e = E_f - \left| \frac{1}{\sqrt{E_x}} \int_{t_1}^{t_2} f x^* \, dt \right|^2 + c^2 E_x - c \sqrt{E_x} \frac{1}{\sqrt{E_x}} \int_{t_1}^{t_2} f^* x \, dt - c \sqrt{E_x} \frac{1}{\sqrt{E_x}} \int_{t_1}^{t_2} f x^* \, dt + \frac{1}{E_x} \int_{t_1}^{t_2} f x^* \, dt \int_{t_1}^{t_2} f^* x \, dt$$

$$E_e = E_f - \left| \frac{1}{\sqrt{E_x}} \int_{t_1}^{t_2} f x^* \, dt \right|^2 + \left(c \sqrt{E_x} - \frac{1}{\sqrt{E_x}} \int_{t_1}^{t_2} f x^* \, dt \right) \left(c \sqrt{E_x} - \frac{1}{\sqrt{E_x}} \int_{t_1}^{t_2} f^* x \, dt \right)$$

complex conjugated

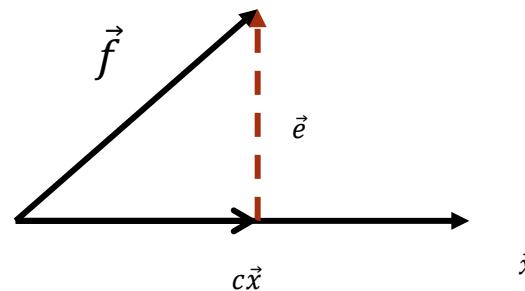
$$E_e = E_f - \left| \frac{1}{\sqrt{E_x}} \int_{t_1}^{t_2} f x^* \, dt \right|^2 + \left| c \sqrt{E_x} - \frac{1}{\sqrt{E_x}} \int_{t_1}^{t_2} f x^* \, dt \right|^2$$

$$\frac{dE_e}{dc} = 0 \Rightarrow$$

$$c = \frac{1}{E_x} \int_{t_1}^{t_2} f(t) x^*(t) dt$$

E_e becomes independent of c if the red term is zero.

Similarity (correlation) of signals



Similarity of vectors:

$$c = \frac{\vec{f} \cdot \hat{x}}{|\hat{x}|^2} = \vec{f} \cdot \hat{x}$$

If c is large and positive, the two vectors are **similar**.

If c is large and negative, the two vectors are **opposite**.

If c is zero, the two vectors have nothing in common. They are **orthogonal**.

A similar measure of similarity can be defined for signals:

$$c = \frac{1}{\sqrt{E_f E_x}} \int_{-\infty}^{\infty} f(t)x^*(t)dt$$

Maximum similarity: $c = 1$

Maximal opposite: $c = -1$

Nothing in common: $c = 0$

In this case the signals are **orthogonal**:

$$\int_{-\infty}^{\infty} f(t)x^*(t)dt = 0$$

Let $z(t)$ be the sum of two complex valued and orthogonal signals:

$$z(t) = x(t) + y(t) \quad x(t) \perp y(t)$$

The energy of $z(t)$ is:

$$\begin{aligned} E_z &= \int_{t_1}^{t_2} |z(t)|^2 dt = \int_{t_1}^{t_2} |x(t) + y(t)|^2 dt \\ &= \int_{t_1}^{t_2} (x(t) + y(t))(x^*(t) + y^*(t)) dt \\ &= \int_{t_1}^{t_2} |x(t)|^2 dt + \int_{t_1}^{t_2} |y(t)|^2 dt + \underbrace{\int_{t_1}^{t_2} x(t)y^*(t) dt}_{=0(\perp)} + \underbrace{\int_{t_1}^{t_2} x^*(t)y(t) dt}_{=0(\perp)} \\ &= E_x + E_y \end{aligned}$$

Observation:

The energy is preserved if we expand a signal in components that are orthogonal.

We have seen that we can use the **inner product** between two signals $x(t)$ and $y(t)$ as a **measure of their similarity**:

Similarity measures are used to detect a signal buried in noise. F. ex. when a **pulse** $x(t)$ is sent out into a medium, and a **reflected echo** $y(t)$ is picked up. The reflected pulse will have a similar shape as the signal sent out but may be distorted due to noise.

Another issue is that the reflected signal $y(t)$ will be delayed compared to the signal $x(t)$ sent out. This means that the two signals may not overlap in time, in which case their product will be zero in the expression showed at the top.

To circumvent this issue, a **correlation function** is defined where the similarity is determined for all possible delays. Here the independent variable t is the time shift of $x(t)$ compared to $y(t)$. At some range of t the shifted $x(t)$ will overlap $y(t)$ and produce a measure of similarity. The degree of similarity is expected to be largest when the two signals overlap completely.

By looking for the similarity, we can find signals buried in noise.

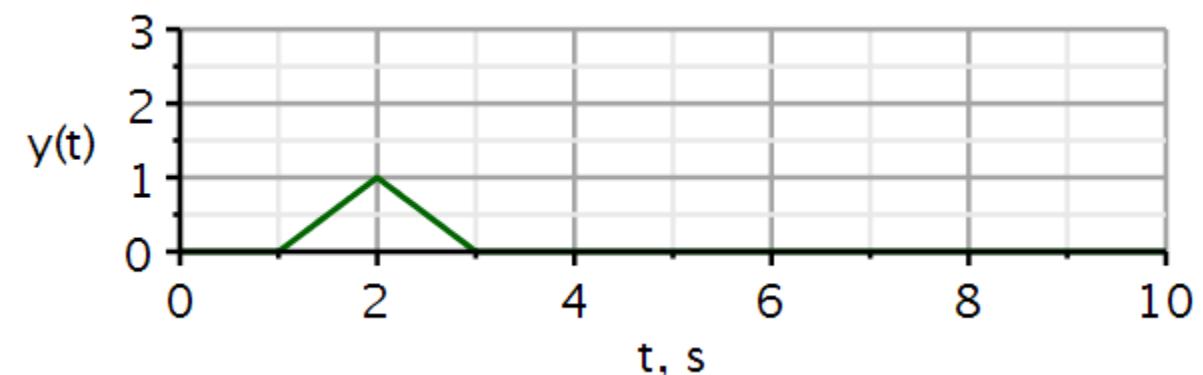
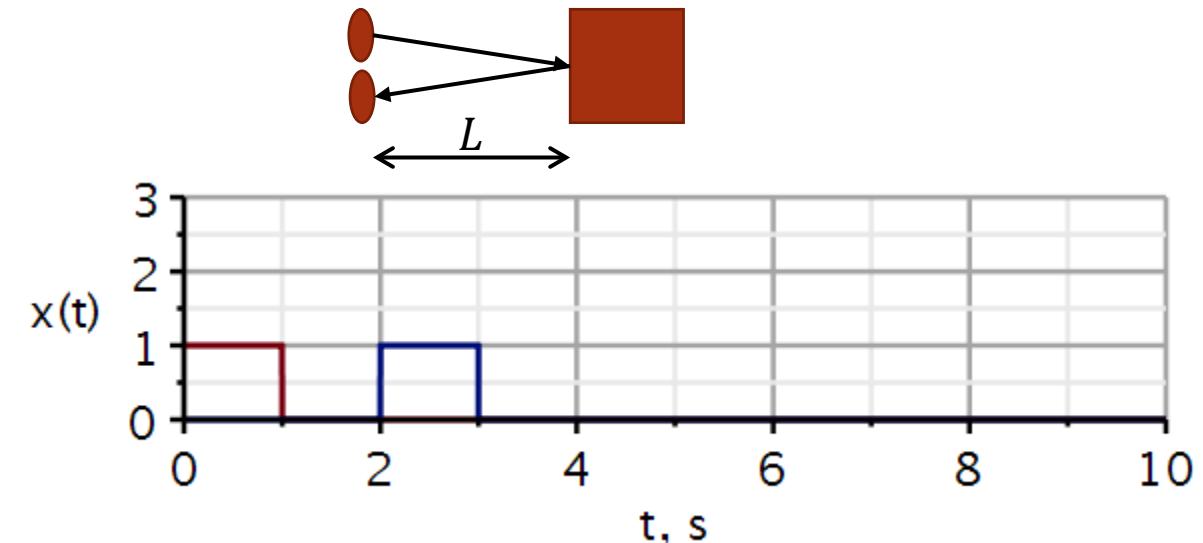
$$\psi = \int_{-\infty}^{\infty} x(t)y^*(t)dt$$

$$\psi_{xy}(t) = \int_{-\infty}^{\infty} x(\tau - t)y^*(\tau)d\tau$$

Correlation, an application of similarity measure

Here a **square pulse (red)** is sent out into a medium, and the **blue pulse is the reflected signal**. We see that the reflected signal takes 2 s to return to the wave detector/transmitter. If we know the wave velocity in the medium, we can calculate the distance between the transmitter and the reflecting object.

Calculating the correlation between the two pulses, we obtain the green curve. It starts at 1 s. This agrees with the observation that we have to shift the red curve 1 s before the red and blue curves start overlapping. At a time-shift of 2 s, the two curves are completely superimposed, giving the highest correlation. From thereon, the overlap will reduce, and the degree of correlation will decrease. Hence the green curve drops off for $t > 2$ s. At a time-shift of 3 s, the red curve no longer overlaps the blue curve, and the correlation is zero.

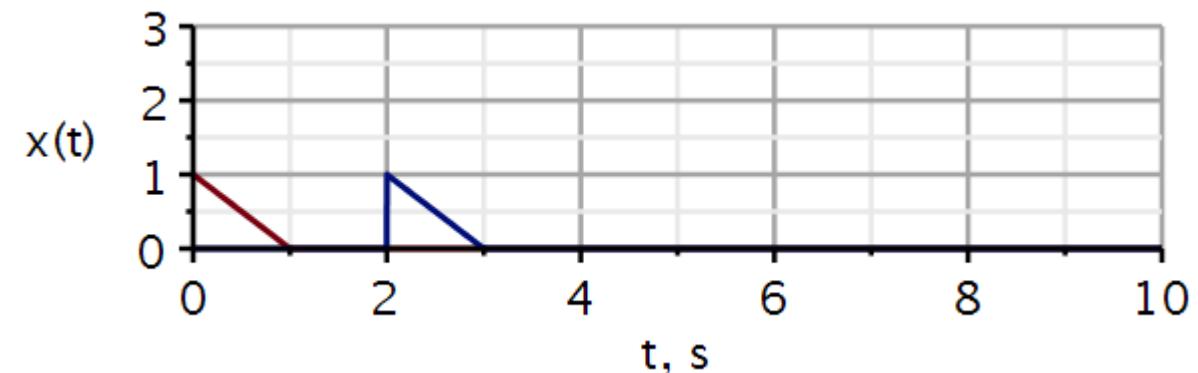


$$L = v \cdot \frac{\Delta t}{2}$$

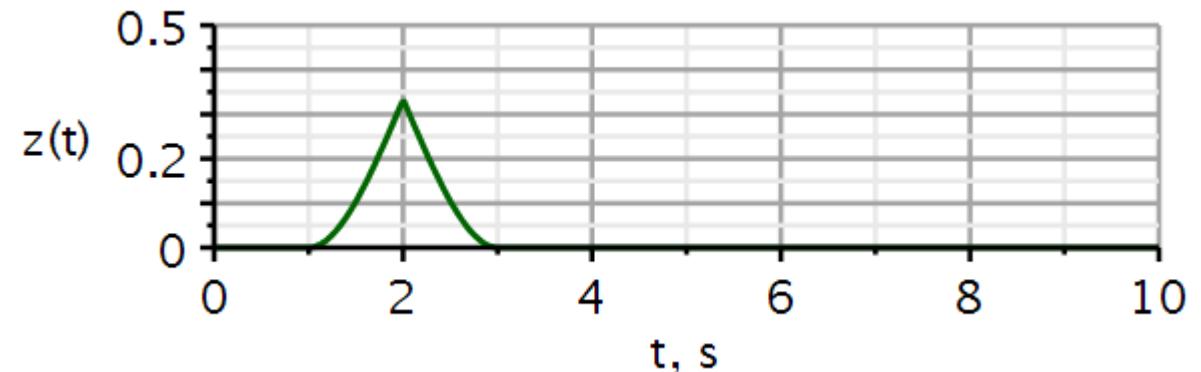
$$L = 1 \frac{m}{s} \cdot \frac{2 \text{ s}}{2} = 1 \text{ m}$$

Correlation, an application of similarity measure

An example like the previous slide. Now the pulse sent out and received is a triangular waveform.



The correlation function indicates clearly that the reflected waveform is delayed 2 s.



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Lathi: Ch. 3.4 – 3.5 incl.

Fourier series

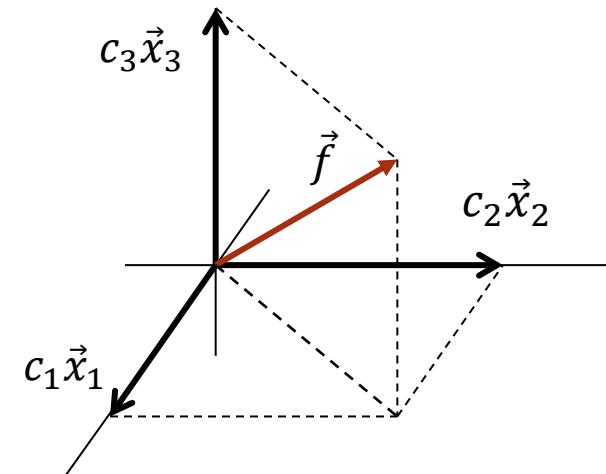
Video 1

Orthogonal expansion in vector space

Orthogonal expansion of vector \vec{f} in terms of a **complete** orthogonal vector basis:

$$\vec{f} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3$$

Determination of component weights:



$$\vec{f} \cdot \vec{x}_1 = \textcolor{red}{c_1} (\vec{x}_1 \cdot \vec{x}_1) + c_2 (\vec{x}_2 \cdot \vec{x}_1) + c_3 (\vec{x}_3 \cdot \vec{x}_1)$$

$$\textcolor{red}{c_1} = \frac{\vec{f} \cdot \vec{x}_1}{\vec{x}_1 \cdot \vec{x}_1}$$

$$\vec{f} \cdot \vec{x}_2 = c_1 (\vec{x}_1 \cdot \vec{x}_2) + \textcolor{red}{c_2} (\vec{x}_2 \cdot \vec{x}_2) + c_3 (\vec{x}_3 \cdot \vec{x}_2)$$

$$\textcolor{red}{c_2} = \frac{\vec{f} \cdot \vec{x}_2}{\vec{x}_2 \cdot \vec{x}_2}$$

$$\vec{f} \cdot \vec{x}_3 = c_1 (\vec{x}_1 \cdot \vec{x}_3) + c_2 (\vec{x}_2 \cdot \vec{x}_3) + \textcolor{red}{c_3} (\vec{x}_3 \cdot \vec{x}_3)$$

$$\textcolor{red}{c_3} = \frac{\vec{f} \cdot \vec{x}_3}{\vec{x}_3 \cdot \vec{x}_3}$$

Orthogonal expansion of signals

We assume that we have a complete set of orthogonal signals (basis signals):

$$\{x_1(t), x_2(t), \dots\} \text{ where } (x_m, x_n) \stackrel{\text{def}}{=} \int_{t_1}^{t_2} x_m(t) x_n^*(t) dt = \begin{cases} 0 & m \neq n (\perp) \\ E_n & m = n (\parallel) \end{cases}$$

By orthogonal expansion of a signal $f(t)$, we understand the representation of $f(t)$ as the weighted sum of orthogonal signals:

$$\begin{aligned} f(t) &= c_1 x_1(t) + c_2 x_2(t) + \dots + c_n x_n(t) + \dots \\ &= \sum_{n=1}^{\infty} c_n x_n(t), \quad t_1 \leq t \leq t_2 \end{aligned}$$

Determination of component weights:

$$(f, x_m) = c_1(x_1, x_m) + c_2(x_2, x_m) + \dots + c_m(x_m, x_m) + \dots$$

$$c_m = \frac{(f, x_m)}{(x_m, x_m)}$$

Signal space

$$c_m = \frac{\vec{f} \cdot \vec{x}_m}{\vec{x}_m \cdot \vec{x}_m}$$

Vector space

The equal sign here must be interpreted carefully.

If we truncate the sum, the error function is the difference between the function and its expansion:

We recall that the coefficients of the expansion are such as to minimize the energy of the error signal:

As more terms are used in the expansion, the energy of the error will be reduced. However, it is the *integral* of a function squared that goes to zero.

This means that the squared error function may be non-zero in a finite set of points, even if the integral is zero. Jump discontinuities are such points.

Parseval's theorem:

$$f(t) = \sum_{n=1}^{\infty} c_n x_n(t), \quad t_1 \leq t \leq t_2$$

$$e(t) = f(t) - \sum_{n=1}^N c_n x_n(t), \quad t_1 \leq t \leq t_2$$

$$E_e = \int_{t_1}^{t_2} |e(t)|^2 dt$$

$$E_e = \int_{t_1}^{t_2} |f(t)|^2 dt - \sum_{n=1}^{\textcolor{red}{N}} c_n^2 E_n$$

Appendix 3A

$$\int_{t_1}^{t_2} |f(t)|^2 dt = \sum_{n=1}^{\infty} c_n^2 E_n$$

For a complete basis set, energy is preserved.

Finality property

Because we are expanding the signal using an orthogonal basis set, we can add more terms to the series expansion without any effect on the previous terms already computed. So, if we want a more accurate approximation, we can just add more terms to the sum we have already computed. We do not have to recalculate all the coefficients.

We would need to do this if we changed an approximation from a first order polynomial fit to a second order polynomial fit.

$$f(t) \approx a_0 + a_1 t$$

$$f(t) \approx b_0 + b_1 t + b_2 t^2$$

$$a_0 \neq b_0, a_1 \neq b_1$$

$$f(t) = \sum_{n=1}^{\infty} c_n x_n(t), \quad t_1 \leq t \leq t_2$$

$$f_{N_1}(t) = \sum_{n=1}^{N_1} c_n x_n(t), \quad t_1 \leq t \leq t_2$$

$$f_{N_2}(t) = \sum_{n=1}^{N_2} c_n x_n(t), \quad t_1 \leq t \leq t_2$$

$$f_{N_2}(t) = f_{N_1}(t) + \sum_{n=N_1+1}^{N_2} c_n x_n(t), \quad t_1 \leq t \leq t_2$$

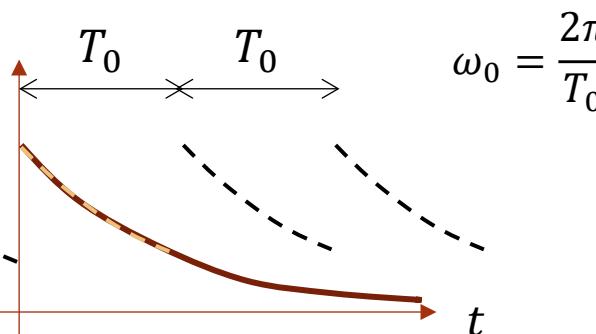
$$c_n = \frac{1}{E_n} \int_{t_1}^{t_2} f(t) x_n^*(t) dt$$

c_n depends only on x_n^* , not on other basis functions and not on other coefficients.

It can be shown that:

The cosine and sine functions form **orthogonal sets**, and we can expand signals in such sets.

The angular frequency ω_0 is the fundamental frequency, having the time period T_0 corresponding to the time duration of the signal we want to expand.



$$\int_{t_1}^{t_1+T_0} \cos(n\omega_0 t) \cos(m\omega_0 t) dt = \begin{cases} 0 & n \neq m (\perp) \\ \frac{T_0}{2} & n = m (\parallel) \end{cases}$$

$$\int_{t_1}^{t_1+T_0} \sin(n\omega_0 t) \sin(m\omega_0 t) dt = \begin{cases} 0 & n \neq m (\perp) \\ \frac{T_0}{2} & n = m (\parallel) \end{cases}$$

$$\int_{t_1}^{t_1+T_0} \sin(n\omega_0 t) \cos(m\omega_0 t) dt = 0, \text{ for all } n \text{ and } m (\perp)$$

$$\underbrace{\begin{aligned} \tilde{f}(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t), \\ &\text{periodic} \end{aligned}}_{\text{aperiodic}}$$

Here we expand $f(t)$ in the time interval from $t = 0$ to $t = T_0$. Outside this interval, the two signals are not equal.

By projecting the signal onto each basis function, we can calculate the coefficient for each basis function:

$$c_m = \frac{(f, x_m)}{(x_m, x_m)}$$

Fundamental frequency:

That frequency ω_0 which has the period T_0 .

Harmonics:

All the other components. Their frequencies are integer multiples of the fundamental frequency. Hence, they all have an integer number of periods with the time interval T_0 .

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) dt : \text{average value within one period}$$

$$a_n = \frac{\int_{t_1}^{t_1+T_0} f(t) \cos(n\omega_0 t) dt}{\int_{t_1}^{t_1+T_0} \cos^2(n\omega_0 t) dt} = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \cos(n\omega_0 t) dt, n = 1, 2, 3, \dots$$

$$b_n = \frac{\int_{t_1}^{t_1+T_0} f(t) \sin(n\omega_0 t) dt}{\int_{t_1}^{t_1+T_0} \sin^2(n\omega_0 t) dt} = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \sin(n\omega_0 t) dt, n = 1, 2, 3, \dots$$

$$\text{ratio of harmonic frequencies} = \frac{n \omega_0}{m \omega_0} = \frac{n}{m} = \text{a rational number}$$

Compact trigonometric Fourier Series

The sum of a cosine and a sine with the same argument can be rewritten as a cosine with a phase angle θ :

We can verify that trigonometric Fourier Series are periodic:

Hence, even if $f(t)$ is aperiodic, the Fourier expansion over a time interval T_0 will be periodic with period T_0 .

If a periodic signal $f(t)$ with period T_1 is expanded over the duration T_0 then the Fourier Series will have a period of T_0 , while $f(t)$ has the period T_1 . In this case, we would normally choose $T_0 = T_1$ such that the expansion has the same period as the periodic signal.

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \quad \text{Compact form}$$

$$C_0 = a_0 \quad C_n = \sqrt{a_n^2 + b_n^2} \quad \theta_n = \tan^{-1} \left(-\frac{b_n}{a_n} \right)$$

$$f(t + T_0) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0(t + T_0) + \theta_n)$$

$$f(t + T_0) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + n\omega_0 T_0 + \theta_n)$$

$$f(t + T_0) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + n2\pi + \theta_n) = f(t)$$

Trigonometric Fourier Series

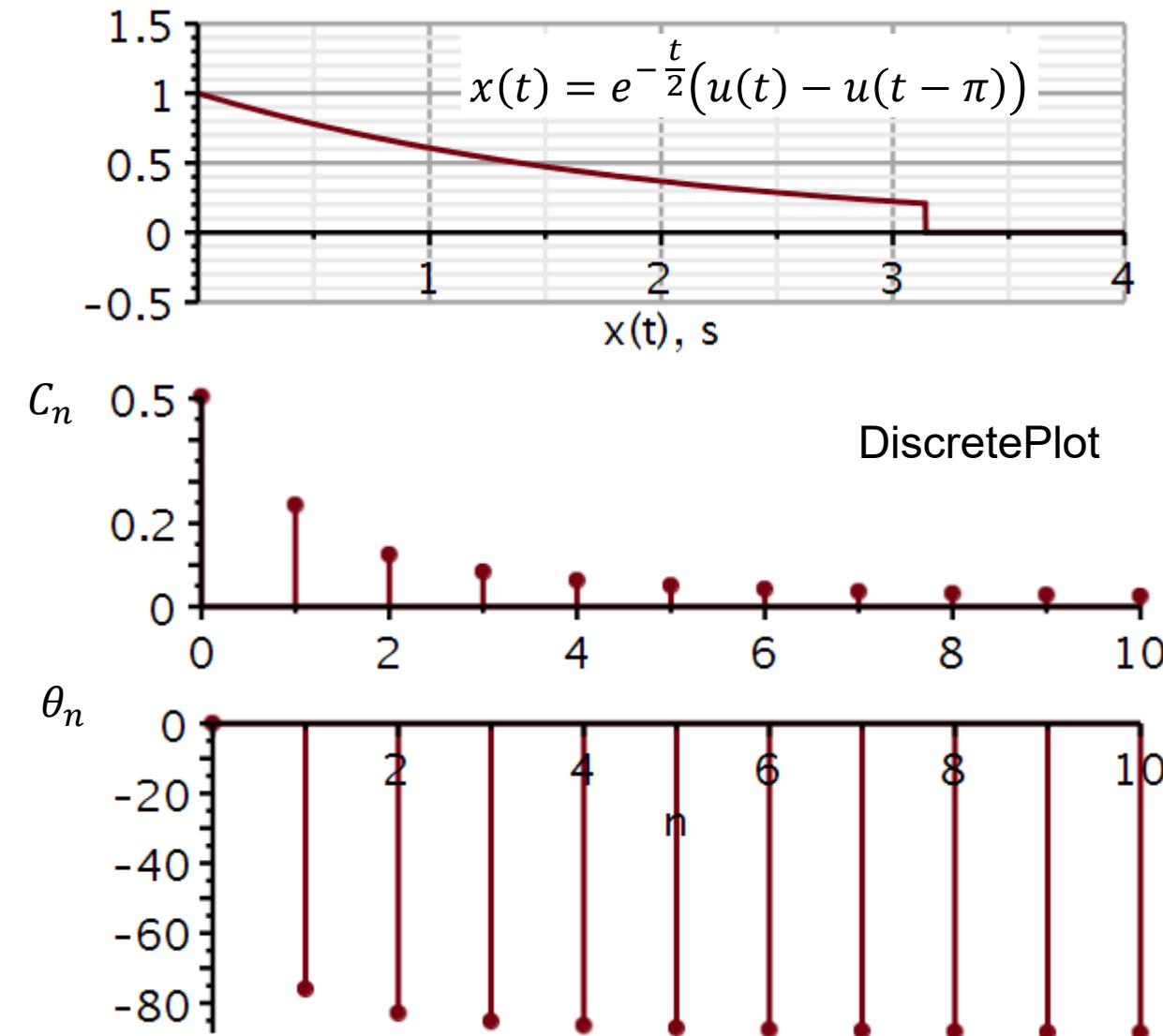
Plotting the Fourier Series coefficients is an informative way of comparing signals. Plotting the a_n and b_n coefficients is not illuminating. However, plotting the coefficients of the *compact form* is very illuminating.

Example: Lathi example 3.5 (3.3 in old book)

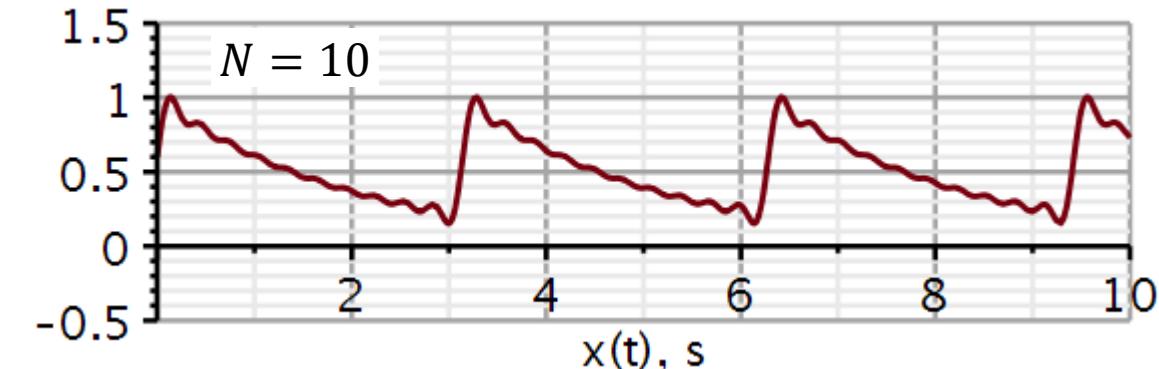
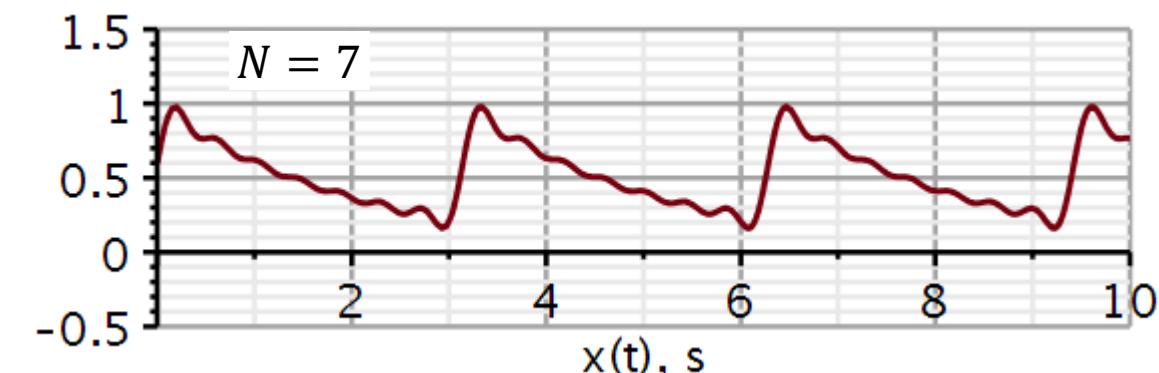
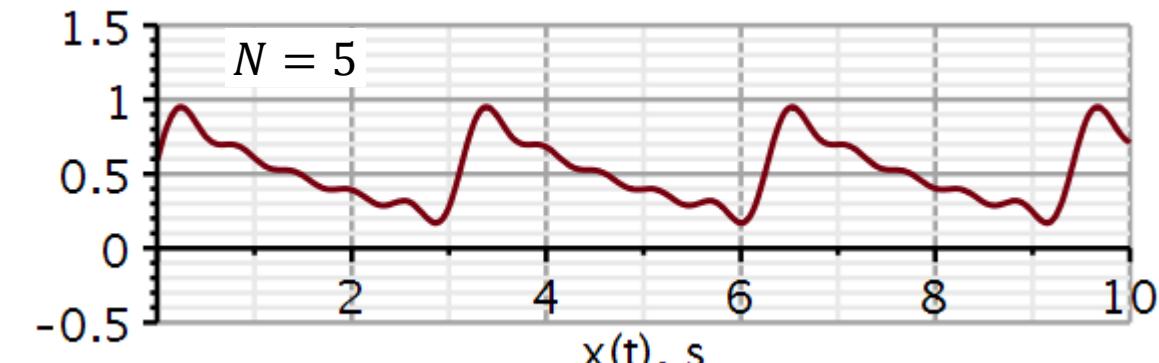
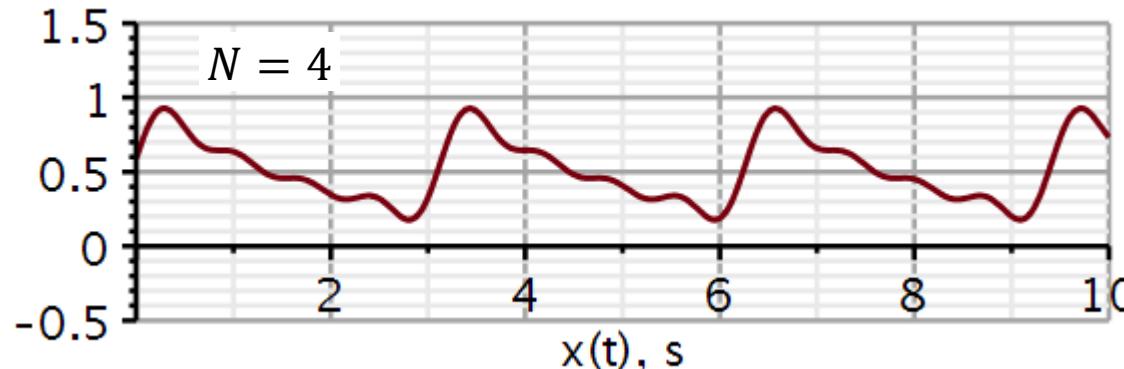
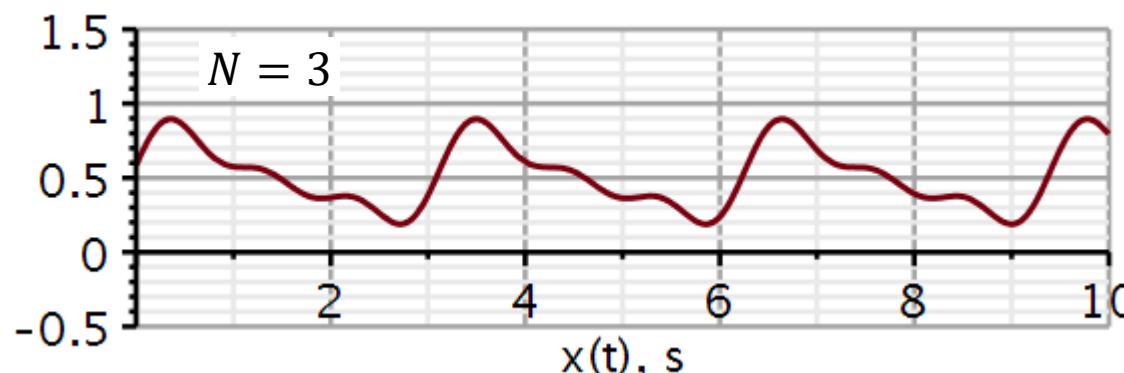
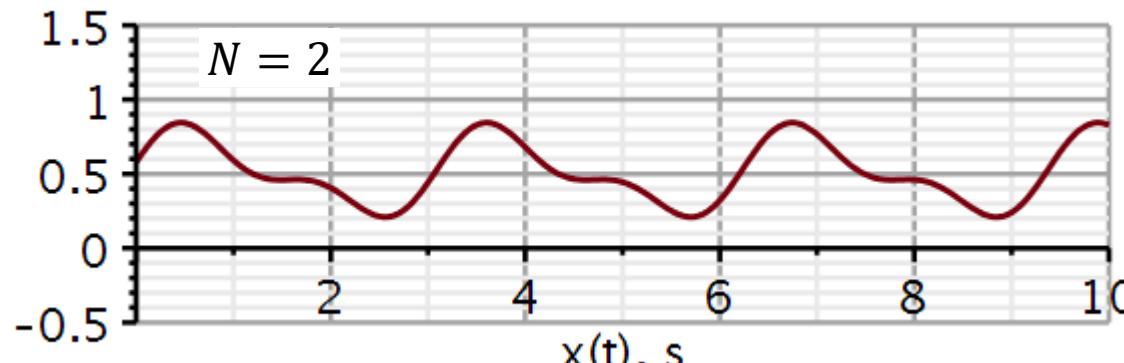
$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

The plot with C_n is the **magnitude plot** and the plot with θ_n is the **phase plot**.

The next slide shows the reconstruction of $x(t)$ using an increasing number of terms in the sum.



Reconstruction of exponential function using trigonometric Fourier Series expansion



The existence of a Fourier Series expansion of a signal $f(t)$ requires that the integrals on the right converge.

$$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) dt : \text{average value within one period}$$

$$a_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \cos n\omega_0 t dt, n = 1, 2, 3, \dots$$

$$b_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \sin n\omega_0 t dt, n = 1, 2, 3, \dots$$

A condition for convergence is that the signal has finite energy:

Must have a finite number of minima and maxima in one period and a finite number of discontinuities in one period.

$$\int_{t_1}^{t_1+T_0} |f(t)|^2 dt < \infty \quad \text{Weak Dirichlet condition}$$

$$\lim_{n \rightarrow \infty} a_n = 0$$

$$\lim_{n \rightarrow \infty} b_n = 0$$

Strong Dirichlet condition

Symmetry in even and odd signals

If $f(t)$ is an odd function in the period, then $a_0 = 0$.

A cosine is an even function. $odd \times even = odd$, hence $a_n = 0$ if $f(t)$ is an odd function.

A sine is an odd function. $even \times odd = odd$, hence $b_n = 0$ if $f(t)$ is an even function.

$$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) dt = \begin{cases} 0 & f \text{ odd} \\ \neq 0 & f \text{ even} \end{cases}$$

$$a_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \cos n\omega_0 t dt = \begin{cases} 0 & f \text{ odd} \\ \neq 0 & f \text{ even} \end{cases}$$

$$b_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \sin n\omega_0 t dt = \begin{cases} 0 & f \text{ even} \\ \neq 0 & f \text{ odd} \end{cases}$$

- Signal representation 3.1 – 3.3
 - Components of signals
 - Signal correlation and orthogonality
 - Orthogonal signal space
- Trigonometric Fourier Series 3.4 – 3.5
 - Orthogonality of trigonometric basis functions
 - Trigonometric Fourier Series.
 - Compact trigonometric Fourier Series.
 - Conditions for existence.
 - Symmetry in even and odd signals
- Complex exponential Fourier Series 3.6
 - Orthogonality of complex exponential basis functions.
 - Determining coefficients
 - Symmetry properties
 - Negative frequencies
 - Power spectrum

Lathi: Ch. 3.6

Complex exponential Fourier series Video 2

Using an Euler identity, we can rewrite the Fourier series expansion into one based on complex exponentials:

$$D_n = \frac{C_n}{2} e^{j\theta_n} = \frac{C_n}{2} (\cos \theta_n + j \sin \theta_n)$$

$$D_{-n} = \frac{C_n}{2} e^{-j\theta_n} = \frac{C_n}{2} (\cos \theta_n - j \sin \theta_n)$$

$$D_n = D_{-n}^*$$

We observe that the coefficients of the complex exponential Fourier series are complex valued.

We also observe that the magnitude of D_n is an even function of n and the phase an odd function of n .

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$\begin{aligned} C_n \cos(n\omega_0 t + \theta_n) &= C_n \left(\frac{e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}}{2} \right) \\ &= \frac{C_n}{2} (e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}) \\ &= \frac{C_n}{2} e^{j\theta_n} e^{jn\omega_0 t} + \frac{C_n}{2} e^{-j\theta_n} e^{-jn\omega_0 t} \\ &= D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t} \end{aligned}$$

$$\begin{aligned} x(t) &= C_0 + \sum_{n=1}^{\infty} (D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t}) \\ &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \end{aligned}$$

Complex exponentials as basis signals

Using complex exponentials, the basis functions are:

$$\phi_n(t) = e^{jn\omega_0 t}, \quad (n = 0, \pm 1, \pm 2, \dots)$$

We expand a periodic signal with period T_0 :

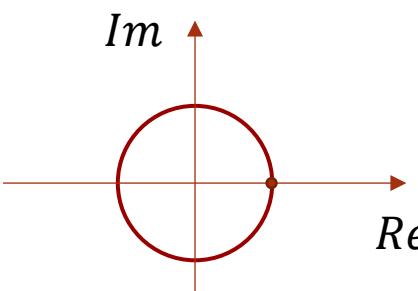
$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}, T_0 = 2\pi/\omega_0$$

The expansion is only useful if the basis signals are orthogonal:

$$I = \int_{t_1}^{t_1+T_0} e^{jm\omega_0 t} (e^{jn\omega_0 t})^* dt = \int_{t_1}^{t_1+T_0} e^{j(m-n)\omega_0 t} dt$$

When $m = n$: $I = \int_{t_1}^{t_1+T_0} e^0 dt = [t]_{t_1}^{t_1+T_0} = T_0$

When $m \neq n$:
$$\begin{aligned} I &= \frac{1}{j(m-n)\omega_0} e^{j(m-n)\omega_0 t} \Big|_{t_1}^{t_1+T_0} \\ &= \frac{1}{j(m-n)\omega_0} e^{j(m-n)\omega_0 t_1} [e^{j(m-n)\omega_0 T_0} - 1] \\ &= \frac{1}{j(m-n)\omega_0} e^{j(m-n)\omega_0 t_1} \left[e^{j(m-n)\omega_0 \frac{2\pi}{\omega_0}} - 1 \right] = 0 \end{aligned}$$
 Why?



Determining expansion coefficients

The complex exponentials form a set of **orthogonal basis signals** over the time-period T_0 .

Fourier series expansion in complex exponentials:

To determine the weights D_n , we form the inner product with the n 'th basis signal:

We have seen that the coefficients D_n can be obtained from C_n from the trigonometric Fourier series. But we can also compute it using the method of projection as shown here.

Coefficients of complex exponential Fourier series:

$$\int_{t_1}^{t_1+T_0} e^{jm\omega_0 t} (e^{jn\omega_0 t})^* dt = \begin{cases} 0 & m \neq n (\perp) \\ T_0 & m = n (\parallel) \end{cases}$$

$$x(t) = \sum_{m=-\infty}^{\infty} D_m e^{jm\omega_0 t}, t_1 \leq t \leq t_1 + T_0, T_0 = \frac{2\pi}{\omega_0}$$

$$\begin{aligned} (x, e^{-jn\omega_0 t}) &= \int_{t_1}^{t_1+T_0} x(t) e^{-jn\omega_0 t} dt & (e^{jn\omega_0 t})^* &= e^{-jn\omega_0 t} \\ &= \int_{t_1}^{t_1+T_0} \sum_{m=-\infty}^{\infty} D_m e^{jm\omega_0 t} e^{-jn\omega_0 t} dt \\ &= \sum_{m=-\infty}^{\infty} D_m \int_{t_1}^{t_1+T_0} e^{jm\omega_0 t} e^{-jn\omega_0 t} dt = D_n T_0 \end{aligned}$$

$$D_n = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} x(t) e^{-jn\omega_0 t} dt$$

- Fourier synthesis

- We construct a signal from orthogonal basis signals

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}, t_1 \leq t \leq t_1 + T_0, T_0 = \frac{2\pi}{\omega_0}$$

- Fourier analysis

- We break the signal up into its components and determine the complex-valued coefficient of each.

$$D_n = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} x(t) e^{-jn\omega_0 t} dt$$

To elucidate the symmetry of the coefficients D_n we expand the **REAL-valued** signal $x(t)$ into its **even** and **odd** components.

We observe, that

- The real part of D_n originates from the even part of the signal.
- The imaginary part of D_n originates from the odd component.

Hence

- If the signal is even, D_n has zero imaginary part.
- If the signal is odd, D_n has zero real part.

$$D_n = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} x(t) e^{-jn\omega_0 t} dt$$

$$\begin{aligned} D_n &= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (x_e(t) + x_o(t))(\cos(n\omega_0 t) - j \sin(n\omega_0 t)) dt \\ &= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (x_e(t))(\cos(n\omega_0 t)) dt && \text{real} \\ &\quad + \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (x_e(t))(-j \sin(n\omega_0 t)) dt && = 0 \\ &\quad + \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (x_o(t))(\cos(n\omega_0 t)) dt && = 0 \\ &\quad + \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (x_o(t))(-j \sin(n\omega_0 t)) dt && \text{imaginary} \end{aligned}$$

We furthermore observe, that

- The real part of D_n is an even function of n .
- The imaginary part of D_n is an odd function of n .

This means that:

- The magnitude of D_n is an even function of n .
- The phase of D_n is an odd function of n .

For complex-valued signals twice as many combinations apply.

Always have these symmetries in mind when forecasting what to expect when you calculate Fourier series coefficients.

$$D_n = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} x(t) e^{-jn\omega_0 t} dt$$

$$\begin{aligned} D_n &= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (x_e(t) + x_o(t)) (\cos(n\omega_0 t) - j \sin(n\omega_0 t)) dt \\ &= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (x_e(t)) (\cos(n\omega_0 t)) dt && \text{even in } n \\ &\quad + \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (x_e(t)) (-j \sin(n\omega_0 t)) dt && = 0 \\ &\quad + \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (x_o(t)) (\cos(n\omega_0 t)) dt && = 0 \\ &\quad + \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (x_o(t)) (-j \sin(n\omega_0 t)) dt && \text{odd in } n \end{aligned}$$

Examples

We will use Maple to carry out a Fourier Series expansion and reconstruction of a periodic square pulse signal.

Including the **DynamicSystems** package allows us to use certain predefined signals, here the `Square()` signal.

```
restart  
with(DynamicSystems) :  
with(plots) :
```

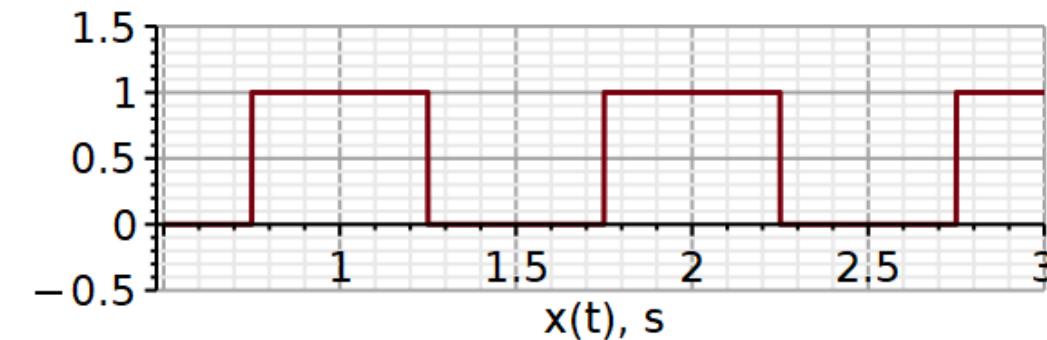
Time domain signal

```
x := t → Square(1, 2π · 1, 1/2, -1/4) :
```

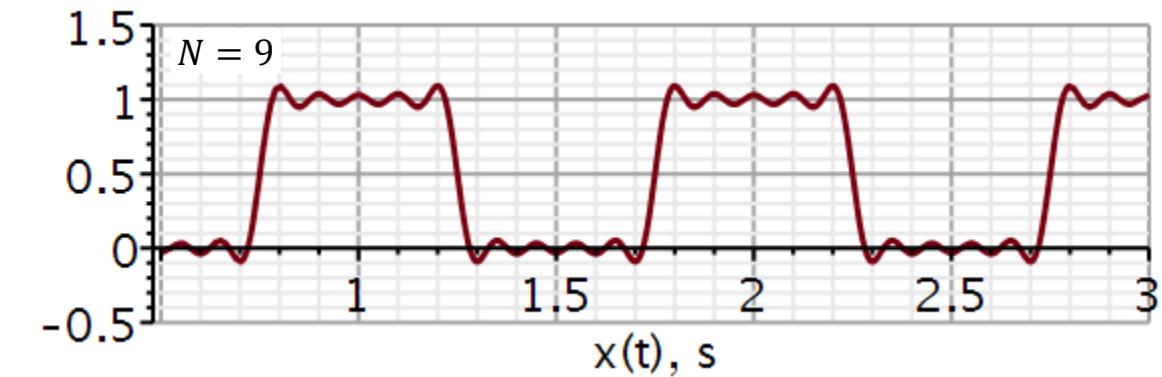
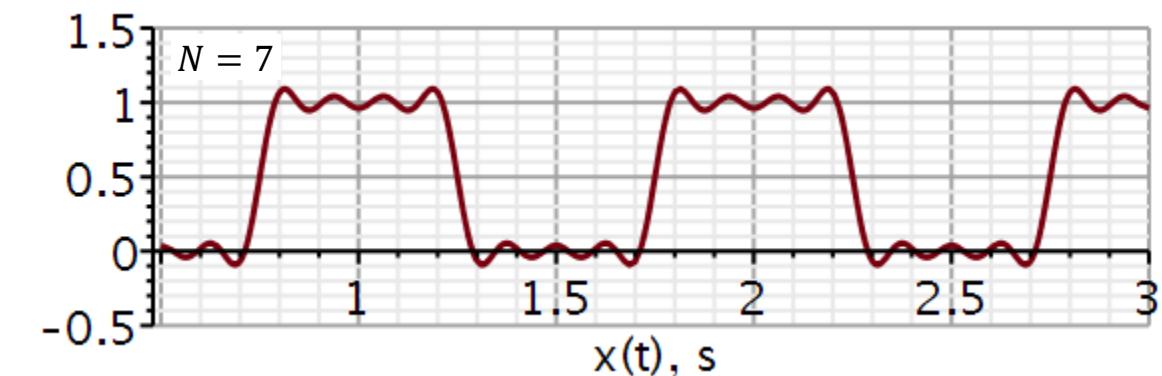
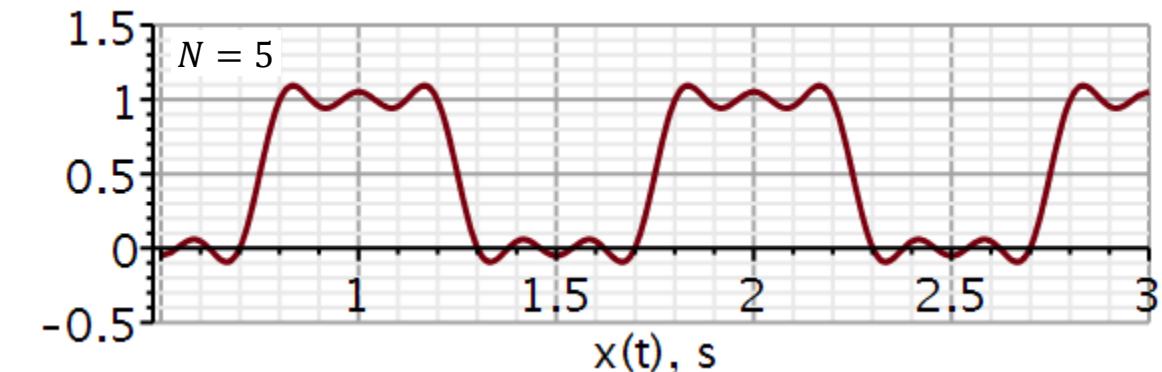
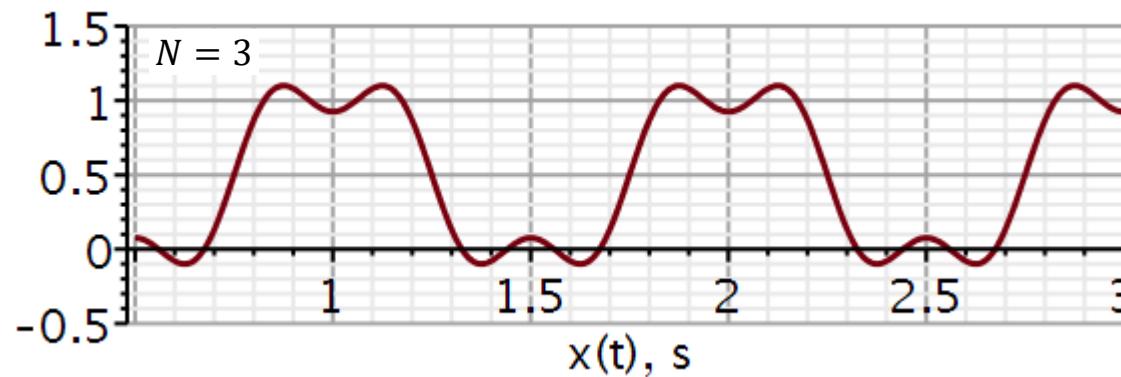
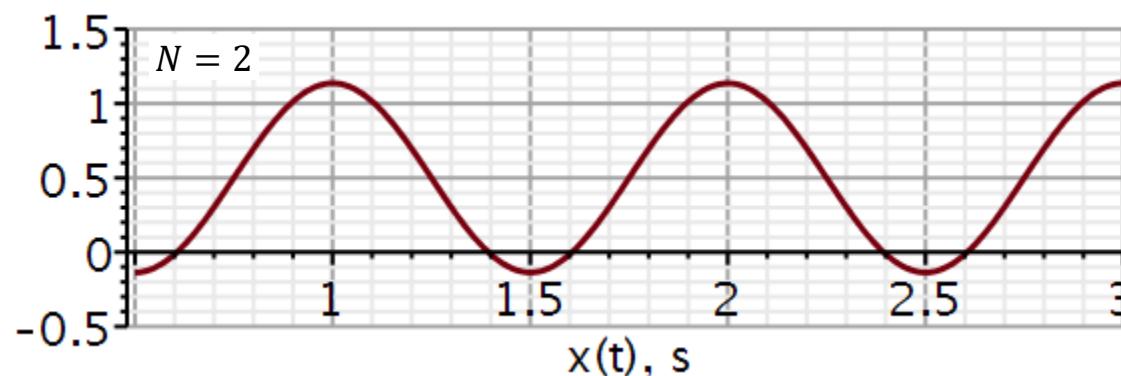
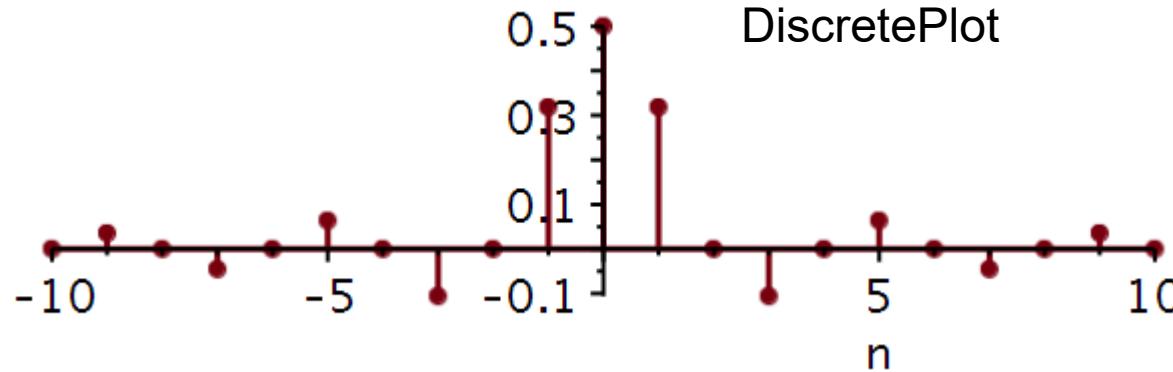
Height
Frequency
Duty cycle
Delay

Plotting signal

```
plot(x(t), t = 0.5 .. 3.0, -0.5 .. 1.5, thickness = 3, axesfont = [Helvetica, "roman", 18],  
axis[2] = [thickness = 2.5], axis[1] = [thickness = 2.5], labels = ["x(t), s",  
" "], labelfont = [Helvetica, 18], numpoints = 100, gridlines, size = [600, 200])
```



Examples



Power spectrum

A cosine with amplitude A has the power $A^2/2$.

We can extend this to the trigonometric Fourier series:

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$P_f = C_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} C_n^2$$

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$D_n = \frac{C_n}{2} e^{j\theta_n} \quad |D_n|^2 = \frac{C_n^2}{4}$$

$$P_f = D_0^2 + 2 \sum_{n=1}^{\infty} |D_n|^2$$

In a **power spectrum plot**, usually only positive frequency stems are plotted. However, the magnitude is doubled (except DC) to include the power of the negative frequencies.

Learning priorities from Problem Solving:

1. Why this method?
2. When is this method valid?
3. How is this method used?
4. What is the result?
5. How do I validate the result?

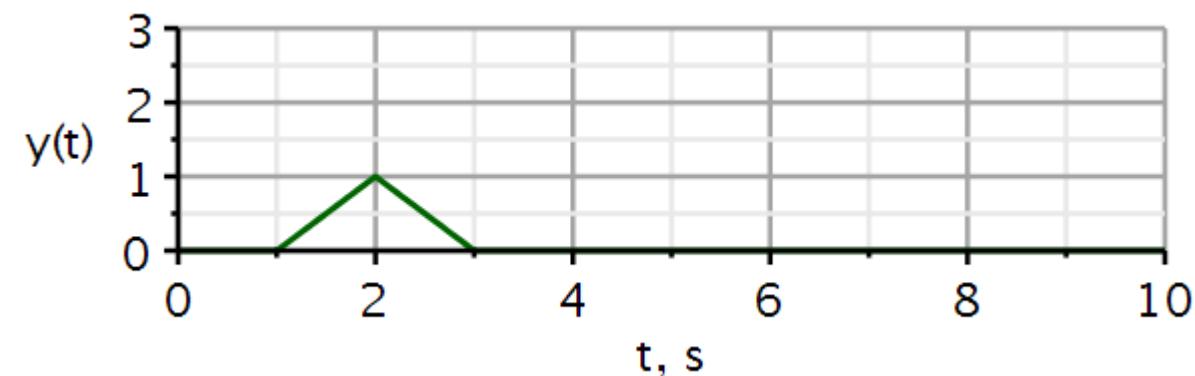
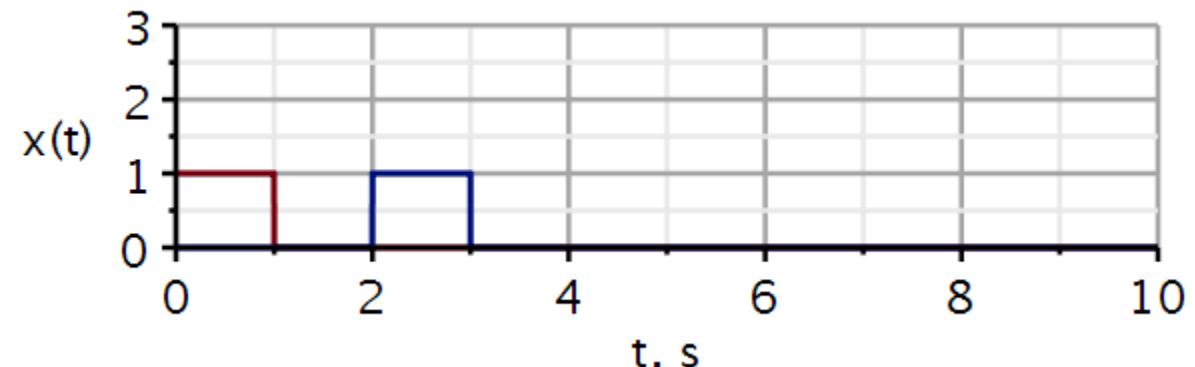
Problems

Problem 1

Here a square pulse (red) is sent out into a medium, and the blue pulse is the reflected signal. We see that the reflected signal takes 2 s to return to the wave detector/transmitter. If we know the wave velocity in the medium, we can calculate the distance between the transmitter and the reflecting object.

Use Maple and:

1. Define the two signals. Hint: use combinations of step functions.
2. Plot them in one graph.
3. Define a correlation integral.
4. Use it to plot the correlation $y(t)$.



If needed, see hints on next slide.

Problem 1 Hint

 $v := t \rightarrow \text{Heaviside}(t) :$
 $\text{corr} := (x, y) \rightarrow t \rightarrow \text{int}(x(\tau - t) * y(\tau), \tau = -10 .. 10);$
 $\text{corr} := (x, y) \mapsto t \mapsto \int_{-10}^{10} x(\tau - t) \cdot y(\tau) d\tau$

x: one signal

y: another signal

t: the time delay of x to make x overlap y

Fixed integration interval. Must be increased if signals extend beyond that.

x1: signal sent out

y1: signal reflected

z1: correlation signal indicating similarity and time delay between the two signals

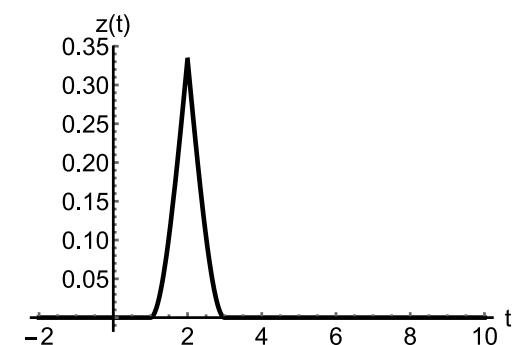
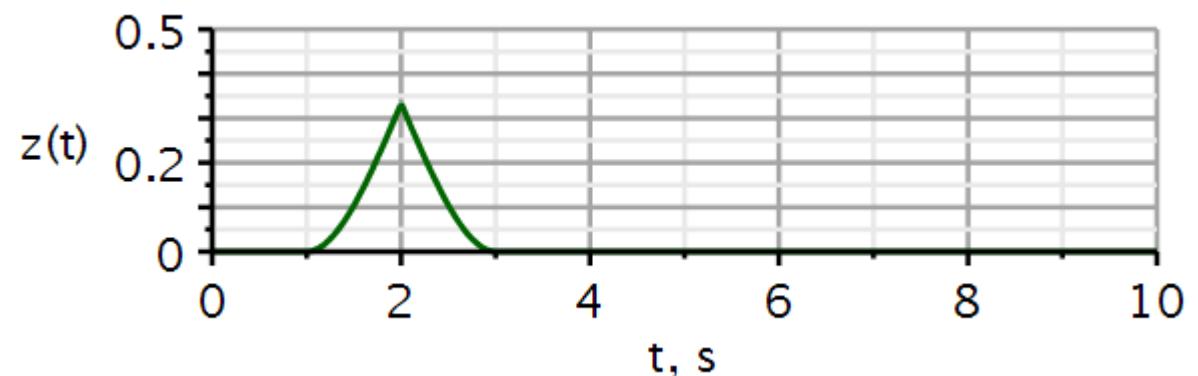
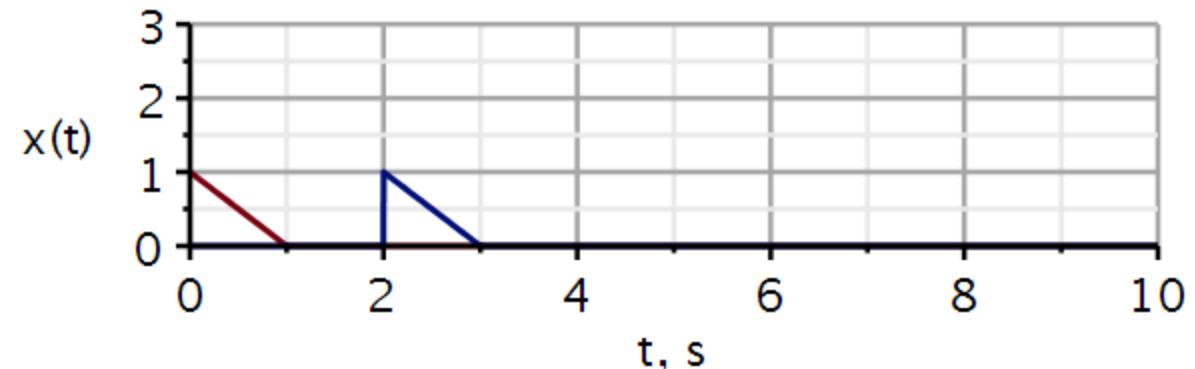
 $x1 := t \rightarrow (v(t) - v(t - 1)) : \quad \# \text{ signal sent out}$
 $y1 := t \rightarrow (v(t - 2) - v(t - 3)) : \quad \# \text{ signal returned}$
 $z1 := t \rightarrow \text{corr}(t \rightarrow x1(t), t \rightarrow y1(t))(t) : \quad \# \text{ correlation signal}$

Problem 2

An example like the previous slide. Now the pulse sent out and received is a triangular waveform.

Use Maple and:

1. Define the two signals.
2. Plot them in one graph.
3. Define a correlation integral.
4. Use it to plot the correlation $z(t)$.



In Mathematica

Problem 3

Example: Lathi example 3.5 (3.3 in old book)

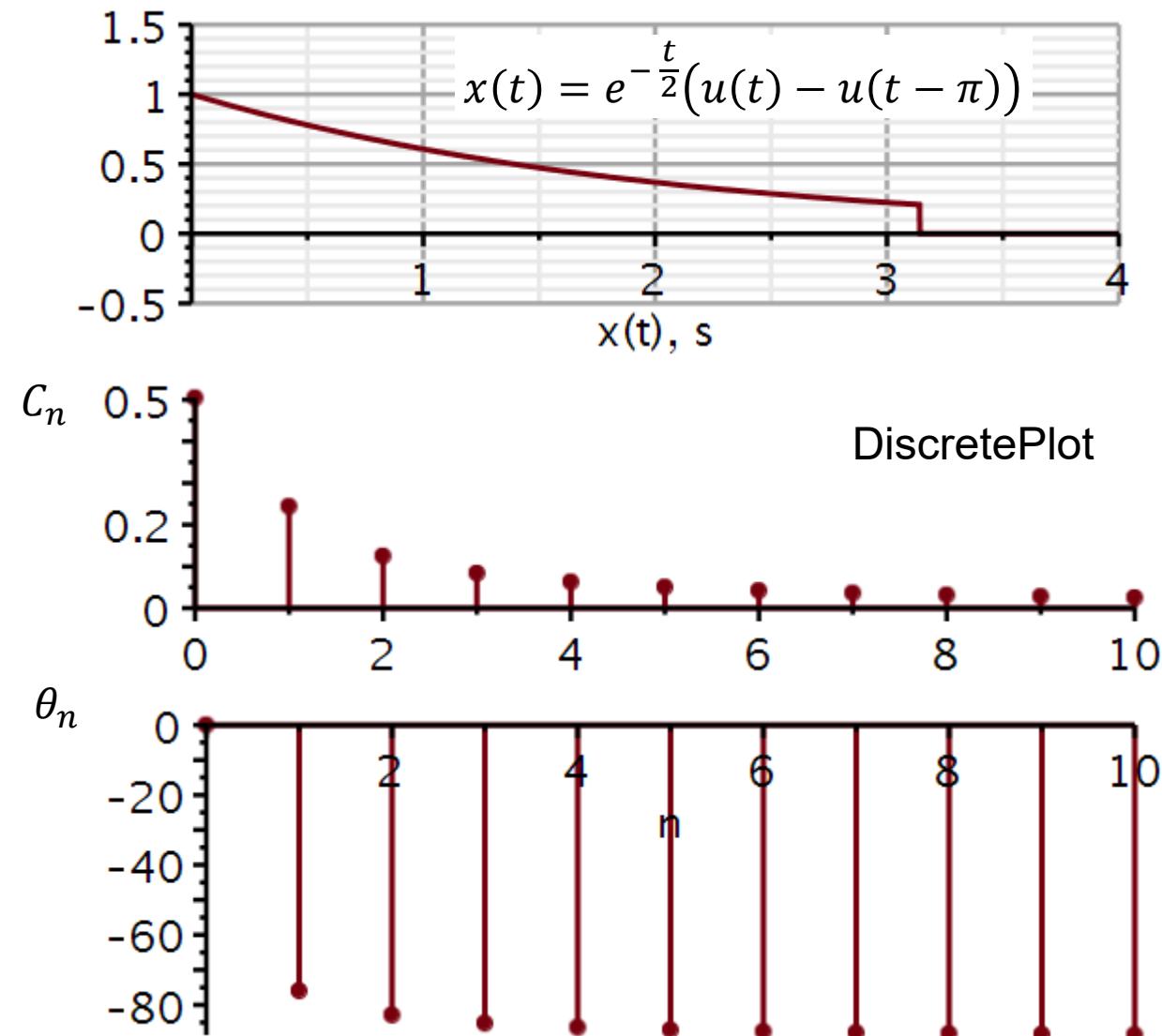
$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

Use Maple to reproduce the plots shown here using the trigonometric Fourier series. Validate that your calculated coefficients agree with those given in example 3.5 (3.3).

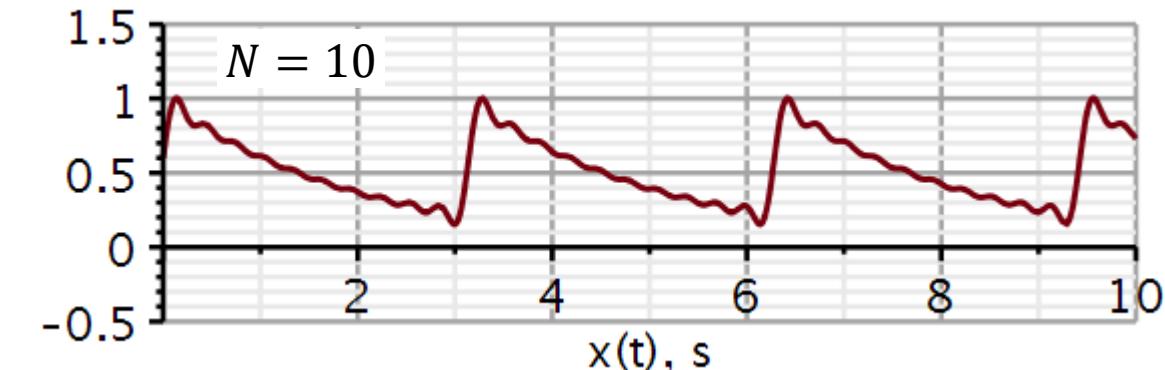
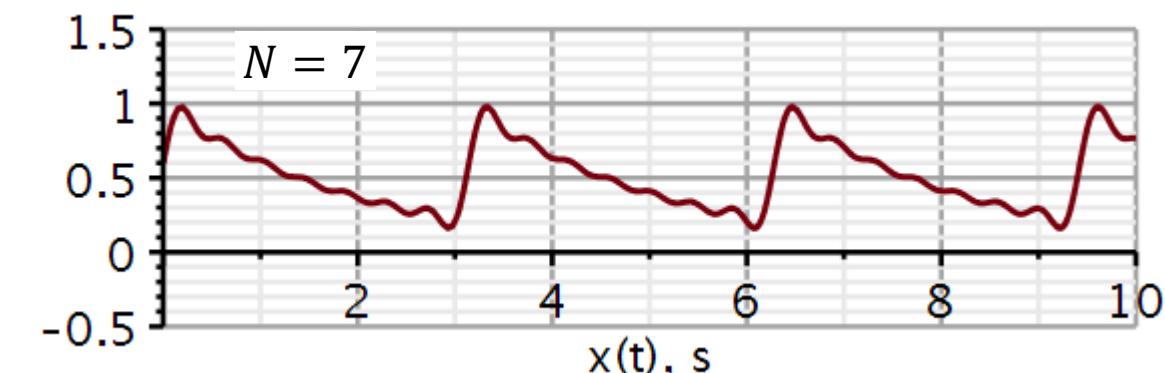
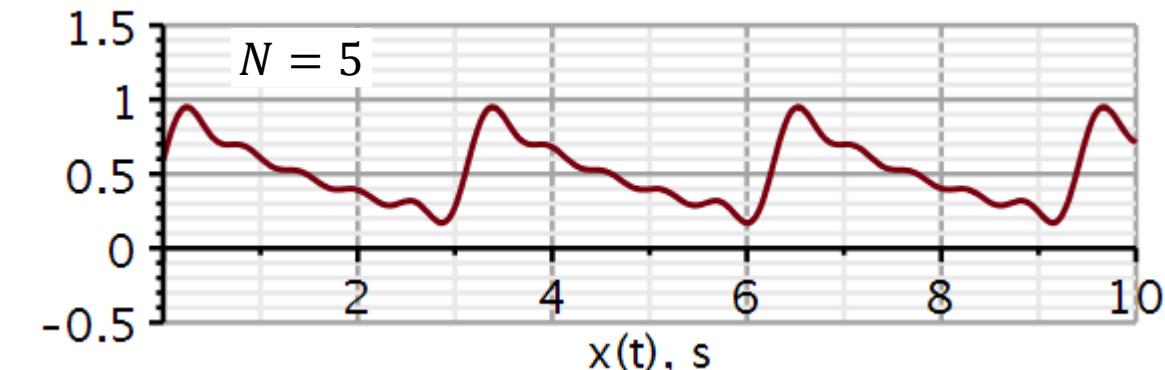
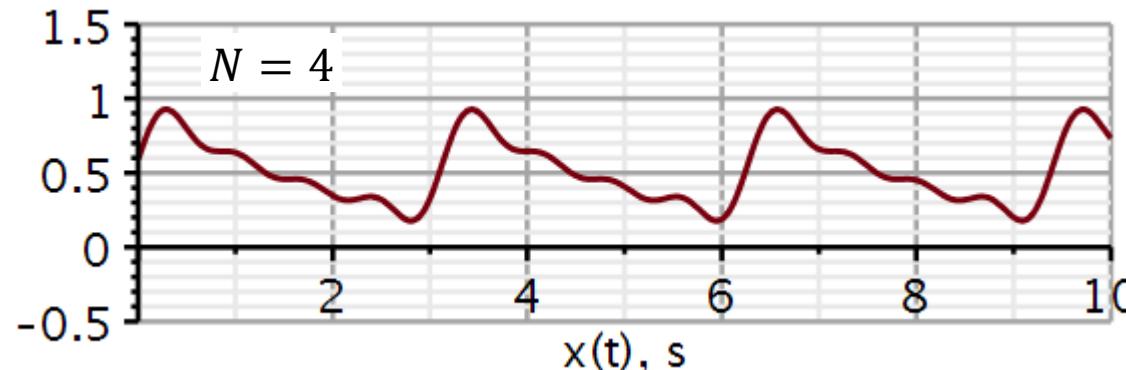
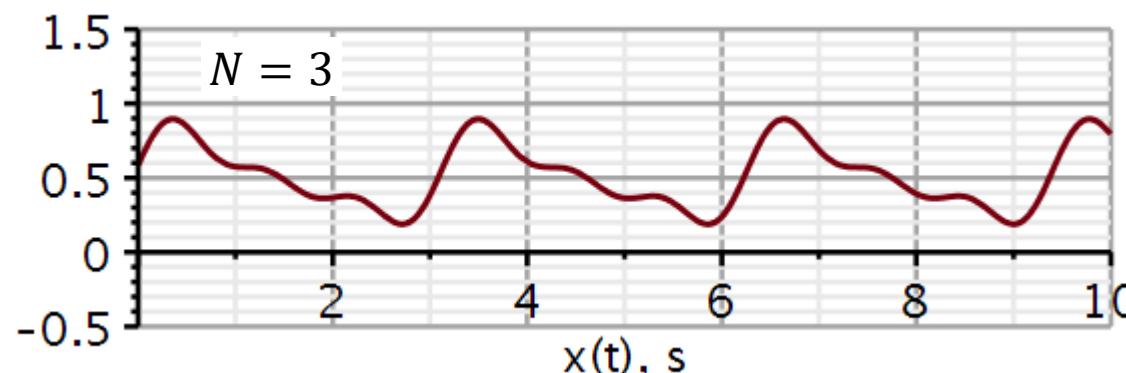
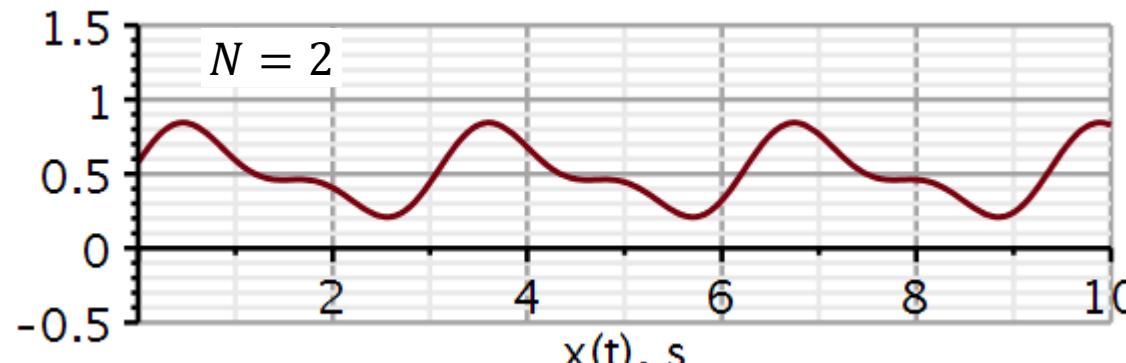
Figure out how to use DiscretePlot. You need the DynamicSystems package.

I organized my data using *Vector*.
Also do the reconstructions shown on the next slide.

See hints below.



Problem 3



Problem 3: Solution

We need to calculate the Fourier series coefficients in Maple. Here is defined three functions that do just that:

Calculation of Fourier Series coefficients:

$$a0 := (n, t1, T0) \rightarrow \frac{1}{T0} \cdot \int_{t1}^{t1 + T0} x(t) \, dt :$$

$$a := (n, t1, T0) \rightarrow \frac{2}{T0} \cdot \int_{t1}^{t1 + T0} x(t) \cdot \cos\left(n \cdot \frac{2 \cdot \pi}{T0} \cdot t\right) \, dt :$$

$$b := (n, t1, T0) \rightarrow \frac{2}{T0} \cdot \int_{t1}^{t1 + T0} x(t) \cdot \sin\left(n \cdot \frac{2 \cdot \pi}{T0} \cdot t\right) \, dt :$$

n: index of Fourier Series coefficient

t1: start time a signal period

T0: Period duration

x(t): The periodic time-signal to be decomposed in a Fourier series. Must be defined before calling this function.

Problem 3: Solution

We need to define the signal:

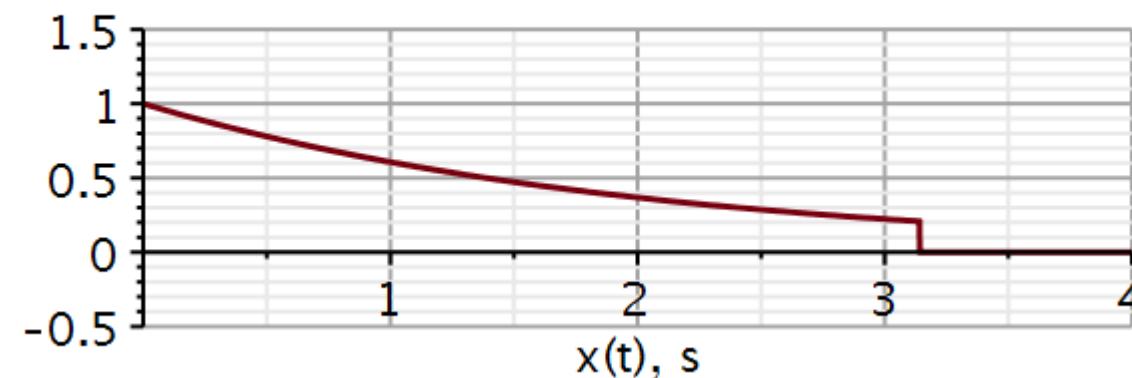
We cannot call the functions on the previous slide until after the signal is defined.

Time domain signal

$$x := t \mapsto e^{-\frac{t}{2}} \cdot (v(t) - v(t-\pi)) : \\ T0 := \pi :$$

Plotting signal

```
plot(x(t), t = 0 .. 4, -0.5 .. 1.5, thickness = 3, axesfont = [Helvetica, "roman", 18], axis[2] = [thickness = 2.5], axis[1] = [thickness = 2.5], labels = ["x(t), s", ""], labelfont = [Helvetica, 18], numpoints = 100, gridlines, size = [600, 200])
```



Problem 3: Solution

When you have defined the integrals, test the code by calculating and printing out a_0 , a_1 , b_1 for example 3.5 (3.3) in Lathi. This is my validation.

Validating function

$$a0 := \text{evalf}(a0(0, 0, T0))$$

$$a0 := 0.5042795237$$

$$a1 := \text{evalf}(a(1, 0, T0))$$

$$a1 := 0.05932700278$$

$$b1 := \text{evalf}(b(1, 0, T0))$$

$$b1 := 0.2373080111$$

$$C1 := \sqrt{a1^2 + b1^2}$$

$$C1 := 0.2446114989$$

$$\theta1 := \frac{180}{\pi} \arctan\left(-\frac{b1}{a1}\right)$$

$$\theta1 := -75.96375653$$

We need many coefficients, so we store them in Maple vectors (see next slide).

Problem 3: Solution

Index vector

 n a_n b_n $T := \text{Vector}(11, n \mapsto n - 1)$ $T :=$

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ \vdots \end{bmatrix}$$

 $A := \text{Vector}(10, n \mapsto \text{evalf}(a(n, 0, T0), 5))$

$$A := \begin{bmatrix} 0.059329 \\ 0.015516 \\ 0.0069555 \\ 0.0039244 \\ 0.0025151 \\ 0.0017479 \\ 0.0012848 \\ 0.00098395 \\ 0.00077760 \\ 0.00062995 \end{bmatrix}$$

$$B := \begin{bmatrix} 0.23731 \\ 0.12413 \\ 0.083464 \\ 0.062793 \\ 0.050304 \\ 0.041951 \\ 0.035975 \\ 0.031487 \\ 0.027995 \\ 0.025198 \end{bmatrix}$$

11 element Vector[column]

Be aware that the first element in an array in Maple starts at index 1.

Problem 3: Solution

When I have my a_n and b_n coefficients saved in vectors $A[n]$ and $B[n]$, I need to make the $C[n]$ and $\theta[n]$ vectors.

I start by making a C Vector with 11 elements, all with the value $a0$.

Then I replace elements 2 to 11 with their proper values:

$$C := \text{Vector}(11, \text{init}, a0)$$

Overwriting all values except $a1$ with $C[n]$ values, $n > 1$.

$$C[2..11] := \text{Vector}(10, n \rightarrow \text{evalf}(\sqrt{(A[n])^2 + (B[n])^2}, 3))$$

Validating the content of C vector

C

$$\begin{bmatrix} 0.5042795237 \\ 0.244 \\ 0.125 \\ 0.0838 \\ 0.0629 \\ 0.0504 \\ 0.0420 \\ 0.0361 \\ 0.0315 \\ 0.0280 \\ \vdots \end{bmatrix}$$

11 element Vector[column]

Problem 3: Solution

A similar technique is used to build the $\theta[n]$ vector.

Initializing vector for phase angles.

```
| θ := Vector(11, init, 0)
```

$$\theta := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

11 element Vector[column]

Filling in all values for $n > 1$

```
θ[2 ..11] := Vector(10, n→evalf(180/π·arctan(-B[n]/A[n])))
```

$$\theta_{2..11} := \begin{bmatrix} -75.96341568 \\ -82.87509722 \\ -85.23624467 \\ -86.42381332 \\ -87.13770836 \\ -87.61413550 \\ -87.95462557 \\ -88.21012343 \\ -88.40893921 \\ -88.56790378 \end{bmatrix}$$

Problem 3: Solution

Here are the two plot statements I use to plot the amplitudes and phase angles as stem-plots.

I used *Vectors* to hold x-axis and y-axis data pairs. Hence T, C, θ are all Maple Vectors.

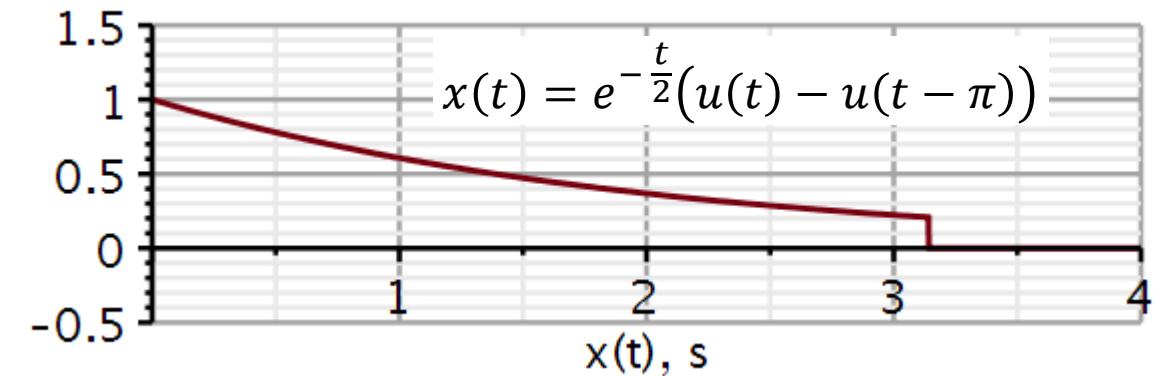
I don't know if that is the only way, but it worked for me.

```
DiscretePlot( T, C, style = stem, thickness = 3, symbol = solidcircle,  
            symbolsize = 18, axesfont = [ Helvetica, roman, 18 ], axis[ 2 ]  
            = [ thickness = 2.5 ], axis[ 1 ] = [ thickness = 2.5 ], labels  
            = [ "n" , " " ], labelfont = [ Helvetica, roman, 18 ], size = [ 600,  
              200 ])
```

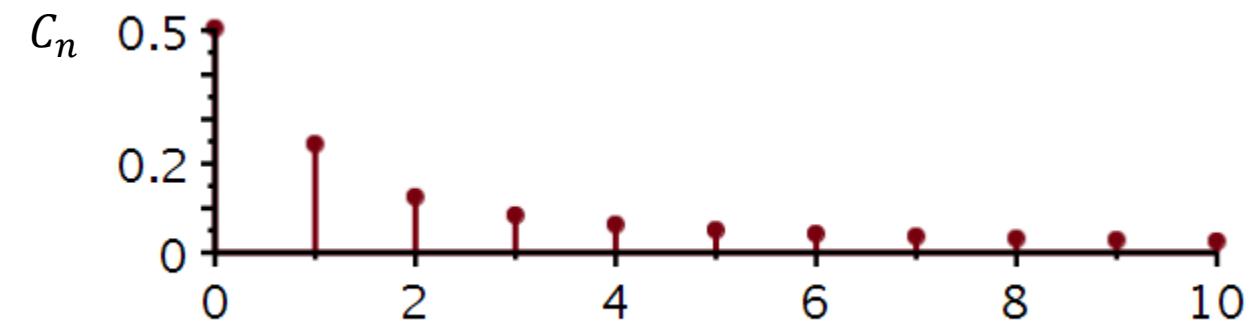
```
DiscretePlot( T, theta, style = stem, thickness = 3, symbol = solidcircle,  
            symbolsize = 18, axesfont = [ Helvetica, roman, 18 ], axis[ 2 ]  
            = [ thickness = 2.5 ], axis[ 1 ] = [ thickness = 2.5 ], labels  
            = [ "n" , " " ], labelfont = [ Helvetica, roman, 18 ], size = [ 600,  
              200 ])
```

Problem 3: Solution

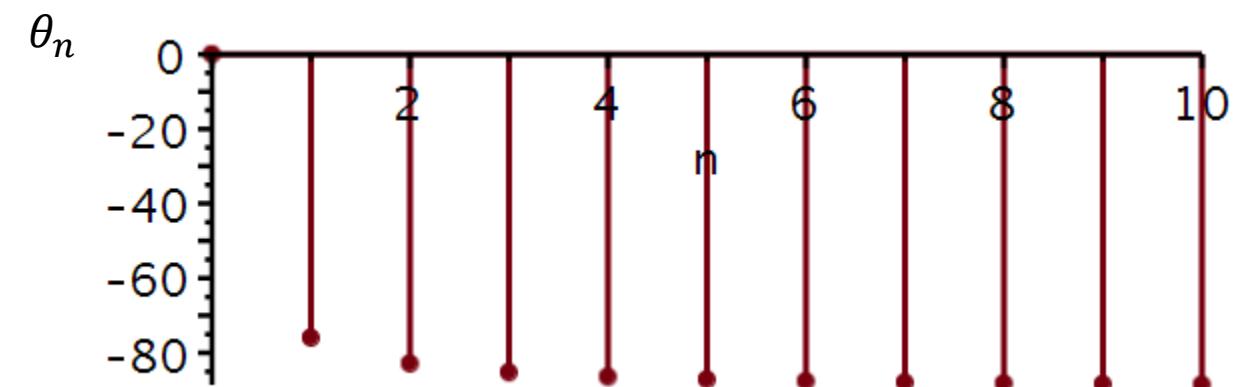
DiscretePlot



Amplitude spectrum:



Phase spectrum:



Here is the Maple code to reconstruct the time signal from its Fourier coefficients:

Synthesis of continuous-time signal from Fourier series coefficients

The coefficients are stored in array A and B starting with index = 1.

$$y := (a0, A, B, T0, K, t) \rightarrow a0 + \sum_{n=1}^K \left('A[n]' \cdot \cos\left(n \cdot \frac{2 \cdot \pi}{T0} \cdot t\right) \right) + \sum_{n=1}^K \left('B[n]' \cdot \sin\left(n \cdot \frac{2 \cdot \pi}{T0} \cdot t\right) \right);$$

A: array of Fourier coefficients. Must be with the apostrophs ('). Some kind of delay.

B: array of Fourier coefficients. Must be with the apostrophs ('). Some kind of delay.

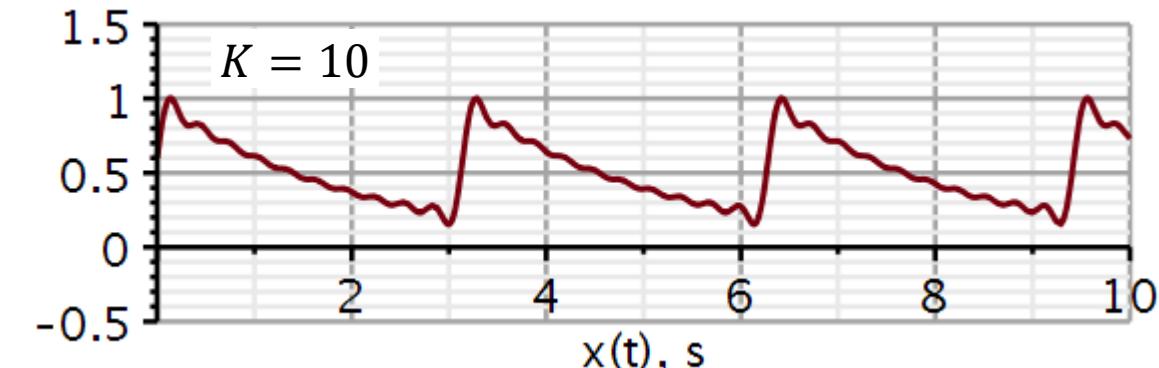
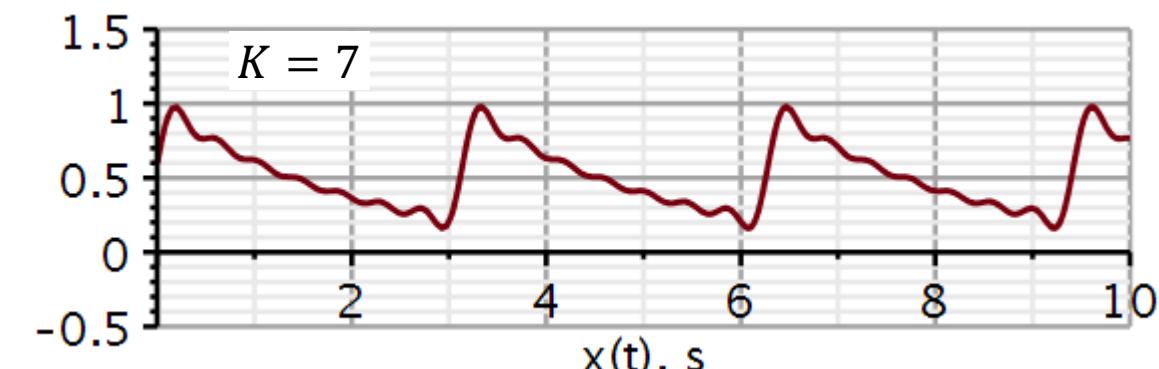
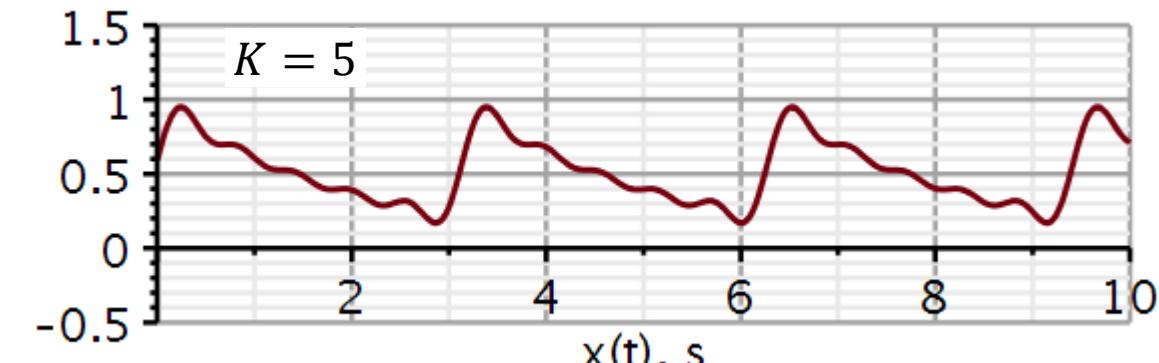
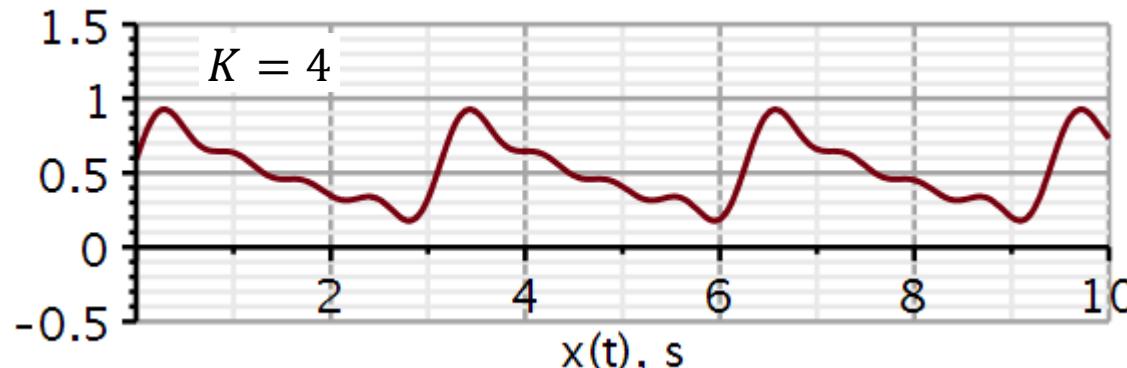
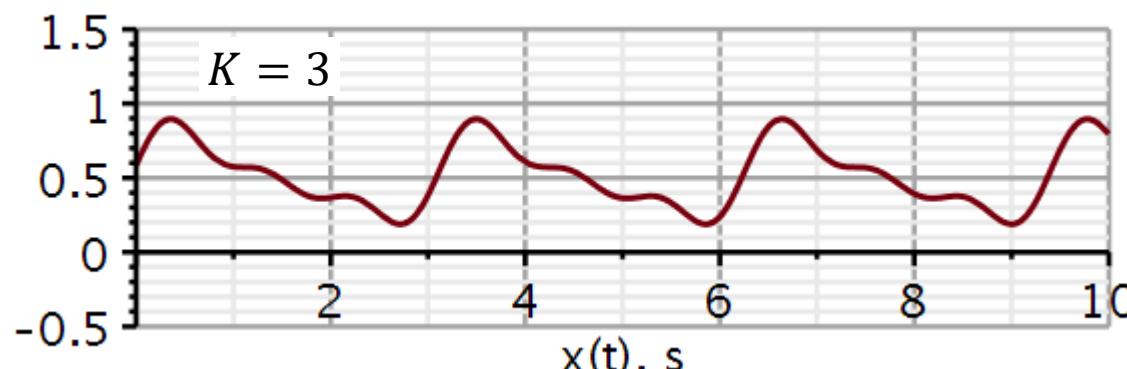
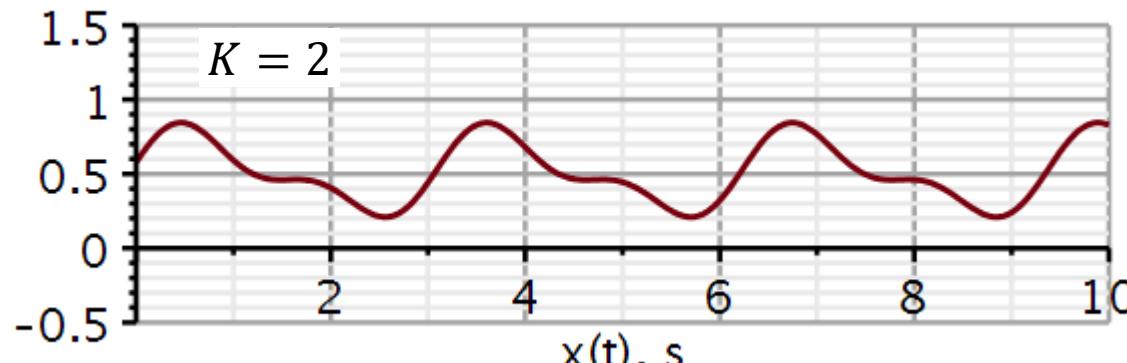
T0: Time period of periodic signal.

K: The number of coefficients we want to include in the reconstruction (synthesis) of the time-signal.

$$K = 1$$

```
plot(y(a0,A,B,T0,1,t),t=0..10.0,-0.5..1.5,thickness=3,font=[ "Helvetica", "roman",  
18],axis[2]=[ thickness=2.5],axis[1]=[ thickness=2.5],labels=[ "x(t)", s", " "],  
labelfont=[ "HELVETICA", 18],numpoints=100,gridlines,size=[ 600, 200])
```

Problem 3: Solution



Problem 4

Use Maple to carry out a complex exponential Fourier Series expansion and reconstruction of a periodic square pulse signal.

Reconstruct the plots on this and the next slide. Note that the plots on the next slide have been copied and pasted into this layout. Don't bother to organize plots in a grid.

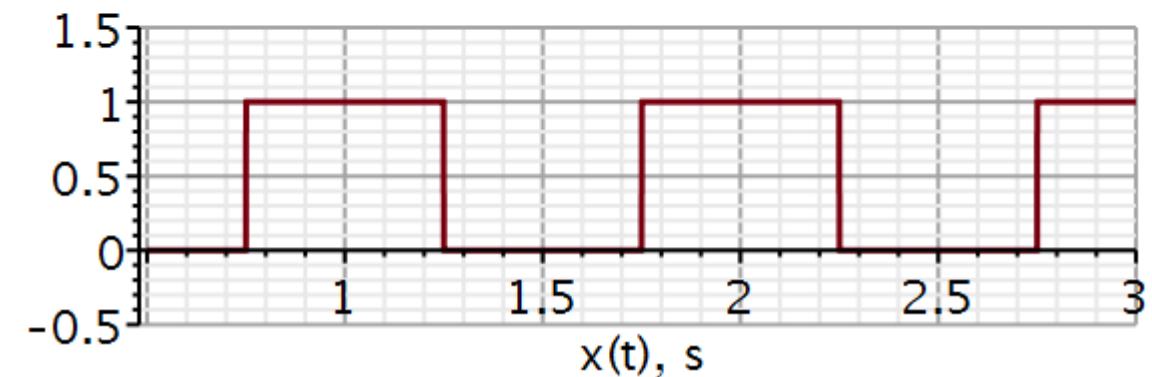
```
restart  
with(DynamicSystems) :  
with(plots) :
```

Time domain signal

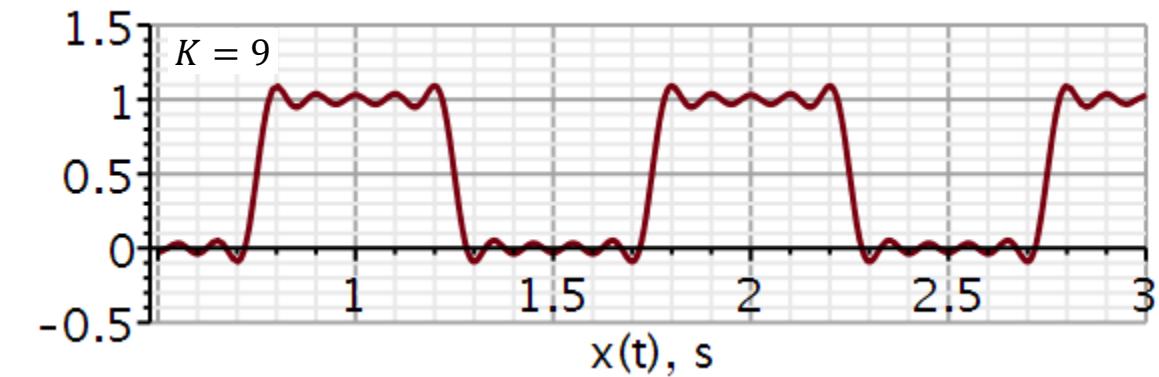
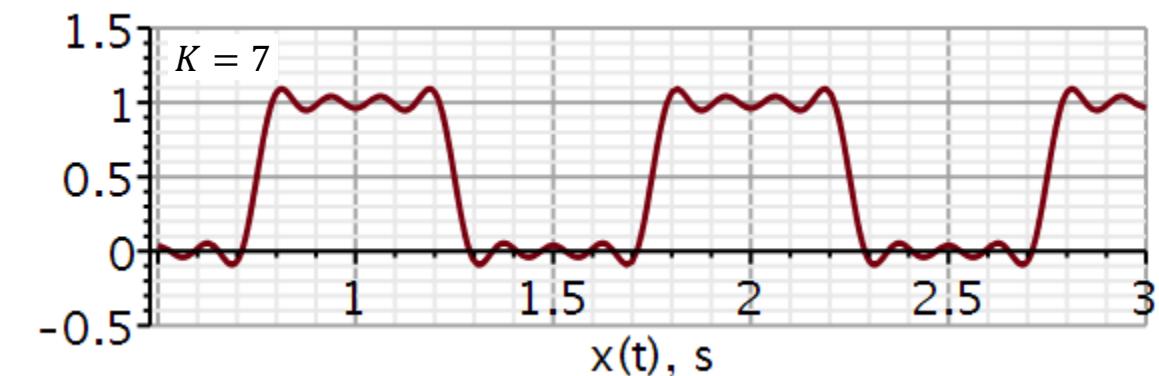
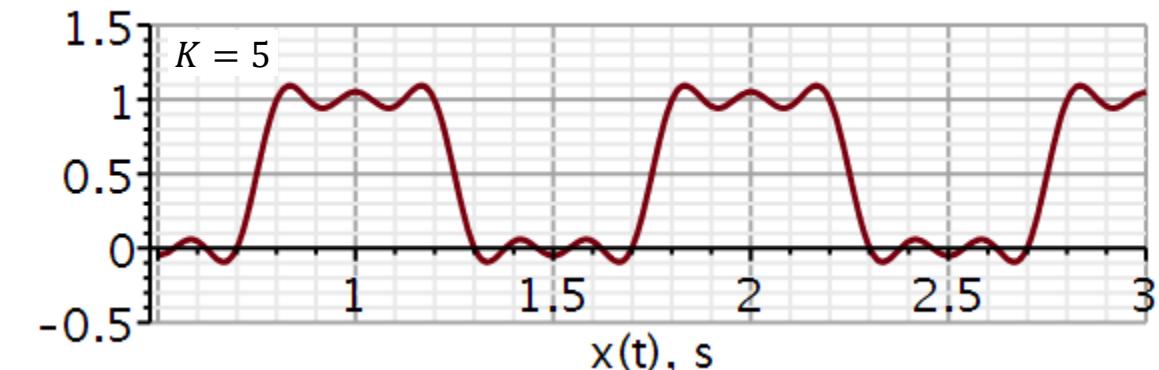
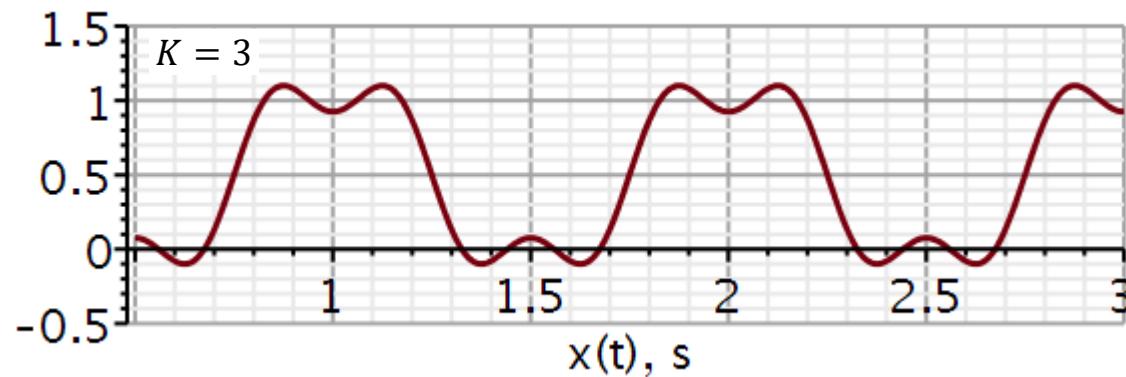
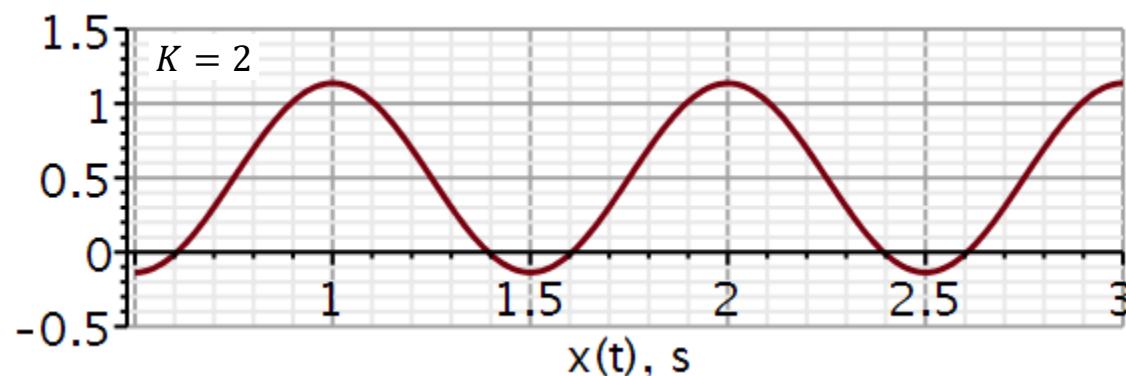
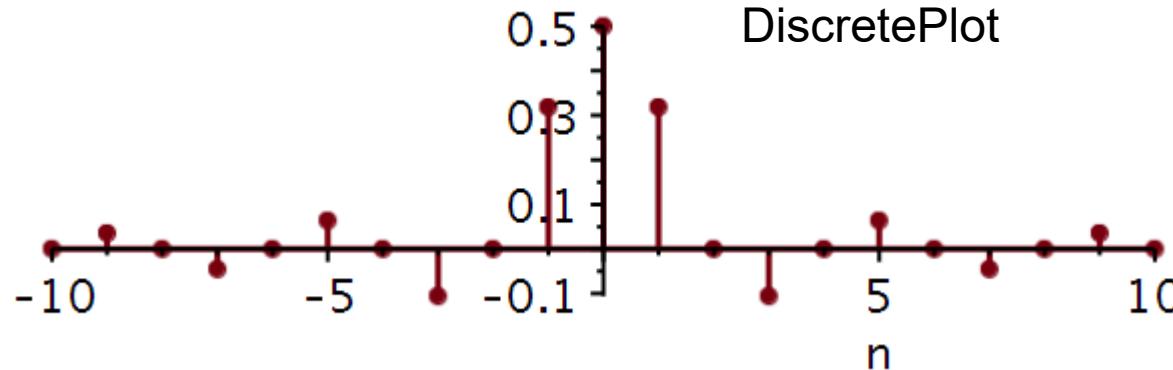
```
x := t → Square(1, 2π · 1, 1/2, -1/4) :
```

Plotting signal

```
plot(x(t), t = 0.5 .. 3.0, -0.5 .. 1.5, thickness = 3, axesfont = [Helvetica, "roman", 18], axis[2] = [thickness = 2.5], axis[1] = [thickness = 2.5], labels = ["x(t), s", " "], labelfont = [Helvetica, 18], numpoints = 100, gridlines, size = [600, 200])
```



Problem 4



Synthesis of continuous-time signal from Fourier series coefficients

The summation is symmetric about zero, however, the coefficients are stored in array A starting with index = 1.

$$y := (A, T_0, M, K, t) \rightarrow \sum_{n=-K}^K \left('A[n+M+1]' \cdot e^{\frac{j \cdot 2 \cdot \pi \cdot n \cdot t}{T_0}} \right) :$$

A: array of Fourier coefficients. Must be with the apostrophes ('). Some kind of delay.

T₀: Time period of periodic signal.

M: Total number of (positive) coefficients calculated. If A holds 21 values (10 negative, 1 zero and 10 positive) then M=10.

K: The number of coefficients we want to include in the reconstruction (synthesis) of the time-signal. K can be less than M to see a worse reconstruction, but cannot be larger than M.

While n runs from $-K$ to K , the coefficients are stored in A from index 1 to index $K + M + 1$. Work out the position of coefficients in the vector by doing example numbers on a piece of paper.

Problem 4: Solution

You will have to figure out yourselves how to define the integrals in Maple to compute the coefficients C_n .

When you have defined the integrals test the code by calculating and printing out $C[0 \dots 3]$. This is my validation.

Validating function

$$\text{evalf}\left(C\left(0, \frac{3}{4}, 1\right)\right)$$

$$0.5000000000$$

$$\text{evalf}\left(C\left(1, \frac{3}{4}, 1\right)\right)$$

$$0.3183098862 + 1.309745927 \cdot 10^{-16} \text{ I}$$

$$\text{evalf}\left(C\left(2, \frac{3}{4}, 1\right)\right)$$

$$2.185976251 \cdot 10^{-17} - 4.882812500 \cdot 10^{-17} \text{ I}$$

$$\text{evalf}\left(C\left(3, \frac{3}{4}, 1\right)\right)$$

$$-0.1061032954 - 6.938893904 \cdot 10^{-17} \text{ I}$$

Problem 4:

Why should we not bother to plot the phase angles in this case?

22050 Continuous-Time Signals and Linear Systems

Kaj-Åge Henneberg

L05

Fourier transformation
Fourier theorems

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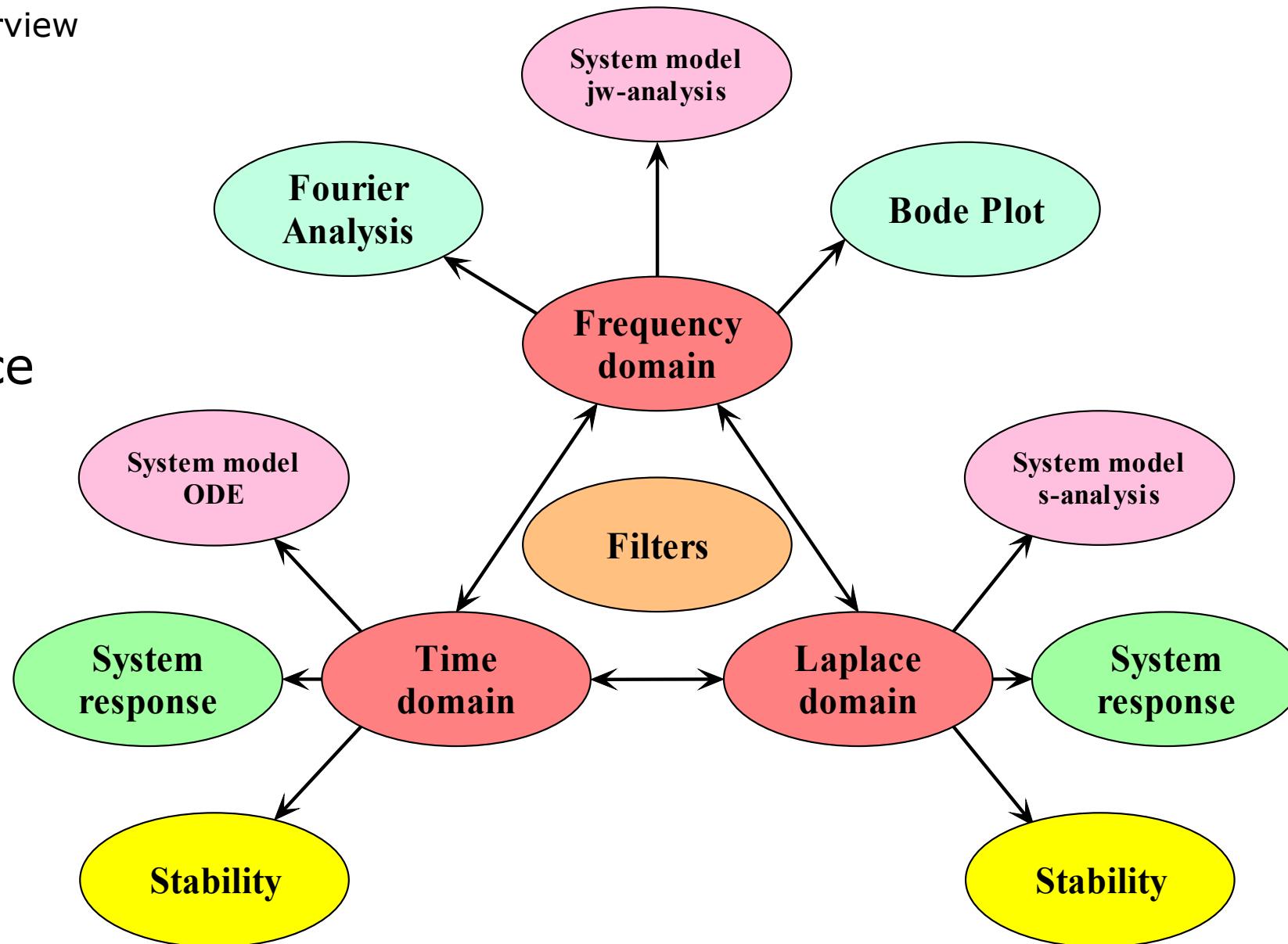
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Course overview

- System
 - Model
 - Response
 - Performance
- Domains
 - Time
 - Frequency
 - s-domain



Lecture plan

- Aperiodic signals
 - Limitation of Fourier series
 - From Fourier series to Fourier transformation
 - Symmetry properties of the Fourier transformed
- Transformation of example signals
- Fourier Theorems
 - Time scaling
 - Time shift
 - Frequency shift and modulation
 - Convolution
 - Integration
 - Differentiation
- Distortion free systems
- The ideal versus the realistic filter

Lathi: Ch. 4.1 – 4.5 incl.

Limitations in the Fourier Series

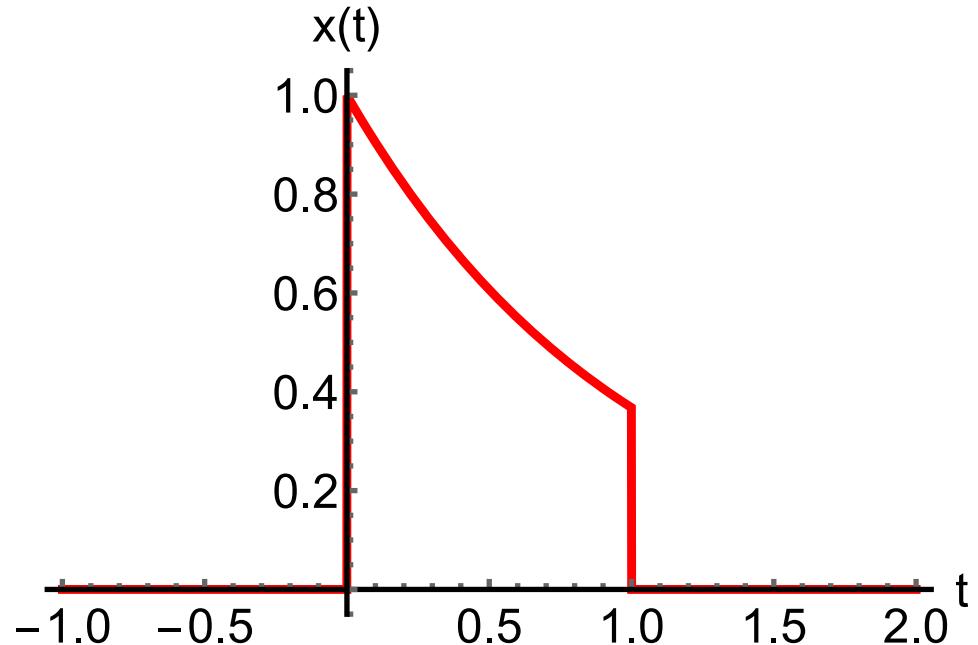
Truncated exponential

$$f_0 = 1/T_0$$

What happens to the Fourier series coefficients if we let $T_0 \rightarrow \infty$?

$$x(t) = \begin{cases} 0 & , \quad t < 0 \\ e^{-at} & , \quad 0 \leq t \leq T_0 \\ 0 & , \quad t > T_0 \end{cases}$$

$$\begin{aligned} D_n &= \frac{1}{T_0} \int_0^{T_0} e^{-at} e^{-j2\pi n f_0 t} dt \\ &= \frac{1}{T_0} \int_0^{T_0} e^{-(a+j2\pi n f_0)t} dt \\ &= \frac{1}{T_0} \frac{-1}{a + j2\pi n f_0} [e^{-(a+j2\pi n f_0)t}]_0^{T_0} \\ D_n &= \frac{1}{T_0} \frac{1}{(a + j2\pi n f_0)} [1 - e^{-(a+j2\pi n f_0)T_0}] \\ &= \frac{1}{T_0} \frac{1}{(a + j2\pi n f_0)} [1 - e^{-aT_0} e^{-j2\pi n}] \\ &= \frac{1}{aT_0 + j2\pi n} [1 - e^{-aT_0}] \end{aligned}$$

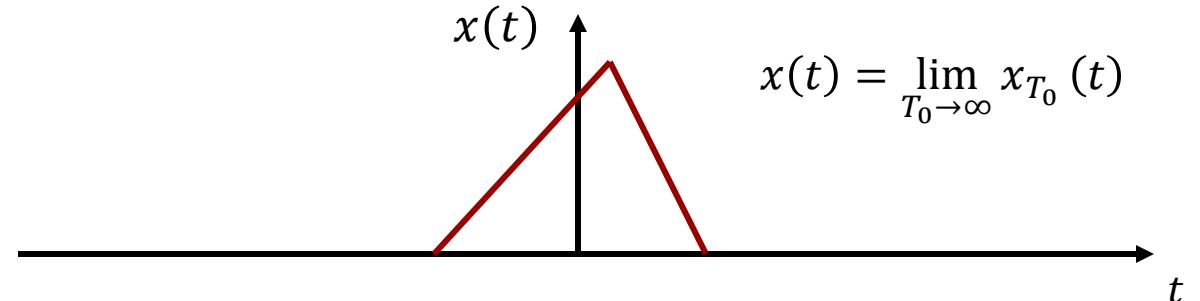


$$\lim_{T_0 \rightarrow \infty} |D_n| = \lim_{T_0 \rightarrow \infty} \frac{1 - e^{-aT_0}}{\sqrt{(aT_0)^2 + (2\pi n)^2}} = 0$$

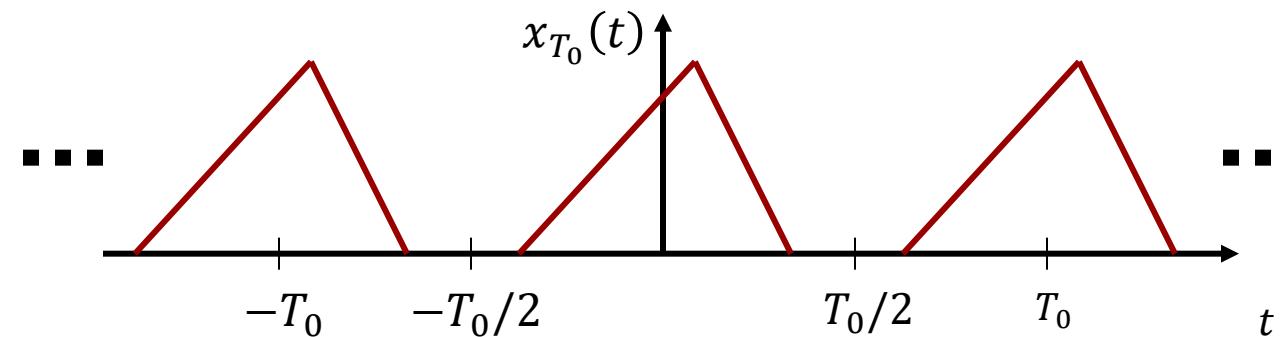
The Fourier Series cannot be used to analyze the spectra of aperiodic functions if they have infinite durations. This is usually the case with the impulse response.

Fourier Transformation

Aperiodic
signal:



Periodic
extension:



Integrating $x_{T_0}(t)$ from $-\frac{T_0}{2}$ to $\frac{T_0}{2}$ yields the same result as integrating $x(t)$ from $-\infty$ to ∞ :

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-jn\omega_0 t} dt$$

$$T_0 D_n = \int_{-\infty}^{\infty} x(t) e^{-jn\omega_0 t} dt$$

Fourier Transformation

If we let $T_0 \rightarrow \infty$, we will get more and more stems placed closer and closer together. In the limit, we must be able to define the amplitude at any frequency, not just in discrete points on the frequency axis.

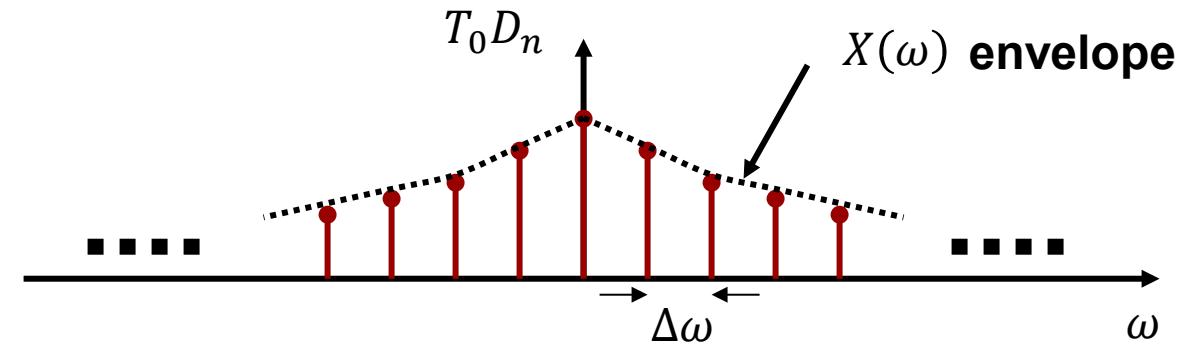
To do so, we define a new function $X(\omega)$:

and we also define a continuous frequency variable:

We can of course evaluate the new function in the discrete frequencies:

We can compare with the Fourier series:

We see that $X(\omega)$ will pass through all the stems of the Fourier series. It is called an **envelope** of the stem plot.



$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \quad \omega_0 = \Delta\omega = \frac{2\pi}{T_0}$$

$$\omega = \lim_{\Delta\omega \rightarrow 0} n\Delta\omega$$

$$X(n\Delta\omega) = \int_{-\infty}^{\infty} x(t)e^{-jn\Delta\omega t} dt$$

$$T_0 D_n = \int_{-\infty}^{\infty} x(t)e^{-jn\omega_0 t} dt = X(n\Delta\omega)$$

$$\lim_{T_0 \rightarrow \infty} |D_n| = \lim_{T_0 \rightarrow \infty} \frac{1 - e^{-aT_0}}{\sqrt{(aT_0)^2 + (2\pi n)^2}} = 0$$

Fourier Synthesis:

$$\omega_0 = \Delta\omega = \frac{2\pi}{T_0}$$

$$\Updownarrow$$

$$\frac{1}{T_0} = \frac{\Delta\omega}{2\pi}$$

We see that if the period $T_0 \rightarrow \infty$, then $\Delta\omega \rightarrow 0$.

In this limit, the sum becomes an integral and the discrete frequencies become a continuous frequency variable $n\Delta\omega \rightarrow \omega$:

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{X(n\Delta\omega)}{T_0} e^{jn\Delta\omega t}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(n\Delta\omega) e^{jn\Delta\omega t} \Delta\omega$$

$$x(t) = \lim_{T_0 \rightarrow \infty} x_{T_0}(t)$$

$$= \lim_{\Delta\omega \rightarrow 0} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(n\Delta\omega) e^{jn\Delta\omega t} \Delta\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

We have now established that an aperiodic signal can be represented by a continuous density function $X(\omega)$. We can think of $X(\omega)$ as the coefficients obtained by expanding the aperiodic signal into an infinite number of orthogonal basis functions the frequencies of which lies infinitely close ($\Delta\omega = 0$).

Transform pair

With this set of transforms we can go back and forth between time domain ($x(t)$) and frequency domain ($X(\omega)$).

We say that $X(\omega)$ is the Fourier transform of $x(t)$, and that $x(t)$ is the inverse Fourier transform of $X(\omega)$.

As $X(\omega)$ is a continuous density distribution of sinusoidal basis functions, $X(\omega)$ is also called a **frequency spectrum**.

The spectrum is a complex valued function of frequency with modulus and angle:<

Forward Transform:	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$	Analysis
Inverse Transform:	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$	Synthesis
$X(\omega) = X(\omega) e^{j\angle X(\omega)}$: Frequency spectrum	
$ X(\omega) $: Amplitude spectrum	
$\angle X(\omega)$: Phase spectrum	

Symmetries of Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

Substituting $-\omega$ for ω :

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t} dt$$

Taking the complex conjugate:

$$X^*(-\omega) = \int_{-\infty}^{\infty} (x(t)e^{j\omega t})^* dt = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t} dt$$

If $x(t)$ is real then $x(t) = x^*(t) \Rightarrow$

$$X^*(-\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = X(\omega)$$

Conjugate symmetry property:

$$X(-\omega) = X^*(\omega)$$

$x(t)$ is real.

Conjugate symmetry property:

$$X(-\omega) = X^*(\omega)$$

$$X_{Re}(-\omega) + jX_{Im}(-\omega) = X_{Re}(\omega) - jX_{Im}(\omega)$$

Real part is even in ω :

$$X_{Re}(\omega) = X_{Re}(-\omega)$$

Imaginary part is odd in ω :

$$X_{Im}(-\omega) = -X_{Im}(\omega)$$

Modulus is even in ω :

$$|X(\omega)| = \sqrt{X_{Re}^2(\omega) + X_{Im}^2(\omega)} = \sqrt{X_{Re}^2(-\omega) + X_{Im}^2(-\omega)} = |X(-\omega)|$$

Angle is odd in ω :

$$\angle X(\omega) = \tan^{-1} \frac{X_{Im}(\omega)}{X_{Re}(\omega)} = \tan^{-1} \frac{-X_{Im}(-\omega)}{X_{Re}(-\omega)} = -\tan^{-1} \frac{X_{Im}(-\omega)}{X_{Re}(-\omega)} = -\angle X(-\omega)$$

Symmetries of Fourier transform

We can also see the symmetry using an alternative approach:

Integrals of odd functions are zero.

The conjugate symmetry property:

We now know what to expect when plotting the amplitude and phase spectra of a real-valued function:

$$\begin{aligned}
 X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \\
 X(\omega) &= \int_{-\infty}^{\infty} (x_e(t) + x_o(t))(\cos \omega t + j \sin \omega t) dt \\
 &= \underbrace{\int_{-\infty}^{\infty} (x_e(t))(\cos \omega t) dt}_{\text{real part is even in } \omega} + \underbrace{\int_{-\infty}^{\infty} (x_e(t))(j \sin \omega t) dt}_{=0} \\
 &\quad + \underbrace{\int_{-\infty}^{\infty} (x_o(t))(\cos \omega t) dt}_{=0} + j \underbrace{\int_{-\infty}^{\infty} (x_o(t))(\sin \omega t) dt}_{\text{imaginary part is odd in } \omega}
 \end{aligned}$$

Real-valued function $x(t)$:

$$\left. \begin{array}{l} \operatorname{Re}\{X(\omega)\} = \operatorname{Re}\{X(-\omega)\} \\ \operatorname{Im}\{X(\omega)\} = -\operatorname{Im}\{X(-\omega)\} \end{array} \right\} \Rightarrow \left\{ \begin{array}{ll} |X(\omega)| = |X(-\omega)| & \text{Even function} \\ \angle X(\omega) = -\angle X(-\omega) & \text{Odd function} \end{array} \right.$$

Fourier Transformation of Exponential function

Do all functions have a Fourier transform? $x(t) = e^{-at} \cdot u(t), a > 0$

This example says no. It depends on the function.

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

$$X(\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$X(\omega) = \frac{-1}{a+j\omega} [e^{-(a+j\omega)t}]_0^{\infty} = \frac{1}{a+j\omega} \left[1 - \lim_{T \rightarrow \infty} e^{-aT} e^{-j\omega T} \right]$$

Using: $|e^{-j\omega T}| = 1$

$$= \frac{1}{a+j\omega} \left[1 - \lim_{T \rightarrow \infty} e^{-aT} \right]$$

$$X(\omega) = \frac{1}{a+j\omega}, \text{ if } a > 0$$

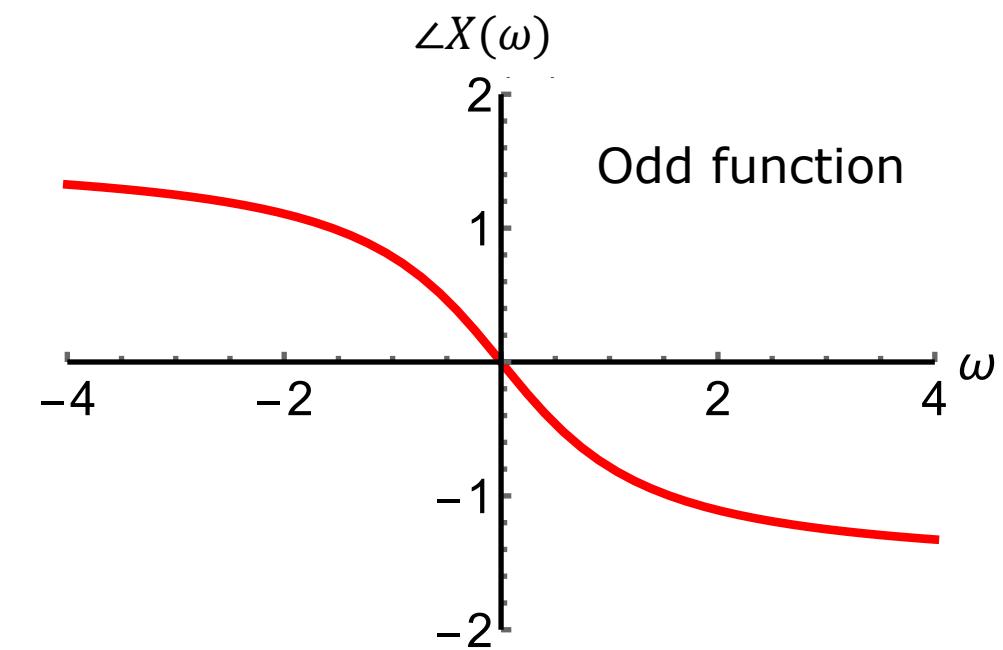
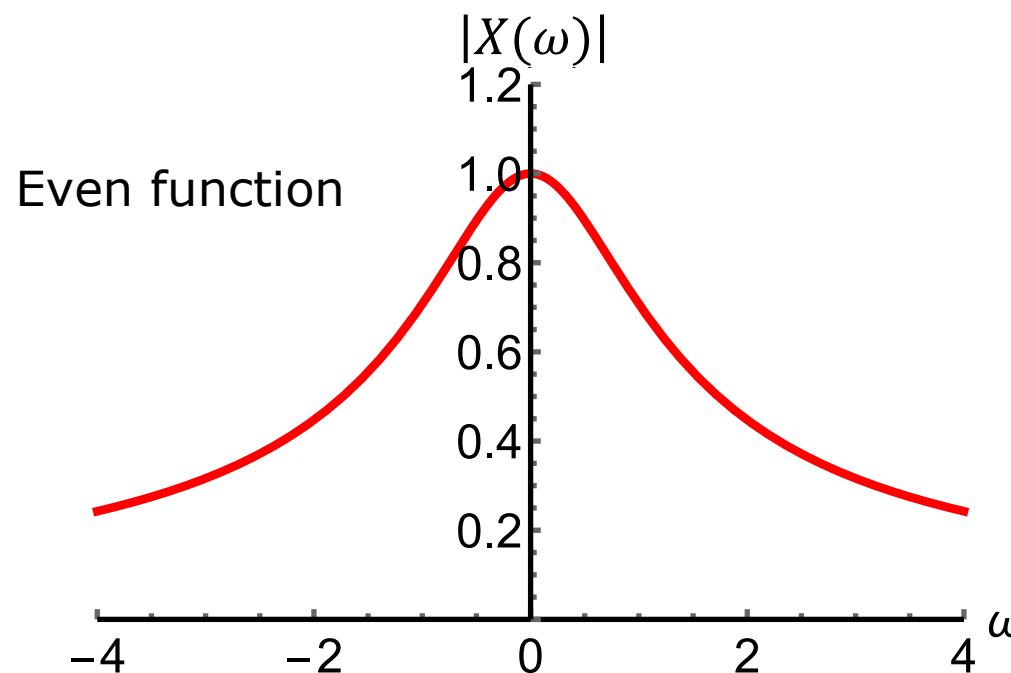
Observation:

The Fourier transform exists for decaying exponentials but not for growing exponentials.

Plotting Frequency Characteristic

Amplitude spectrum and phase spectrum for $a = 1$:

$$X(\omega) = \frac{1}{a + j\omega} = \frac{1}{\sqrt{a^2 + \omega^2}} \cdot e^{-j \tan^{-1}\left(\frac{\omega}{a}\right)}$$



$$\left. \begin{array}{l} \operatorname{Re}\{X(\omega)\} = \operatorname{Re}\{X(-\omega)\} \\ \operatorname{Im}\{X(\omega)\} = -\operatorname{Im}\{X(-\omega)\} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} |X(\omega)| = |X(-\omega)| \\ \angle X(\omega) = -\angle X(-\omega) \end{array} \right. \quad \begin{array}{l} \text{Even function} \\ \text{Odd function} \end{array}$$

Existence of Fourier transform

$$|X(\omega)| = \left| \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right| \leq \int_{-\infty}^{\infty} |x(t)| |e^{-j\omega t}| dt$$

$$|e^{-j\omega t}| = 1 \quad |X(\omega)| \leq \int_{-\infty}^{\infty} |x(t)| dt < \infty \quad \text{Dirichlet condition}$$

A function must satisfy the Dirichlet condition to have a Fourier transform. In addition, it may only have a finite number of maxima and minima, and a finite number of jump discontinuities.

All physically possible signals obey the Dirichlet condition.

If the function $x(t)$ has jump discontinuities, the integral on the right will equal the average of the amplitudes approaching the jump from left and right.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

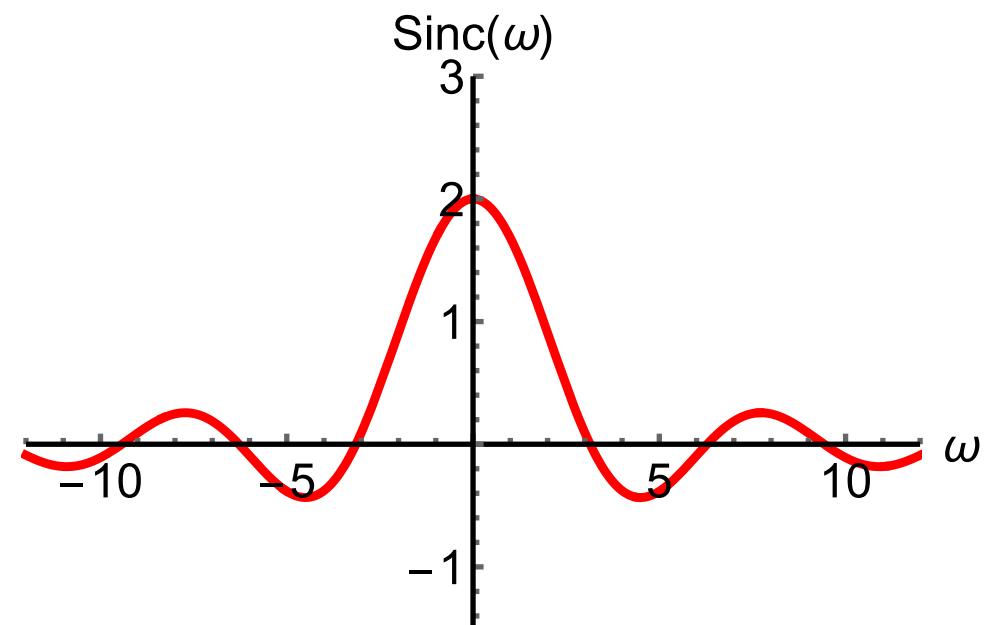
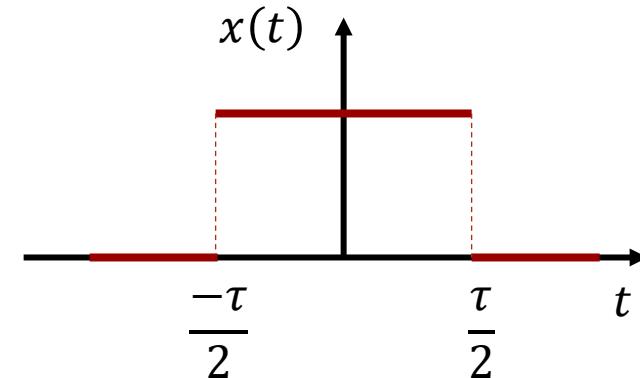
Pulse Function – Lathi: Ex. 4.2

$$x(t) = \text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 0 & , t > |\tau/2| \\ 1/2 & , t = |\tau/2| \\ 1 & , t < |\tau/2| \end{cases}$$

$$\begin{aligned} X(\omega) &= \int_{-\tau/2}^{\tau/2} x(t)e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt \\ &= \frac{-1}{j\omega} [e^{-j\omega t}]_{-\tau/2}^{\tau/2} = \frac{1}{j\omega} \left[e^{\frac{j\omega\tau}{2}} - e^{-\frac{j\omega\tau}{2}} \right] \\ &= \frac{2 \sin(\omega \tau/2)}{\omega} = \tau \frac{\sin(\omega \tau/2)}{\omega \tau/2} \\ &\stackrel{\text{def}}{=} \tau \text{sinc}(\omega \tau/2) \end{aligned}$$

For a square pulse of width τ , where are the zeros of $\text{sinc}(\omega \tau/2)$?

$$\begin{aligned} \text{sinc}(\omega_n \tau/2) &= 0 \\ \frac{\omega_n \tau}{2} &= n\pi \\ \omega_n &= \frac{n \cdot 2\pi}{\tau} \end{aligned}$$



Fourier Transform in Maple

→ *restart*
with(inttrans) :
with(DynamicSystems) :
with(plots) :

Useful functions:

dB := ω → 20 · log10(|H(ω)|) : # converts to decibel
angle := ω → argument(H(ω)) · $\frac{180}{\pi}$: # converts from radians to degrees

Calculation of Fourier transform:

fourier(expr, t, w)

- expr - expression, equation, or set of equations and/or expressions to be transformed
- t - variable expr is transformed with respect to t
- w - parameter of transform
- opt - option to run this under (optional)

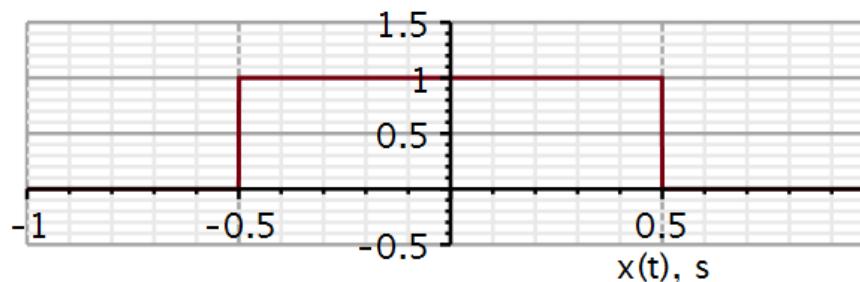
Fourier Transform in Maple

Time domain signal

```
x := (Heaviside(t + 0.5) - Heaviside(t-0.5)) :
```

Plotting signal

```
plot(x, t=-1 ..1, -0.5 ..1.5, thickness = 3, axesfont = [Helvetica, "roman", 18], axis[2] = [thickness = 2.5], axis[1] = [thickness = 2.5], labels = ["x(t), s", " "], labelfont = [Helvetica, 18], numpoints = 100, gridlines, size = [600, 200])
```

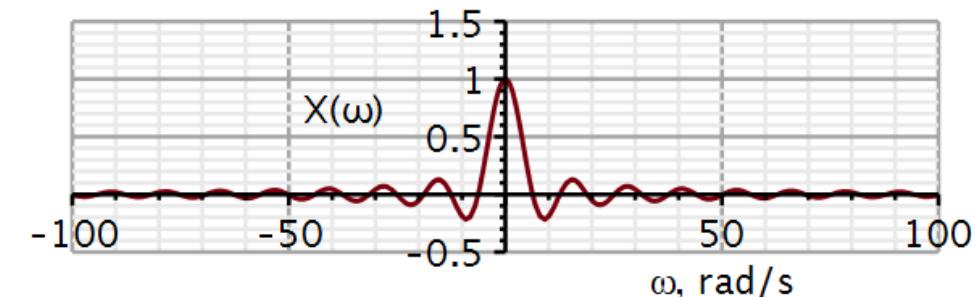


Calculating Fourier transform

```
X := fourier(x, t, ω)
```

$$X := \frac{2 \cdot \sin(0.5000000000 \omega)}{\omega}$$

```
plot(X, ω=-100 ..100, -0.5 ..1.5, thickness = 3, axesfont = [Helvetica, "roman", 18], axis[2] = [thickness = 2.5], axis[1] = [thickness = 2.5], labels = ["ω, rad/s", "X(ω)"], labelfont = [Helvetica, 18], numpoints = 100, gridlines, size = [600, 200])
```



In this case the Fourier transform is a real function. In general, it is complex-valued, and we need to plot the modulus and the angle in separate plots.

If I define a function as $x := t \rightarrow$ something it does not do the Fourier transform.

Other Useful functions – Lathi (Ex. 4.3 – 4.6)

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = ? \quad 1$$

$$F^{-1}\{\delta(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = ? \quad \frac{1}{2\pi}$$

$$F^{-1}\{2\pi\delta(\omega)\} = \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = ? \quad 1$$

$$F^{-1}\{2\pi\delta(\omega - \omega_0)\} = \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = ? \quad e^{j\omega_0 t}$$

$$F^{-1}\{\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))\} = ? \quad \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} = \cos(\omega_0 t)$$

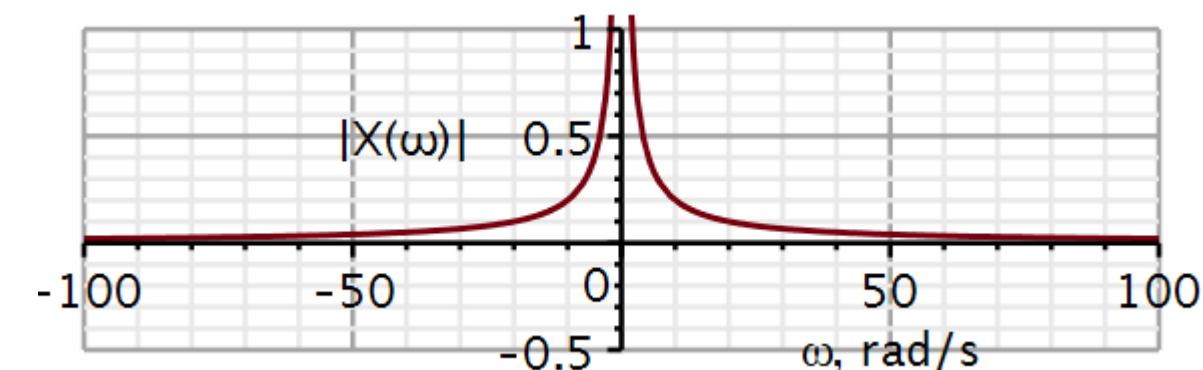
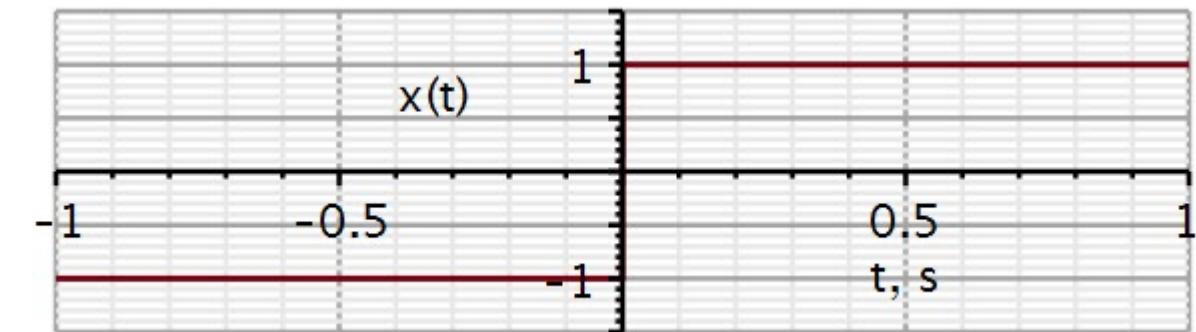
Signum function – Lathi (Ex. 4.10)

$$\text{sgn}(t) = \begin{cases} -1 & t < 0 \\ 1 & t > 0 \end{cases} = \lim_{a \rightarrow 0} [e^{-at}u(t) - e^{-a(-t)}u(-t)]$$

$$F\{e^{-at}u(t)\} = \frac{1}{a + j\omega}$$

$$\begin{aligned} \int_{-\infty}^{\infty} e^{at} u(-t) e^{-j\omega t} dt &= \int_{-\infty}^0 e^{(a-j\omega)t} dt \\ &= \frac{1}{a-j\omega} [e^{(a-j\omega)t}]_{-\infty}^0 = \frac{1}{a-j\omega} \end{aligned}$$

$$F\{\text{sgn}(t)\} = \lim_{a \rightarrow 0} \left[\frac{1}{a + j\omega} - \frac{1}{a - j\omega} \right] = \frac{2}{j\omega}$$

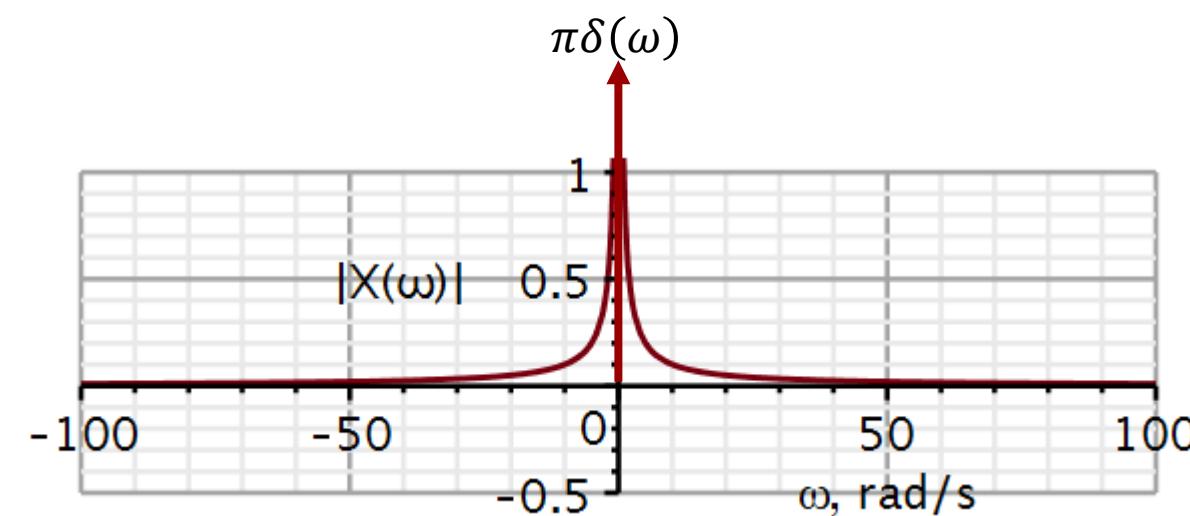
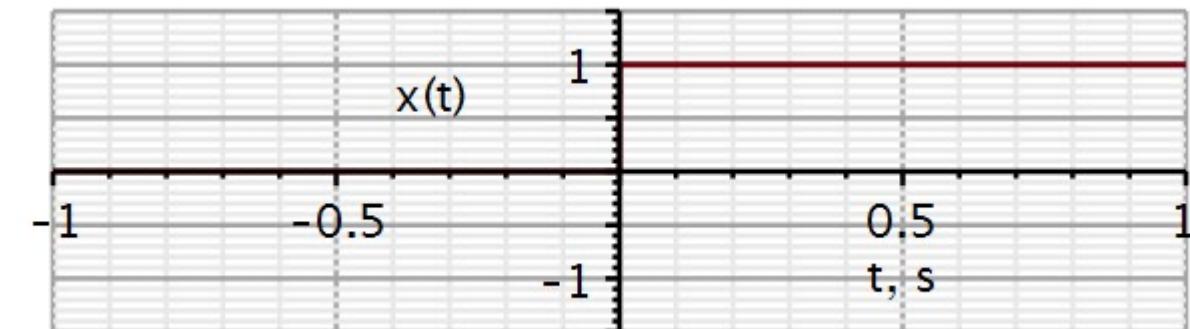


Step function – Lathi (Ex. 4.9)

$$u(t) = \frac{1}{2} + \frac{1}{2} \operatorname{sgn}(t)$$

$$F\{u(t)\} = \frac{1}{2} F\{1\} + \frac{1}{2} F\{\operatorname{sgn}(t)\}$$

$$\begin{aligned} F\{u(t)\} &= \frac{1}{2} \cdot 2\pi\delta(\omega) + \frac{1}{2} \cdot \frac{2}{j\omega} \\ &= \pi\delta(\omega) + \frac{1}{j\omega} \end{aligned}$$



Fourier theorems

Lathi 4.3

Video 2

Duality

If we have a known function $x(t)$, we can calculate its Fourier transform $X(\omega)$. This way we have established a transform relationship between two functions.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$
$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$
$$x(t) \leftrightarrow X(\omega)$$

We have obtained a function X . What would be the Fourier transform of this function, if we know consider this function to be a function of time?

In other words, what is $\mathcal{F}\{X(t)\}$?

Duality

First, we calculate the Fourier transform:

Now we use that same function as a time function. Using variable substitution, we have an expression for $X(t)$:

We can calculate the Fourier transform of this new time function:

Changing order of integrals:

We need to figure out the red integral.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(s) e^{-j\omega s} ds$$

$$X(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(s) e^{-jst} ds$$

$$\mathcal{F}\{X(t)\} = \mathcal{F} \left\{ \int_{-\infty}^{\infty} x(s) e^{-jst} ds \right\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(s) e^{-jst} ds e^{-j\omega t} dt$$

$$\mathcal{F}\{X(t)\} = \int_{-\infty}^{\infty} x(s) \int_{-\infty}^{\infty} e^{-j(s+\omega)t} dt ds$$

Duality

The inverse Fourier transform of $\delta(\omega)$ is:

It follows that the forward Fourier integral of 1 is:

We can now compute the red integral:

$$\mathcal{F}^{-1}\{2\pi\delta(\omega)\} = \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = 1$$

$$\mathcal{F}\{1\} = \int_{-\infty}^{\infty} e^{-j\omega t} dt = 2\pi\delta(\omega)$$

$$\int_{-\infty}^{\infty} e^{-j(s+\omega)t} dt = 2\pi\delta(s + \omega)$$

$$\begin{aligned} \mathcal{F}\{X(t)\} &= \int_{-\infty}^{\infty} x(s) \int_{-\infty}^{\infty} e^{-j(s+\omega)t} dt ds \\ &= \int_{-\infty}^{\infty} x(s) 2\pi\delta(s + \omega) ds = 2\pi x(-\omega) \end{aligned}$$

If $X(\omega)$ is the forward Fourier transform of $x(t)$ then
 $2\pi x(-\omega)$ is the forward Fourier transform of $X(t)$

$$2\pi x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad x(t) \leftrightarrow X(\omega)$$

Exchange t with $-t$

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega) e^{-j\omega t} d\omega$$

Exchange t and ω

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt \quad X(t) \leftrightarrow 2\pi x(-\omega)$$

If $X(\omega)$ is the forward Fourier transform of $x(t)$ then
 $x(-\omega)$ is the forward Fourier transform of $X(t)$

Duality - example

We have calculated the transform fair

$$x(t) = \text{rect}\left(\frac{t}{\tau}\right) \leftrightarrow \tau \text{sinc}\left(\frac{\tau}{2} \omega\right) = X(\omega)$$

Using the Duality property:

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

We obtain:

$$\tau \text{sinc}\left(\frac{\tau}{2} t\right) \leftrightarrow 2\pi \text{rect}\left(\frac{\omega}{\tau}\right)$$

We can check by doing the math:

$$X(\omega) = 2\pi \text{rect}\left(\frac{\omega}{\tau}\right) = 2\pi \begin{cases} 0 & , \omega > |\tau/2| \\ 1/2 & , \omega = |\tau/2| \\ 1 & , \omega < |\tau/2| \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\tau/2}^{\tau/2} 2\pi e^{j\omega t} d\omega = \frac{1}{jt} [e^{j\omega t}]_{-\tau/2}^{\tau/2} = \frac{e^{j\frac{\tau}{2}t} - e^{-j\frac{\tau}{2}t}}{jt}$$

$$x(t) = \frac{2 \sin\left(\frac{\tau}{2}t\right)}{t} = \tau \frac{\sin\left(\frac{\tau}{2}t\right)}{\frac{\tau}{2}t} = \tau \text{sinc}\left(\frac{\tau}{2}t\right)$$

Time Scaling

Using variable substitution:

$$a > 0$$

$$s \stackrel{\text{def}}{=} at$$

$$\frac{ds}{a} = dt$$

$$\int_{-\infty}^{\infty} x(at)e^{-j\omega t} dt = \frac{1}{a} \int_{-\infty}^{\infty} x(s)e^{-j\frac{\omega}{a}s} ds \\ = \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

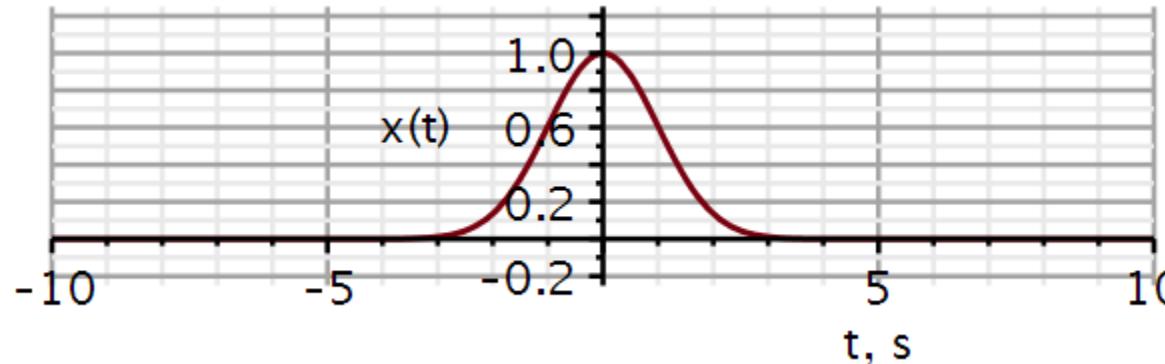
$$a < 0$$

$$\int_{-\infty}^{\infty} x(at)e^{-j\omega t} dt = \frac{1}{a} \int_{-\infty}^{\infty} x(s)e^{-j\frac{\omega}{a}s} ds \\ = \frac{-1}{a} X\left(\frac{\omega}{a}\right)$$

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Time Scaling

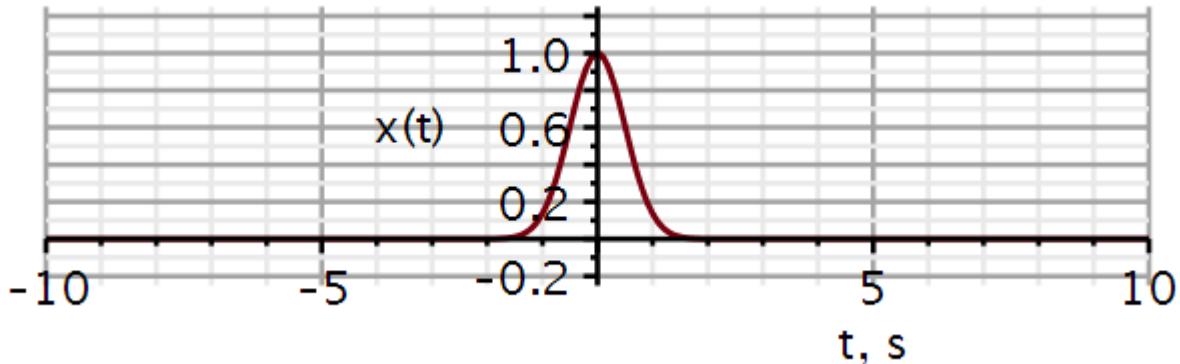
$$x(t) = e^{-t^2/2}$$



$$X1 := \text{fourier}(x1(t), t, \omega)$$

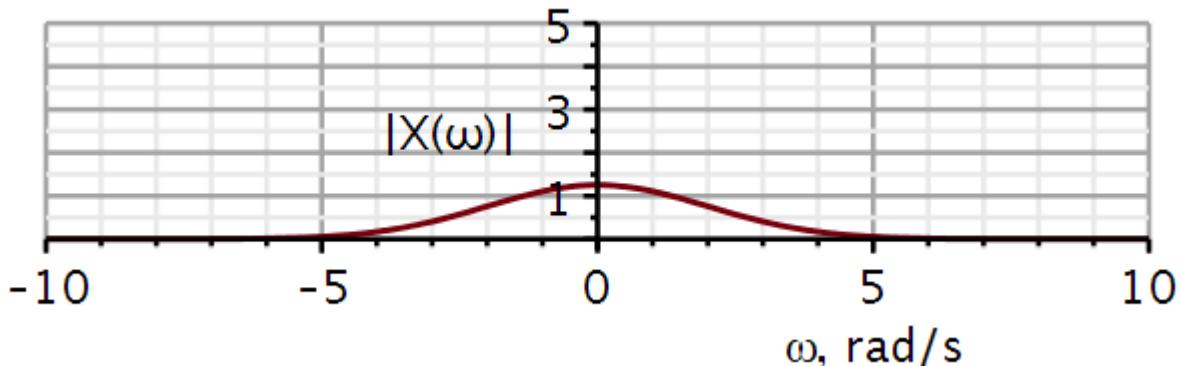
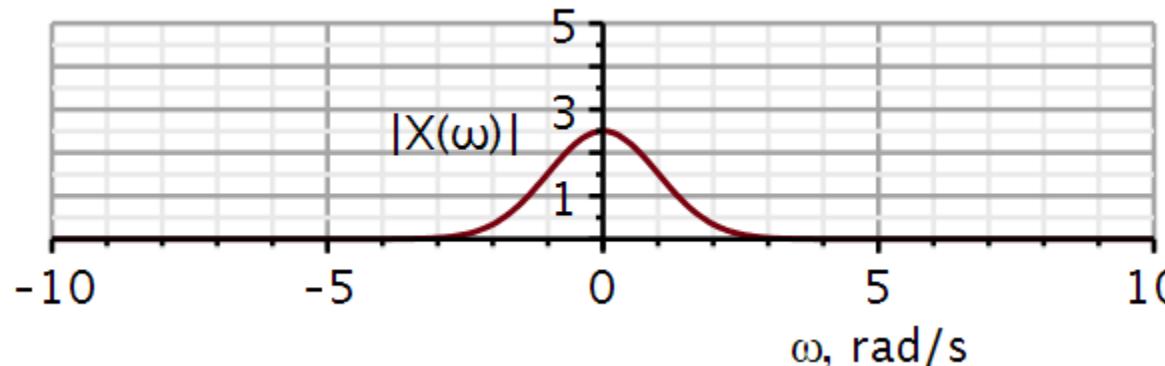
$$\underline{X1} := e^{-\frac{\omega^2}{2}} \sqrt{2} \sqrt{\pi}$$

$$x(t) = e^{-(2t)^2/2}$$



$$X2 := \text{fourier}(x1(2 \cdot t), t, \omega)$$

$$\underline{X2} := \frac{e^{-\frac{\omega^2}{8}} \sqrt{2} \sqrt{\pi}}{2}$$



Time Shift

Using variable substitution:

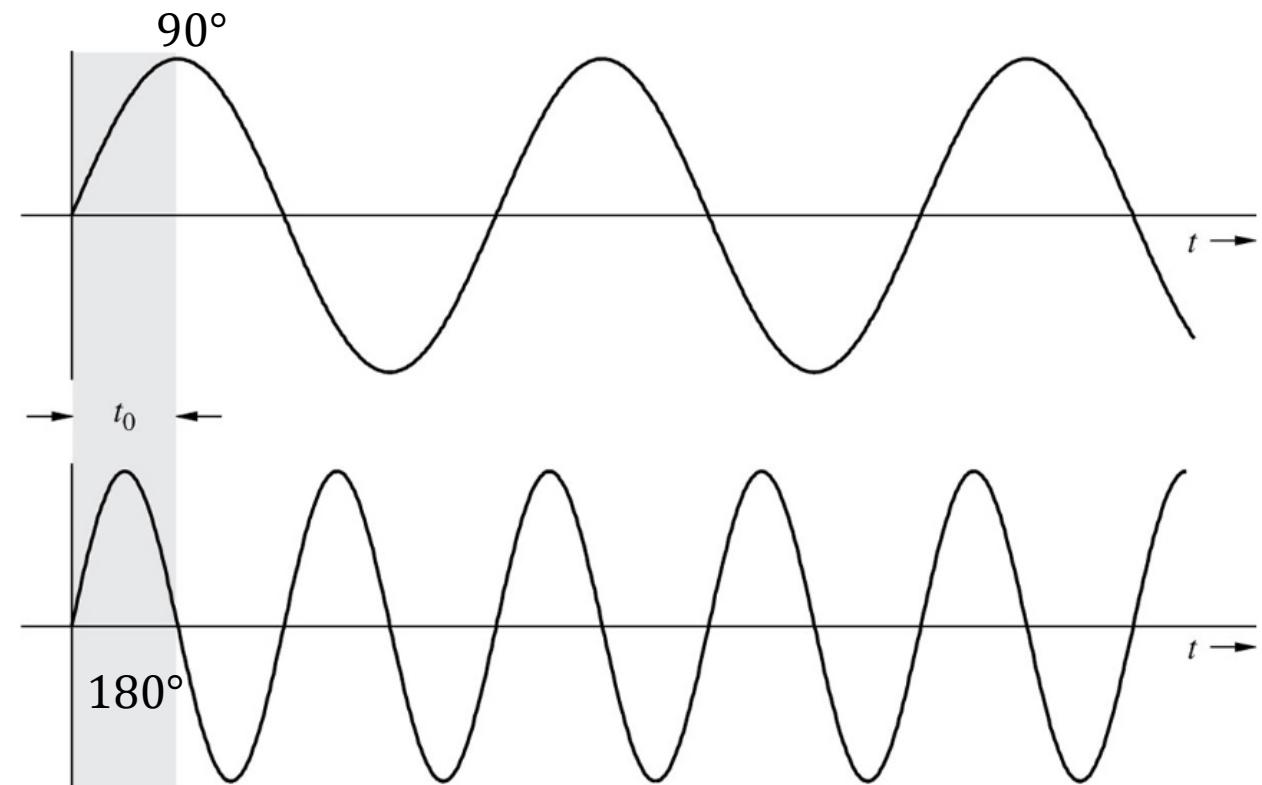
$$\begin{aligned}s &= t - t_0 \\ ds &= dt\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt &= \int_{-\infty}^{\infty} x(s) e^{-j\omega(s+t_0)} ds = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(s) e^{-j\omega s} ds \\ &= e^{-j\omega t_0} X(\omega) = X(\omega) e^{j\theta}\end{aligned}$$

When a signal is delayed in the time domain, a phase angle is introduced in the frequency spectrum.

We notice that in the case of a time delay of the entire signal, the phase angle is a linear function of frequency:

$$\theta = -\omega \cdot t_0$$



Using variable substitution:

$$\begin{aligned}s &= t - t_0 \\ ds &= dt\end{aligned}$$

$$\int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0} X(\omega)$$

Example:

$$\cos(\omega_0(t - t_d)) = \cos(\omega_0 t - \omega_0 t_d) = \cos(\omega_0 t + \theta)$$

$$t_d = \frac{-\theta}{\omega_0}$$

We set:

$$\theta = -\pi/2$$

$$t_d = \frac{\pi}{2\omega_0}$$

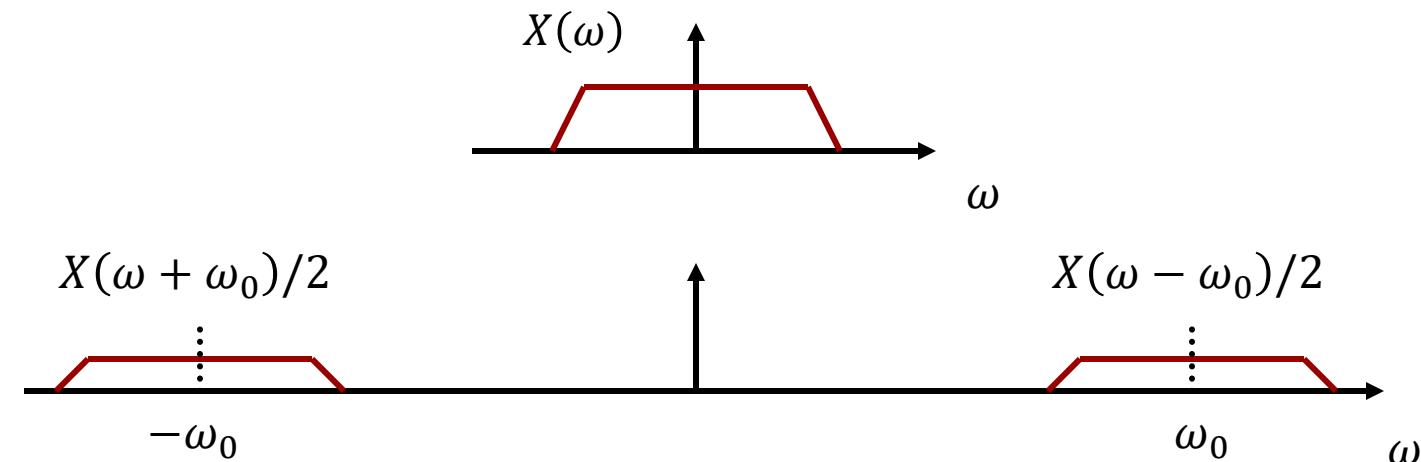
$$\begin{aligned}\cos(\omega_0(t - t_d)) &\leftrightarrow \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \cdot e^{-j\omega t_d} \\ &= \pi[\delta(\omega - \omega_0) \cdot e^{-j\omega_0 t_d} + \delta(\omega + \omega_0) \cdot e^{j\omega_0 t_d}] \\ &= \pi[\delta(\omega - \omega_0) \cdot e^{-j\pi/2} + \delta(\omega + \omega_0) \cdot e^{j\pi/2}] \\ \text{Why? } &= j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)] \\ &\leftrightarrow \sin(\omega_0 t)\end{aligned}$$

Frequency shift: $x(t)e^{j\omega_0 t} \leftrightarrow \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega - \omega_0)t}dt$

$= X(\omega - \omega_0)$

Modulation:

$$\begin{aligned}x(t) \cos(\omega_0 t) &= \frac{1}{2} x(t) e^{j\omega_0 t} + \frac{1}{2} x(t) e^{-j\omega_0 t} \\&\leftrightarrow \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0)\end{aligned}$$

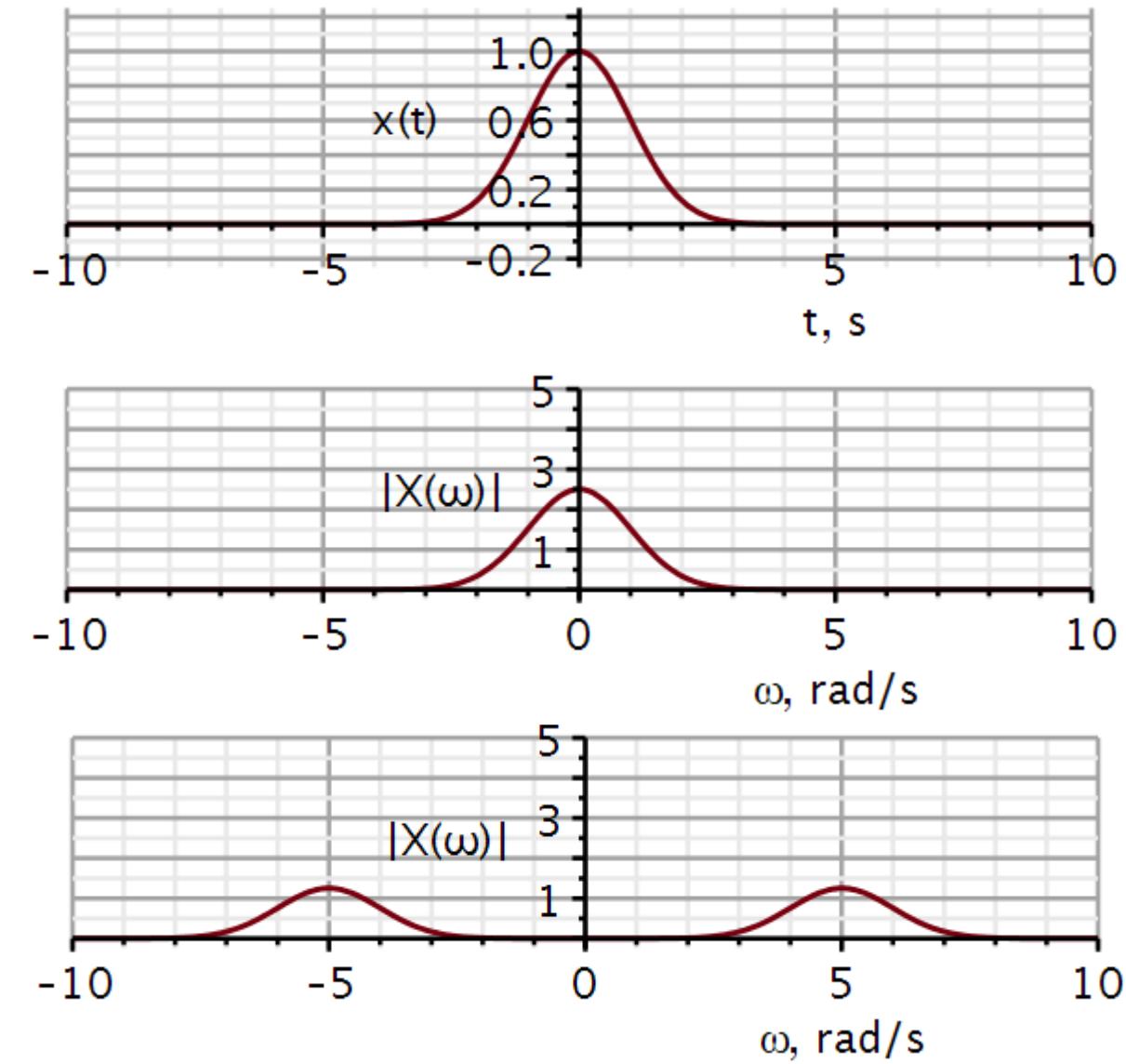


Modulation

Here we modulate the Gauss function with a modulation frequency $\omega_0 = 5 \text{ rad/s}$.

$X3 := \text{fourier}(x1(t) \cdot \cos(5 \cdot t), t, \omega)$

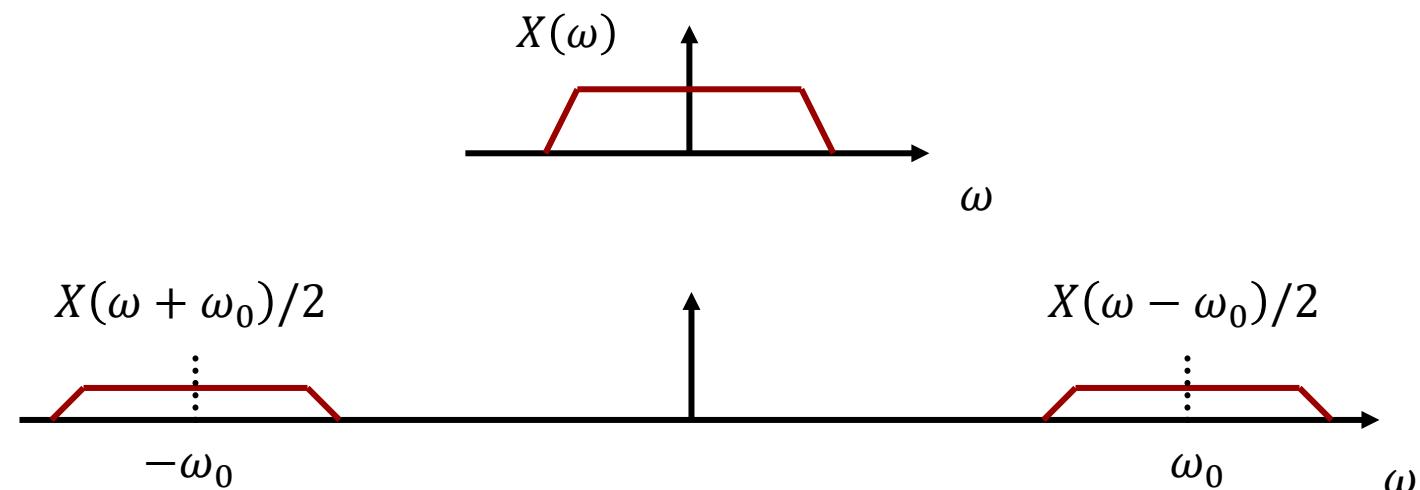
$$X3 := \sqrt{2} \sqrt{\pi} \cosh(5\omega) e^{-\frac{25}{2} - \frac{\omega^2}{2}}$$



Frequency Shift and Modulation

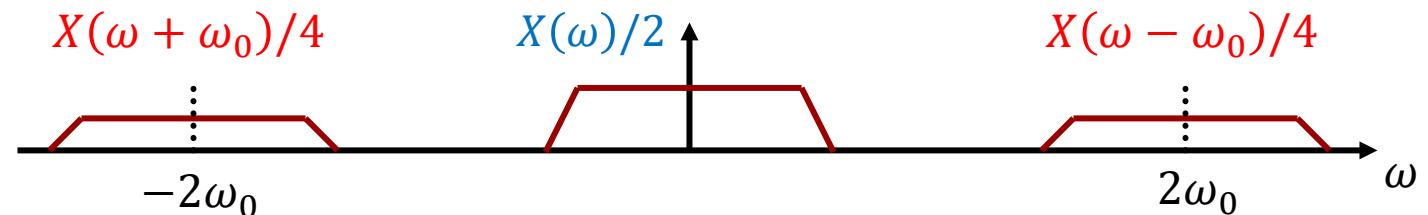
Modulation:

$$\begin{aligned} x(t) \cos(\omega_0 t) &= \frac{1}{2} x(t) e^{j\omega_0 t} + \frac{1}{2} x(t) e^{-j\omega_0 t} \\ &\leftrightarrow \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0) \end{aligned}$$



Demodulation:

$$\begin{aligned} x(t) \cos(\omega_0 t) \cos(\omega_0 t) &= \left(\frac{1}{2} x(t) e^{j\omega_0 t} + \frac{1}{2} x(t) e^{-j\omega_0 t} \right) \left(\frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t} \right) = \left(\frac{1}{2} + \frac{1}{2} \cos(2\omega_0 t) \right) x(t) \\ &\leftrightarrow \frac{1}{4} X(\omega - 2\omega_0) + \frac{1}{4} X(\omega + 2\omega_0) + \frac{1}{2} X(\omega) \end{aligned}$$



Convolution in the **time** domain

We find an extremely useful property by deriving the Fourier transform of the convolution integral:

Convolution of two functions in the time domain corresponds to multiplication of their spectra in the frequency domain

When you need to convolve two functions, it is often faster to Fourier transform the two functions, multiply their spectra, and then inverse Fourier transform the product. The Fast Fourier Transformation (FFT) uses $n \times \log_2 n$ operations.

$$\begin{aligned} \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau &\leftrightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h(t - \tau)e^{-j\omega t}dt d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)H(\omega)e^{-j\omega\tau}d\tau \\ &= H(\omega) \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \\ &= H(\omega) X(\omega) \end{aligned}$$

$$\begin{aligned} y_{zs}(t) &= h(t) * x(t) \\ \uparrow_3 & \quad \downarrow_1 \quad \downarrow_2 \quad n \times \log_2 n \\ Y_{zs}(\omega) &= H(\omega) \cdot X(\omega) \end{aligned}$$

$$1024 \times 10 \approx \underbrace{10^4}_{FFT} \ll \underbrace{10^6}_{convolution}$$

Convolution in the **frequency** domain

We also find an extremely useful property by deriving the inverse Fourier transform of the convolution integral:

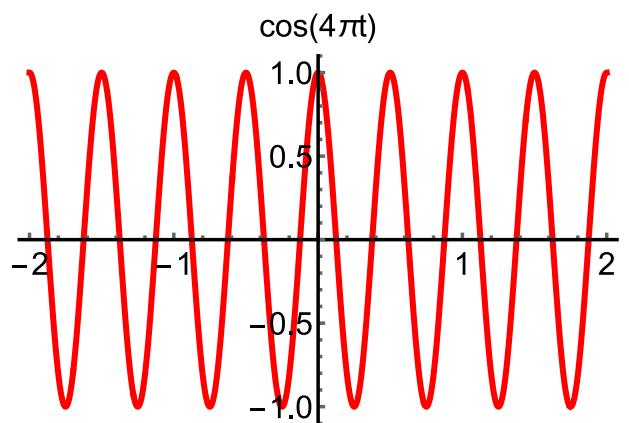
Convolution of two spectra in the frequency domain corresponds to multiplication of the functions in the time domain

When you need to convolve two spectra, it is often faster to inverse Fourier transform the two spectra, multiply their time functions, and then Fourier transform the product.

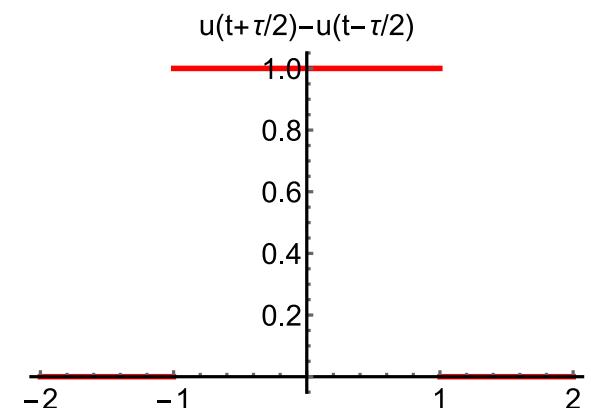
$$\begin{aligned}
 \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega_0) H(\omega - \omega_0) d\omega_0 &\leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega_0) H(\omega - \omega_0) d\omega_0 e^{j\omega t} d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega_0) \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega - \omega_0) e^{j\omega t} d\omega d\omega_0 \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega_0) h(t) e^{j\omega_0 t} d\omega_0 \\
 &= h(t) \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega_0) e^{j\omega_0 t} d\omega_0 \\
 &= h(t) x(t)
 \end{aligned}$$

$$\begin{aligned}
 y_{zs}(t) &= 2\pi(h(t) \cdot x(t)) \\
 Y_{zs}(\omega) &= H(\omega) * X(\omega)
 \end{aligned}$$

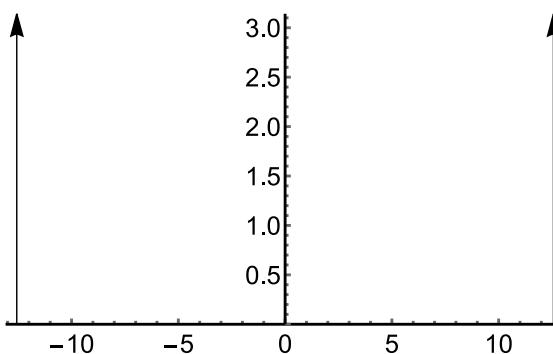
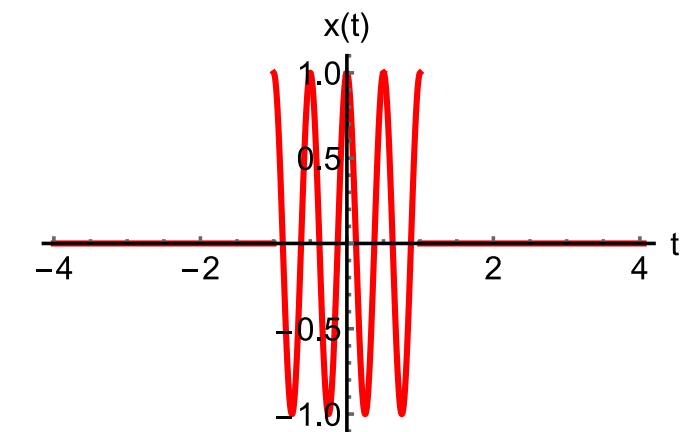
Convolution in the frequency domain



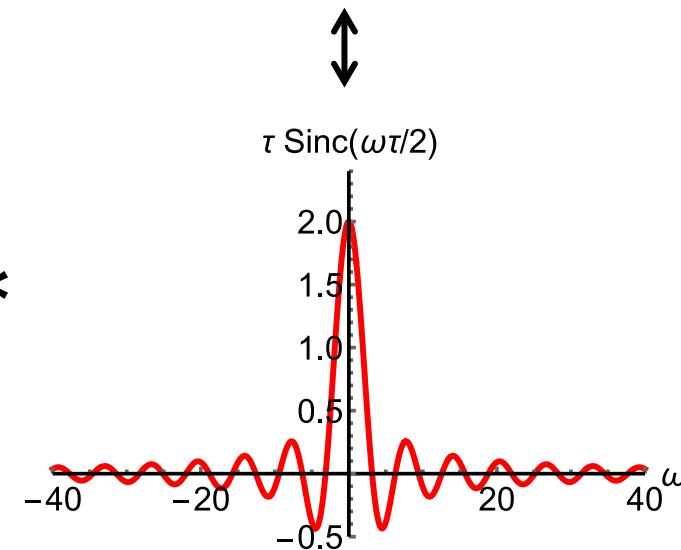
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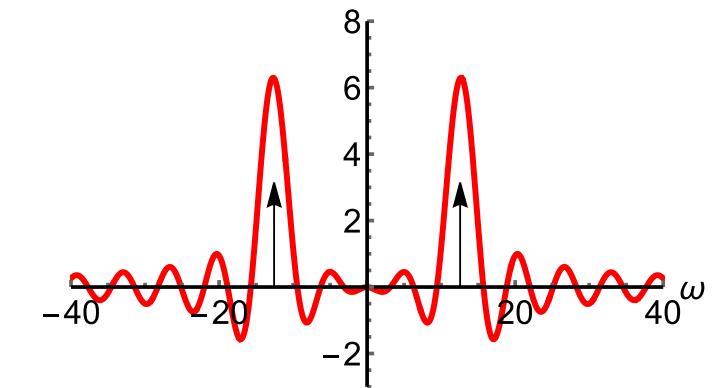
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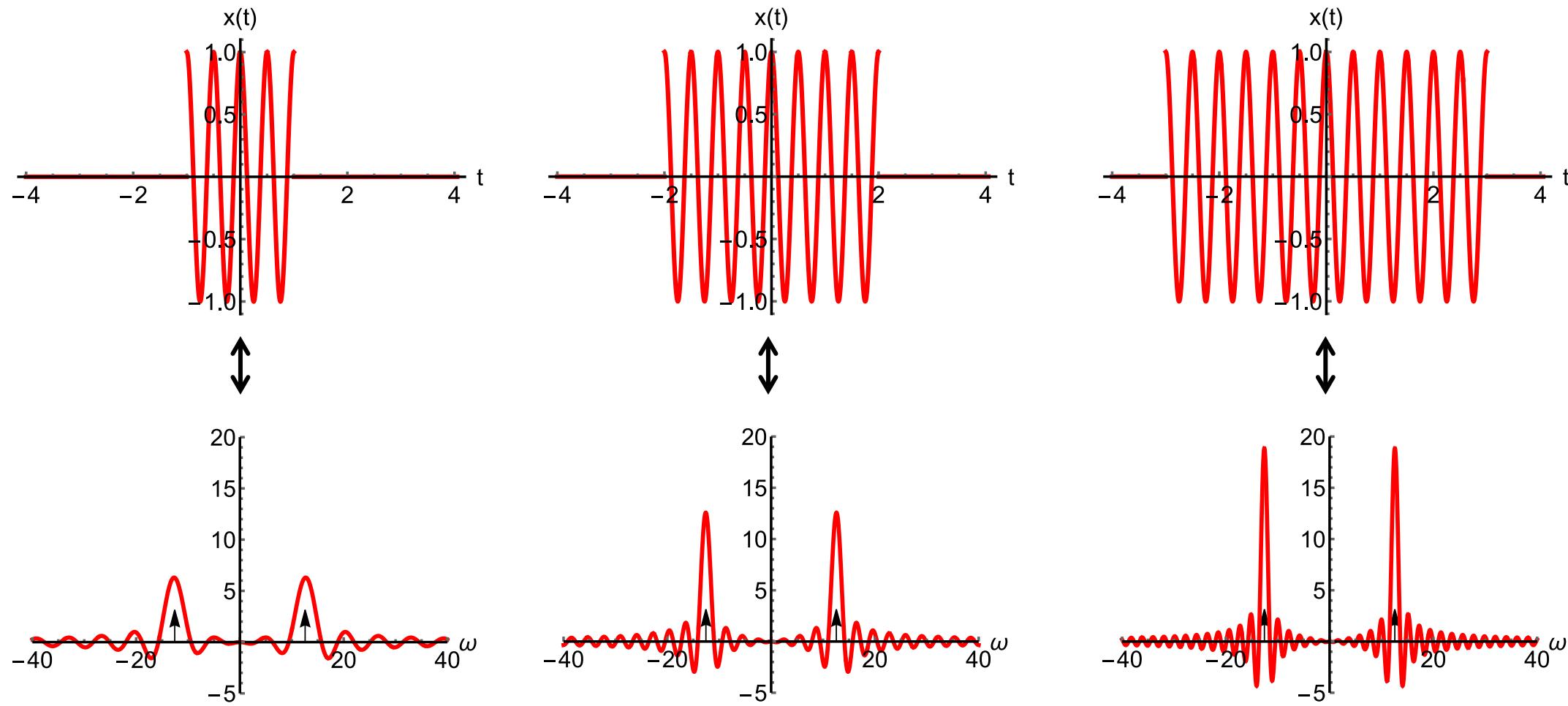


$$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\tau \cdot \text{sinc}(\omega\tau/2)$$

$$\begin{aligned} & \pi\tau \cdot \text{sinc}((\omega - \omega_0)\tau/2) \\ & + \pi\tau \cdot \text{sinc}((\omega + \omega_0)\tau/2) \end{aligned}$$

Convolution in the frequency domain



Leakage of energy

Temporal Integration

$$\int_{-\infty}^t x(\tau) d\tau = \int_{-\infty}^{\infty} x(\tau) u(t - \tau) d\tau = x(t) * u(t)$$

$$\leftrightarrow X(\omega) \cdot \left[\pi\delta(\omega) + \frac{1}{j\omega} \right] = \pi X(0)\delta(\omega) + \frac{X(\omega)}{j\omega}$$

Example: Capacitor voltage

$$v_C(t) = \frac{1}{C} \int_{-\infty}^t i_C(\tau) d\tau$$

$$V_C(\omega) = \frac{\pi I_C(0)\delta(\omega)}{C} + \frac{I_C(\omega)}{j\omega C}$$

In case of zero DC, e.g. sinusoidal signal:

$$V_C(\omega) = \frac{I_C(\omega)}{j\omega C}$$

Impedance of capacitor:

$$Z_C(\omega) \stackrel{\text{def}}{=} \frac{V_C(\omega)}{I_C(\omega)} = \frac{1}{j\omega C} = \frac{1}{|\omega|C} e^{-j\pi/2}$$

Temporal Differentiation

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \frac{d}{dt} e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} [j\omega X(\omega)] e^{j\omega t} d\omega \leftrightarrow j\omega X(\omega)\end{aligned}$$

Example: Inductor voltage:

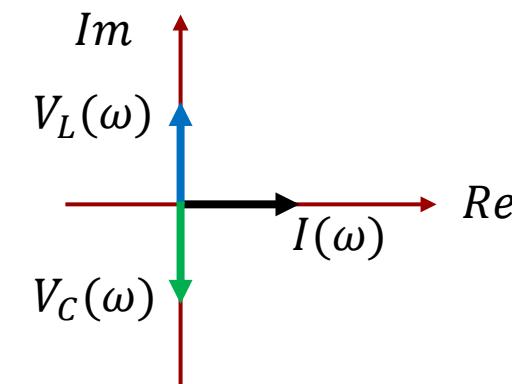
$$v_L(t) = L \frac{di_L}{dt} \leftrightarrow V_L(\omega) = j\omega L \cdot I_L(\omega)$$

Inductor impedance:

$$Z_L(\omega) = j\omega L = |\omega|L e^{j\pi/2}$$

The voltage drop across an inductor is 90 degrees ahead of the current. "Lead"

The voltage drop across a capacitor is 90 degrees behind the current. "Lag"



Application of the transform pair

$$\frac{d}{dt} \leftrightarrow j\omega$$

$$\ddot{y} + a_1\dot{y} + a_0y = b_2\ddot{x} + b_1\dot{x} + b_0x$$

$$F\{y(t)\} = Y(\omega) \quad F\{\dot{y}(t)\} = j\omega Y(\omega) \quad F\{\ddot{y}(t)\} = (j\omega)^2 Y(\omega)$$

$$[(j\omega)^2 + a_1 \cdot (j\omega) + a_0]Y(\omega) = [b_2 \cdot (j\omega)^2 + b_1 \cdot (j\omega) + b_0]X(\omega)$$

System frequency characteristic:

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{b_2 \cdot (j\omega)^2 + b_1 \cdot (j\omega) + b_0}{(j\omega)^2 + a_1 \cdot (j\omega) + a_0} = \frac{P(j\omega)}{Q(j\omega)}$$

We now have the means to Fourier transform a differential equation in order to obtain the frequency characteristic of the system.

Conversely, if we have the frequency characteristic, we can translate it into the differential equation for the system. In many cases it might be faster to derive $H(\omega)$ from an electric circuit diagram first and then obtain the differential equation from $H(\omega)$.

In cases where we are only interested in $H(\omega)$, we do not need to derive the differential equation.

Calculation of system response:

Time domain approach

$$x(t) \rightarrow * h(t) \rightarrow y(t)$$

Frequency domain approach

$$X(\omega) \rightarrow \cdot H(\omega) \rightarrow Y(\omega)$$

The impulse response $h(t)$ is of importance when we want to calculate system response in the time domain.

$$y(t) = h(t) * x(t) \quad \text{convolution}$$

If the purpose of the system is to attenuate certain frequencies, then the Fourier transform of the impulse response ($H(\omega)$) is a lot more useful.

$$Y(\omega) = X(\omega) \cdot H(\omega) = |X(\omega)| \cdot |H(\omega)| e^{j[\angle X(\omega) + \angle H(\omega)]}$$

Its modulus is the gain of the system as a function of frequency.

$$|Y(\omega)| = |X(\omega)| \cdot |H(\omega)| \quad \text{Amplitude response:}$$

Its angle reveals the time delay as a function of frequency.

$$\angle Y(\omega) = \angle X(\omega) + \angle H(\omega) \quad \text{Phase response:}$$

Distortion-less systems

A distortion-less system is one that only scales and delays the signal as it passes through the system:

Here $y(t)$ is a scaled and delayed copy of $x(t)$. Apart from these changes, the output is identical to the input, and the system is distortion free.

Time domain approach



Frequency domain approach

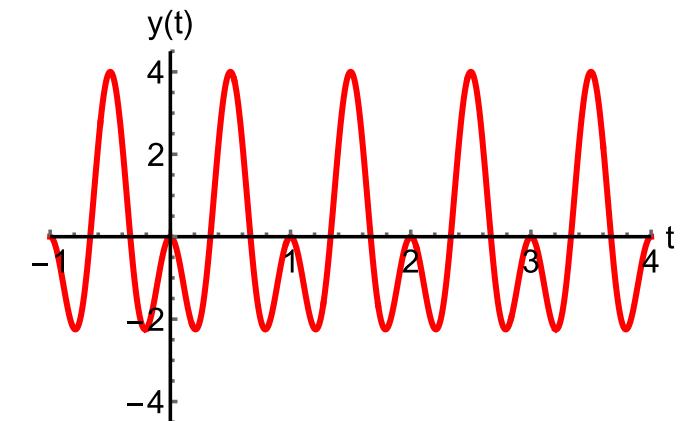
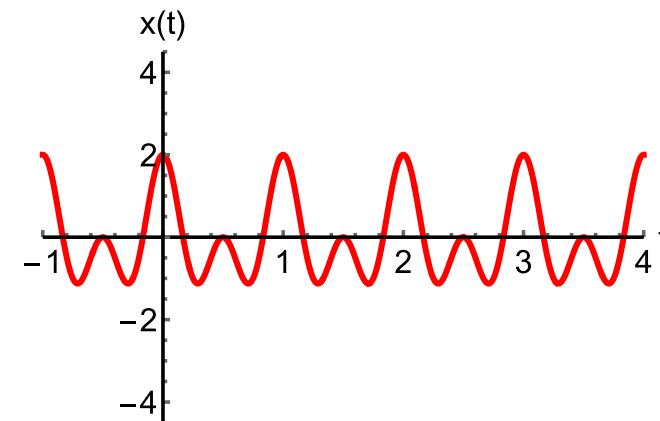


$$y(t) = k \cdot x(t - t_d)$$

F. ex.

$$x(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$$

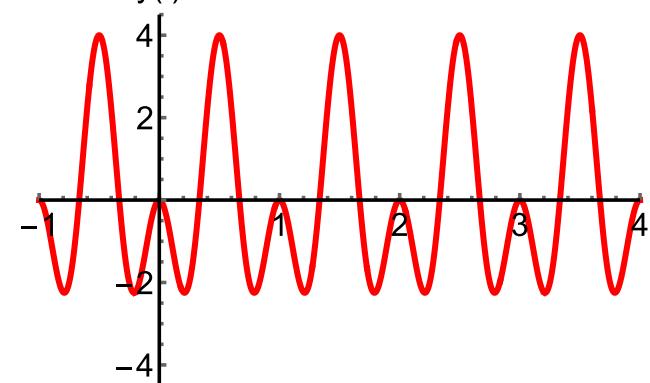
$$y(t) = k \cdot (\cos(\omega_1(t - t_d)) + \cos(\omega_2(t - t_d)))$$



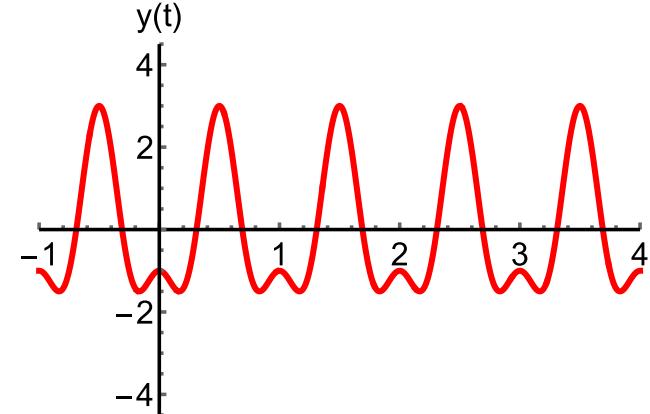
Amplitude and phase distortion

Amplitude distortion

$$y(t) = 2 \cdot \left(\cos\left(2\pi \cdot 1 \cdot \left(t - \frac{1}{2}\right)\right) + \cos\left(2\pi \cdot 2 \cdot \left(t - \frac{1}{2}\right)\right) \right)$$

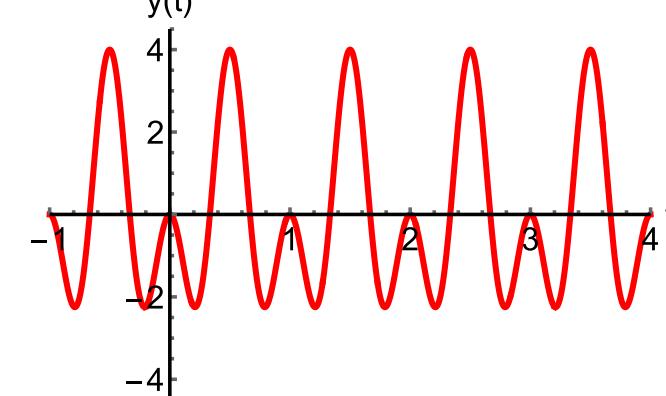


$$y(t) = 2 \cos\left(2\pi \cdot 1 \cdot \left(t - \frac{1}{2}\right)\right) + \cos\left(2\pi \cdot 2 \cdot \left(t - \frac{1}{2}\right)\right)$$

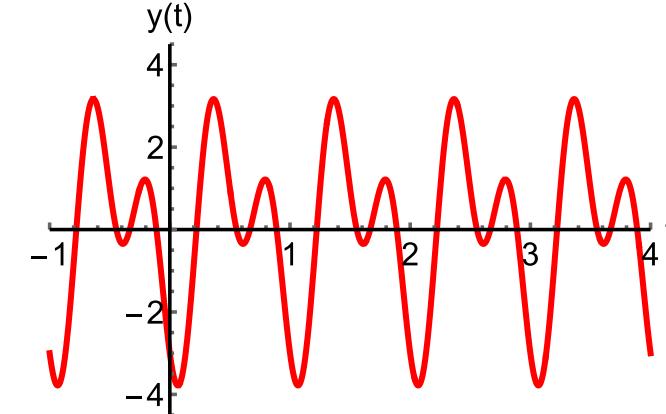


Phase distortion

$$y(t) = 2 \cdot \left(\cos\left(2\pi \cdot 1 \cdot \left(t - \frac{1}{2}\right)\right) + \cos\left(2\pi \cdot 2 \cdot \left(t - \frac{1}{2}\right)\right) \right)$$



$$y(t) = 2 \cdot \left(\cos\left(2\pi \cdot 1 \cdot \left(t - \frac{1}{2}\right)\right) + \cos\left(2\pi \cdot 2 \cdot \left(t - \frac{2}{3}\right)\right) \right)$$



A system with no amplitude distortion will have an amplitude characteristic:

$$|H(\omega)| = k$$

We also need to understand what is required of a system to avoid phase distortion, i.e., to delay all frequencies by the same constant time delay.

We introduce the phase angle into the expression for the signal and match it with the time delay.

We observe that the time delay is the negative derivative of the phase angle with respect to angular frequency.

To have constant time delay for all frequencies, the phase angle must be a linear function of frequency. The system is said to have a **linear phase characteristic**.

$$y(t) = k \cdot (\cos(\omega_1(t - t_d)) + \cos(\omega_2(t - t_d)))$$

$$y(t) = k \cdot (\cos(\omega_1 t + \theta(\omega_1)) + \cos(\omega_2 t + \theta(\omega_2)))$$

$$\theta(\omega) = -\omega t_d \Rightarrow \frac{d\theta(\omega)}{d\omega} = -t_d$$

$$t_d = -\frac{d\theta(\omega)}{d\omega}$$

$$t_d = -\frac{d\theta(\omega)}{d\omega} = \text{constant} \Rightarrow \theta(\omega) = \underbrace{\text{constant} \cdot \omega}_{\text{linear}}$$

Ideal and Practical Filters

Ideal filter: $H(\omega) = \text{rect}\left(\frac{\omega}{2W}\right) \cdot e^{-j\omega t_d}$

The ideal filter has a gain of 1 in the pass band, a gain of 0 in the stop band, and a linear phase.

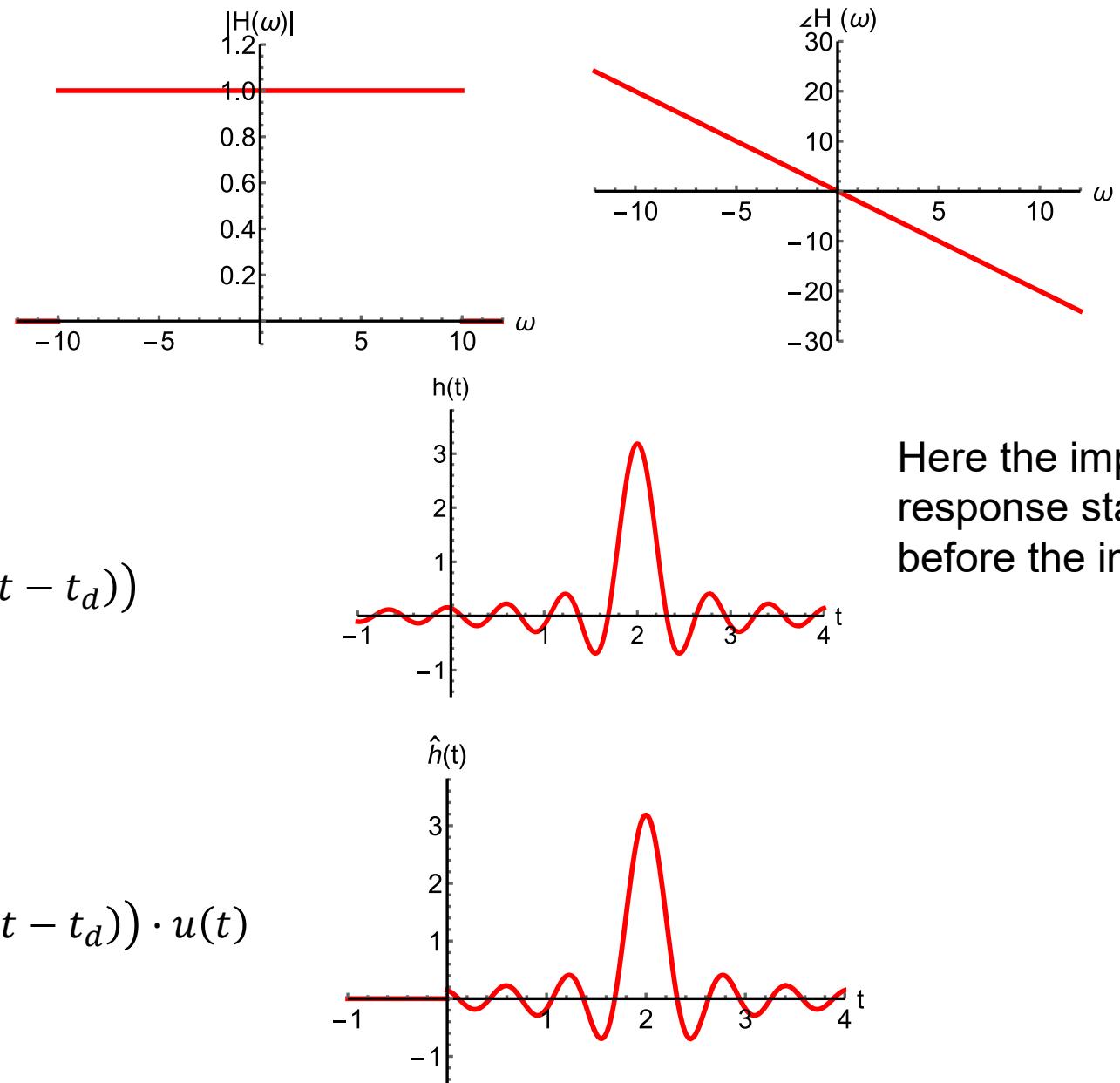
Non-causal filter is unrealizable:

$$h(t) = \frac{W}{\pi} \text{sinc}(W(t - t_d))$$

Practical filter:

Causal filter is realizable:

$$\hat{h}(t) = \frac{W}{\pi} \text{sinc}(W(t - t_d)) \cdot u(t)$$

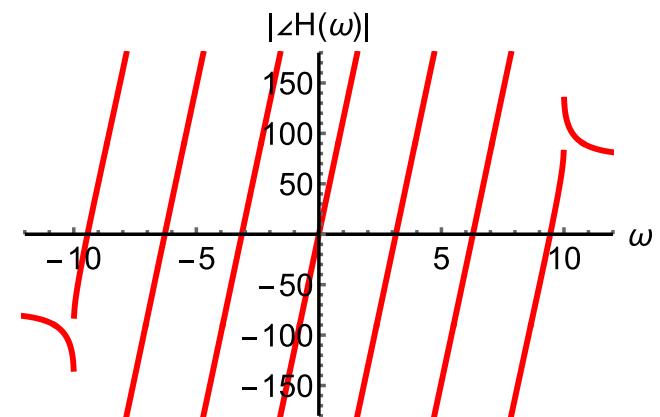
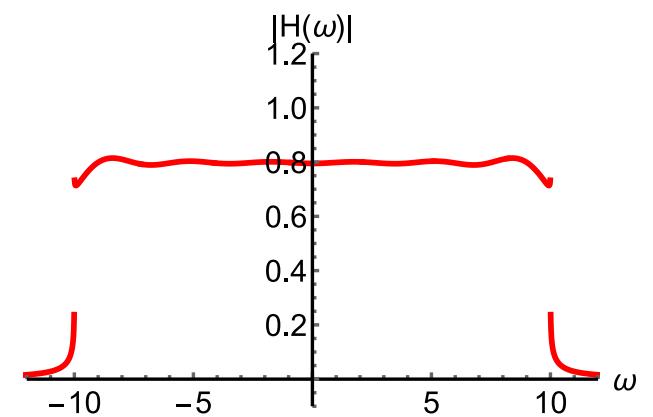
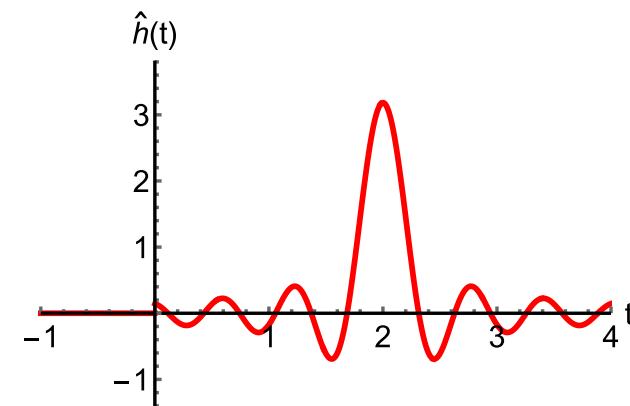


Here the impulse response starts before the impulse.

$$\hat{h}(t) = \frac{W}{\pi} \operatorname{sinc}(W(t - t_d)) \cdot u(t)$$

The consequence of truncating the impulse response at $t = 0$ is shown here.

We get ripple on the amplitude characteristic and the phase characteristic is not linear.



Problems

Problem 1

Derive these results
using pen and paper:

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = ?$$

Answer

1

$$F^{-1}\{\delta(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = ?$$

 $\frac{1}{2\pi}$

$$F^{-1}\{2\pi\delta(\omega)\} = \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = ?$$

1

$$F^{-1}\{2\pi\delta(\omega - \omega_0)\} = \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = ?$$

 $e^{j\omega_0 t}$

$$F^{-1}\{\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))\} = ?$$

$$\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} = \cos(\omega_0 t)$$

Problem 2

Derive these results
using Maple/Python:

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = ?$$

Answer

1

$$F^{-1}\{\delta(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = ?$$

 $\frac{1}{2\pi}$

$$F^{-1}\{2\pi\delta(\omega)\} = \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = ?$$

1

$$F^{-1}\{2\pi\delta(\omega - \omega_0)\} = \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = ?$$

 $e^{j\omega_0 t}$

$$F^{-1}\{\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))\} = ?$$

$$\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} = \cos(\omega_0 t)$$

Problem 3

Observe the two functions plotted here.

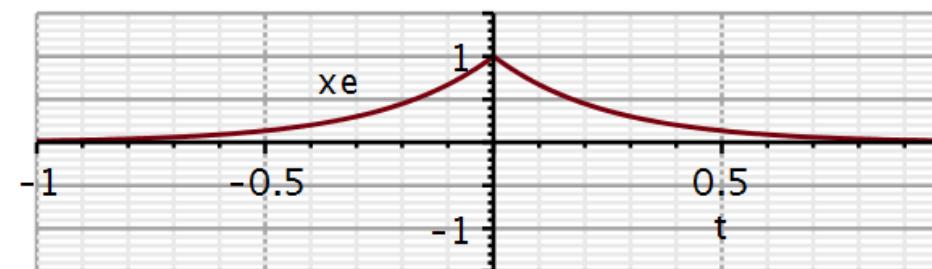
x_e is even and x_o is odd.

1. What symmetries do you expect in the Fourier transforms?
2. Calculate their Fourier transforms and verify your expectations.

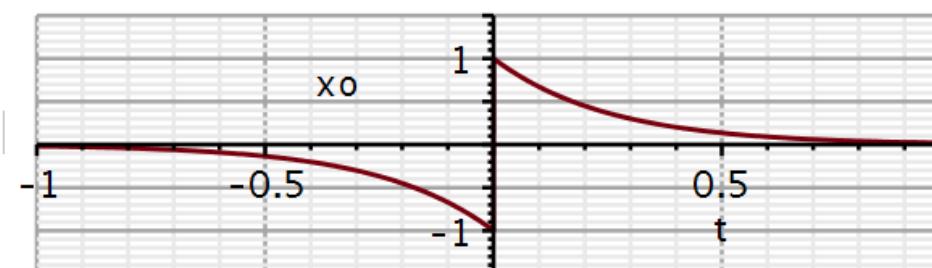
$$xe := e^{-4 \cdot |t|} ;$$

$$xo := -e^{-4 \cdot |t|} \cdot \text{Heaviside}(-t) + e^{-4 \cdot |t|} \cdot \text{Heaviside}(t) ;$$

```
plot(xe, t=-1..1, -1.5..1.5, thickness = 3, axesfont = [Helvetica, "roman", 18], axis[2] = [thickness = 2.5], axis[1] = [thickness = 2.5], labels = ["t", "xe"], labelfont = [Helvetica, 18], numpoints = 100, gridlines, size = [600, 200])
```



```
plot(xo, t=-1..1, -1.5..1.5, thickness = 3, axesfont = [Helvetica, "roman", 18], axis[2] = [thickness = 2.5], axis[1] = [thickness = 2.5], labels = ["t", "xo"], labelfont = [Helvetica, 18], numpoints = 100, gridlines, size = [600, 200])
```



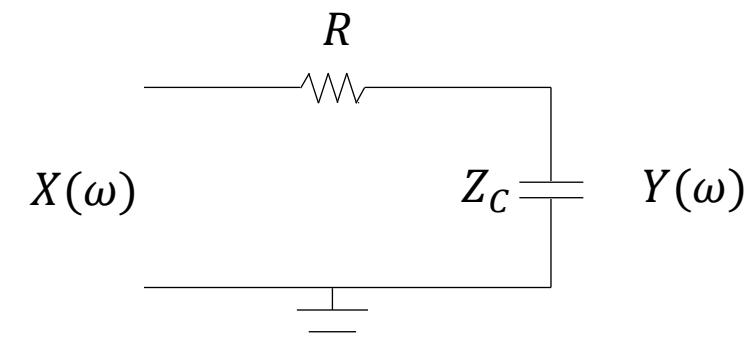
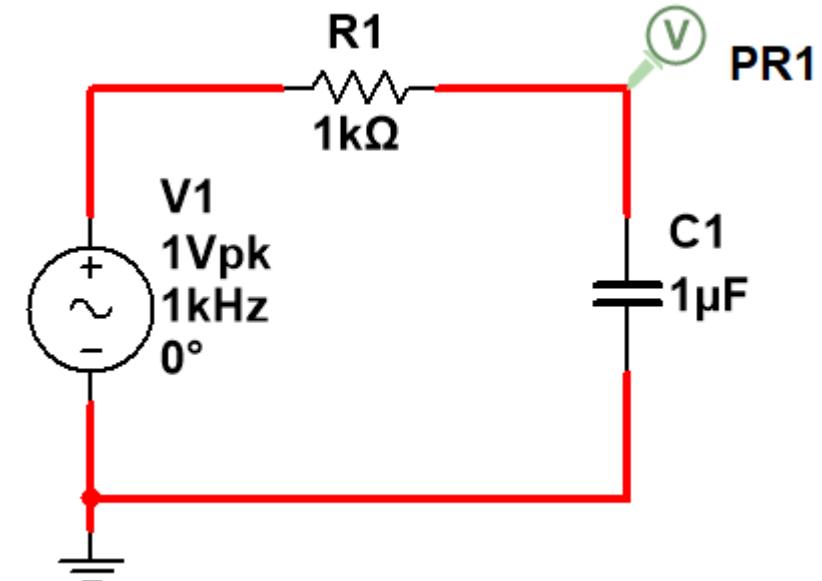
Problem 4. RC circuit analysis in frequency domain

1. Sketch this RC circuit in KiCad/Spice.
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3. With pen and paper, redraw the circuit with the capacitor replaced by its impedance.
4. Show that the input-output relationship can be expressed as a voltage divider:

$$Y(\omega) = \frac{Z_C(\omega)}{R + Z_C(\omega)} X(\omega)$$

5. Insert the expression for the impedance of a capacitor and show that:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{j\omega RC + 1} = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}$$



Problem 4. RC circuit analysis in frequency domain

6. With pen and paper, determine the symmetry properties of the amplitude characteristic and the phase characteristic.
7. Plot the magnitude and phase plots in Maple/Python
8. Notice how different the curves in KiCad/Spice looks. Always take notice of the type of axes used (linear, log, decibel).
9. Use semilogplot to plot the spectra on logarithmic frequency axes in Hz. In this case, plot only positive frequencies. Plot amplitude in decibels.
10. Calculate the inverse Fourier transform of $H(\omega)$ and plot it.
11. Replace the 1k resistor with a 500 Ohm resistor. Repeat questions 7 and 10.
12. Compare the two cases and explain using the time scaling property of Fourier transformation.
13. Plot the time delay as a function of frequency in Maple/Python

Problem solutions

Problem 1

Derive these results
using pen and paper:

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = ?$$

Answer

1

$$F^{-1}\{\delta(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = ?$$

 $\frac{1}{2\pi}$

$$F^{-1}\{2\pi\delta(\omega)\} = \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = ?$$

1

$$F^{-1}\{2\pi\delta(\omega - \omega_0)\} = \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = ?$$

 $e^{j\omega_0 t}$

$$F^{-1}\{\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))\} = ?$$

$$\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} = \cos(\omega_0 t)$$

Problem 1 (sol)

Derive these results
using pen and paper:

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega_0 t} dt = 1$$

$$F^{-1}\{\delta(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega_0 t} d\omega = \frac{1}{2\pi}$$

$$F^{-1}\{2\pi\delta(\omega)\} = \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega_0 t} d\omega = 1$$

$$F^{-1}\{2\pi\delta(\omega - \omega_0)\} = \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega_0 t} d\omega = e^{j\omega_0 t}$$

$$F^{-1}\{\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))\} = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} = \cos(\omega_0 t)$$

Problem 2

Derive these results
using Maple:

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = ?$$

Answer

1

$$F^{-1}\{\delta(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = ?$$

 $\frac{1}{2\pi}$

$$F^{-1}\{2\pi\delta(\omega)\} = \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = ?$$

1

$$F^{-1}\{2\pi\delta(\omega - \omega_0)\} = \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = ?$$

 $e^{j\omega_0 t}$

$$F^{-1}\{\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))\} = ?$$

$$\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} = \cos(\omega_0 t)$$

Problem 2 (sol)

Answer

Derive these results using Maple:

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = ?$$

→ $x := (\text{Heaviside}(t + 0.5) - \text{Heaviside}(t - 0.5)) :$
 $\delta := t \rightarrow \text{Dirac}(t) :$
 $\text{sgn} := t \rightarrow 2 \cdot \text{Heaviside}(t) - 1 :$

$$F^{-1}\{\delta(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = ?$$

fourier($\text{Dirac}(t), t, \omega$)
 $\frac{1}{2\pi}$

$$F^{-1}\{2\pi\delta(\omega)\} = \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = ?$$

invfourier($2\pi \cdot \delta(\omega), \omega, t$)
 $\frac{1}{2\pi}$

$$F^{-1}\{2\pi\delta(\omega - \omega_0)\} = \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = ?$$

invfourier($2\pi \cdot \delta(\omega - \omega_0), \omega, t$)
 $e^{j\omega_0 t}$

$$F^{-1}\{\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))\} = ?$$

invfourier($\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)), \omega, t$)
 $\cos(\omega_0 t)$

Problem 3

Observe the two functions plotted here.

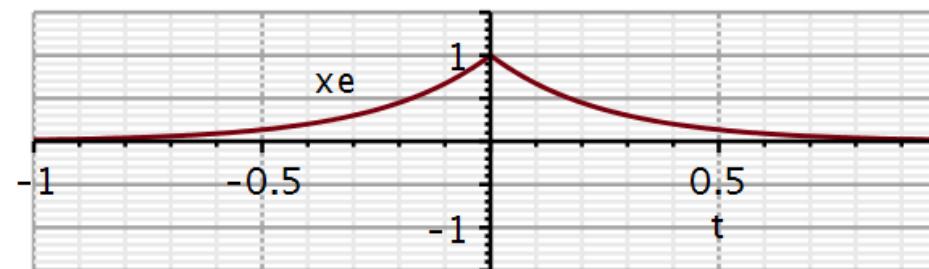
x_e is even and x_o is odd.

1. What symmetries do you expect in the Fourier transforms?
2. Calculate their Fourier transforms and verify your expectations.

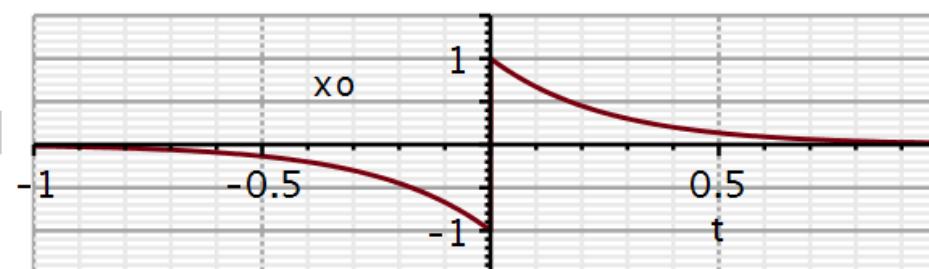
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$$xo := -e^{-4 \cdot |t|} \cdot \text{Heaviside}(-t) + e^{-4 \cdot |t|} \cdot \text{Heaviside}(t) :$$

```
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= 100, gridlines, size = [600, 200])
```



```
plot(xo, t=-1..1, -1.5..1.5, thickness = 3, axesfont = [Helvetica,
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```



Problem 3 (sol)

Theory predicts real and even functions to have a Fourier transform that is **real and even** in ω .

Theory also predicts real and odd functions to have a Fourier transform that is **imaginary and odd** in ω .

$$\begin{aligned}
 X(\omega) &= \int_{-\infty}^{\infty} (x_e(t) + x_o(t))(\cos \omega t + j \sin \omega t) dt \\
 &= \underbrace{\int_{-\infty}^{\infty} (x_e(t))(\cos \omega t) dt}_{\text{real part is even in } \omega} + \underbrace{\int_{-\infty}^{\infty} (x_e(t))(j \sin \omega t) dt}_{=0} \\
 &\quad + \underbrace{\int_{-\infty}^{\infty} (x_o(t))(\cos \omega t) dt}_{=0} + j \underbrace{\int_{-\infty}^{\infty} (x_o(t))(\sin \omega t) dt}_{\text{imaginary part is odd in } \omega}
 \end{aligned}$$

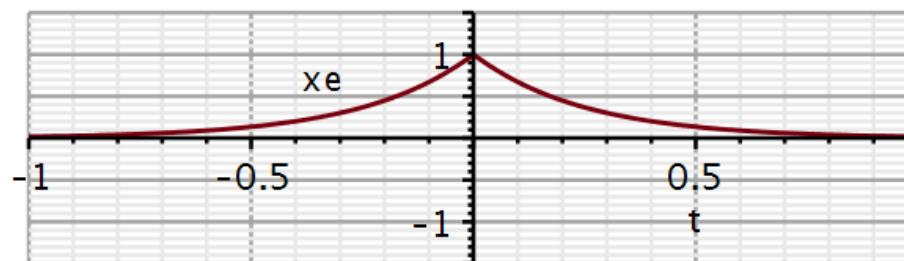
$$\left. \begin{array}{l} Re\{X(\omega)\} = Re\{X(-\omega)\} \\ Im\{X(\omega)\} = -Im\{X(-\omega)\} \end{array} \right\} \Rightarrow \left\{ \begin{array}{ll} |X(\omega)| = |X(-\omega)| & \text{Even function} \\ \angle X(\omega) = -\angle X(-\omega) & \text{Odd function} \end{array} \right.$$

Problem 3 (sol)

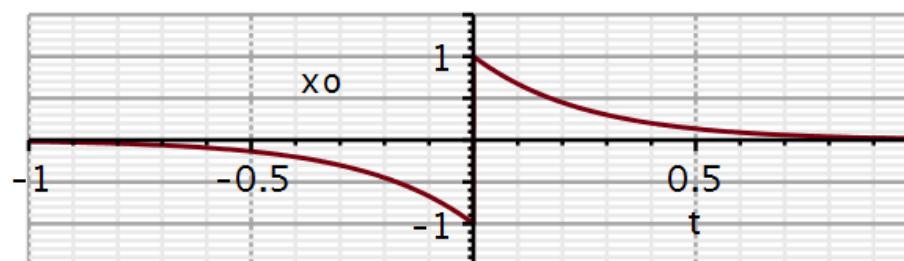
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```
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    = 2.5], labels = ["t", "xo"], labelfont = [Helvetica, 18], numpoints
    = 100, gridlines, size = [600, 200])
```



Calculating Fourier transform

$$X := \text{fourier}(xe, t, \omega)$$

$$X := \frac{8}{\omega^2 + 16}$$

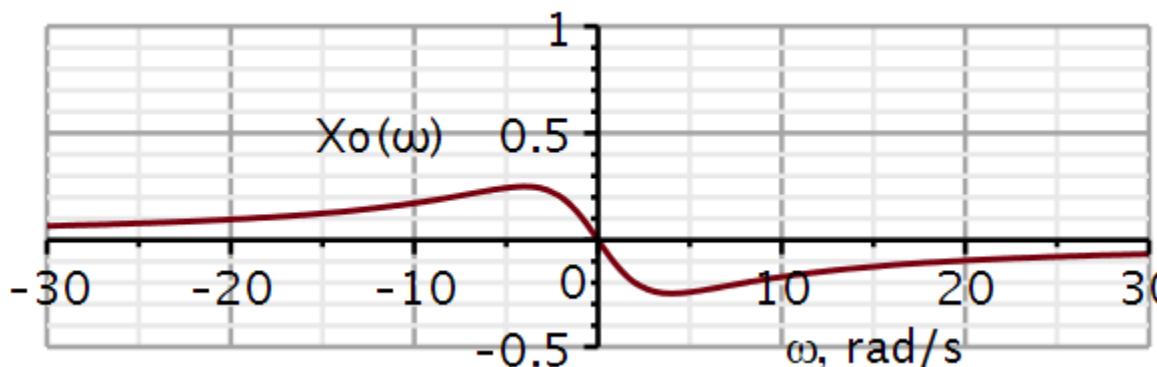
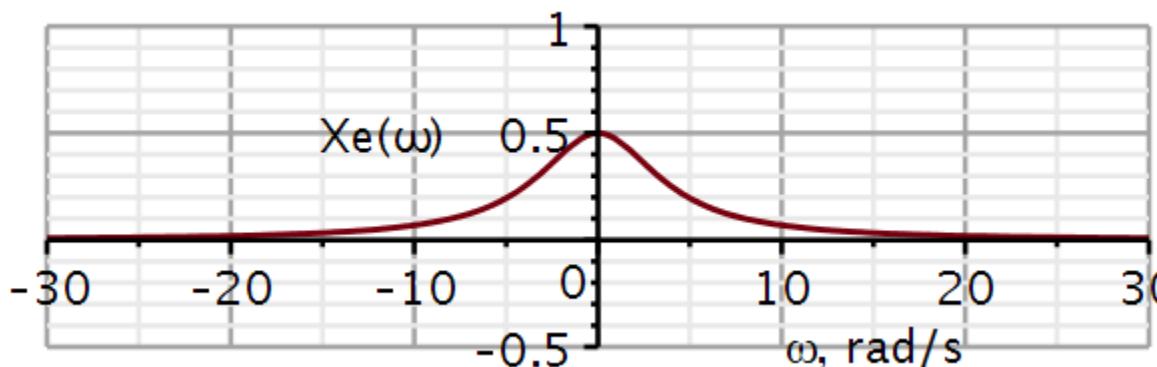
$$X := \text{fourier}(xo, t, \omega)$$

$$X := \frac{-2 \text{I} \omega}{\omega^2 + 16}$$

For an even function, the *FT* is real and even.

For an odd function, the *FT* is imaginary and odd.

Problem 3 (sol)

**Calculating Fourier transform**

$$X := \text{fourier}(xe, t, \omega)$$

$$X := \frac{8}{\omega^2 + 16}$$

$$X := \text{fourier}(xo, t, \omega)$$

$$X := \frac{-2 \operatorname{I} \omega}{\omega^2 + 16}$$

For an even function, the *FT* is real and even.

For an odd function, the *FT* is imaginary and odd.

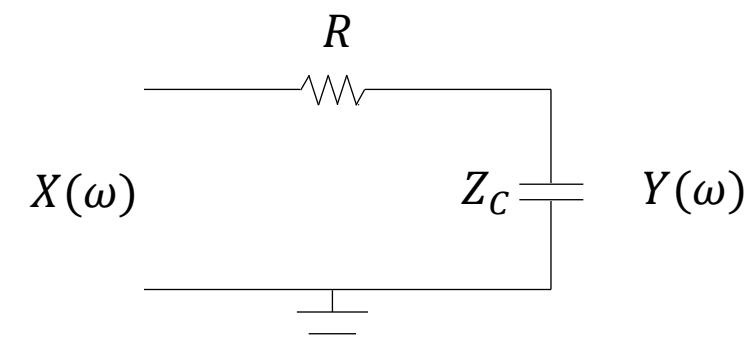
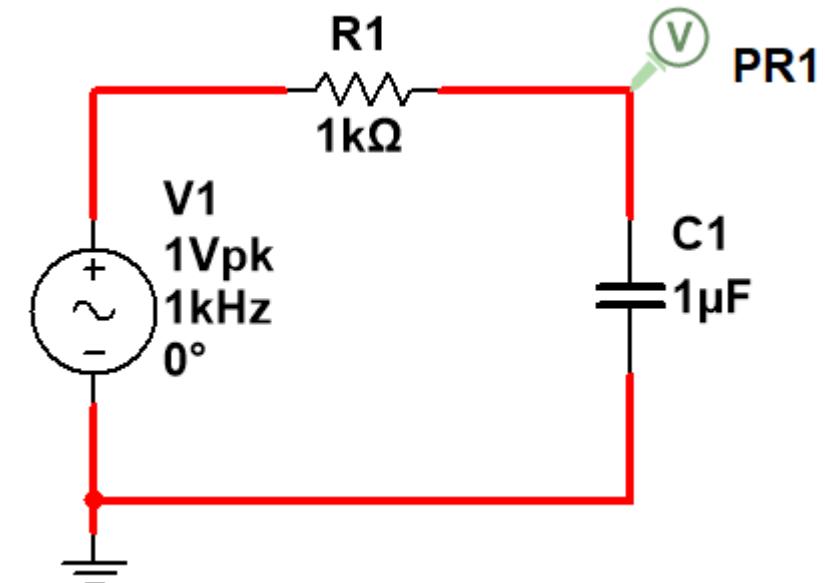
Problem 4. RC circuit analysis in frequency domain

1. Sketch this RC circuit in KiCad/Spice.
2. Run an AC sweep in the frequency range from 1 Hz to 10 kHz and plot magnitude in decibel and phase in degrees.
3. With pen and paper, redraw the circuit with the capacitor replaced by its impedance.
4. Show that the input-output relationship can be expressed as a voltage divider:

$$Y(\omega) = \frac{Z_C(\omega)}{R + Z_C(\omega)} X(\omega)$$

5. Insert the expression for the impedance of a capacitor and show that:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{j\omega RC + 1} = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}$$

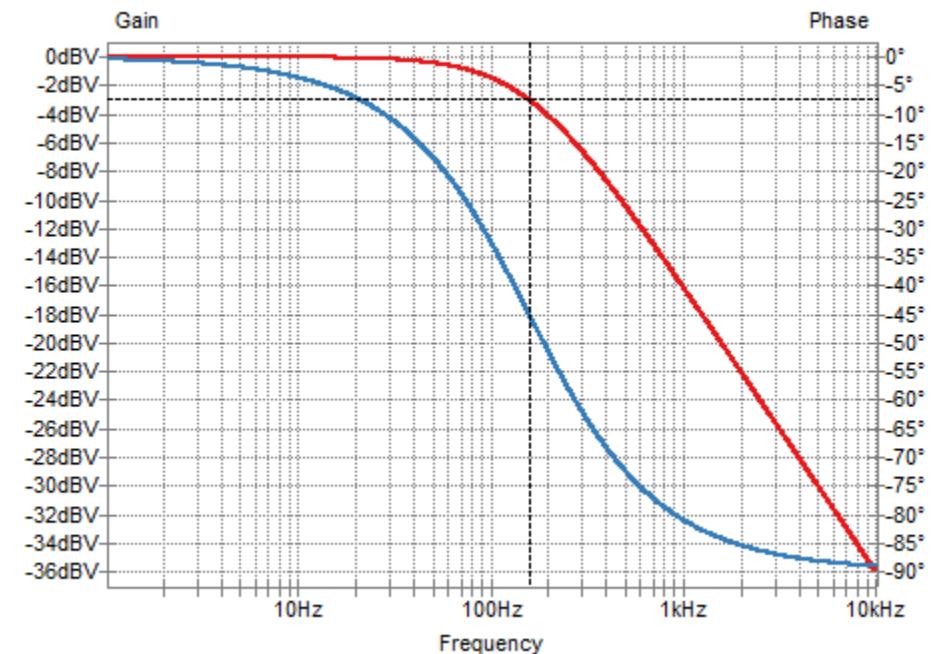
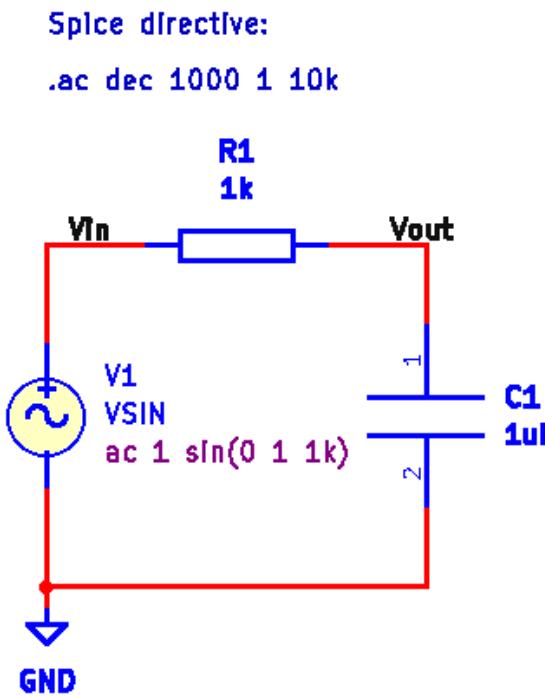


Problem 4. RC circuit analysis in frequency domain

6. With pen and paper, determine the symmetry properties of the amplitude characteristic and the phase characteristic.
7. Plot the magnitude and phase plots in Maple/Python.
8. Notice how different the curves in KiCad/Spice looks. Always take notice of the type of axes used (linear, log, decibel).
9. Use semilogplot to plot the spectra on logarithmic frequency axes in Hz. In this case, plot only positive frequencies. Plot amplitude in decibels.
10. Calculate the inverse Fourier transform of $H(\omega)$ and plot it.
11. Replace the 1k resistor a 500 Ohm resistor. Repeat questions 7 and 10.
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Problem 4. RC circuit analysis in frequency domain (sol)

1. Sketch this RC circuit in KiCad/Spice.
2. Run an AC sweep in the frequency range from 1 Hz to 10 kHz and plot magnitude in decibel and phase in degrees.



Problem 4. RC circuit analysis in frequency domain (sol)

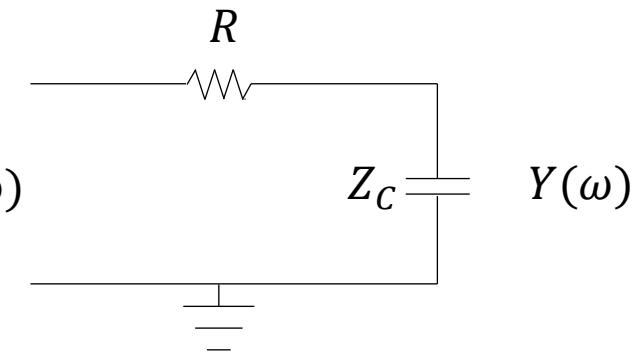
If two resistors in series (R_1, R_2) are set up to divide a total voltage drop of V_{in} , then the voltage output across R_2 is:

$$V_{out} = \frac{R_2}{R_1 + R_2} V_{in}$$

Here R_2 is replaced by the impedance of a capacitor:

$$4: Y(\omega) = \frac{Z_C(\omega)}{R + Z_C(\omega)} X(\omega)$$

Inserting $Z_C(\omega) = \frac{1}{j\omega C}$



$$Y(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{j\omega RC + 1}$$

$$5: H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}$$

Problem 4. RC circuit analysis in frequency domain (sol)

6. With pen and paper, determine the symmetry properties of the amplitude characteristic and the phase characteristic.
7. Plot the magnitude and phase plots in Maple/Python as functions of ω and using linear x- and y axes.
8. Notice how different the curves in KiCad/Spice looks. Always take notice of the type of axes used (linear, log, decibel).
9. Use semilogplot to plot the spectra on logarithmic frequency axes in Hz. In this case, plot only positive frequencies. Plot amplitude in decibels.
10. Calculate the inverse Fourier transform of $H(\omega)$ and plot it.

$$\frac{e^{j\theta_n}}{e^{j\theta_d}} = e^{j(\theta_n - \theta_d)}$$

$$5: H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}$$

$$6: |H(\omega)| = \frac{\left| \frac{1}{RC} \right|}{\left| j\omega + \frac{1}{RC} \right|} = \frac{1}{|j\omega RC + 1|} = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

$$6: \angle H(\omega) = \angle 1 - \angle(j\omega RC + 1)$$

$$6: \angle H(\omega) = 0 - \tan^{-1} \frac{\omega RC}{1}$$

$$6: \angle H(\omega) = -\tan^{-1} \omega RC$$

Tan and inv Tan are odd functions

Problem 4. RC circuit analysis in frequency domain (sol)

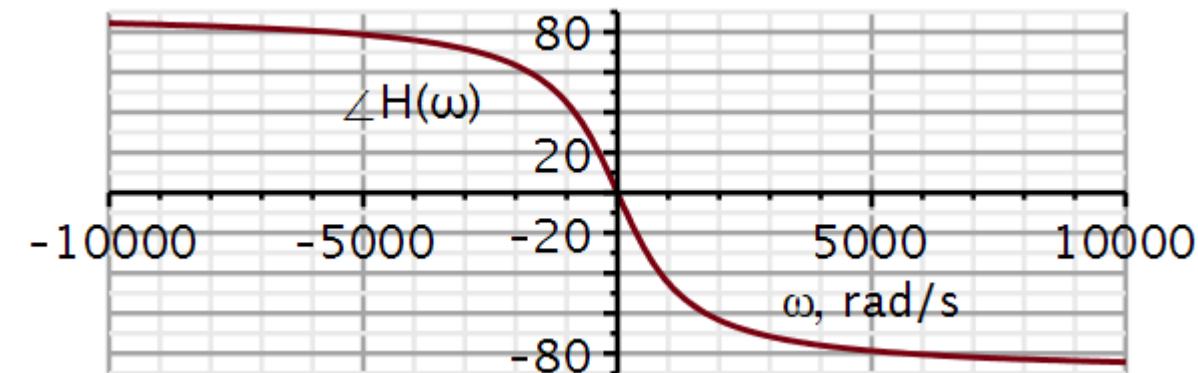
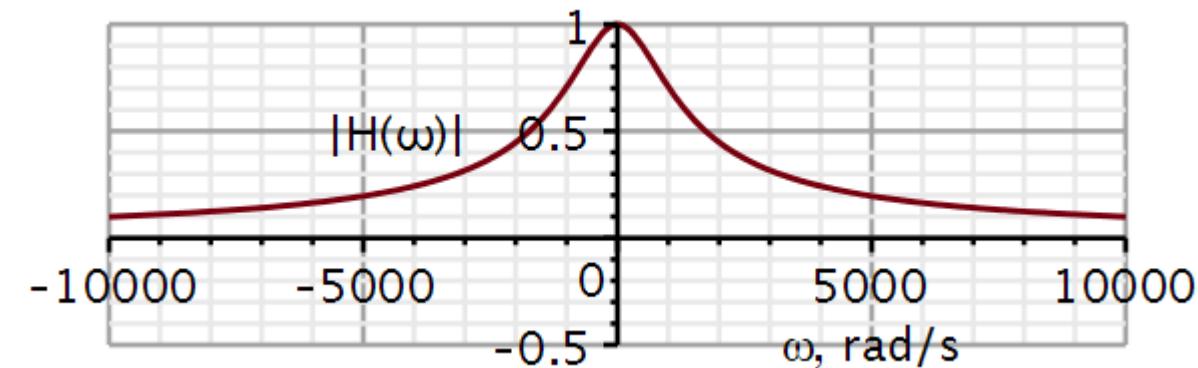
Question 7.

Transfer function

$$H := \omega \rightarrow \frac{1}{j \cdot \omega \cdot R \cdot C + 1} :$$

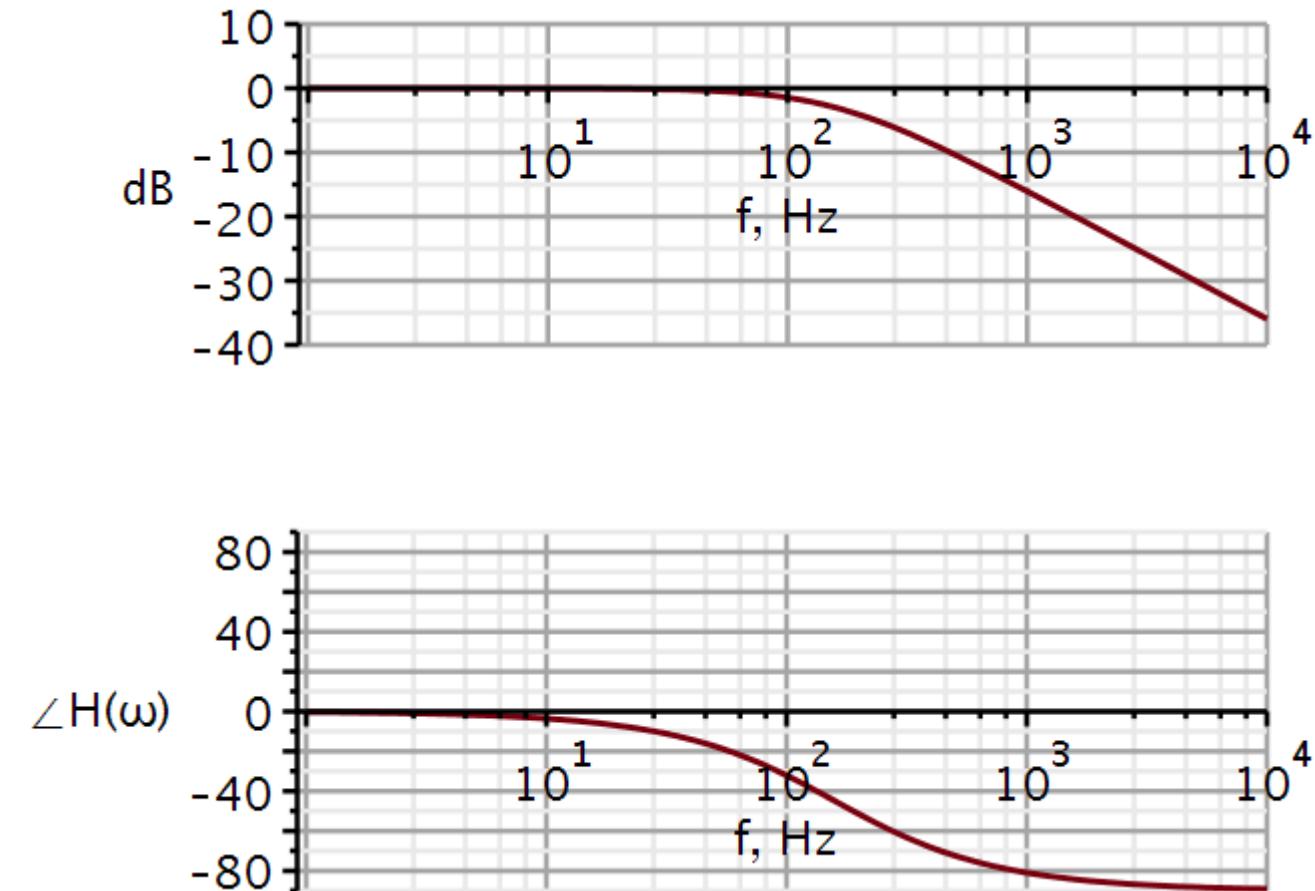
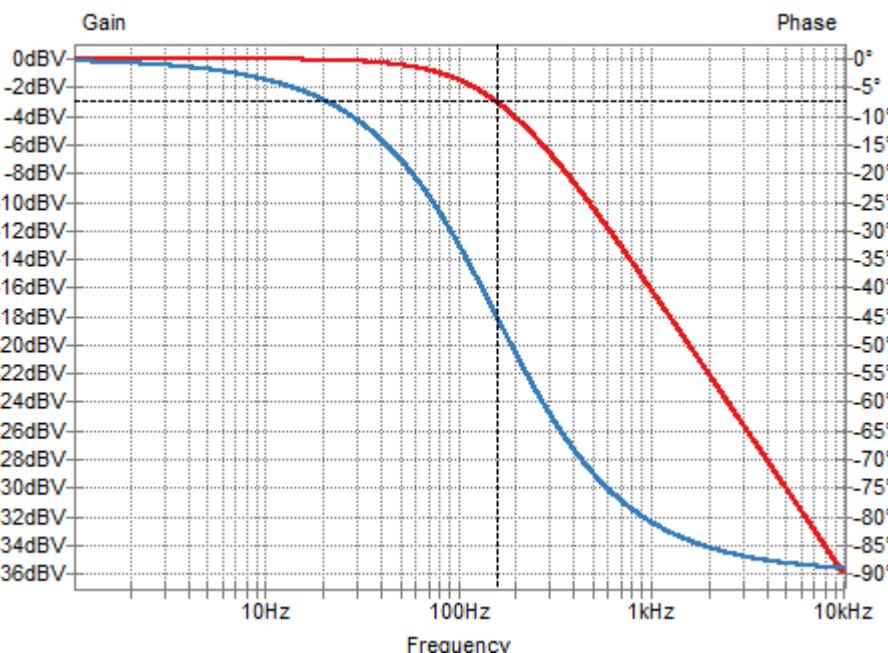
$$R := 1000 ;; C := 1E-6 ;;$$

Because we have plotted on linear axes, the curves looks very different from those obtained in KiCad/Spice.



Problem 4. RC circuit analysis in frequency domain (sol)

9. Use semilogplot to plot the spectra on logarithmic frequency axes in Hz. In this case, plot only positive frequencies. Plot amplitude in decibels.

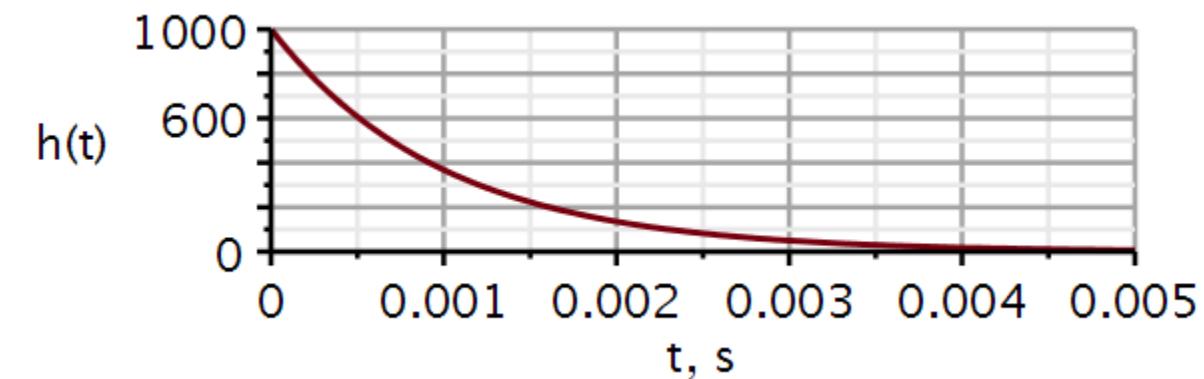


Problem 4. RC circuit analysis in frequency domain (sol)

10. Calculate the inverse Fourier transform of $H(\omega)$ and plot it.

$$h := \text{invfourier}\left(\frac{1}{j\cdot\omega\cdot R\cdot C + 1}, \omega, t\right)$$

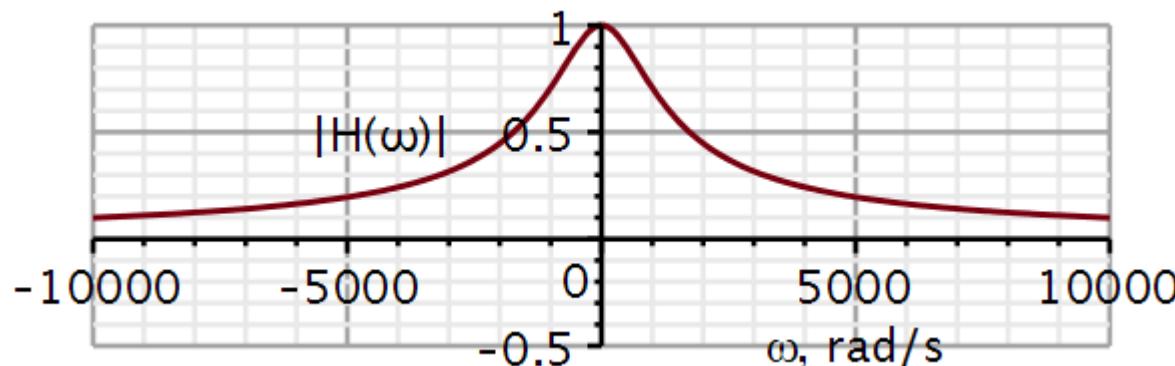
$$h := 1000 \cdot e^{-1000 \cdot t} \text{Heaviside}(t)$$



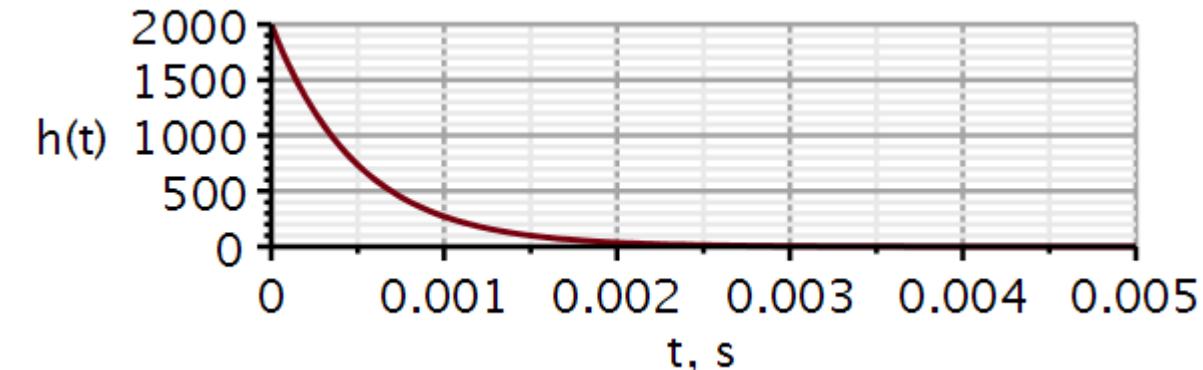
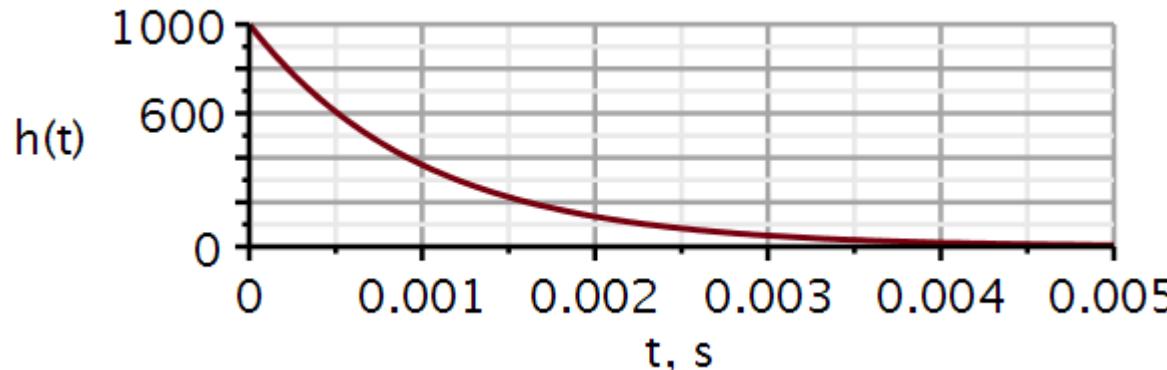
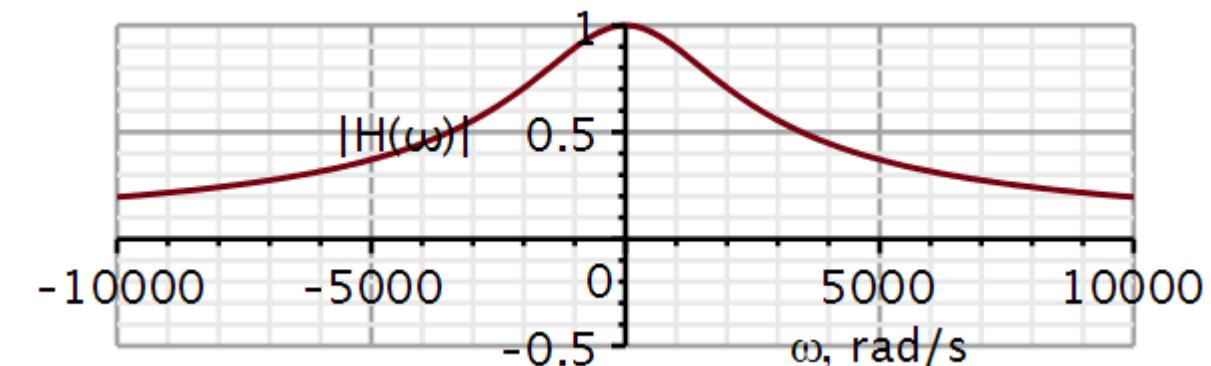
Problem 4. RC circuit analysis in frequency domain (sol)

11. Replace the 1k resistor with a 500 Ohm resistor. Repeat questions 7 and 10.

$$R = 1k\Omega$$



$$R = 500 \Omega$$



Problem 4. RC circuit analysis in frequency domain (sol)

12. Compare the two cases and explain using the time scaling property of Fourier transformation.

$$R_0 = 1k\Omega$$

$$R = 500\Omega = \frac{R_0}{2}$$

The impulse response is:

$$h_0(t) = \frac{1}{R_0 C} e^{-t/R_0 C}$$

$$h_1(t) = \frac{2}{R_0 C} e^{-2t/R_0 C}$$

The Fourier transform is:

$$H_0(\omega) = \frac{1}{j \omega R_0 C + 1}$$

$$H_1(\omega) = \frac{1}{j \omega \frac{R_0 C}{2} + 1} = \frac{1}{j \frac{\omega}{2} R_0 C + 1}$$

$$f(at) \leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

$$|a|f(at) \leftrightarrow F\left(\frac{\omega}{a}\right)$$

The time scaling property is very clear. If we scale the time by a factor of 2, then ω is divided by a factor of 2. Hence a wider band width (500Ω) in the frequency domain yields a shorter lasting response in the time domain, - as illustrated in the previous slide.

Problem 4. RC circuit analysis in frequency domain (sol)

13. Plot the time delay as a function of frequency in Maple/Python.

The derivative of arctan():

$$z := \arctan(a t)$$

$$\dot{z}$$

$$z := \arctan(a t)$$

$$\frac{a}{a^2 t^2 + 1}$$

At 0 Hz we get a time delay of $RC = 0.5\text{ms}$, i.e., the full effect of the capacitor. As $f \rightarrow \infty$, the capacitor becomes a short circuit, i.e., a wire, and we are left with a resistor circuit with no time delaying properties.

At the cutoff frequency ($\omega = 1/RC$), we get

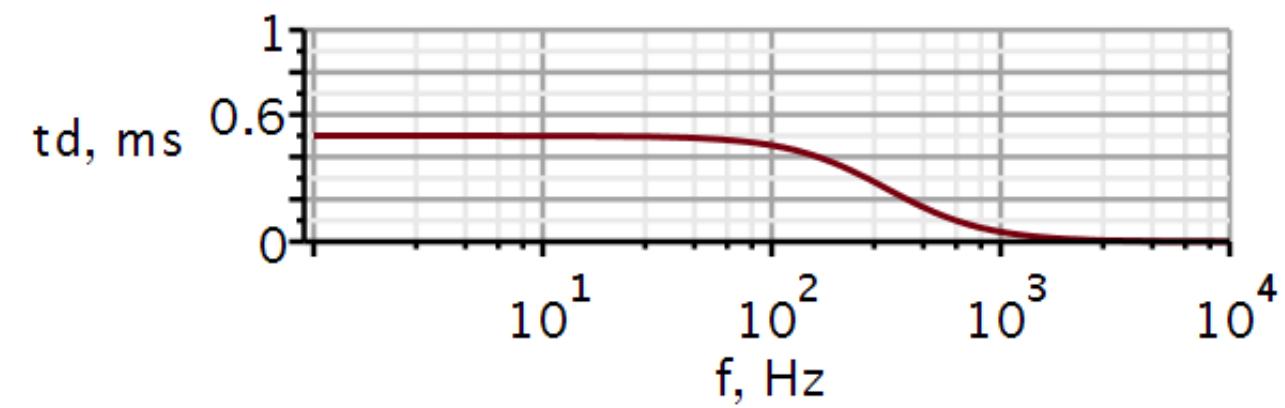
$$t_d = \frac{RC}{2} = 0.25\text{ms}$$

$$6: \angle H(\omega) = -\tan^{-1} \omega RC$$

$$t_d = -\frac{d}{d\omega} \angle H(\omega) = -\frac{d}{d\omega} (-\tan^{-1} \omega RC)$$

$$t_d = \frac{d}{d\omega} (\tan^{-1} \omega RC) = \frac{1}{(\omega RC)^2 + 1} \cdot \frac{d}{d\omega} (\omega RC)$$

$$t_d = \frac{d}{d\omega} (\tan^{-1} \omega RC) = \frac{RC}{(\omega RC)^2 + 1}$$



22050 Continuous-Time Signals and Linear Systems

Kaj-Åge Henneberg

L06

Impulse trains

Sampling

Applications of Fourier transformation

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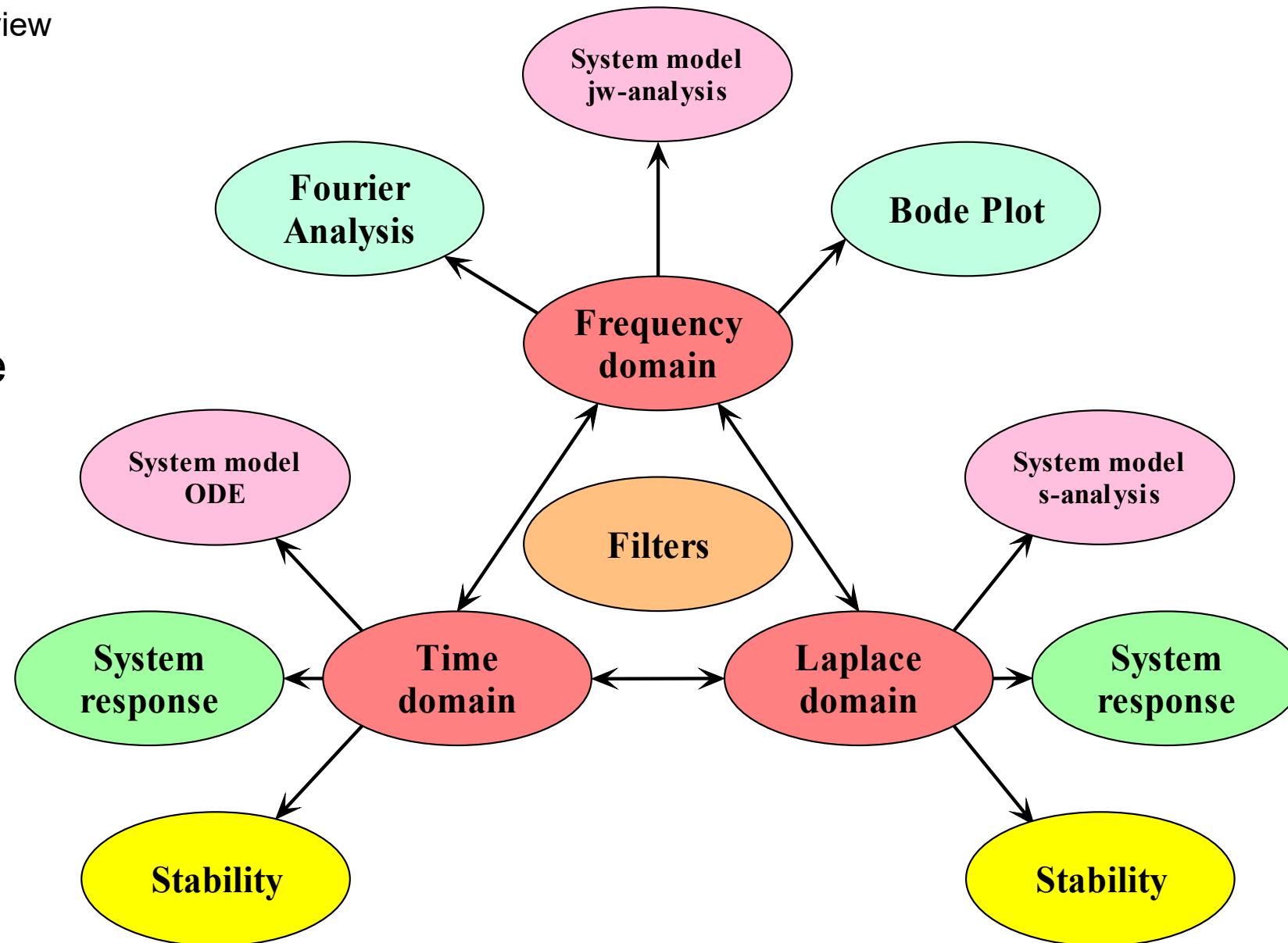
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Course overview

- System
 - Model
 - Response
 - Performance
- Domains
 - Time
 - Frequency
 - s-domain



- Review of Fourier transformation
- Fourier transform of the impulse train
- Convolution with an impulse train
- Multiplication with an impulse train (sampling)
- Ideal and practical sampling
- Ideal and practical reconstruction from samples
- Amplitude and phase spectra
- Fourier analysis of filter circuits
 - First order lowpass filter
 - Nodal equations in frequency domain
 - Asymptote analysis
 - Frequency scaling
 - Impedance scaling
 - AC sweep
- First order highpass
 - Nodal equations in frequency domain
 - Asymptote analysis
 - Frequency scaling
 - Impedance scaling
 - AC sweep

Review: Transform pair

With this set of transforms we can go back and forth between time domain ($x(t)$) and frequency domain ($X(\omega)$).

We say that $X(\omega)$ is the Fourier transform of $x(t)$, and that $x(t)$ is the inverse Fourier transform of $X(\omega)$.

As $X(\omega)$ is a continuous density distribution of sinusoidal basis functions, $X(\omega)$ is also called a **frequency spectrum**.

The spectrum is a complex valued function of frequency with modulus and angle:

Forward
Transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Analysis

Inverse
Transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Synthesis

$$X(\omega) = |X(\omega)| e^{j\angle X(\omega)} \quad : \text{Frequency spectrum}$$

$$|X(\omega)| \quad : \text{Amplitude spectrum}$$

$$\angle X(\omega) \quad : \text{Phase spectrum}$$

Time scaling:

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Time shift:

$$x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$$

Frequency shift:

$$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

Time domain convolution:

$$x(t) * h(t) \leftrightarrow H(\omega) X(\omega)$$

Frequency domain convolution:

$$\frac{1}{2\pi} X(\omega) * H(\omega) \leftrightarrow h(t)x(t)$$

Time domain integration:

$$\int_{-\infty}^t x(\tau) d\tau \leftrightarrow \pi X(0)\delta(\omega) + \frac{X(\omega)}{j\omega}$$

Time domain differentiation:

$$\frac{dx}{dt} \leftrightarrow j\omega X(\omega)$$

Application of the transform pair

$$\frac{d}{dt} \leftrightarrow j\omega$$

$$\ddot{y} + a_1\dot{y} + a_0y = b_2\ddot{x} + b_1\dot{x} + b_0x$$

$$[(j\omega)^2 + a_1 \cdot (j\omega) + a_0]Y(\omega) = [b_2 \cdot (j\omega)^2 + b_1 \cdot (j\omega) + b_0]X(\omega)$$

System frequency characteristic:

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{b_2 \cdot (j\omega)^2 + b_1 \cdot (j\omega) + b_0}{(j\omega)^2 + a_1 \cdot (j\omega) + a_0} = \frac{P(j\omega)}{Q(j\omega)}$$

We now have the means to Fourier transform a differential equation in order to obtain the frequency characteristic of the system.

Conversely, if we have the frequency characteristic, we can translate it into the differential equation for the system. In many cases it might be faster to derive $H(\omega)$ from an electric circuit diagram first and then obtain the differential equation from $H(\omega)$.

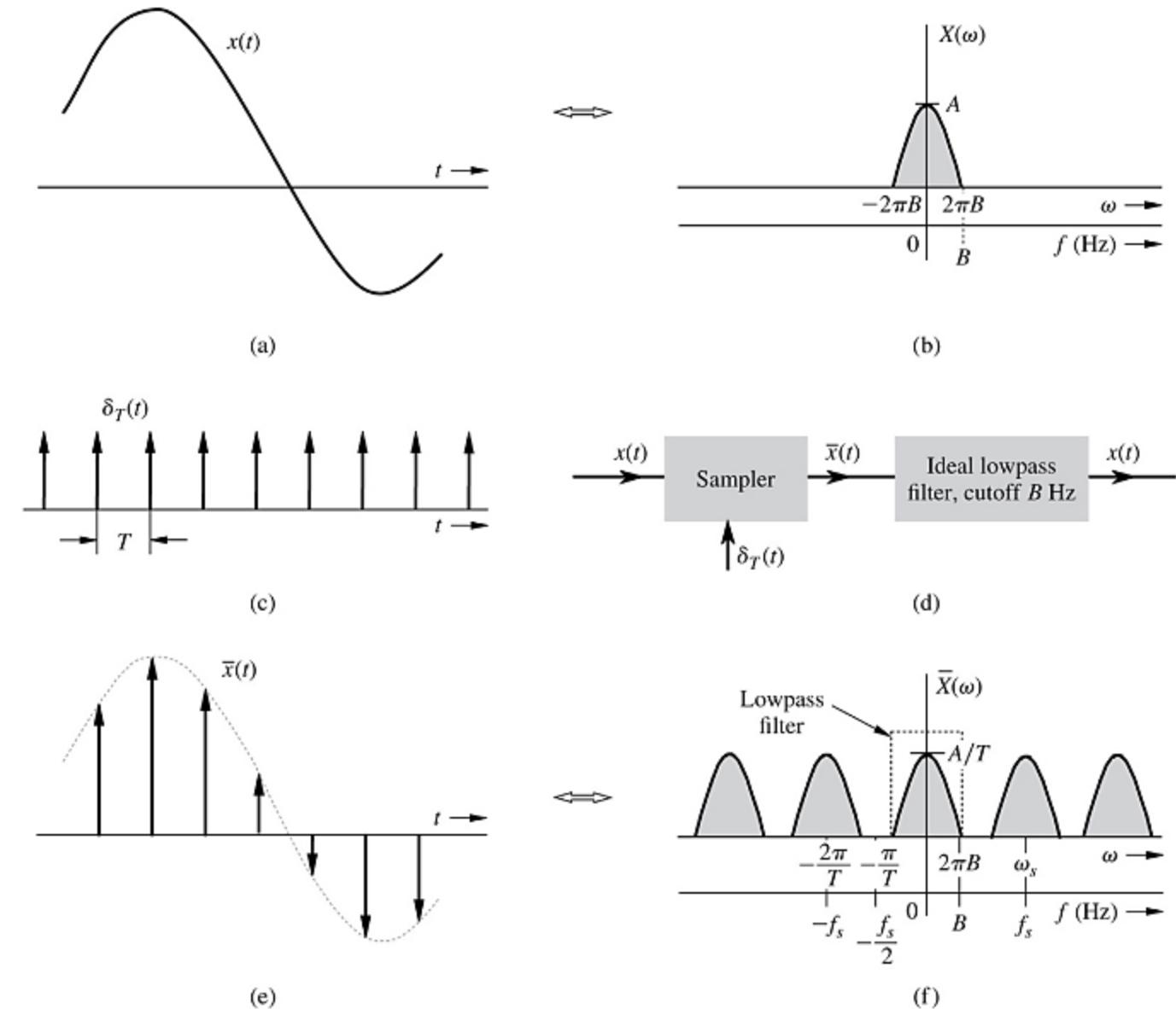
In cases where we are only interested in $H(\omega)$, we do not need to derive the differential equation.

Sampling

Fig. 5.1 in the textbook shows that if we multiply a time signal $x(t)$ by a train of impulse functions $\delta_T(t)$, a new signal is obtained $\bar{x}(t)$ which is zero everywhere except for impulses located at discrete times nT . The strength of each impulse equals the amplitude of $x(nT)$.

We will call $\bar{x}(t)$ a **sampled version** of $x(t)$.

The amplitude spectrum of the original signal is shown at the top on the right-hand side. The amplitude spectrum of $\bar{x}(t)$ is shown at the bottom on the right-hand side.



Fourier Transformation of Periodic Signals

Fourier Series synthesis
of periodic signal:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

Fourier Transformation:

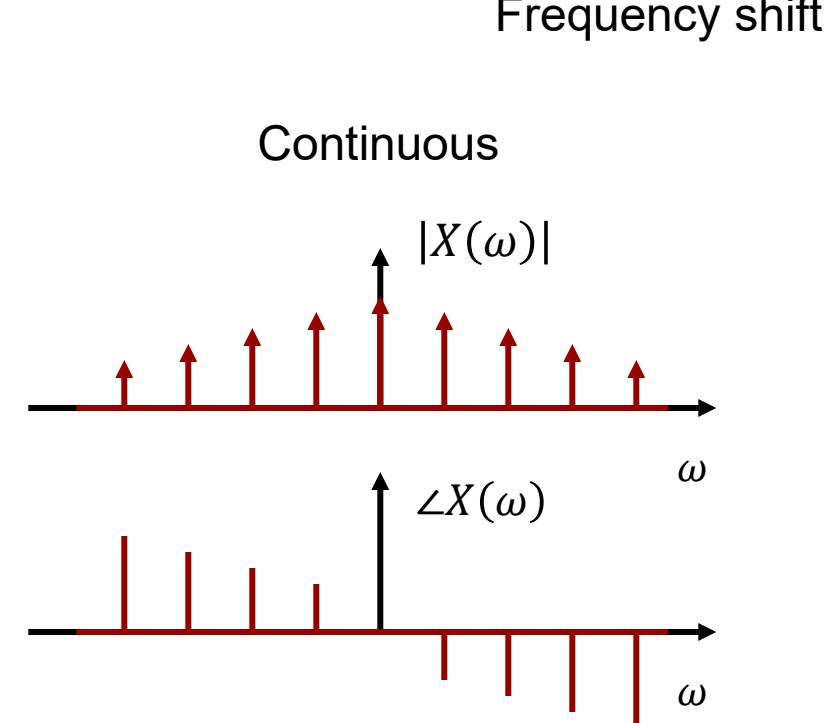
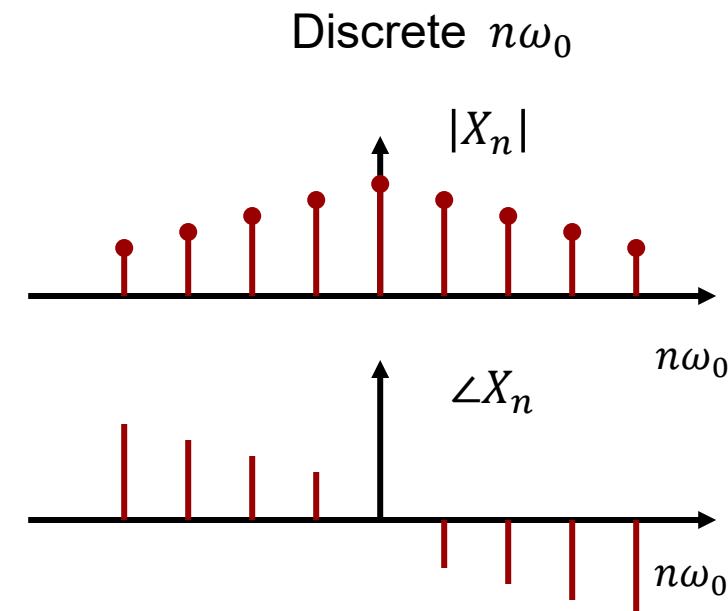
$$X(\omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \right\} = \sum_{n=-\infty}^{\infty} \mathcal{F}\{X_n e^{jn\omega_0 t}\} = \sum_{n=-\infty}^{\infty} 2\pi X_n \delta(\omega - n\omega_0)$$

$$A \leftrightarrow A2\pi\delta(\omega)$$

$$X_n \leftrightarrow X_n 2\pi\delta(\omega)$$

$$X_n e^{jn\omega_0 t} \leftrightarrow 2\pi X_n \delta(\omega - n\omega_0)$$

Discrete and continuous
functions of frequency:
The former is not defined
between stems. The latter
is defined everywhere.



Example

Fourier Series synthesis of periodic signal:

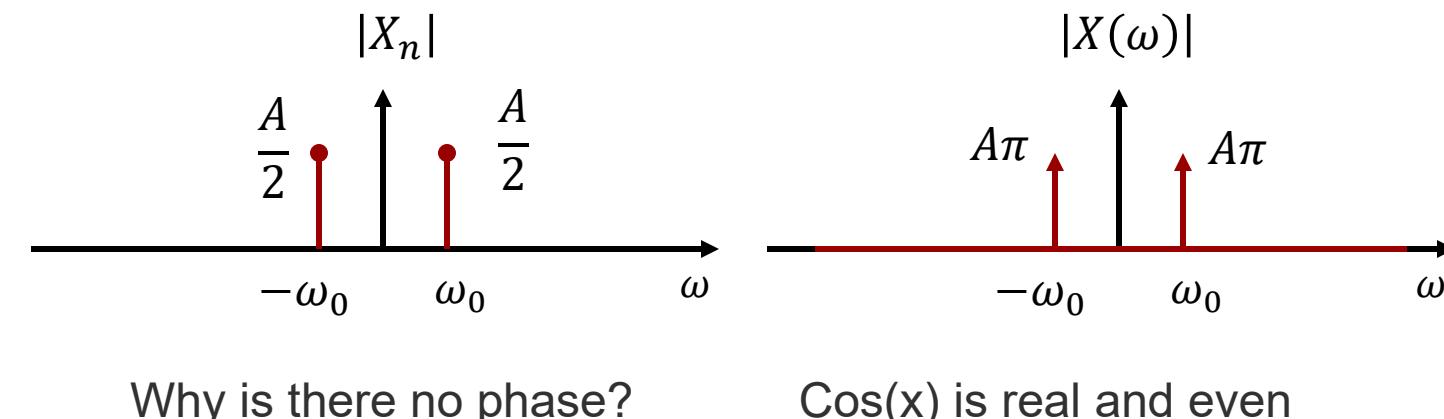
$$A \cos(\omega_0 t) = A \left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \right] = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$X_n = \begin{cases} A/2 & n = \pm 1 \\ 0 & n \neq \pm 1 \end{cases}$$

Fourier Transformation:

$$X(\omega) = \sum_{n=-\infty}^{\infty} 2\pi X_n \delta(\omega - n\omega_0) = A\pi\delta(\omega + \omega_0) + A\pi\delta(\omega - \omega_0)$$

Discrete and continuous functions of frequency:



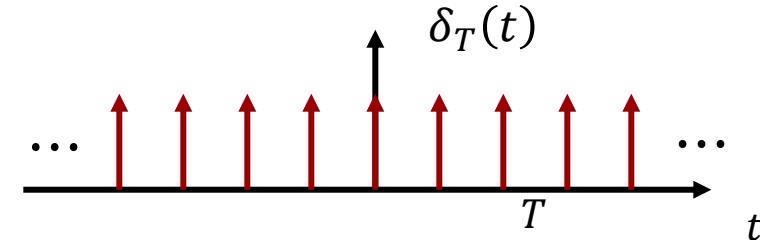
Impulse train

An impulse train is a periodic repetition of impulses, hence a periodic signal.

We can expand it into a complex exponential Fourier series:

Fourier Series coefficients:

Fourier Series expansion:



$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \omega_s = \frac{2\pi}{T}$$

$$D_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta_T(t) e^{-jn\omega_s t} dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta_T(t) e^{-jn\omega_s 0} dt = \frac{1}{T}$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_s t} = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\omega_s t}$$

Impulse train

Fourier Series expansion:

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\omega_s t} \quad \omega_s = \frac{2\pi}{T}$$

Fourier Transformation:

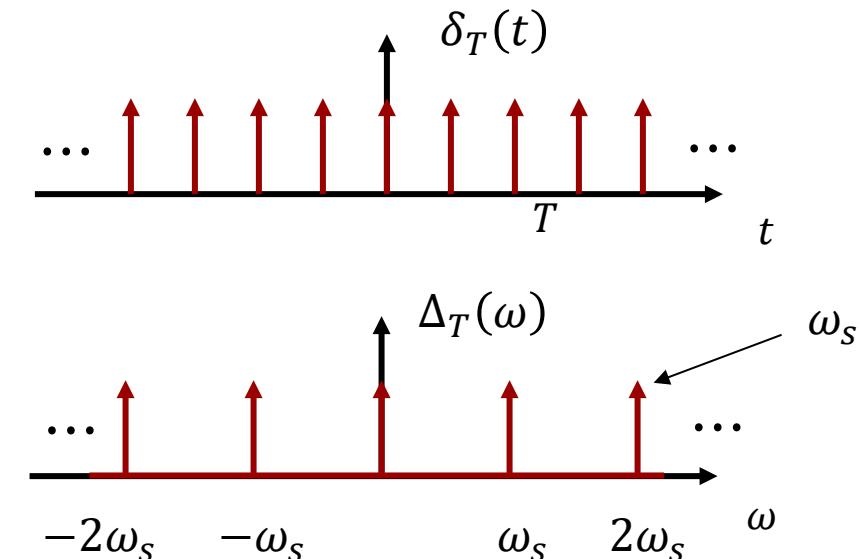
$$\begin{aligned} \Delta_T(\omega) &= \int_{-\infty}^{\infty} \delta_T(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\omega_s t} e^{-j\omega t} dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{T} e^{jn\omega_s t} \right) e^{-j\omega t} dt \\ &= \sum_{n=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - n\omega_s) = \sum_{n=-\infty}^{\infty} \omega_s \delta(\omega - n\omega_s) \end{aligned}$$

$$A \leftrightarrow 2\pi A \delta(\omega)$$

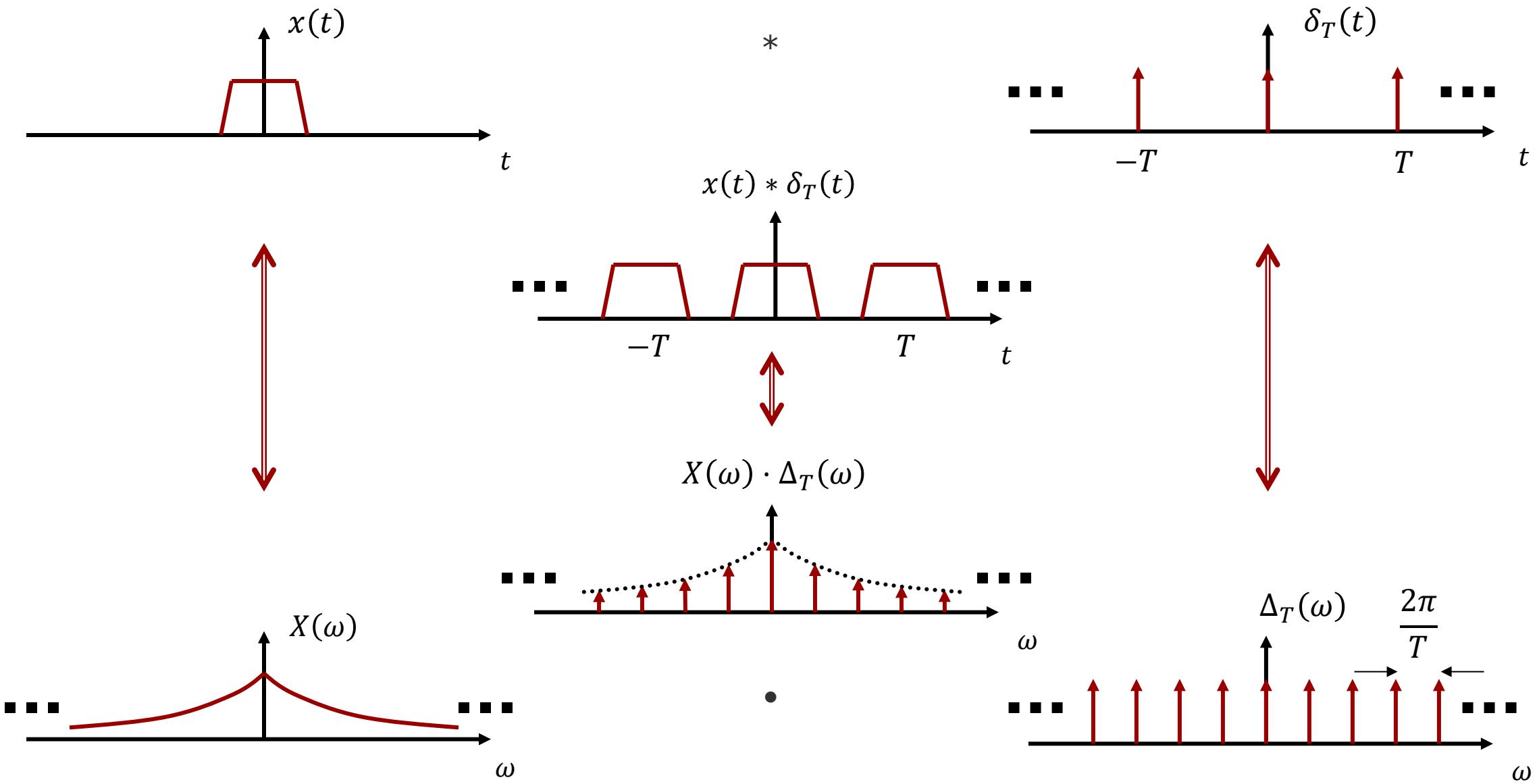
$$Ae^{j\omega_0 t} \leftrightarrow 2\pi A \delta(\omega - \omega_0)$$

$$\frac{1}{T} e^{jn\omega_0 t} \leftrightarrow \frac{2\pi}{T} \delta(\omega - n\omega_0)$$

The spectrum of an impulse train is periodic in ω with period ω_s :



Impulse Train Convolution with Aperiodic Signal

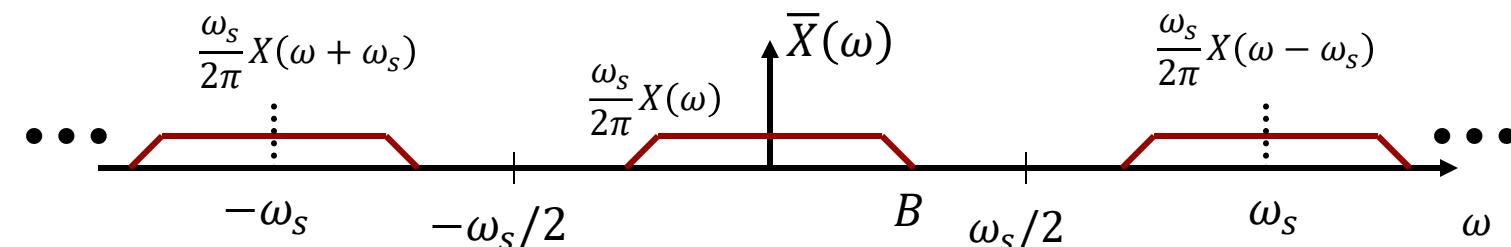
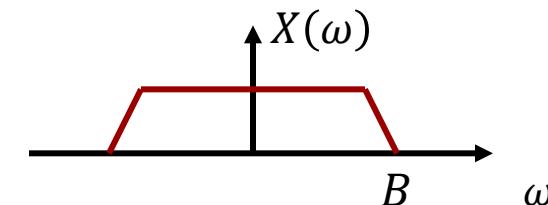


Aperiodic signal multiplied by impulse train

Multiplication in time: $\bar{x}(t) = x(t) \cdot \delta_T(t)$

Convolution in frequency domain:

$$\bar{X}(\omega) = \frac{1}{2\pi} X(\omega) * \Delta_T(\omega) = \frac{1}{2\pi} X(\omega) * \sum_{n=-\infty}^{\infty} \omega_s \delta(\omega - n\omega_s) = \frac{\omega_s}{2\pi} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

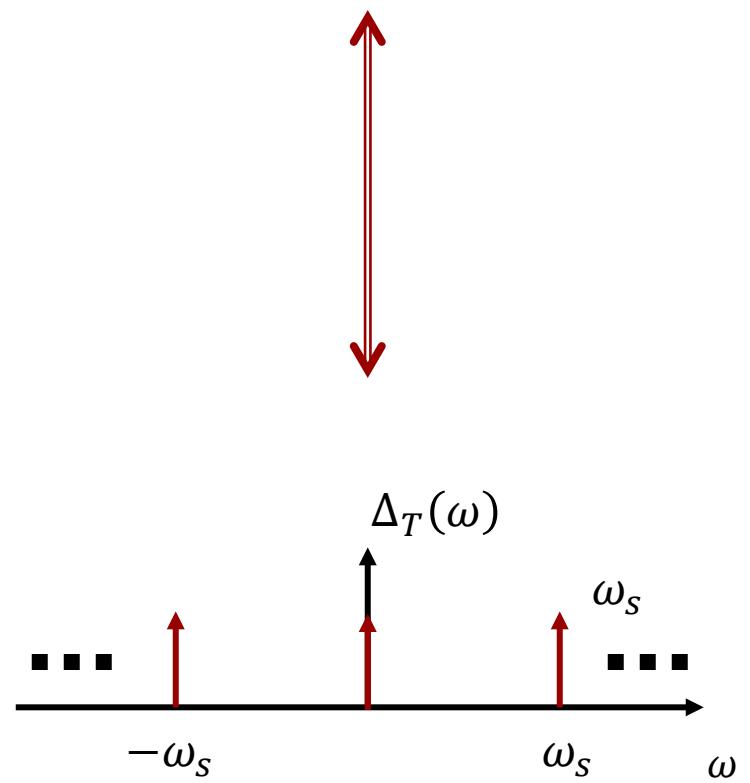
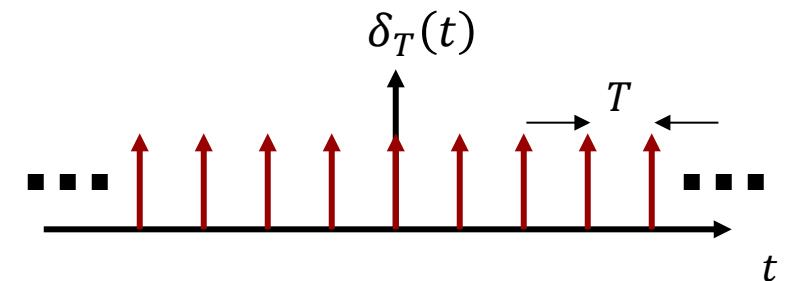
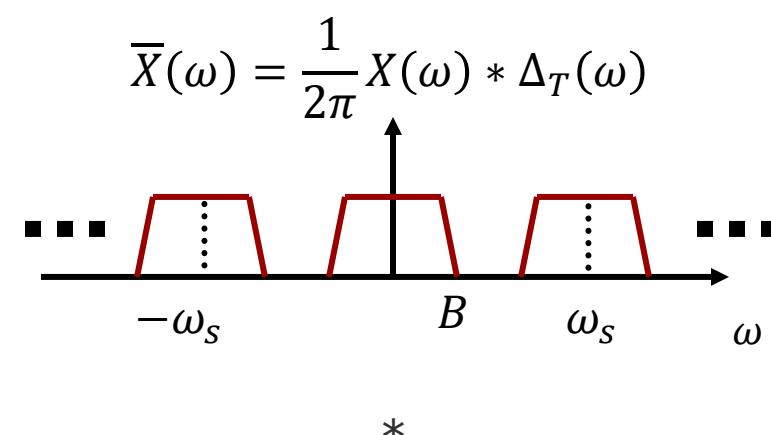
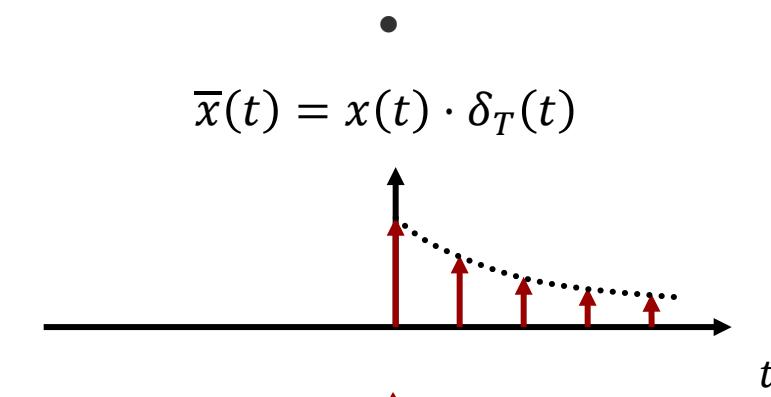
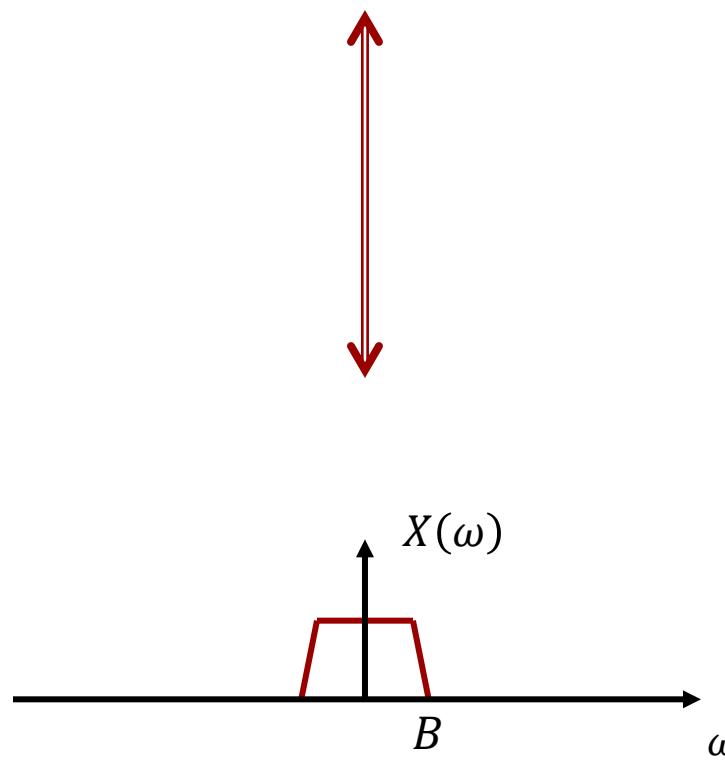
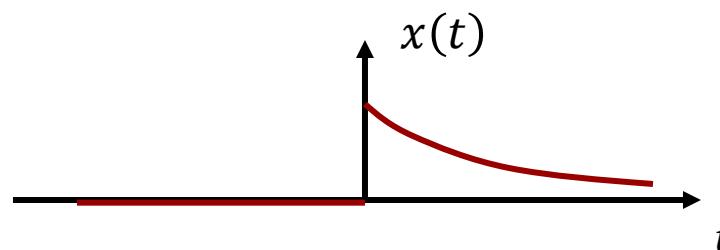


Nyquist theorem:

$$\omega_s > 2 \cdot B$$

If the sampling frequency is too small, adjacent extensions of the frequency spectrum overlap and "irreversible" aliasing occurs.

Mathematical sampling of Continuous-Time Signals



Sampling

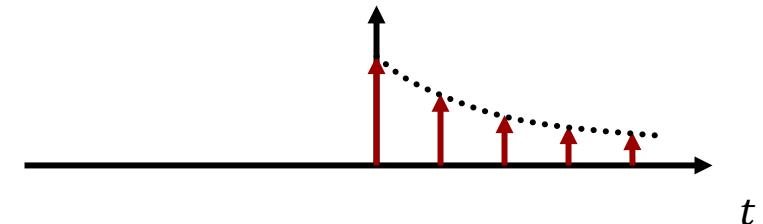
A sampled continuous-time signal is a series of equally spaced **impulses** with a strength determined by its amplitude.

This type of sampling is a mathematical process, not a process done by an analog-digital converter (ADC).

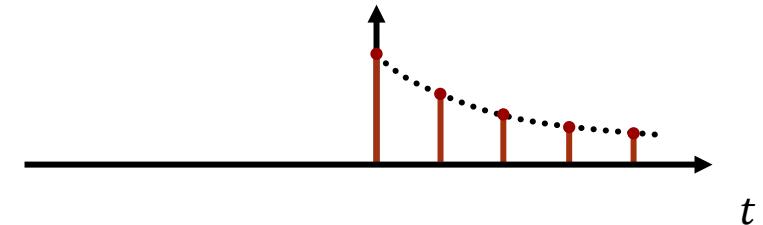
A discrete-time signal is a series of equally spaced **numbers** representing the same information.

This sequence of numbers is produced by an ADC.

$$\bar{x}(t) = x(t) \cdot \delta_T(t)$$

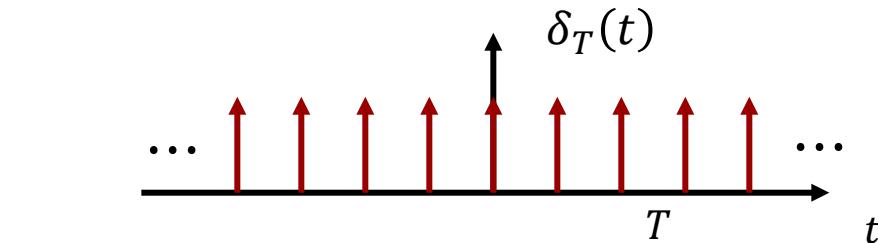


$$x(n\Delta t)$$



We just obtained the frequency characteristic of the mathematically sampled CT signal $\bar{x}(t)$ using convolution in the frequency domain. An alternative approach is to use the **modulation theorem**.

The trigonometric Fourier series:



$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \omega_s = \frac{2\pi}{T}$$

$$\delta_T(t) = \frac{1}{T} (1 + 2 \cos \omega_s t + 2 \cos 2\omega_s t + 2 \cos 3\omega_s t + \dots)$$

$$\bar{x}(t) = x(t)\delta_T(t) = \frac{1}{T} (x(t) + 2x(t) \cos \omega_s t + 2x(t) \cos 2\omega_s t + 2x(t) \cos 3\omega_s t + \dots)$$

$$\bar{X}(\omega) = \frac{1}{T} (X(\omega) + X(\omega - \omega_s) + X(\omega + \omega_s) + X(\omega - 2\omega_s) + X(\omega + 2\omega_s) + \dots)$$

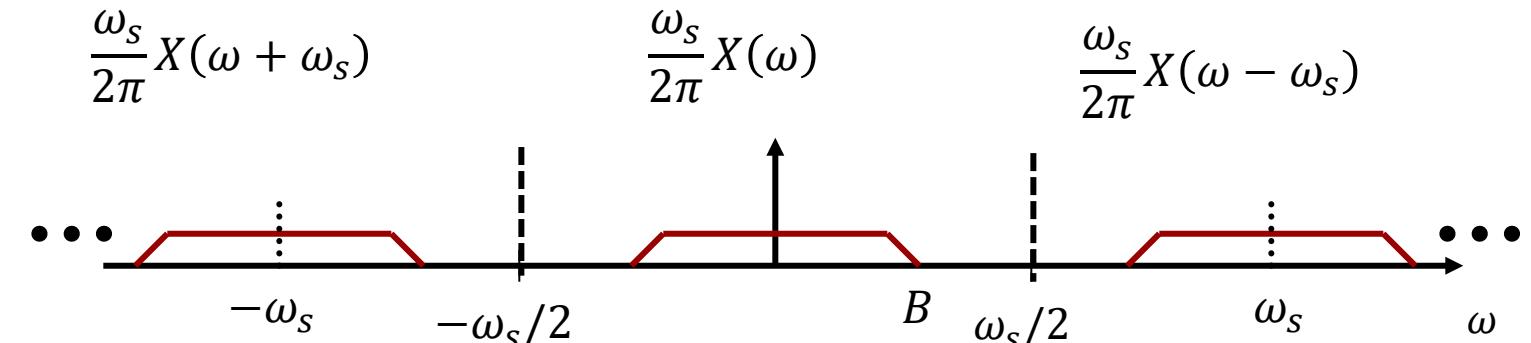
$$\bar{X}(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

Nyquist sampling theorem

The frequency spectrum $\bar{X}(\omega)$ is periodic, with periodicity ω_s .

We can reconstruct $X(\omega)$ from $\bar{X}(\omega)$ only if there is no overlap between the periodic repetitions of $X(\omega)$ in $\bar{X}(\omega)$.

The conditions for reconstructing $X(\omega)$ from $\bar{X}(\omega)$ has been formulated by several scientists and forgotten again. It was rediscovered by Nyquist and is now known as the **Nyquist sampling theorem**.



Expressed in radians/s

$$B \leq \frac{\omega_s}{2} \quad \omega_s \geq 2B$$

Expressed in Hz

$$B \leq \frac{f_s}{2} \quad f_s \geq 2B \quad T = \frac{1}{f_s}$$

Nyquist sampling rate:

$$f_s = 2B$$

Some signals require:

$$f_s > 2B$$

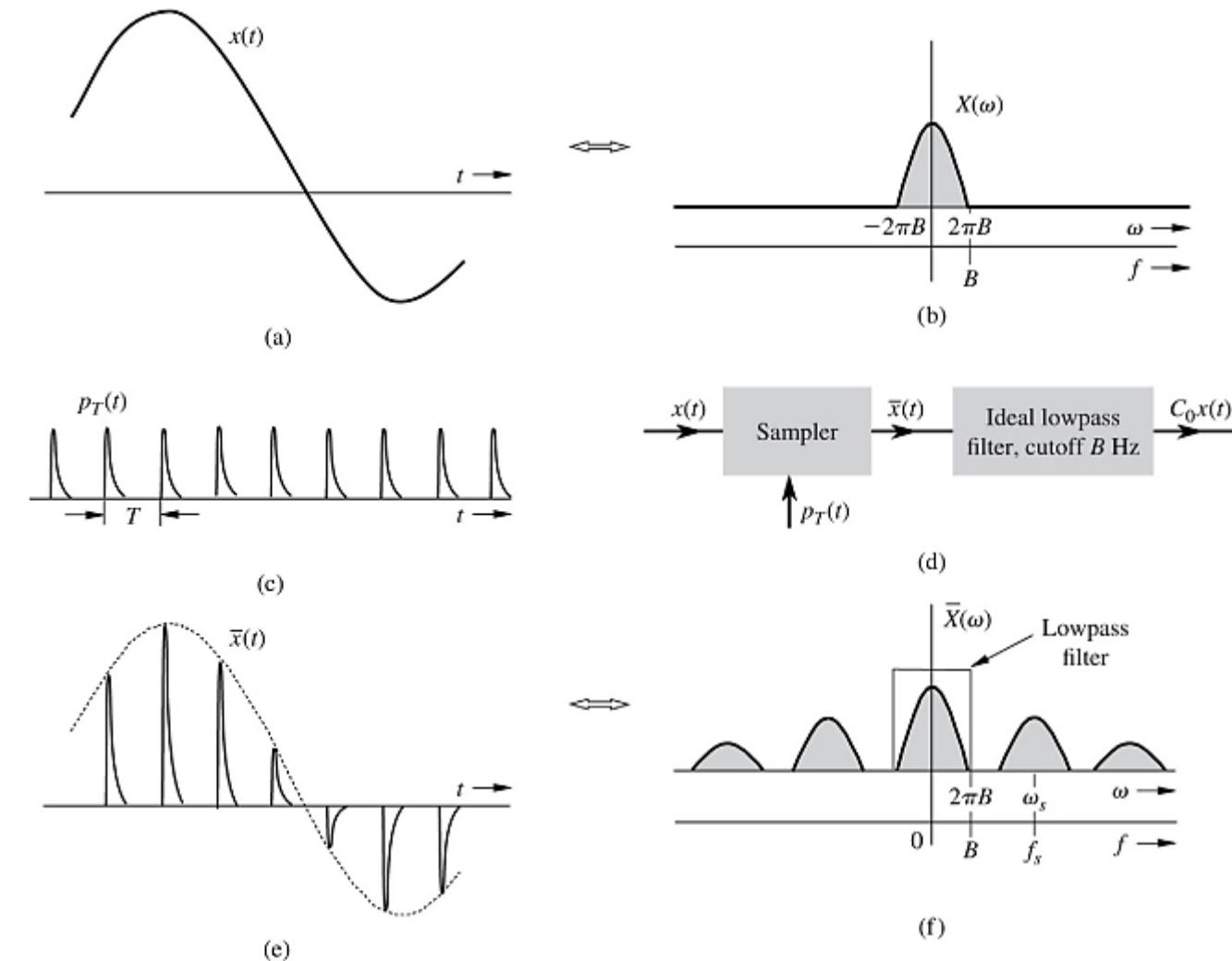
Practical sampling

Fig. 5.3 in the textbook illustrates the practical sampling process where it is not possible to produce an impulse. Instead, a practically realizable pulse $p_T(t)$ is multiplied onto the signal to be sampled.

The pulse must be periodic with period T and the duration of each pulse must be less than T .

It is shown that the resulting frequency spectrum still has periodic repetitions of $X(\omega)$ but that each repetition now is scaled in magnitude. The exact scaling depends on the shape of the pulse.

When the sampling pulse was an ideal impulse, all spectral repetition had the same scaling.

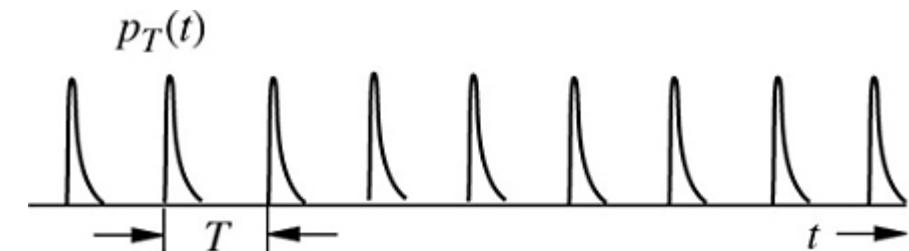


Practical sampling

The train of practically realizable pulses can be expressed as a compact Fourier series. Remember that the Fourier series is a periodic function, hence the Fourier series represent the entire train of pulses, not just one pulse.

The practically sampled signal $\bar{x}(t)$ can now be expressed as a Fourier series expansion:

Each term in the series is a modulation of the CT signal $x(t)$, but each repetition of the spectrum is scaled by $C_n/2$.



$$p_T(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_s t + \theta_n)$$

$$\bar{x}(t) = x(t)p_T(t) = x(t) \left(C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_s t + \theta_n) \right)$$

$$= \textcolor{blue}{C}_0 x(t) + \sum_{n=1}^{\infty} \textcolor{green}{C}_n x(t) \cos(n\omega_s t + \theta_n)$$

$$\bar{X}(\omega) = \frac{1}{T} \left(\textcolor{blue}{C}_0 X(\omega) + \frac{\textcolor{green}{C}_1}{2} (X(\omega - \omega_s) + X(\omega + \omega_s)) + \frac{\textcolor{green}{C}_2}{2} (X(\omega - 2\omega_s) + X(\omega + 2\omega_s)) + \dots \right)$$

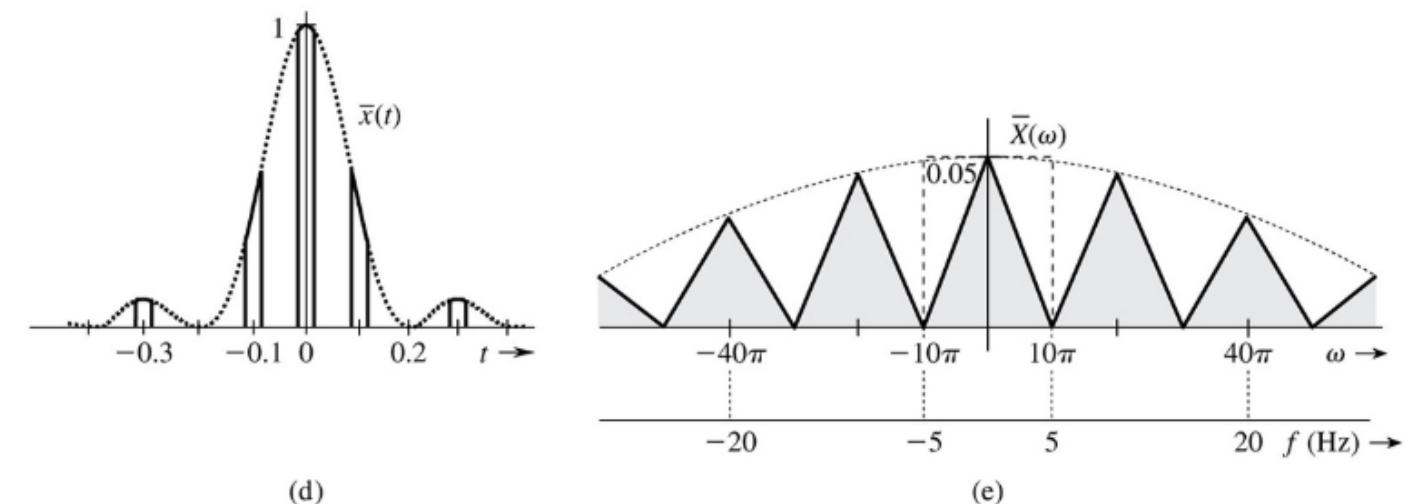
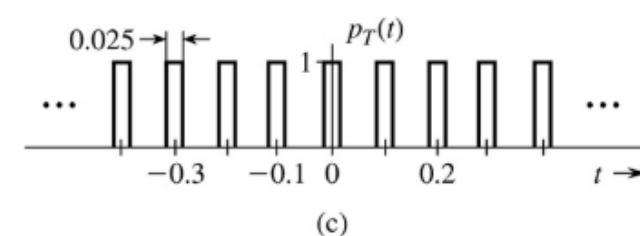
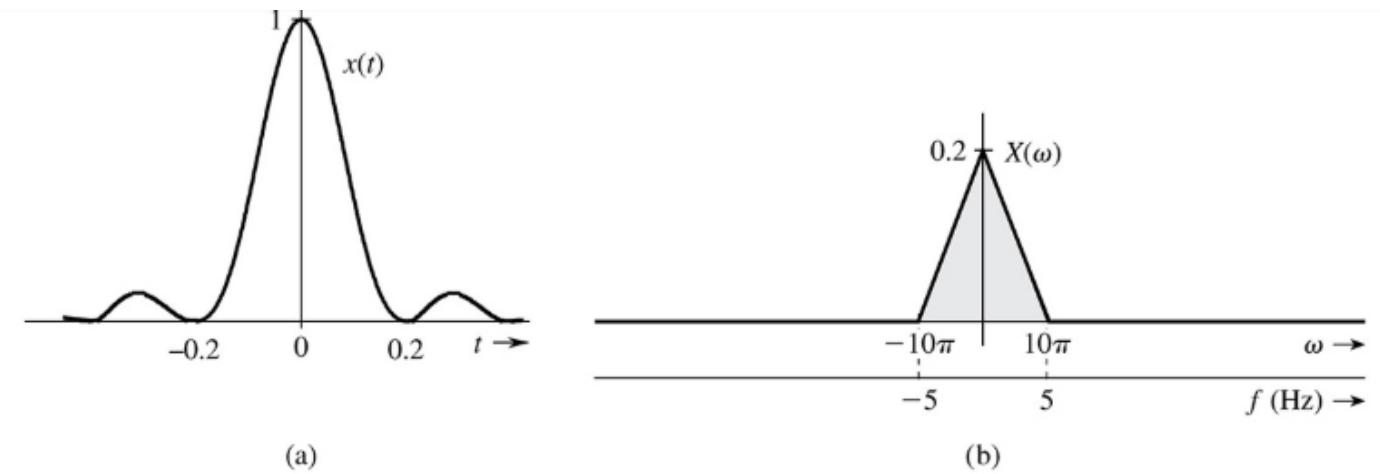
Practical sampling

Now we would like to consider a sampling pulse train for which we can derive its Fourier transform.

The sampling pulse train is a train of square pulses. They are spaced apart by T , i.e. the time interval between samples.

Each pulse has a height of 1 and a width of T_1 .

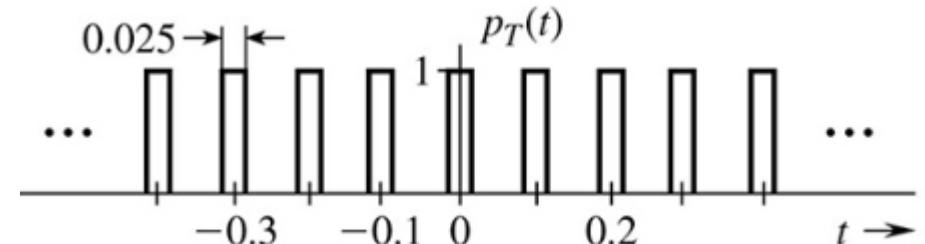
We will first approach this as done in the textbook, ch. 5.1. Then we will use an alternative approach.



Fourier series of square pulse

We will assume that each pulse has a duration T_1 and a periodicity of T .

$$\omega_s = \frac{2\pi}{T}$$



$$a_0 = \frac{1}{T} \int_{-T_1/2}^{T_1/2} p_T(t) dt = \frac{T_1}{T}$$

$$p_T(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_s t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_s t)$$

$$a_n = \frac{2}{T} \int_{-T_1/2}^{T_1/2} p_T(t) \cos n\omega_s t dt = \frac{2}{T} \int_{-T_1/2}^{T_1/2} \cos n\omega_s t dt = \frac{2}{Tn\omega_s} [\sin n\omega_s t]_{-T_1/2}^{T_1/2} = \frac{2}{Tn\omega_s} \left(\sin \frac{n\omega_s T_1}{2} - \sin \frac{-n\omega_s T_1}{2} \right)$$

$$a_n = \frac{2T_1}{T} \frac{2}{n\omega_s T_1} \sin \frac{n\omega_s T_1}{2} = \frac{2T_1}{T} \operatorname{sinc}\left(\frac{n\omega_s T_1}{2}\right) = \frac{2T_1}{T} \operatorname{sinc}\left(\frac{n\pi T_1}{T}\right)$$

$$b_n = \frac{2}{T} \int_{-T_1/2}^{T_1/2} p_T(t) \sin n\omega_s t dt = 0, p_T \text{ even}$$

Practical sampling

Inserting the Fourier series coefficients, we obtain:

$$\bar{x}(t) = x(t)p_T(t) = \mathcal{C}_0 x(t) + \sum_{n=1}^{\infty} \mathcal{C}_n x(t) \cos(n\omega_s t + \theta_n)$$

Modulation theorem:

$$x(t) \cos(\omega_0 t) \leftrightarrow \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0)$$

$$\begin{aligned} \bar{x}(t) &= x(t)p_T(t) = x(t) \left(\frac{T_1}{T} + \sum_{n=1}^{\infty} \frac{2T_1}{T} \text{sinc}\left(\frac{n\pi T_1}{T}\right) \cos(n\omega_s t) \right) \\ &= \frac{T_1}{T} x(t) + \frac{2T_1}{T} \sum_{n=1}^{\infty} \text{sinc}\left(\frac{n\pi T_1}{T}\right) x(t) \cos(n\omega_s t) \end{aligned}$$

$$\begin{aligned} \bar{X}(\omega) &= \frac{T_1}{T} X(\omega) \\ &\quad + \frac{T_1}{T} \text{sinc}\left(\frac{1\pi T_1}{T}\right) (X(\omega - \omega_s) + X(\omega + \omega_s)) \\ &\quad + \frac{T_1}{T} \text{sinc}\left(\frac{2\pi T_1}{T}\right) (X(\omega - 2\omega_s) + X(\omega + 2\omega_s)) \\ &\quad + \dots \end{aligned}$$

Practical sampling – Alternative approach

Let us approach the same analysis using a different approach where we use the Fourier Transform theorems.

The main idea here is to consider the practical sampling pulse to be the result of convolving a train of ideal impulse functions $\delta_T(t)$ with a single square pulse $p(t)$.

We make the Fourier Transform of the time-domain expression and then find the Fourier Transform of all three functions.

We can then get the product:

$$\bar{x}(t) = x(t)p_T(t) = x(t)(\delta_T(t) * p(t))$$

$$\bar{X}(\omega) = \frac{1}{2\pi} X(\omega) * (\Delta_T(\omega)P(\omega))$$

$$x(t) \Leftrightarrow X(\omega)$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \Leftrightarrow \Delta_T(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

$$p(t) = \text{rect}\left(\frac{t}{T_1}\right) \Leftrightarrow P(\omega) = T_1 \text{sinc}\left(\frac{\omega T_1}{2}\right)$$

$$\Delta_T(\omega)P(\omega) = \frac{2\pi T_1}{T} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n\omega_s T_1}{2}\right) \delta(\omega - n\omega_s)$$

Practical sampling – Alternative approach

From the previous slide we have:

$$\bar{X}(\omega) = \frac{1}{2\pi} X(\omega) * (\Delta_T(\omega) P_T(\omega))$$

$$\Delta_T(\omega) P_T(\omega) = \frac{2\pi T_1}{T} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n\omega_s T_1}{2}\right) \delta(\omega - n\omega_s)$$

Next, we calculate the convolution:

$$\bar{X}(\omega) = \frac{1}{2\pi} X(\omega) * \frac{2\pi T_1}{T} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n\omega_s T_1}{2}\right) \delta(\omega - n\omega_s)$$

$$\bar{X}(\omega) = \frac{T_1}{T} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n\omega_s T_1}{2}\right) X(\omega - n\omega_s)$$

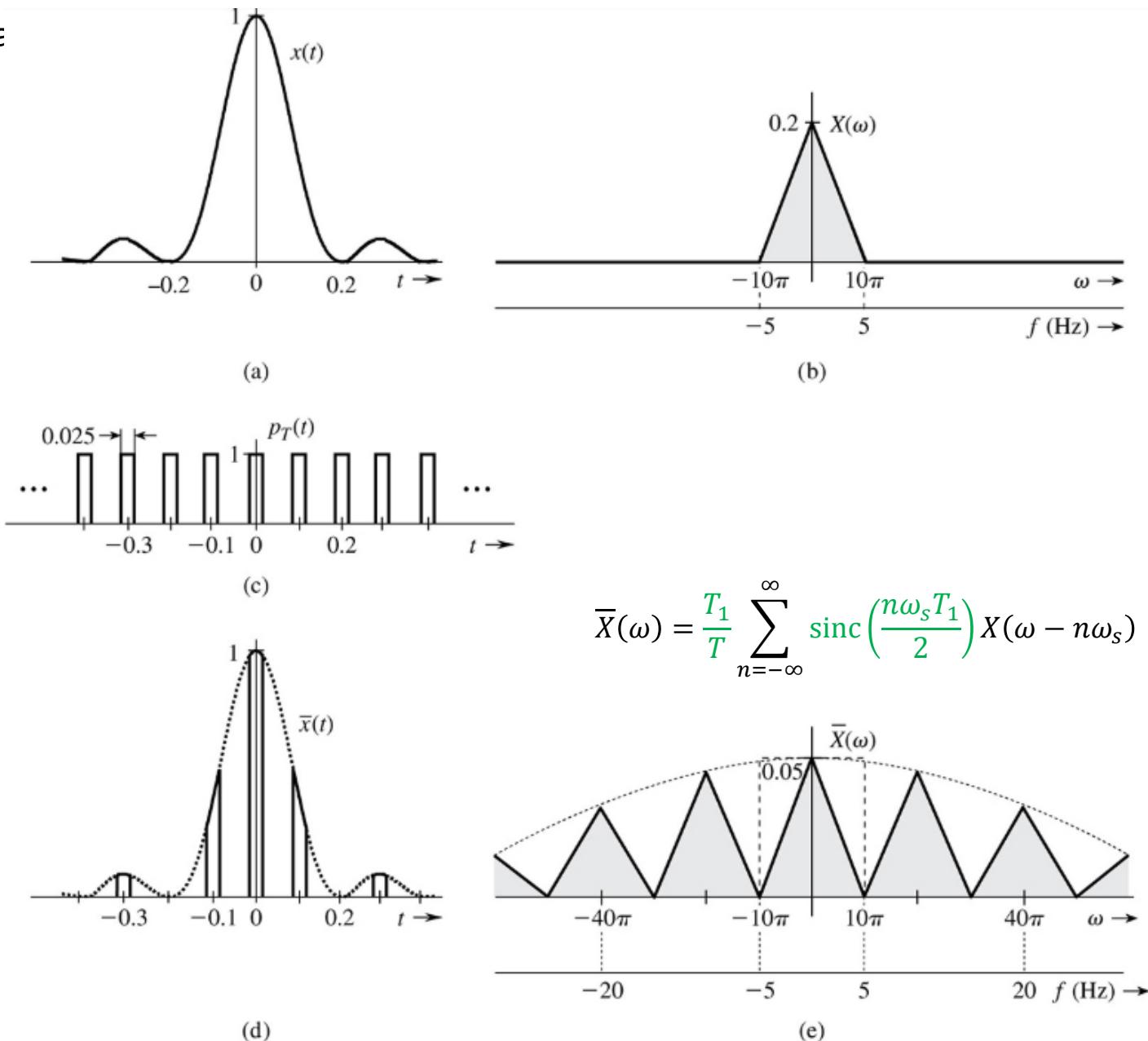
Practical sampling – Alternative approach

Observations:

Although we replaced the ideal sampling function (the impulse) with a square pulse, the frequency spectrum $X(\omega)$ of the CT signal is still contained within the periodic spectrum $\bar{X}(\omega)$.

It is scaled in amplitude depending on the width of the sampling pulse, but provided the Nyquist sampling theorem is not violated, we can recover the spectrum $X(\omega)$ by lowpass filtering $\bar{X}(\omega)$.

We see that spectrum scaling is determined by the pulse width T_1 . Letting $T_1 \rightarrow 0$ would not work.

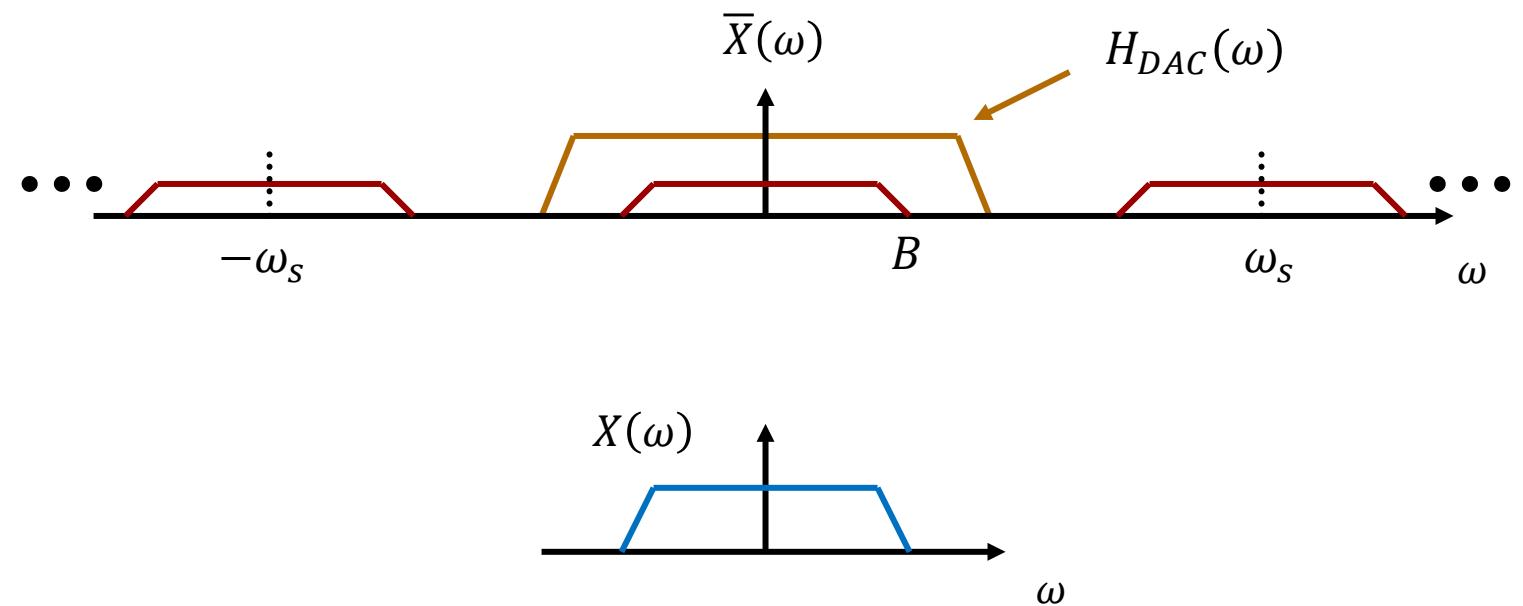


We have just learned that we can reconstruct the spectrum $X(\omega)$ by lowpass filtering $\bar{X}(\omega)$.

Ideal reconstruction:

We call this an ideal reconstruction because the periodic repetitions of spectra do not overlap.
This is due to the Nyquist theorem not being violated.

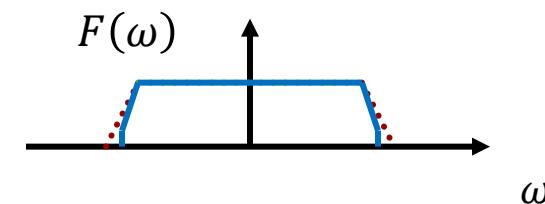
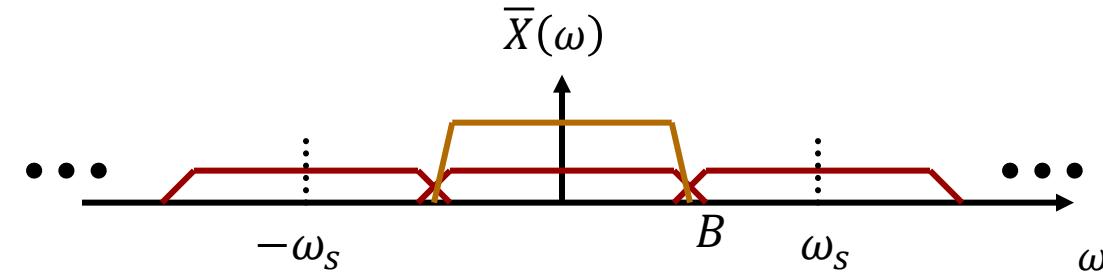
If the spectrum $X(\omega)$ is preserved by the sampling, it means that it is possible to reconstruct $x(t)$ from its samples.



Reconstruction of Continuous-Time Signal

Here the Nyquist sampling theorem has been violated ($\omega_s < 2B$).

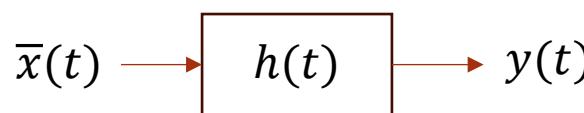
Imperfect reconstruction:



In this case, the spectra overlap and in fact their overlapping parts add up and modify the spectrum in a region centered around $\omega_s/2$. This error is called **aliasing**.

Aliasing is the irreversible consequence of violating the Nyquist sampling theorem.

Reconstruction of Continuous-Time signal from its samples



$$h(t) = \text{rect}\left(\frac{t}{T}\right)$$

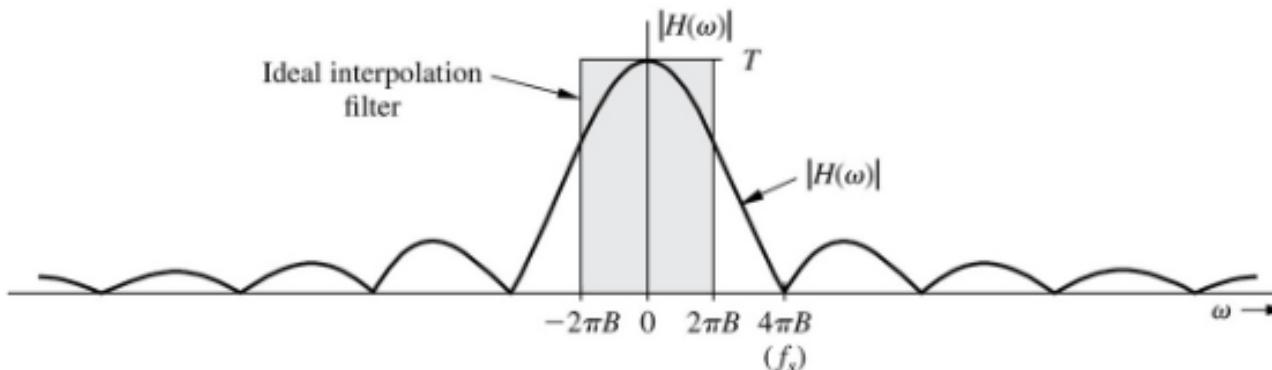
$$\bar{x}(t) = x(t) \cdot \delta_T(t) \quad \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$y(t) = \bar{x}(t) * h(t) = x(t) \cdot (\delta_T(t) * h(t))$$

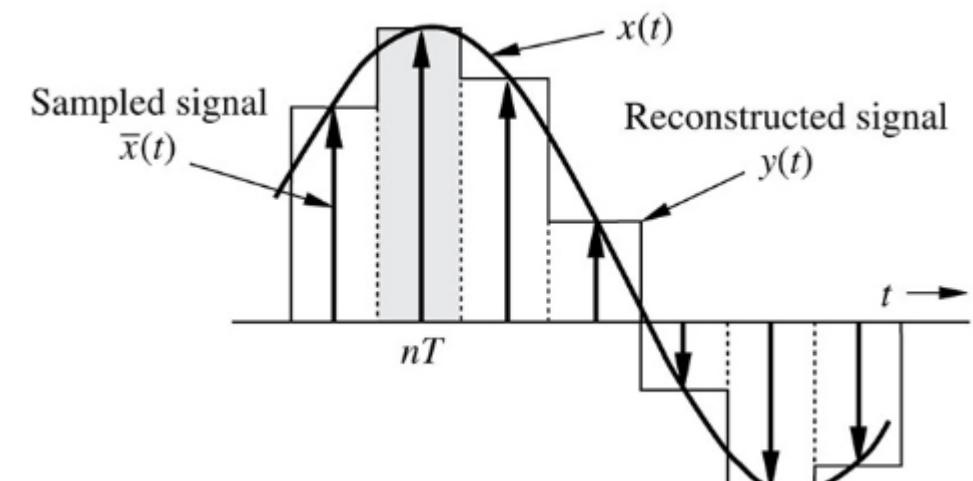
To create a CT signal from a set of samples, interpolation must be used.

If we use a filter with a pulse as its impulse response to interpolate between samples, we get a staircase approximation $y(t)$.

The reason for the poor interpolation is that we are multiplying the spectrum with a sinc.



$$= \sum_{n=-\infty}^{\infty} x(t) \text{rect}\left(\frac{t - nT}{T}\right)$$



Reconstruction of Continuous-Time signal from its samples by ideal interpolation

To isolate the spectrum $X(\omega)$ from $\bar{X}(\omega)$ we could multiply with a rect function in the frequency domain, i.e. an ideal filter.

$$X(\omega) = \bar{X}(\omega) H(\omega)$$

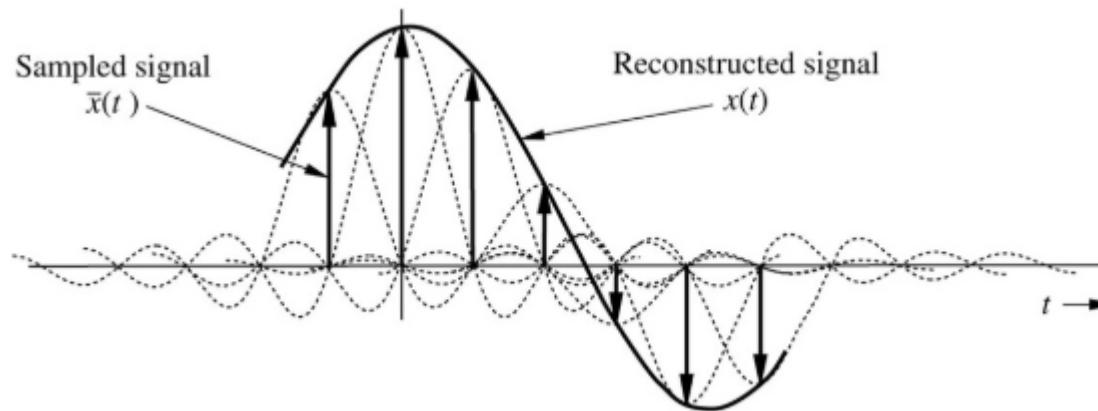
$$H(\omega) = T \operatorname{rect}\left(\frac{\omega}{4\pi B}\right)$$

$$h(t) = \operatorname{sinc}(2\pi B t)$$

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)h(t - nT)$$

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)\operatorname{sinc}(2\pi B(t - nT))$$

$$y(nT) = x(nT)$$



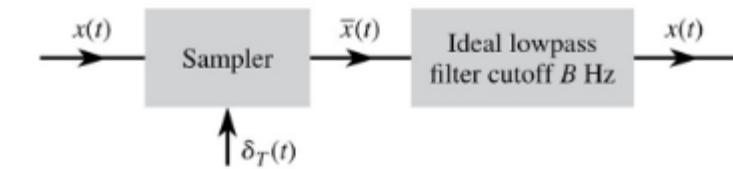
The reconstructed signal $y(t) = x(t)$ because they are both equal to $\mathcal{F}^{-1}\{\bar{X}(\omega) H(\omega)\}$

The ideal interpolation requires a non-causal filter:

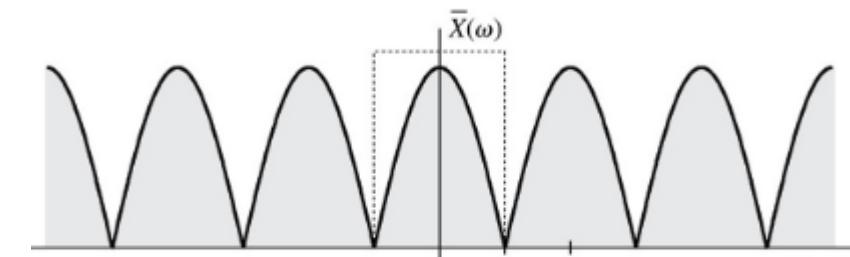
$$h(t) = \text{sinc}(2\pi B t)$$

If the sampling frequency is increased than $\frac{\omega_s}{2} > 2\pi B$, or $\frac{f_s}{2} > B$, then we can use a lowpass filter with a finite slope to attenuate all periodically repeated cycles.

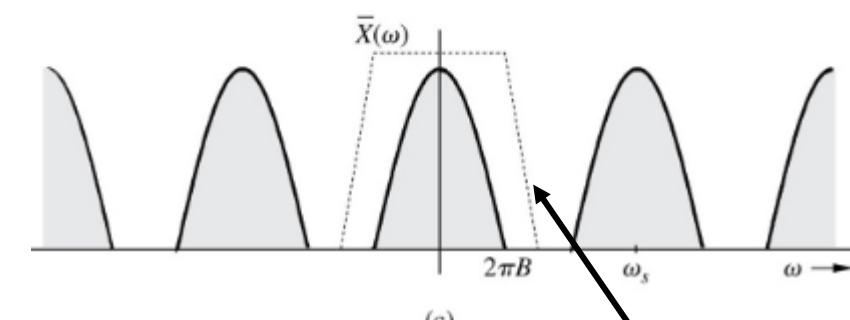
Ideally such a filter should have zero gain for all frequencies above $f_s/2$. No practical realizable filter can achieve this. The best we can aim for is some feasible compromise between increasing the sampling frequency and increasing the slope of the filter cut-off.



(a)



(b)



(c)

Filter cut-off slope

Non-bandlimited signal

Time-limited signals have infinite bandwidth.

Let us make a signal $y(t)$ time-limited by multiplying it with a pulse of duration T_1 .

This means that the frequency spectrum of $y(t)$ will be the spectrum of $x(t)$ convolved with a never ending sinc function.

All practically realizable signals are time-limited.

Hence, all practically realizable signals have infinite bandwidth.

$$y(t) \stackrel{\text{def}}{=} x(t)p(t)$$

$$p(t) = \text{rect}\left(\frac{t}{T_1}\right) \Leftrightarrow P(\omega) = T_1 \text{sinc}\left(\frac{\omega T_1}{2}\right)$$

$$Y(\omega) = X(\omega) * T_1 \text{sinc}\left(\frac{\omega T_1}{2}\right)$$

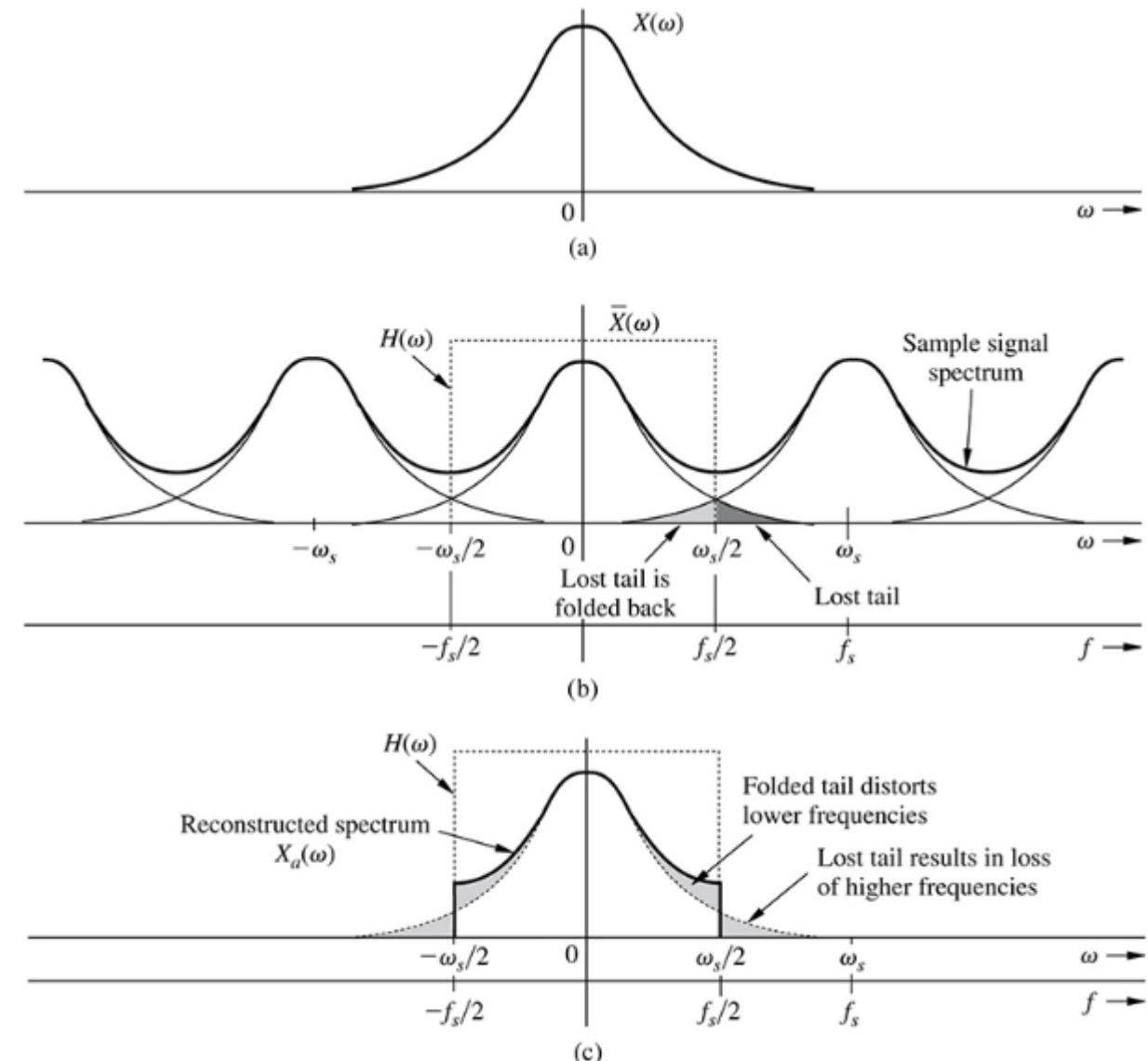
Non-bandlimited signal

Time-limited signals have infinite bandwidth.

When sampled, the frequency spectrum

$\bar{X}(\omega)$ will show overlapping tails, as shown on the sketch. The tail above $f_s/2$ is also found reflected below $f_s/2$. It looks as if the upper tail has been folded about $f_s/2$, which is therefore termed the **folding frequency**.

The folded tail adds to the spectrum and distorts it irreversibly.

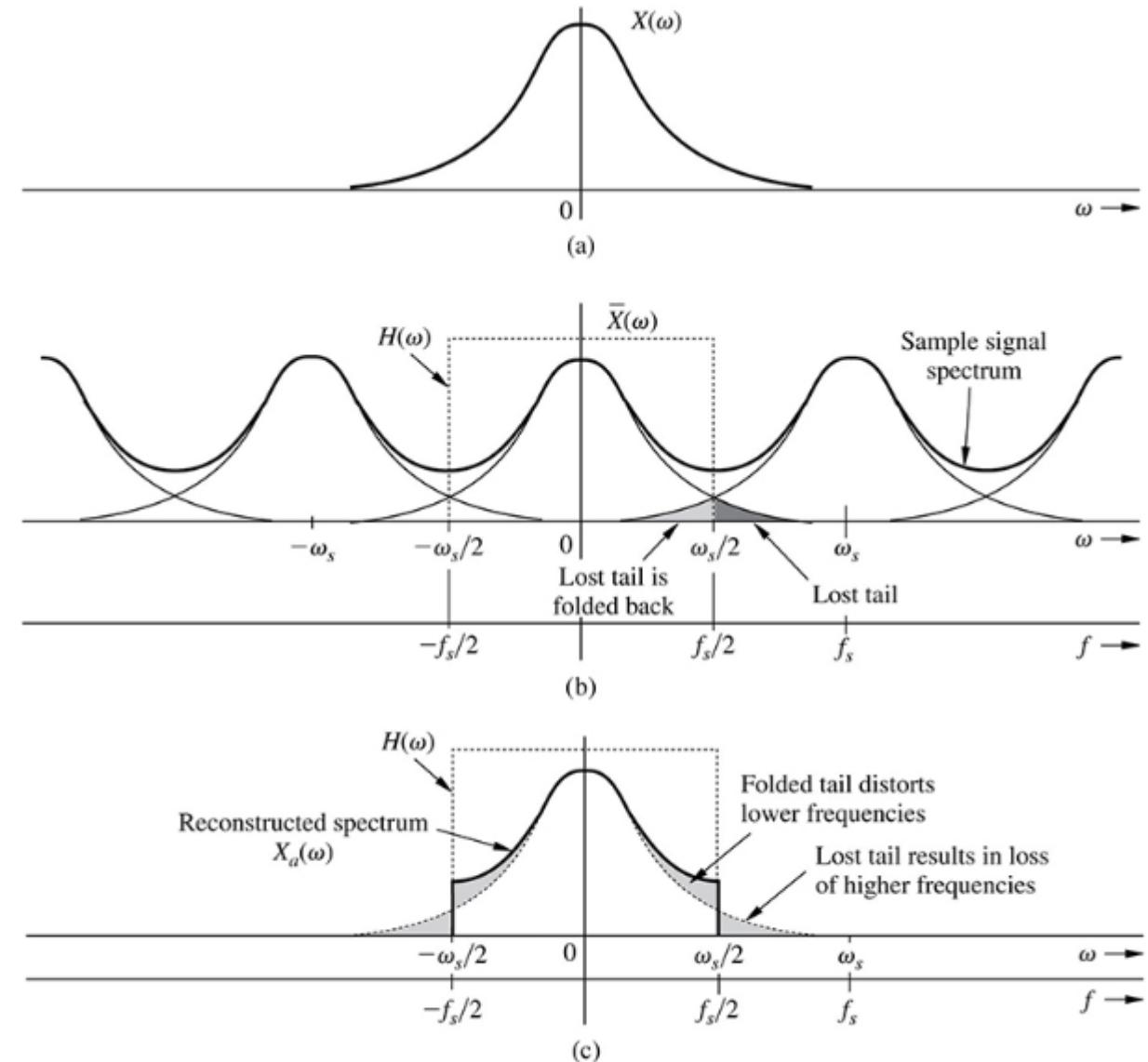


Non-bandlimited signal

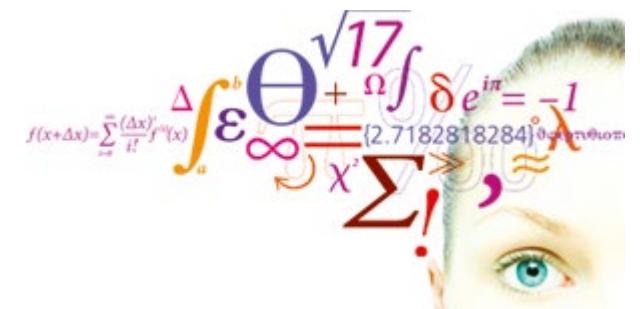
Strictly speaking Nyquist sampling theorem can never be satisfied for practically realizable signals because the sampling frequency would need to be infinite.

If the frequencies near the folding frequency originates from noise, we can use a lowpass filter to reduce its magnitude to an insignificant level. In such cases the distortion of the frequency spectrum is minimal at frequencies originating from noise.

Such a lowpass filter is called an **anti-aliasing filter**. It is always placed in front of an ADC to minimize the effect of frequencies folding across the folding frequency.



Break



Fourier analysis of filter circuits

Video 2

We will often derive the frequency characteristic $H(\omega)$ of a filter. It is in general a complex-valued function, and to visualize it, we need to plot its modulus (amplitude) and argument (phase).

Here we show that the modulus of $H(\omega)$ is the modulus of the numerator divided by the modulus of the denominator.

Comparing the exponentials on the left and right sides shows that the phase angle of $H(\omega)$ is the phase angle of the numerator minus the phase angle of the denominator:

$$H(\omega) = \frac{P(j\omega)}{Q(j\omega)}$$

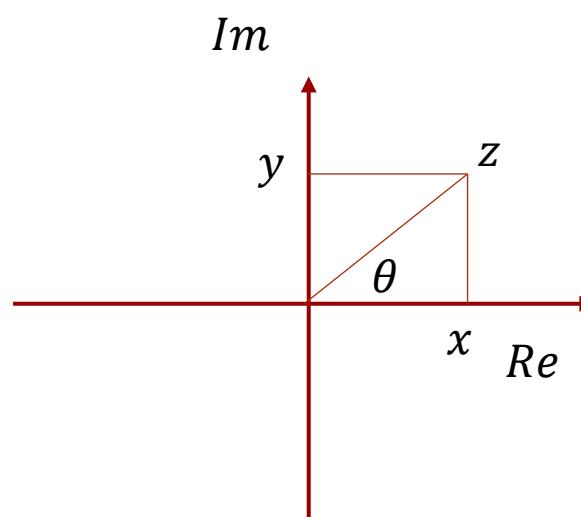
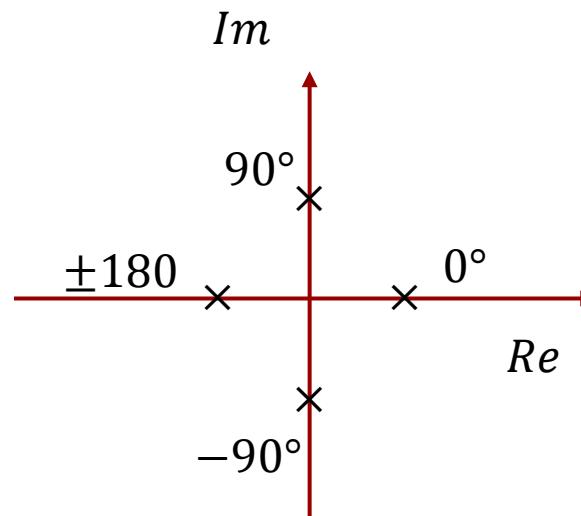
$$H(\omega) = \underbrace{|H(\omega)|e^{j\angle H(\omega)}}_{\text{Polar form}} = \frac{P(j\omega)}{Q(j\omega)} = \underbrace{\frac{|P(\omega)|e^{j\angle P(\omega)}}{|Q(\omega)|e^{j\angle Q(\omega)}}}_{\text{Polar form}}$$

$$|H(\omega)| = \frac{|P(\omega)|}{|Q(\omega)|}$$

$$e^{j\angle H(\omega)} = \frac{e^{j\angle P(\omega)}}{e^{j\angle Q(\omega)}} = e^{j\angle P(\omega)} e^{-j\angle Q(\omega)} = e^{j(\angle P(\omega) - \angle Q(\omega))}$$

$$\angle H(\omega) = \underbrace{\angle P(\omega)}_{\substack{\text{numerator} \\ \text{angle}}} - \underbrace{\angle Q(\omega)}_{\substack{\text{denominator} \\ \text{angle}}}$$

Amplitude and phase



$$z = x + jy$$

$$|z| = |x + jy| = \sqrt{x^2 + y^2}$$

$$y = x \tan \theta$$

$$\angle z = \theta = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{\Im(z)}{\Re(z)}$$

First order active filters

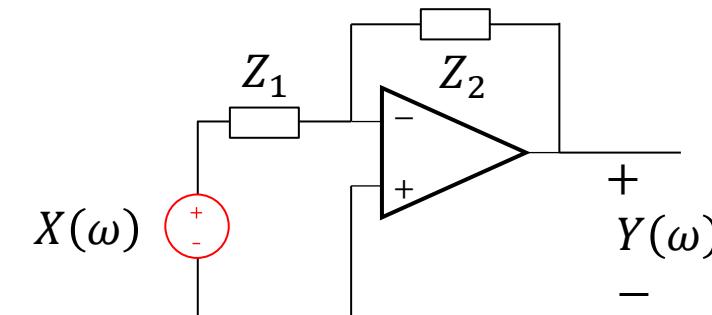
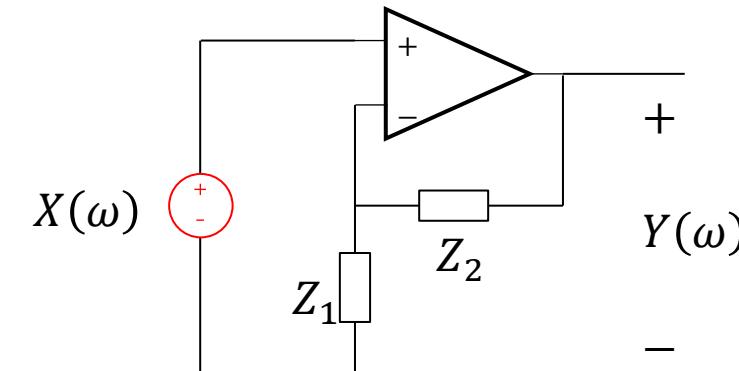
A non-inverting 1st order filter:

$$Y(\omega) = \left(1 + \frac{Z_2(\omega)}{Z_1(\omega)}\right)X(\omega)$$

We simply replace resistors with impedances in the gain equations.

An inverting 1st order filter:

$$Y(\omega) = -\frac{Z_2(\omega)}{Z_1(\omega)}X(\omega)$$



It is the frequency dependent nature of the impedances that determines, if the two circuits are lowpass, highpass, bandpass or bandstop filters.

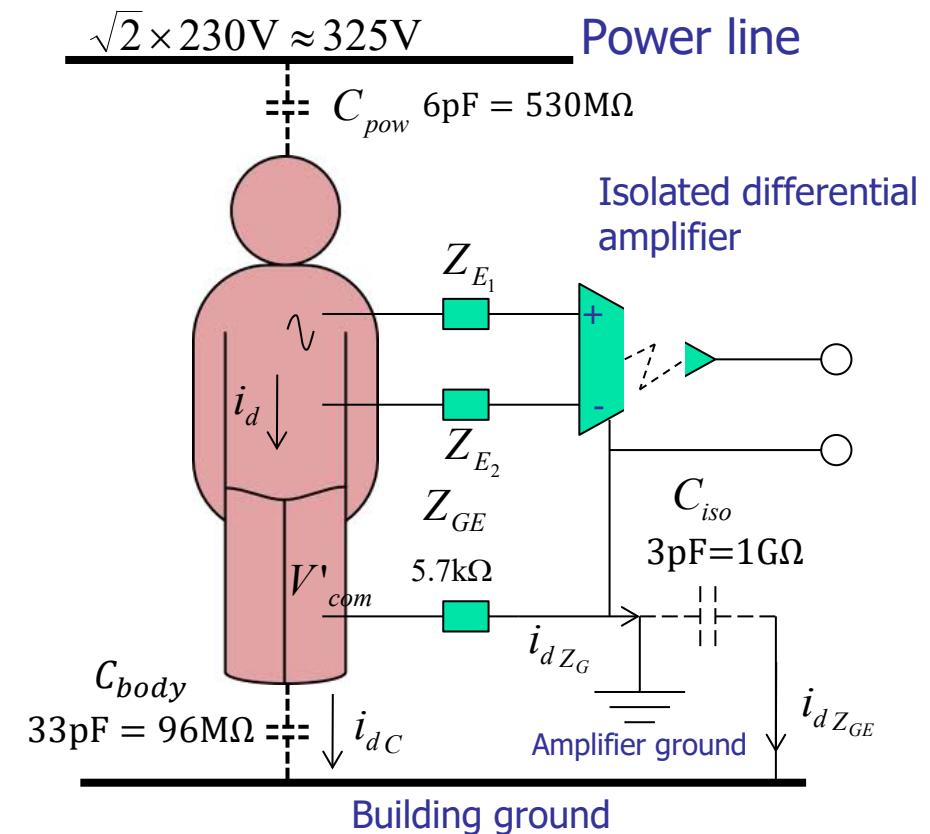
Example of real-world application of non-inv. amplifier as a filter

The sketch to the right shows the challenge of amplifying the electrocardiograph (ECG) while also suppressing 50Hz noise from powerlines.

Stray capacitors form between a patient and powerlines (C_{pow}) and also between patient and building ground (C_{body}). This produces a large common mode 50Hz noise voltage which superimpose on the ECG signal.

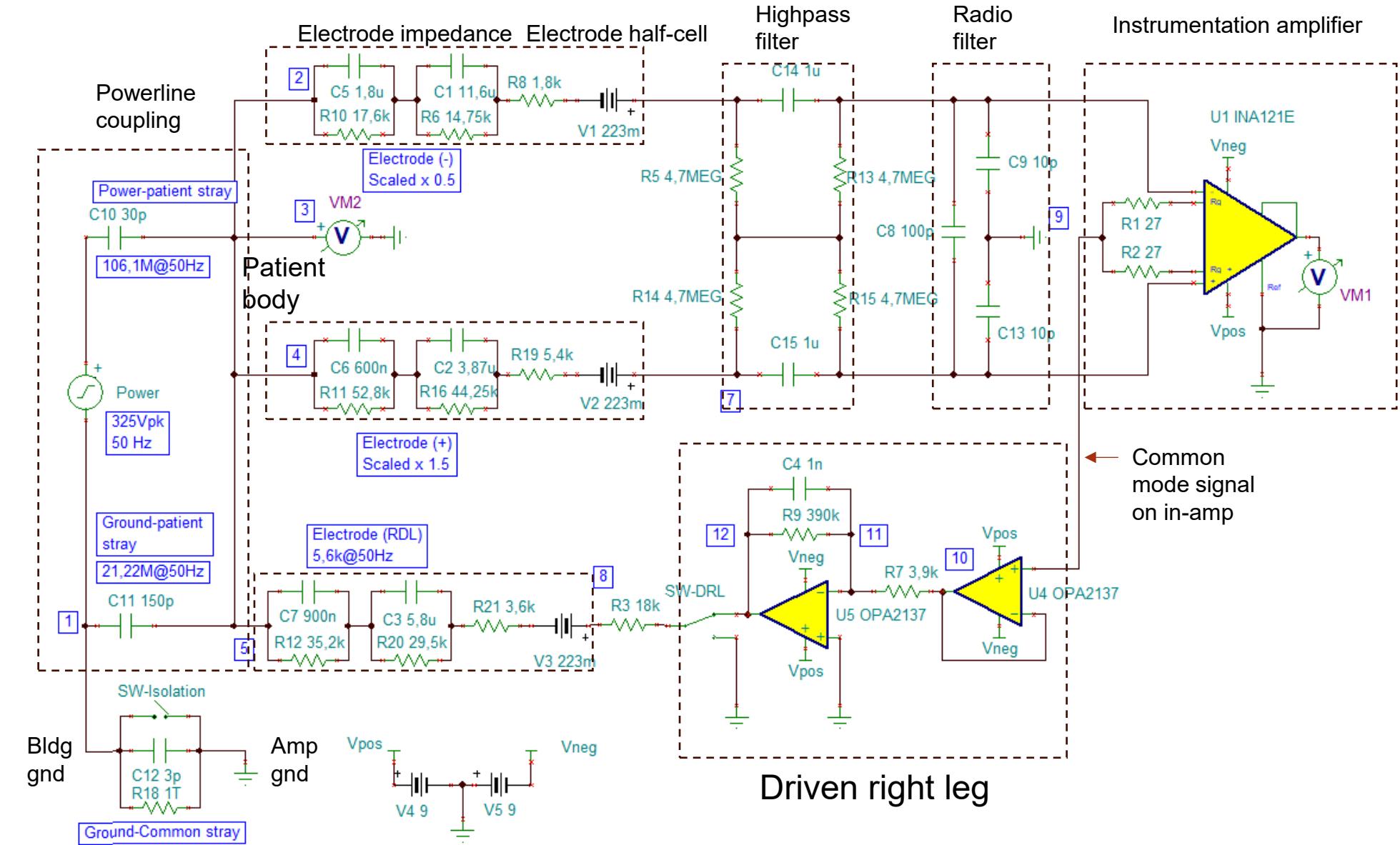
For safety reasons the amplifier must be isolated from building ground, fx by being driven by batteries. In this case, there will also be a stray capacitance between amplifier ground and building ground (C_{iso}).

A small capacitively coupled current (i_d) will run through the patient. The fraction which runs through the foot electrode impedance will create a common mode voltage V_{com} on the patient. By using a negative feedback circuit, we can send a proportional current in the opposite direction, thereby reducing V_{com} . Such a technique is called **Driven Right Leg**.



Electrocardiogram amplifier

The driven right leg circuit takes a voltage from a common mode point on the instrumentation amplifier, buffers it to avoid interfering with the gain resistors (R_1, R_2). It then sends V_{com} through a lowpass filter (U5) with a DC gain of 100. This will reduce V_{com} by 100. There is a risk of the feedback to become unstable.

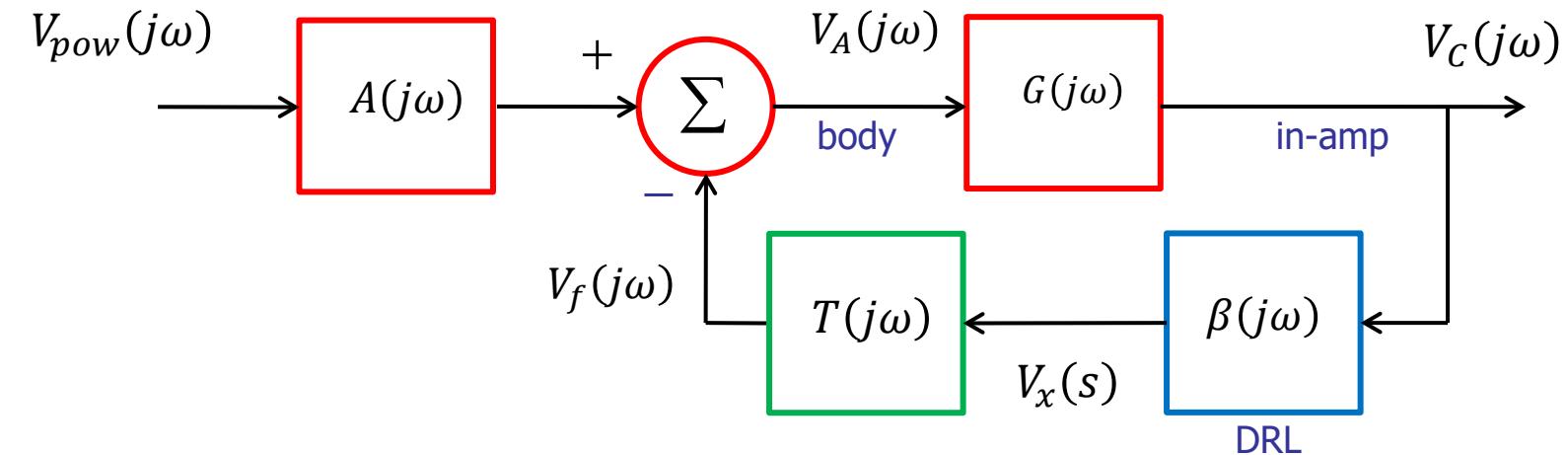


We can draw a block diagram of the system as is always done in control systems

We can redraw the circuit using a block diagram. Our objective is to reduce $V_A(j\omega)$.

If the frequency characteristic of the feedback loop (open loop) $H_{ol}(j\omega)$ has a phase change of more than 180° while the open loop gain is larger than 1, then the feedback circuit continues to add a growing feedback to the input signal. The system becomes unstable and will saturate.

We can investigate this by studying the open loop frequency characteristic .



$$V_A(j\omega) = A(j\omega)V_{pow}(j\omega) - T(j\omega)\beta(j\omega)G(j\omega)V_A(j\omega)$$

$$\frac{V_f(j\omega)}{V_A(j\omega)} = H_{ol}(j\omega) \stackrel{\text{def}}{=} T(j\omega)\beta(j\omega)G(j\omega)$$

$$V_f(j\omega) = H_{ol}(j\omega)V_A(j\omega)$$

$$V_c(j\omega) = G(j\omega)V_A(j\omega)$$

Open loop frequency characteristic

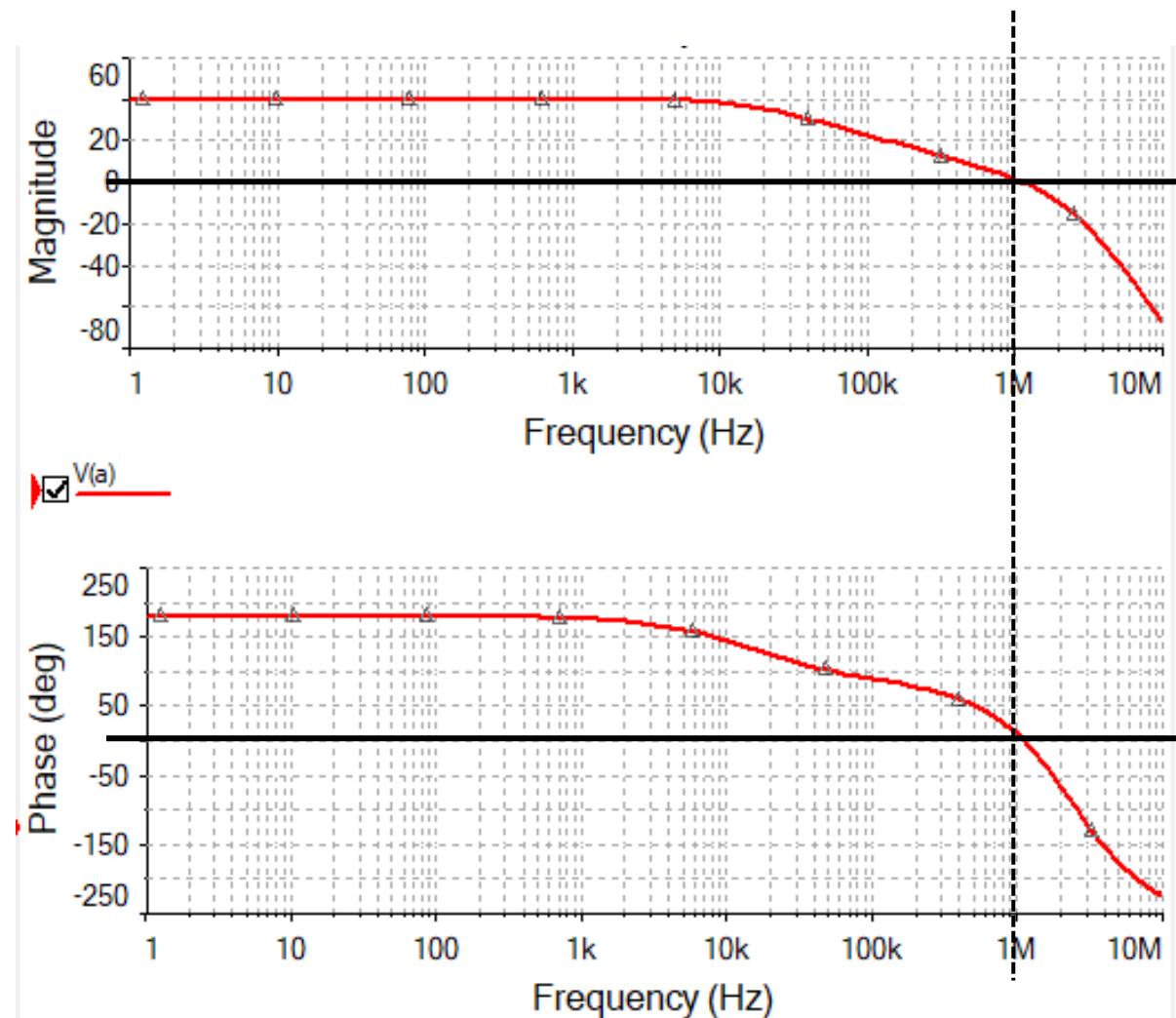
Case: isolated amplifier, no C4

We observe that at around 1MHz, the gain is 1 while the phase change is only slightly smaller than 180° .

The phase margin is about 10° .

This is too small.

$$H_{ol}(j\omega) \stackrel{\text{def}}{=} T(j\omega)\beta(j\omega)G(j\omega)$$



Open loop frequency characteristic - reduced bandwidth

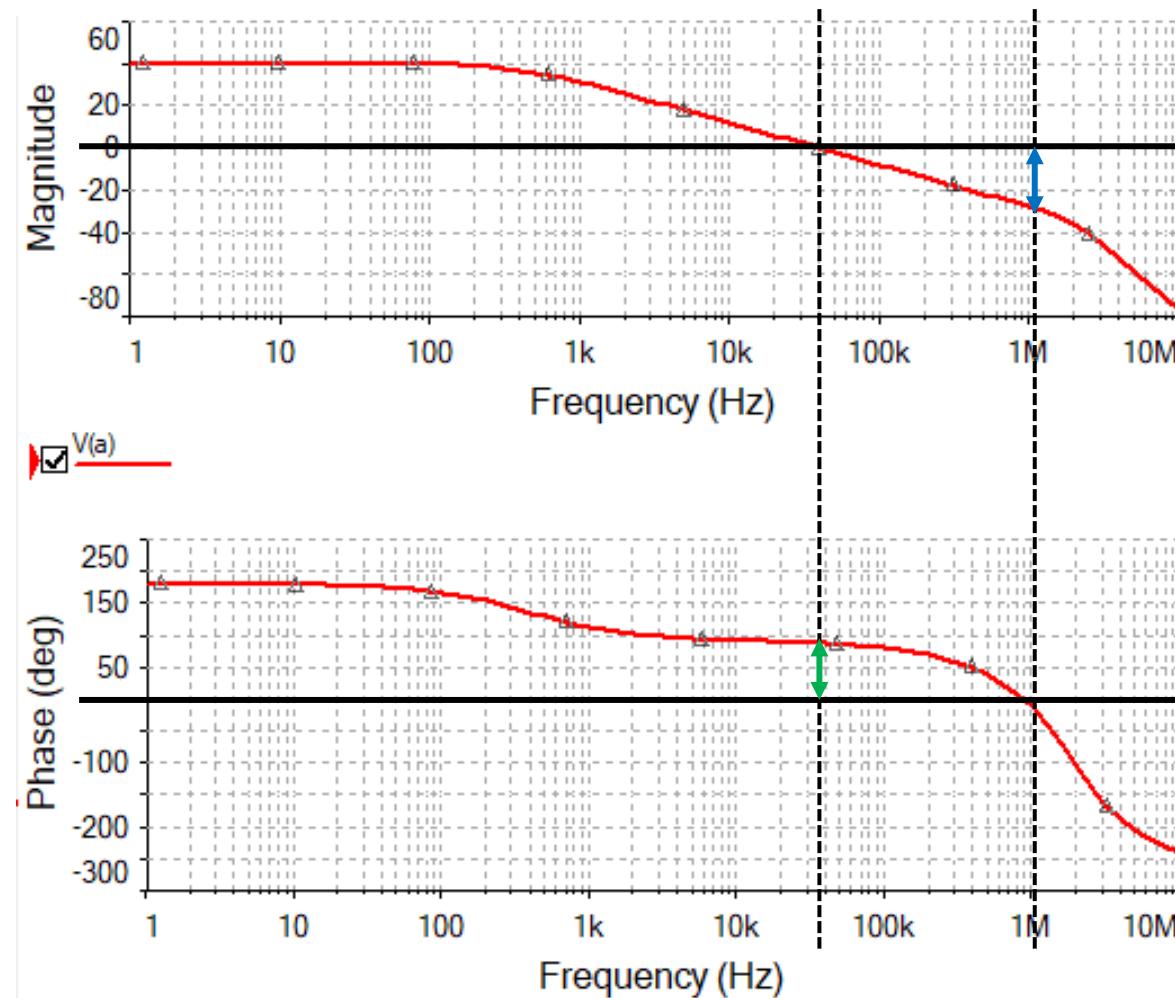
Case: Reduced bandwidth; $C_4 = 1\text{nF}$

The problem is that the DRL circuit has too much gain at high frequencies. We can make the DRL amplifier a lowpass filter by placing a capacitor C_4 in parallel with R_9 .

With the reduced bandwidth the open loop gain has dropped to 0dB at 40kHz, and the phase change at 40kHz is only about 90° . Hence the **phase margin** is 90° (green arrow).

At 1MHz the phase has changed 180° and the open loop gain is -30dB . Hence the **gain margin** is 30dB .

$$H_{ol}(j\omega) \stackrel{\text{def}}{=} T(j\omega)\beta(j\omega)G(j\omega)$$



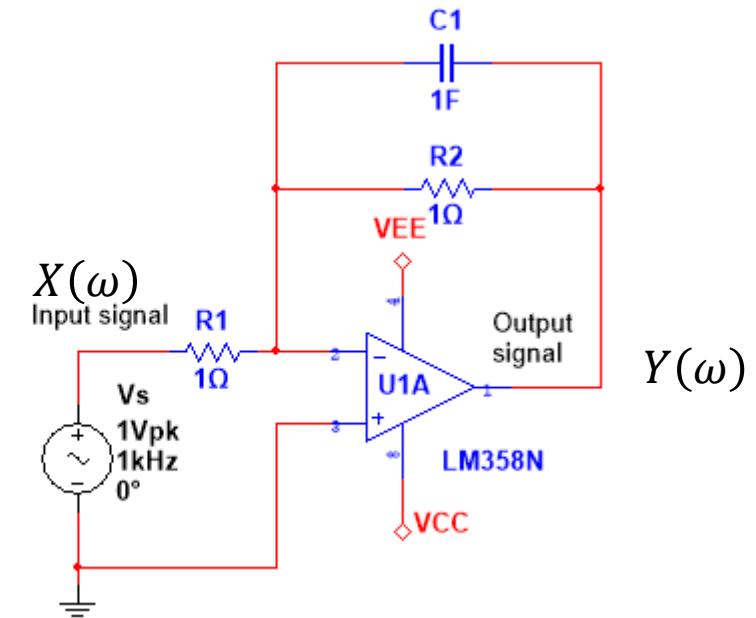
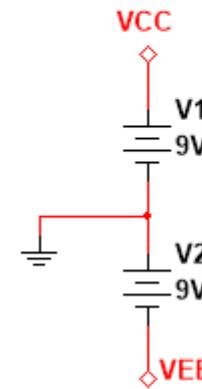
First order active low pass filter

Now that we have seen an interesting example of using an inverting amplifier as a lowpass filter, let us look at how we design such a filter.

Gain equation for inverting amplifier.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = -\frac{Z_2(\omega)}{Z_1(\omega)}$$

phasors



Component impedances:

$$Z_1(\omega) = R_1$$

$$Z_2(\omega) = R_2 || Z_{C_1}(\omega) = \frac{R_2 Z_{C_1}(\omega)}{R_2 + Z_{C_1}(\omega)} = \frac{R_2 \cdot \frac{1}{j\omega C_1}}{R_2 + \frac{1}{j\omega C_1}} = \underbrace{\frac{R_2}{1 + j\omega R_2 C_1}}_{\substack{C_1 \text{ and } R_2 \\ \text{in parallel}}} \quad \underbrace{\text{polynomial form}}$$

Substitute impedances into gain equation:

Impedances

$$ZCI := \omega \rightarrow \frac{1}{j\omega C1} :$$

Transfer function

$$H := \omega \rightarrow -\frac{\text{Par}(R2, ZCI(\omega))}{R1} :$$

$$\frac{\text{simplify}(\text{evalf}(H(\omega)))}{(R2 \omega C1 - 1.1) R1}$$

Choice of components

$$C1 := 1 ; R1 := 1 ; R2 := 1 ;$$

$$\frac{Y(\omega)}{X(\omega)} = -\frac{Z_2(\omega)}{Z_1(\omega)} = \frac{j R_2}{(R_2 \omega C_1 - j) R_1} = -\frac{\frac{R_2}{R_1}}{\underbrace{1 + j\omega R_2 C_1}_{\text{polynomial form}}} =$$

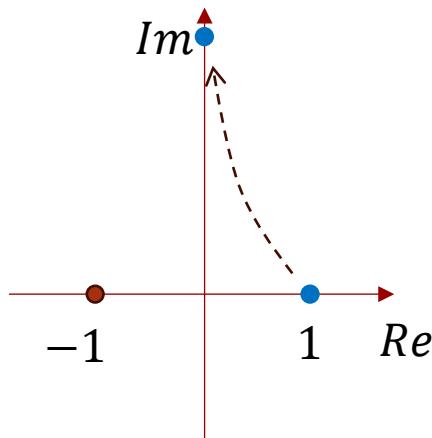
$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = -\frac{1}{1 + j\omega} \quad \text{Component values inserted}$$

$$H_{LFasymptote}(\omega) = -\frac{1}{1} = -1 = 1 e^{\pm j \pi}$$

$$H_{HFasymptote}(\omega) = -\frac{1}{j\omega} = \frac{j}{\omega} = \frac{1}{\omega} e^{j\pi/2}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + \omega^2}} = \frac{1}{\sqrt{2}} \Rightarrow \underbrace{\omega_c = 1 \frac{\text{rad}}{s}}_{\text{cut-off magnitude}} \Rightarrow \underbrace{f_c = \frac{1}{2\pi}}_{\text{cut-off frequency}} = 0.159\text{Hz}$$

First order active low pass filter



$$\angle H(\omega) = \underbrace{\angle P(\omega)}_{\substack{\text{numerator} \\ \text{angle}}} - \underbrace{\angle Q(\omega)}_{\substack{\text{denominator} \\ \text{angle}}}$$

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = -\frac{1}{1 + j\omega}$$

$$\theta(\omega) = \angle H(\omega) = 180^\circ - \tan^{-1} \omega$$

Low frequency asymptote

$$\theta(0) = \angle H(0) = 180^\circ - \tan^{-1} 0 = 180^\circ$$

High frequency asymptote

$$\theta(\infty) = \angle H(\infty) = 180^\circ - \tan^{-1} \infty = 180^\circ - 90^\circ = 90^\circ$$

A filter will let frequencies pass through in some frequency interval (**pass band**) and attenuate the output signal in other frequency intervals (**stop band**). The frequency that marks the transition between pass band and stop band is called the **cut-off frequency** ω_c or f_c .

A quantitative definition for the cut-off frequency is that frequency where the power of the output signal is 50% of the power of the input signal. Power is $V \cdot I = V^2/R$.

It follows that the cut-off frequency is that frequency, where the amplitude characteristic is $1/\sqrt{2}$:

For some amplitude characteristics, this is equivalent to a 3 dB attenuation. The cut-off frequency is therefore also called the **3-dB frequency**.

Observation:

The equations often use radians ω , but in graphs, f is used. Design specifications are usually given in Hertz. This must then be translated to radians before deriving component values.

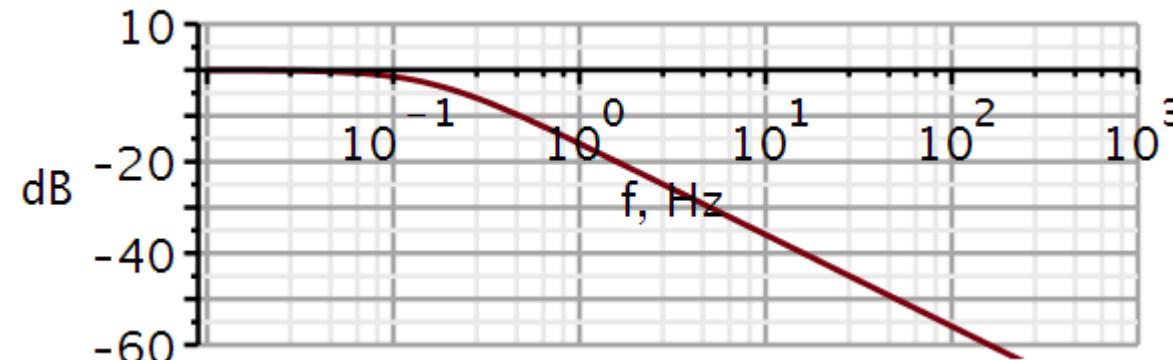
$$\underbrace{|H(\omega_c)|^2 = \frac{|Y(\omega_c)|^2}{|X(\omega_c)|^2} = \frac{1}{2}}_{\text{cut-off magnitude}}$$

$$\underbrace{|H(\omega_c)| = \frac{1}{\sqrt{2}}}_{\text{cut-off magnitude}}$$

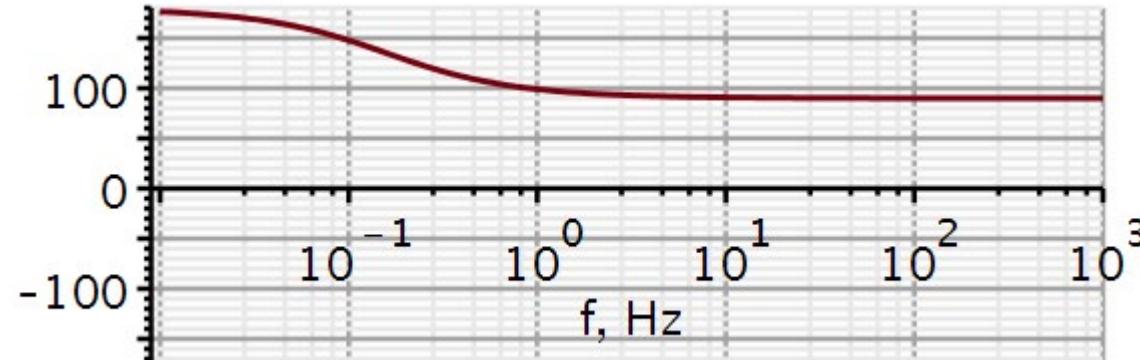
$$\underbrace{|H(\omega_c)|_{dB} = 20 \cdot \log_{10} \frac{1}{\sqrt{2}} = -3 \text{ dB}}_{\text{cut-off magnitude}}$$

First order active low pass filter

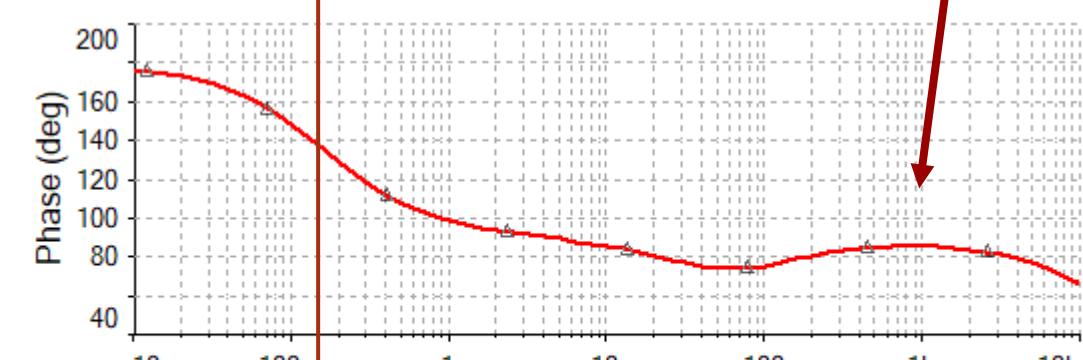
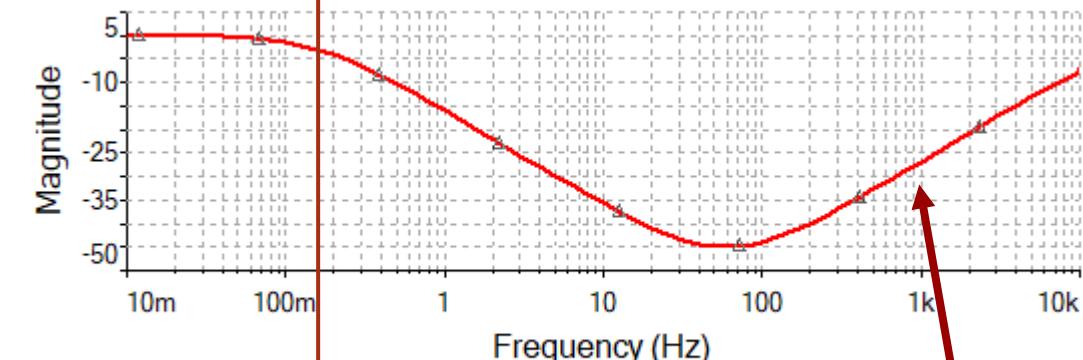
Amplitude characteristic



Phase characteristic



Pass
band Stop
band



Frequency scaling

Let us now change the cut-off frequency from 0.159 Hz to 200 Hz. We do **frequency scaling** as follows:

$$\omega = K_F \cdot \omega_1$$

$$K_F = \frac{200\text{Hz}}{0.159\text{Hz}} = 1258 \Rightarrow C_1 = \frac{1\text{F}}{1258} = 795\mu\text{F}$$

We can also do an **impedance scaling**, where we scale all component with the same factor K_Z , but without changing time constants.

If we desire a compact capacitor, we would like it to be smaller than 1μF.

$$\frac{Y(\omega)}{X(\omega)} = - \frac{\frac{R_2}{R_1}}{1 + j(K_F \cdot \omega)R_2 \frac{C_1}{K_F}}$$

$$\tau = R \cdot C = K_Z R \cdot C / K_Z$$

$$\frac{Y(\omega)}{X(\omega)} = - \frac{\frac{K_Z R_2}{K_Z R_1}}{1 + j(K_F \cdot \omega)K_Z R_2 \frac{C_1}{K_F K_Z}}$$

Let us aim for a capacitor of 47nF.

So here we let the impedance scaling factor K_Z be decided from a desired magnitude of the capacitor value.

Then the resistor values are determined based on this.

Capacitors are usually available in the E6 series. We may then have to fit odd values of resistors or use E48 or E96 resistors.

It is much cheaper to get a high precision resistor than a high precision capacitor.

$$C_1'' = \frac{C_1'}{K_Z} = \frac{C_1}{K_F K_Z}$$

$$K_Z = \frac{795 \mu\text{F}}{47 \text{ nF}} = 16915$$

$$R_1 = 16915 \cdot 1\Omega \approx 15\text{k}\Omega$$

$$R_2 = 16915 \cdot 1\Omega \approx 15\text{k}\Omega$$

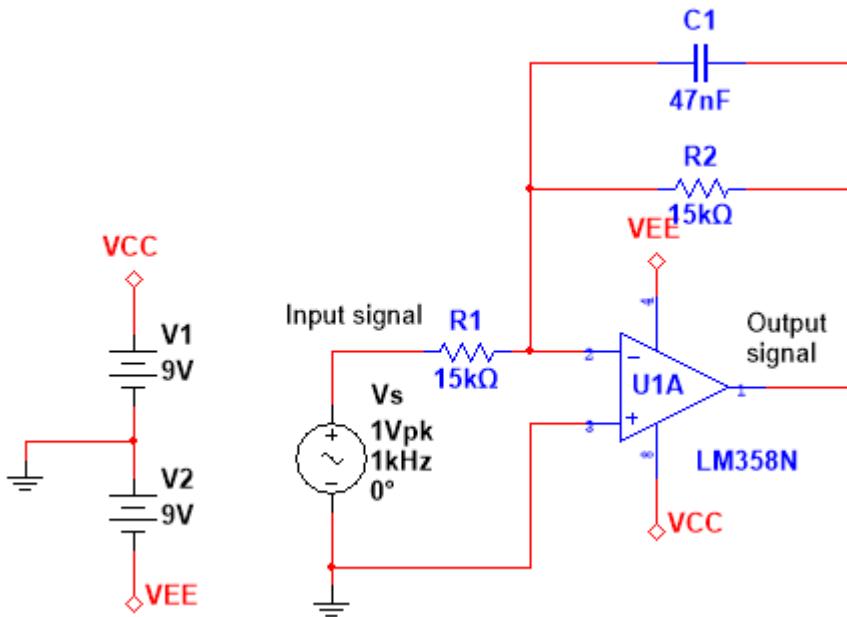
$$C_1 = 47\text{nF}$$

$$R_1 C_1 = 1\Omega \cdot 795\mu\text{F} = 0.795\text{ms} \quad f_c = 200\text{Hz}$$

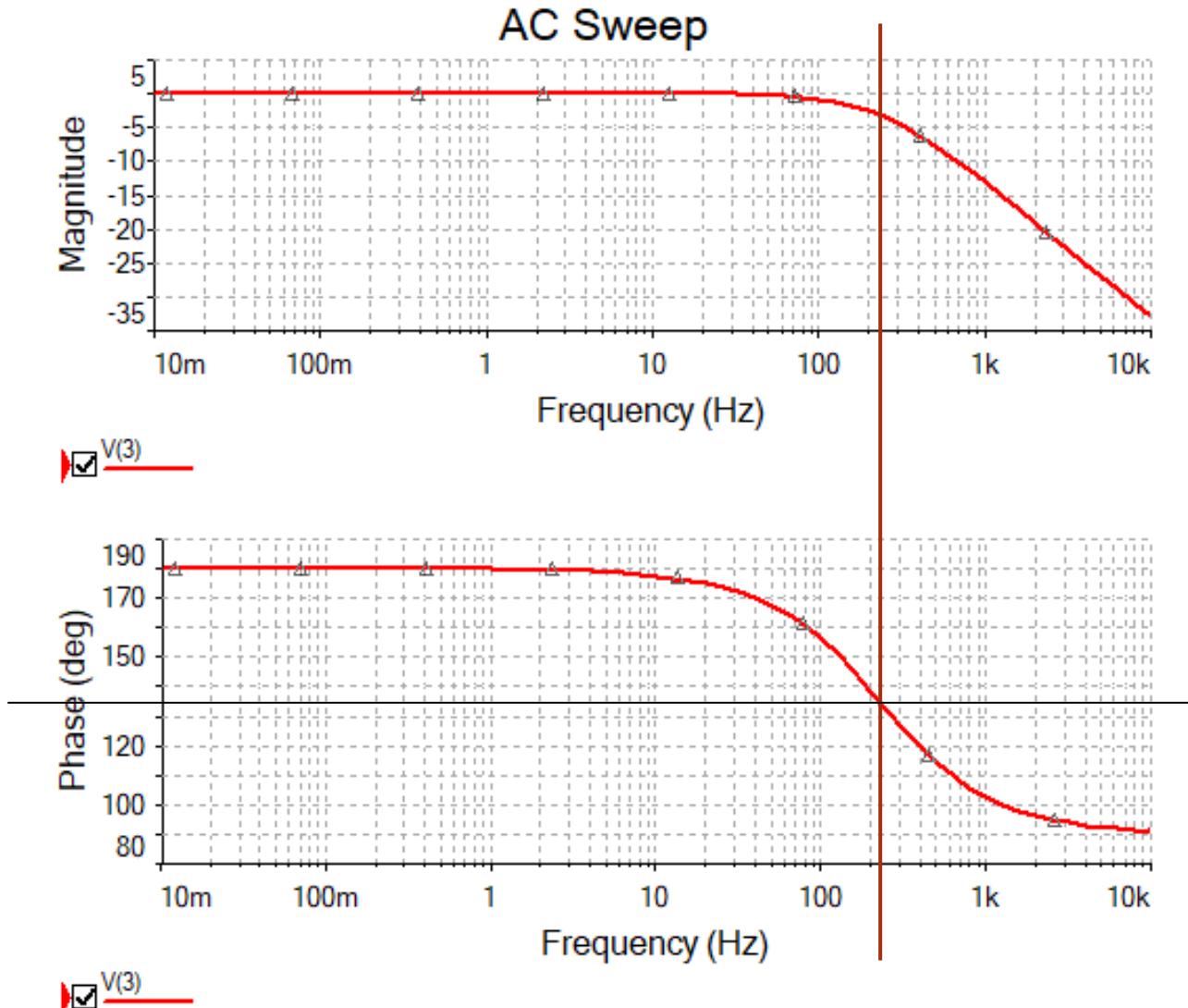
$$R_1 C_1 \approx 15\text{k}\Omega \cdot 47\text{nF} = 0.705\text{ms} \quad f_c = 226\text{Hz}$$

C values E6:	10
	15
	22
	33
	47
	68

Result of scaling



We have now a finished lowpass filter. We can read the cut-off frequency at -3dB on the magnitude plot or at half the phase angle change (here 135°) on the phase plot.



Using a lookup table

We just arbitrarily chose a capacitor value of 47nF. We then had to round-off the resistor value.

Using a look-up table, we can spot the best combination.

Here 8.160 is the number closest to 7.95.

$$\tau = R_1 C_1 = 1\Omega \cdot 795\mu\text{F} = 0.795\text{ms} \text{ (ideal)}$$

$$\tau = R_1 C_1 \approx 15\text{k}\Omega \cdot 47\text{nF} = 0.705\text{ms}$$

$$R_1 C_1 = 1.2 \times 10^x \cdot 6.8 \times 10^y = 0.816\text{ms}$$

$$f_c = 195\text{Hz}$$

This look-up table is made in Excel.

The numbers of interest are often found in a narrow diagonal.

R\C	$\tau = R \cdot C$														
	E6	E12	1	1,2	1,5	1,8	2,2	2,7	3,3	3,9	4,7	5,6	6,8	8,2	10
1	1,000	1,200	1,500	1,800	2,200	2,700	3,300	3,900	4,700	5,600	6,800	8,200	10,000		
1,2	1,200	1,440	1,800	2,160	2,640	3,240	3,960	4,680	5,640	6,720	8,160	9,840	12,000		
1,5	1,500	1,800	2,250	2,700	3,300	4,050	4,950	5,850	7,050	8,400	10,200	12,300	15,000		
1,8	1,800	2,160	2,700	3,240	3,960	4,860	5,940	7,020	8,460	10,080	12,240	14,760	18,000		
2,2	2,200	2,640	3,300	3,960	4,840	5,940	7,260	8,580	10,340	12,320	14,960	18,040	22,000		
2,7	2,700	3,240	4,050	4,860	5,940	7,290	8,910	10,530	12,690	15,120	18,360	22,140	27,000		
3,3	3,300	3,960	4,950	5,940	7,260	8,910	10,890	12,870	15,510	18,480	22,440	27,060	33,000		
3,9	3,900	4,680	5,850	7,020	8,580	10,530	12,870	15,210	18,330	21,840	26,520	31,980	39,000		
4,7	4,700	5,640	7,050	8,460	10,340	12,690	15,510	18,330	22,090	26,320	31,960	38,540	47,000		
5,6	5,600	6,720	8,400	10,080	12,320	15,120	18,480	21,840	26,320	31,360	38,080	45,920	56,000		
6,8	6,800	8,160	10,200	12,240	14,960	18,360	22,440	26,520	31,960	38,080	46,240	55,760	68,000		
8,2	8,200	9,840	12,300	14,760	18,040	22,140	27,060	31,980	38,540	45,920	55,760	67,240	82,000		

Using a lookup table

The look-up table gives the base value, but not the scale. We need to determine the multipliers for R and C .

We get one equation and two scale parameters. We need to choose one and calculate the other:

We can aim for $C_1 = 6.8\text{nF}$, i.e., $y = -9$:

In this case we will get a resistor

$$R_1 = 1.2 \times 10^5 \Omega = 120\text{k}\Omega$$

Alternatively, we could use:

$$R_1 C_1 = 1.2 \times 10^x \cdot 6.8 \times 10^y = 0.816\text{ms}$$

$$1.2 \times 10^x \cdot 6.8 \times 10^y = 8.16 \times 10^{x+y} = 8.16 \times 10^{-4}$$

$$x + y = -4$$

$$x = -y - 4 = 9 - 4 = 5$$

$$R_1 = 120\text{k}\Omega, C_1 = 6.8\text{nF}$$

$$R_1 = 12\text{k}\Omega, C_1 = 68\text{nF}$$

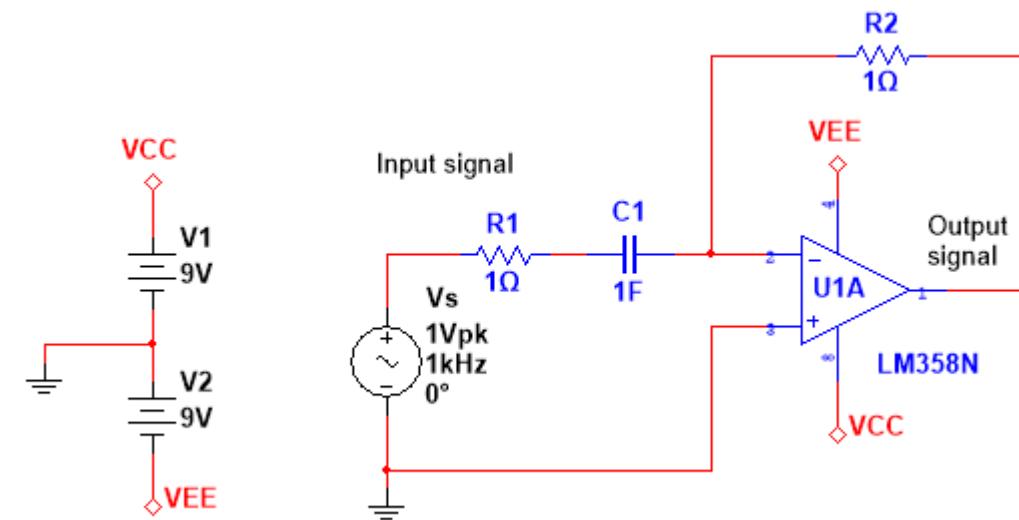
Observation:

Picking an arbitrary value for C and then calculating the resistor value will not give the best approximation to the time constant and cut-off frequency. Using a look-up table is an easy approach to spot the best combination of R and C values. Look-up tables are easily constructed in Excel.

First order high pass filter

Gain equation for inverting amplifier.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = -\frac{Z_2(\omega)}{Z_1(\omega)}$$



Component impedances:

$$Z_1(\omega) = R_1 + \frac{1}{j\omega C_1}$$

$$Z_2(\omega) = R_2$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = -\frac{R_2}{R_1 + \frac{1}{j\omega C_1}} = -\frac{j\omega R_2 C_1}{1 + j\omega R_1 C_1}$$

polynomial form

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = -\frac{j\omega}{1 + j\omega}$$

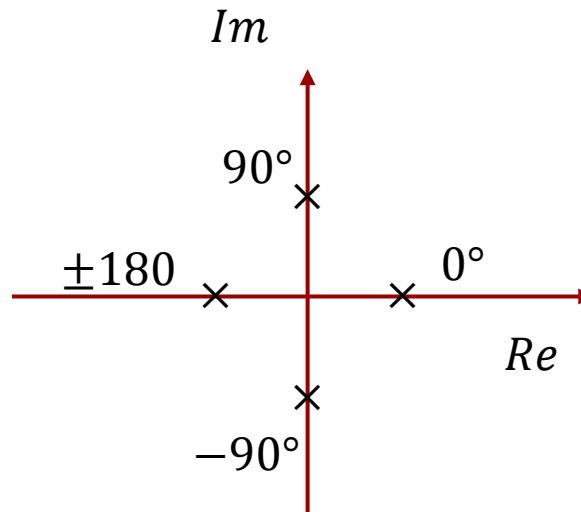
Component values inserted

$$H_{LFasymptote}(\omega) = -\frac{j\omega}{1} = -j\omega = \omega e^{-j\pi/2}$$

$$H_{HFasymptote}(\omega) = -\frac{j\omega}{j\omega} = 1e^{-j\pi}$$

A highpass filter with a phase starting in -90 degrees and ending in -180 degrees.

First order high pass filter



$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = -\frac{j\omega}{1 + j\omega}$$

$$\angle H(\omega) = \underbrace{\angle P(\omega)}_{\substack{\text{numerator} \\ \text{angle}}} - \underbrace{\angle Q(\omega)}_{\substack{\text{denominator} \\ \text{angle}}}$$

$$\theta(\omega) = \angle H(\omega) = -180^\circ + 90^\circ - \tan^{-1} \omega$$

$$\theta(0) = \angle H(0) = -90^\circ - \tan^{-1} 0 = -90^\circ$$

$$z = x + jy$$

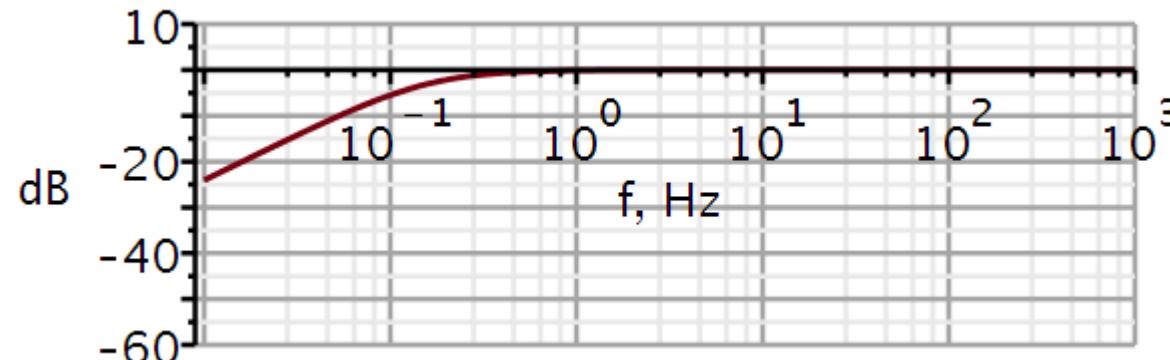
$$\theta(\infty) = \angle H(\infty) = -90^\circ - \tan^{-1} \infty = -90^\circ - 90^\circ = -180^\circ$$

$$|z| = |x + jy| = \sqrt{x^2 + y^2}$$

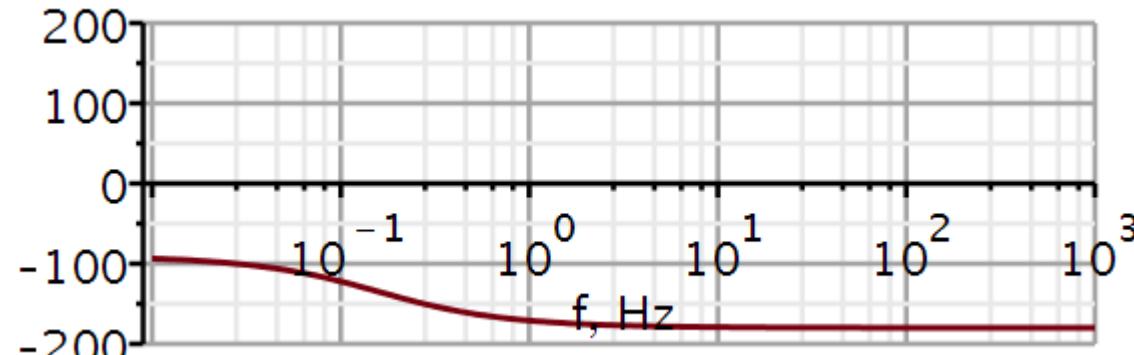
$$\angle z = \tan^{-1} \frac{y}{x}$$

High pass filter with normalized cut-off frequency

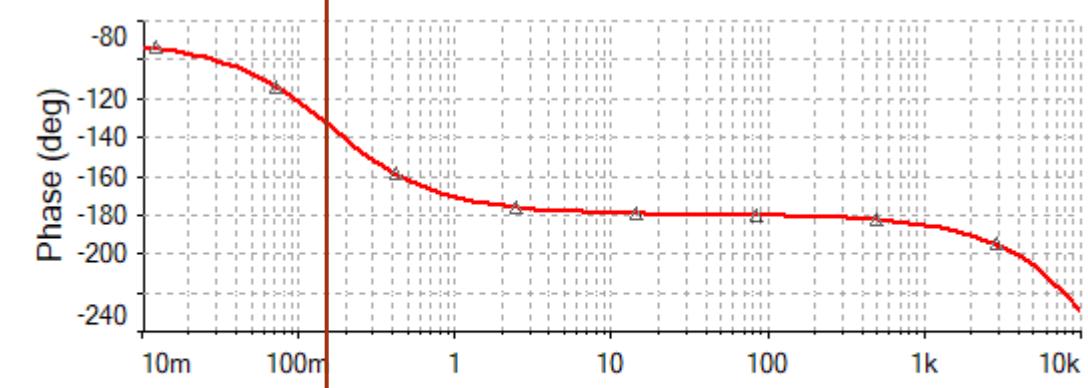
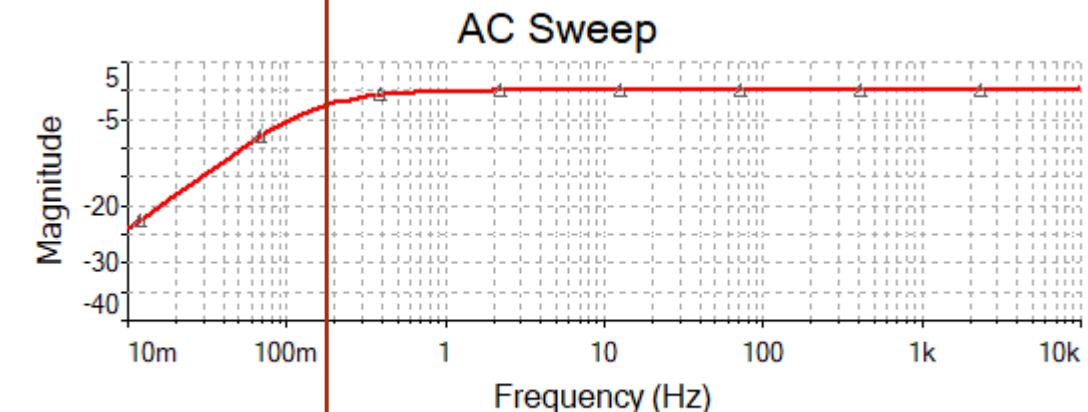
Amplitude characteristic



Phase characteristic



Stop band Pass band



Frequency scaling

Let us now scale the cut-off frequency from 0.159 to 20 Hz. We can **scale the frequency characteristic** as follows:

$$\omega = K_F \cdot \omega_1$$

$$K_F = \frac{20\text{Hz}}{0.159\text{Hz}} = 125.8 \Rightarrow C_1 = \frac{1\text{F}}{125.8} = 7950 \mu\text{F}$$

We can also do an **impedance scaling**, where we scale all component with the same factor K_Z , but without changing time constants.

If we desire a compact capacitor, we would like it to be smaller than $1\mu\text{F}$.

$$H(\omega) = -\frac{j K_F \cdot \omega R_2 \frac{C_1}{K_F}}{1 + j K_F \cdot \omega R_1 \frac{C_1}{K_F}}$$

$$\tau = R \cdot C = K_Z R \cdot \frac{C}{K_Z} = 1\Omega \cdot 7950\mu\text{F} = 7.95\text{ms}$$

$$H(\omega) = -\frac{j (K_F \cdot \omega)(K_Z \cdot R_2) \frac{C_1}{K_F K_Z}}{1 + j (K_F \cdot \omega)(K_Z \cdot R_1) \frac{C_1}{K_F K_Z}}$$

Using a lookup table

$$\tau = R_1 C_1 = 1\Omega \cdot 7950\mu\text{F} = 7.95\text{ms (ideal)}$$

Using a look-up table, we can spot the best combination of R_1 and C_1 . Here 8.160 is the number closest to 7.95.

$$R_1 C_1 = 1.2 \times 10^x \cdot 6.8 \times 10^y = 8.16\text{ms}$$

This look-up table is made in Excel.

The numbers of interest are often found in a narrow diagonal.

R\C	$\tau = R \cdot C$														
	E6	E12	1	1,2	1,5	1,8	2,2	2,7	3,3	3,9	4,7	5,6	6,8	8,2	10
1	1,000	1,200	1,500	1,800	2,200	2,700	3,300	3,900	4,700	5,600	6,800	8,200	10,000		
1,2	1,200	1,440	1,800	2,160	2,640	3,240	3,960	4,680	5,640	6,720	8,160	9,840	12,000		
1,5	1,500	1,800	2,250	2,700	3,300	4,050	4,950	5,850	7,050	8,400	10,200	12,300	15,000		
1,8	1,800	2,160	2,700	3,240	3,960	4,860	5,940	7,020	8,460	10,080	12,240	14,760	18,000		
2,2	2,200	2,640	3,300	3,960	4,840	5,940	7,260	8,580	10,340	12,320	14,960	18,040	22,000		
2,7	2,700	3,240	4,050	4,860	5,940	7,290	8,910	10,530	12,690	15,120	18,360	22,140	27,000		
3,3	3,300	3,960	4,950	5,940	7,260	8,910	10,890	12,870	15,510	18,480	22,440	27,060	33,000		
3,9	3,900	4,680	5,850	7,020	8,580	10,530	12,870	15,210	18,330	21,840	26,520	31,980	39,000		
4,7	4,700	5,640	7,050	8,460	10,340	12,690	15,510	18,330	22,090	26,320	31,960	38,540	47,000		
5,6	5,600	6,720	8,400	10,080	12,320	15,120	18,480	21,840	26,320	31,360	38,080	45,920	56,000		
6,8	6,800	8,160	10,200	12,240	14,960	18,360	22,440	26,520	31,960	38,080	46,240	55,760	68,000		
8,2	8,200	9,840	12,300	14,760	18,040	22,140	27,060	31,980	38,540	45,920	55,760	67,240	82,000		

Let us aim for a capacitor of 680nF.

$$C_1'' = \frac{C_1'}{K_Z} = \frac{C_1}{K_F K_Z}$$

C values

$$K_Z = \frac{7950 \mu\text{F}}{680 \text{ nF}} = 11691$$

E6:

10

15

22

33

47

68

$$R_1 = 11691 \cdot 1\Omega \approx 12\text{k}\Omega$$

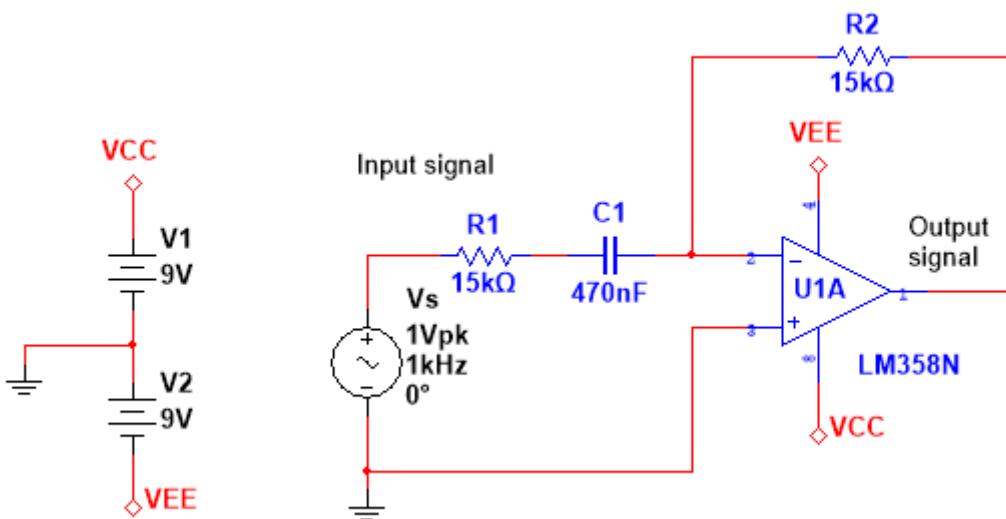
$$R_2 = 11691 \cdot 1\Omega \approx 12\text{k}\Omega$$

$$C_1 = 680\text{nF}$$

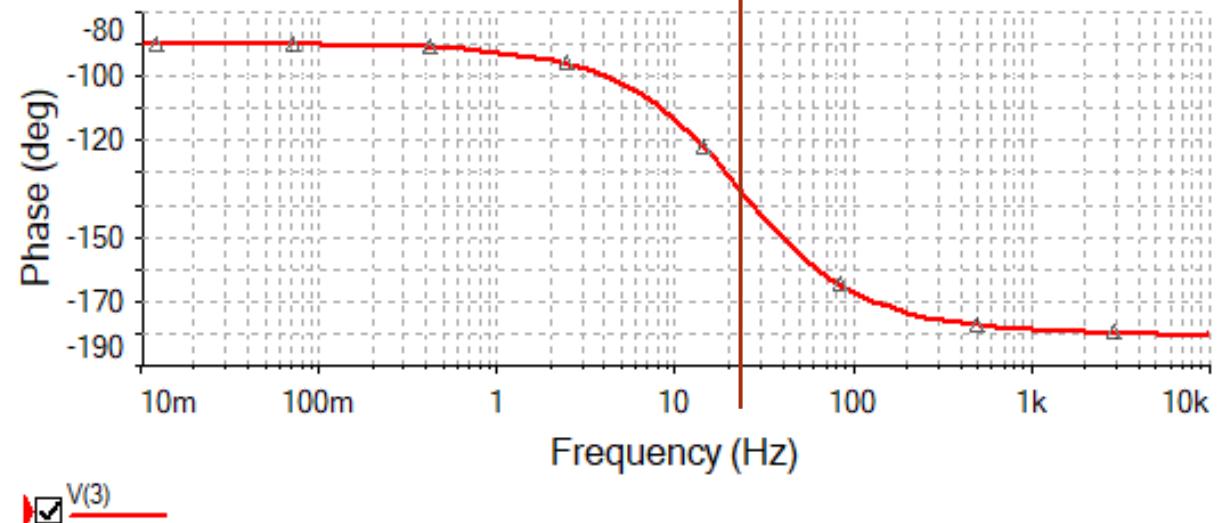
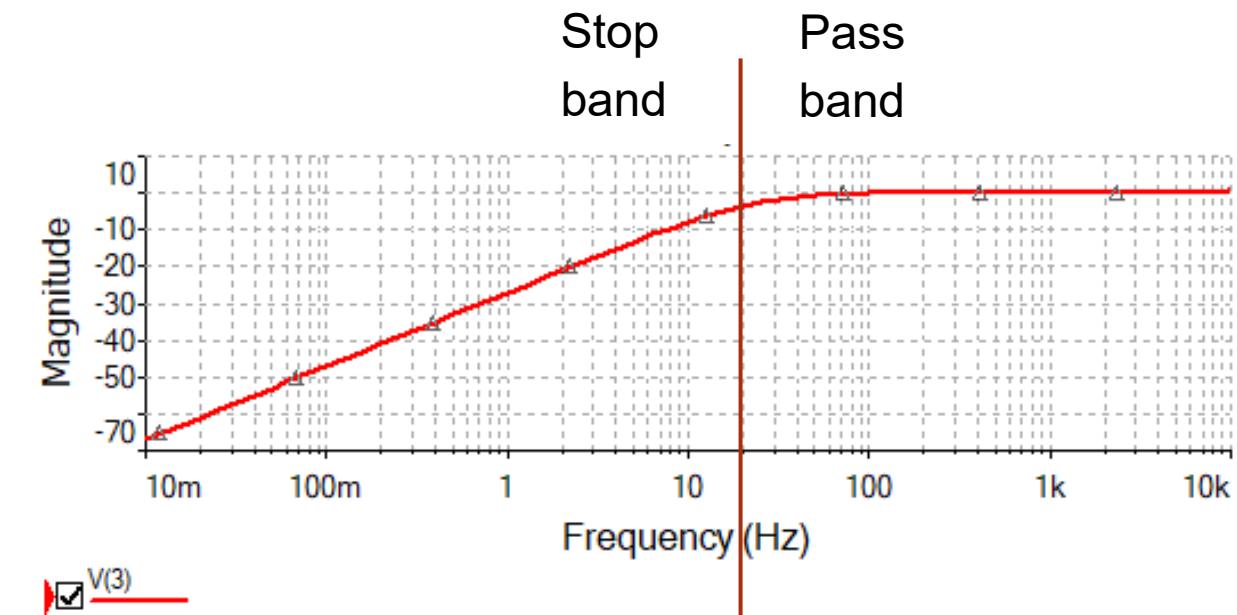
So again, we let the impedance scaling factor K_Z be decided from a desired magnitude of the capacitor value.

Then the resistor values are determined based on this.

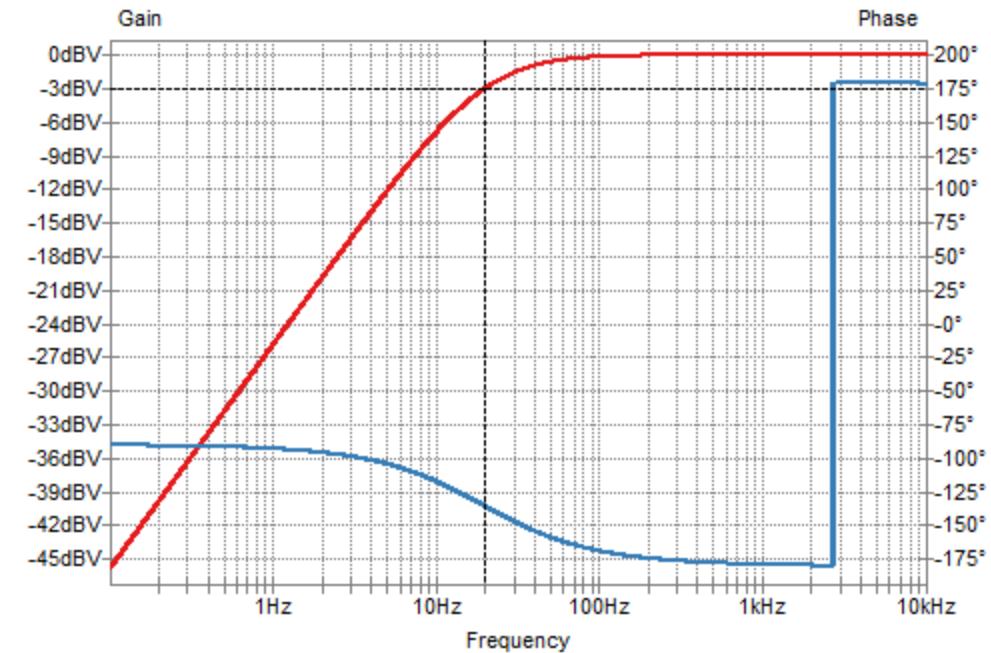
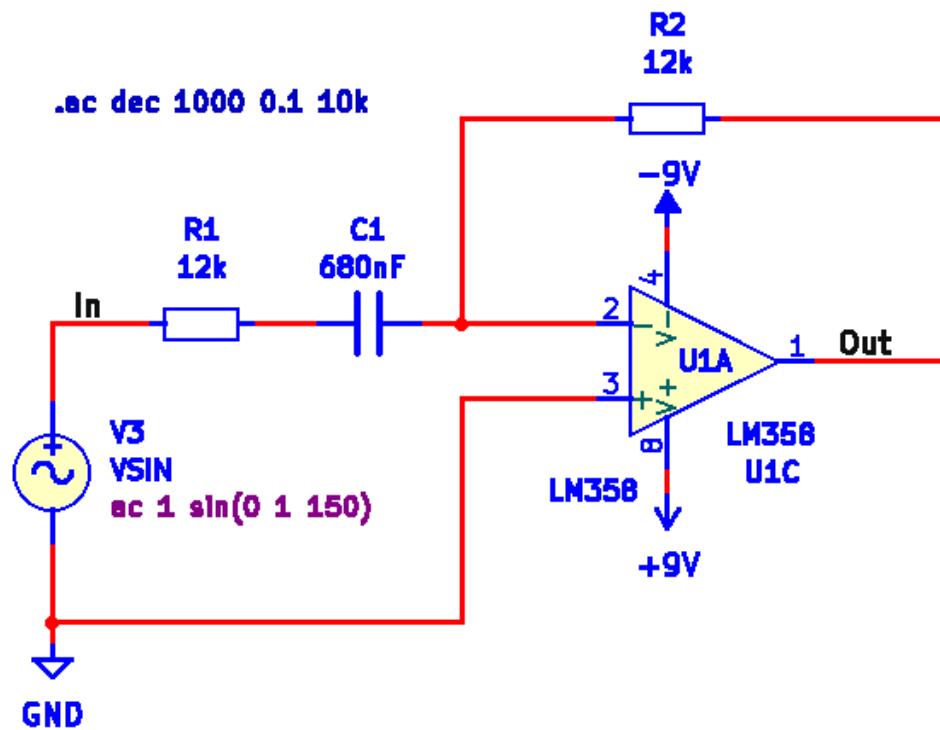
Scaled high pass filter



Simulation in Multisim



Scaled high pass filter



Simulation in KiCad

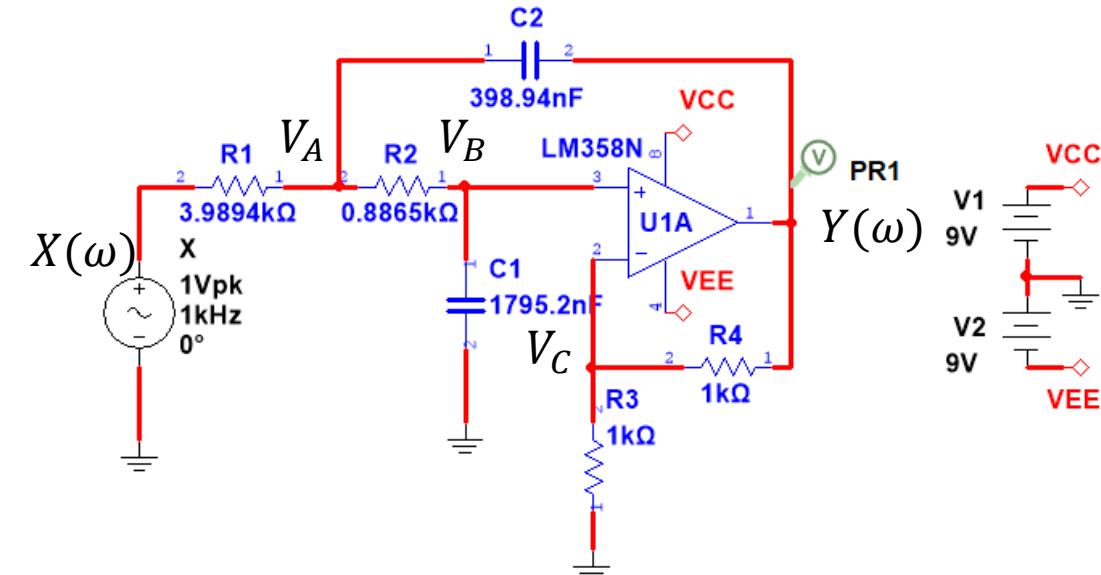
Problems

Video 3

Filter 10 – Lowpass filter

A Sallen-Key filter is shown here. You are asked to analyse this circuit and:

- Derive the equation for the frequency characteristic $H(\omega)$. Use Maple/Python if possible. Verify the numerical values of the coefficients (a_1, a_0) with those calculated in Lecture 1.
- Derive the differential equation.
- Derive the equation for the amplitude characteristic $|H(\omega)|$ and determine Pass band gain and Stop band asymptote slope (dB/decade).
- Derive the equation for phase angle $\angle H(\omega)$ and predict phase angles at low and high frequencies.
- Plot the amplitude and phase characteristics in Maple/Python.
- Draw the circuit in KiCad/Spice, run an AC sweep and plot the amplitude and phase characteristics.
- Validate the mathematical analysis by comparing Maple/Python graphs with KiCad/Spice graphs.

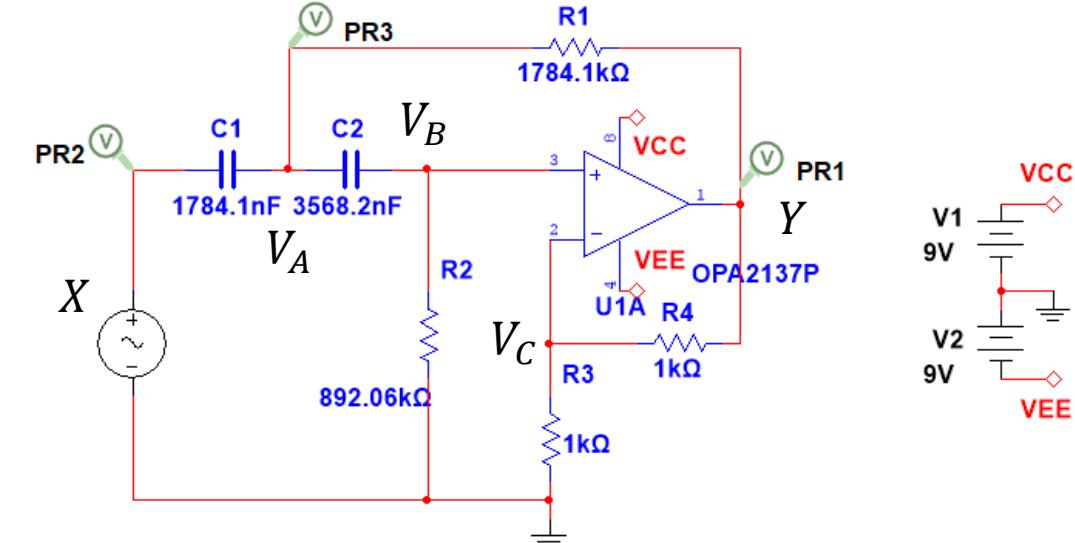


C values	
E6:	10
	15
	22
	33
	47
	68

Filter 11 – Highpass filter

A Sallen-Key filter is shown here. You are asked to analyse this circuit and:

- Derive the equation for the frequency characteristic $H(\omega)$. Use Maple/Python if possible. Verify the numerical values of the coefficients (a_1, a_0) with those calculated in Lecture 1.
- Derive the differential equation.
- Derive the equation for the amplitude characteristic $|H(\omega)|$ and determine Pass band gain and Stop band asymptote slope (dB/decade).
- Derive the equation for phase angle $\angle H(\omega)$ and predict phase angles at low and high frequencies.
- Plot the amplitude and phase characteristics in Maple/Python.
- Draw the circuit in KiCad/Spice, run an AC sweep and plot the amplitude and phase characteristics.
- Validate the mathematical analysis by comparing Maple/Python graphs with KiCad/Spice graphs.

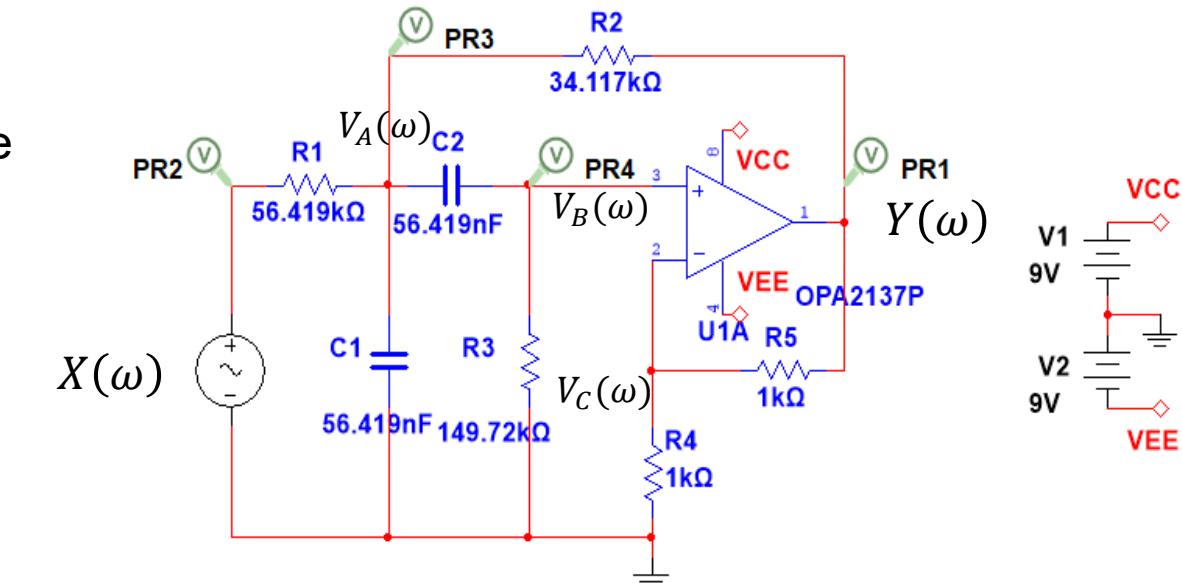


C values
E6:
10
15
22
33
47
68

Filter 12 – Bandpass filter

A Sallen-Key filter is shown here. You are asked to analyse this circuit and:

- Derive the equation for the frequency characteristic $H(\omega)$. Use Maple/Python if possible. Verify the numerical values of the coefficients (a_1, a_0) with those calculated in Lecture 1.
- Derive the differential equation.
- Derive the equation for the amplitude characteristic $|H(\omega)|$ and determine Pass band gain and Stop band asymptote slope (dB/decade).
- Derive the equation for phase angle $\angle H(\omega)$ and predict phase angles at low and high frequencies.
- Plot the amplitude and phase characteristics in Maple/Python.
- Draw the circuit in KiCad/Spice, run an AC sweep and plot the amplitude and phase characteristics.
- Validate the mathematical analysis by comparing Maple/Python graphs with KiCad/Spice graphs.



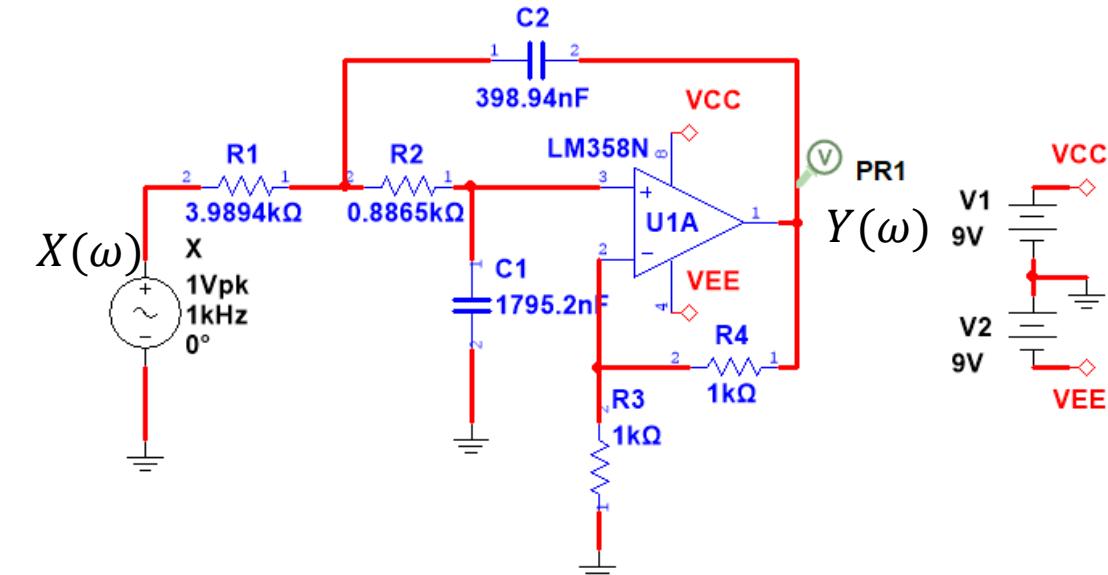
C values
E6:
10
15
22
33
47
68

Solutions

Filter 10 – Lowpass filter

A Sallen-Key filter is shown here. You are asked to analyse this circuit and:

- Derive the equation for the frequency characteristic $H(\omega)$. Use Maple/Python if possible. Verify the numerical values of the coefficients (a_1, a_0) with those calculated in Lecture 1.
- Derive the differential equation.
- Derive the equation for the amplitude characteristic $|H(\omega)|$ and determine Pass band gain and Stop band asymptote slope (dB/decade).
- Derive the equation for phase angle $\angle H(\omega)$ and predict phase angles at low and high frequencies.
- Plot the amplitude and phase characteristics in Maple/Python.
- Draw the circuit in KiCad/Spice, run an AC sweep and plot the amplitude and phase characteristics.
- Validate the mathematical analysis by comparing Maple/Python graphs with KiCad/Spice graphs.



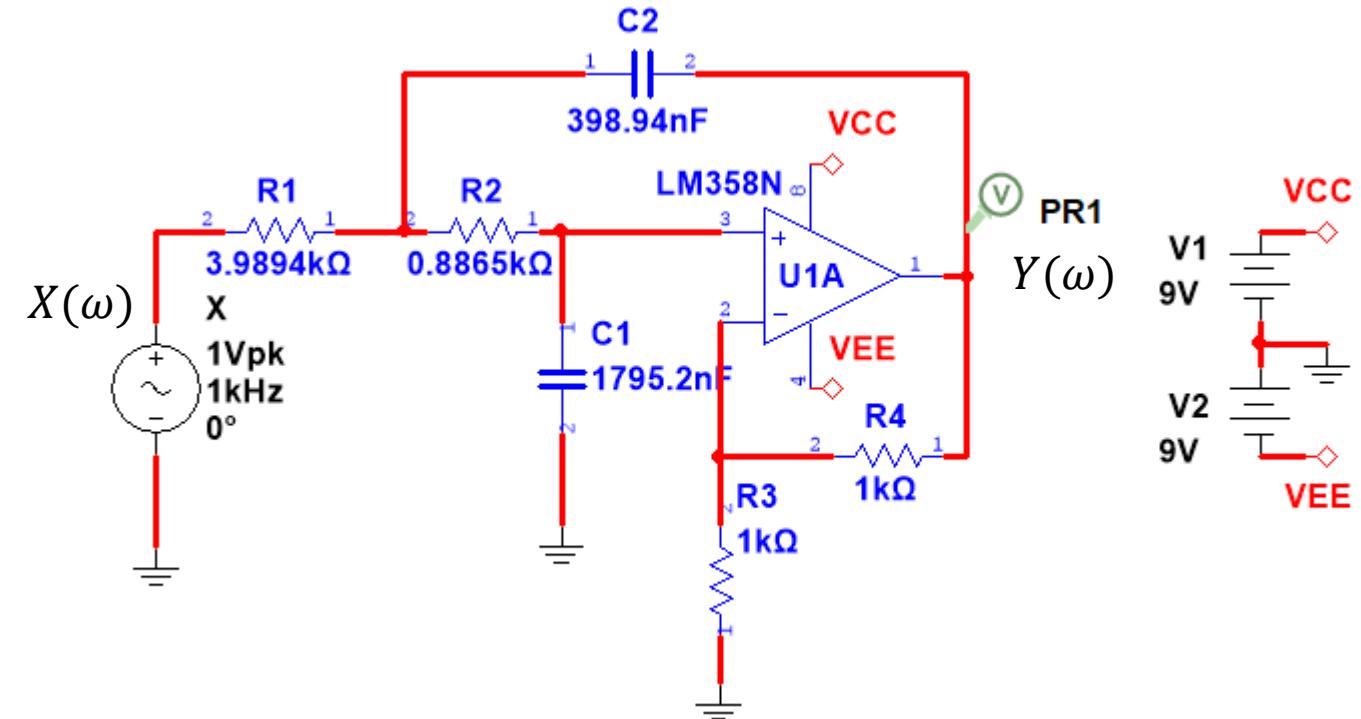
C values
E6:
10
15
22
33
47
68

Filter 10 – Lowpass filter (sol)

We now want to analyse the Sallen-Key filter from a frequency domain perspective.

We want to:

1. Obtain the equation for the frequency characteristic.
2. Check the values of the equation coefficients.
3. Obtain the differential equation from the frequency characteristic
4. Predict the main features of the amplitude and phase characteristics
5. Plot the amplitude and phase characteristics in Maple/Python.



Filter 10 – Lowpass filter (sol)

Nodal symbols have been added.

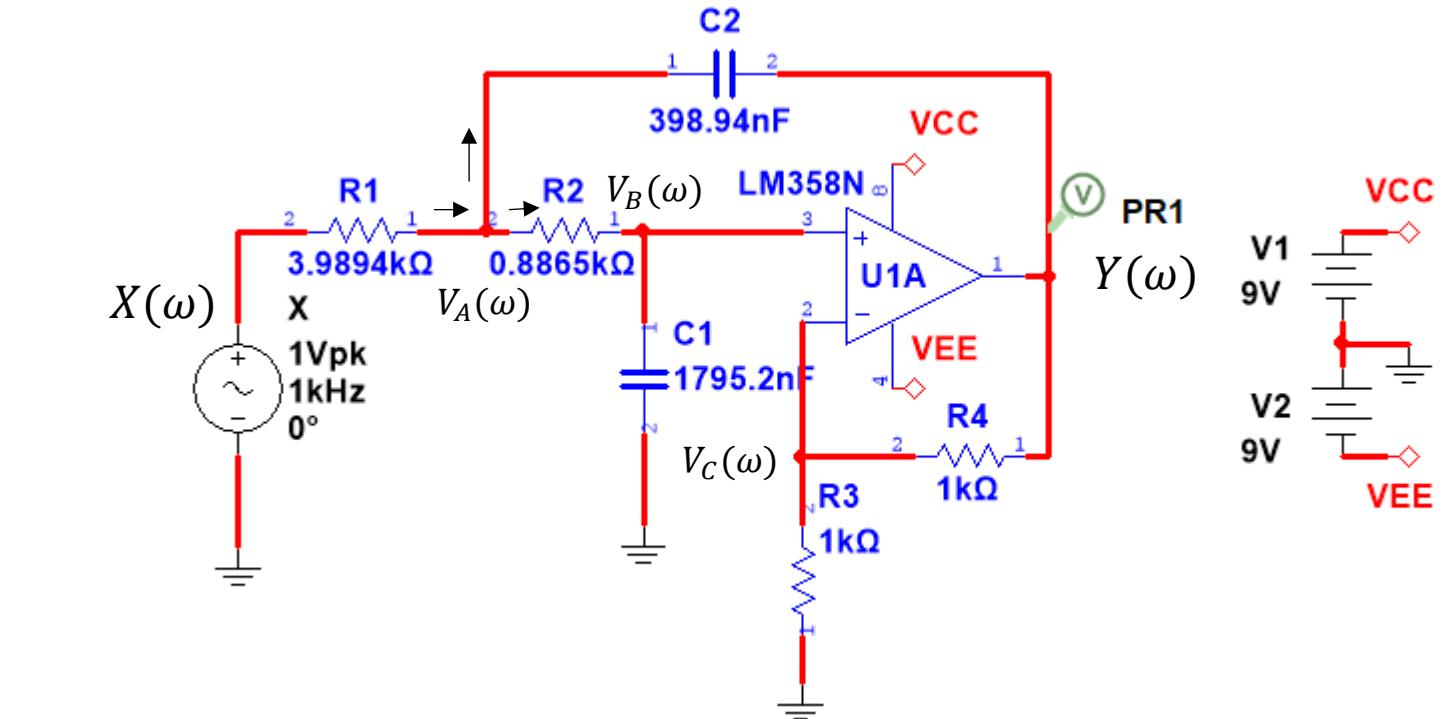
We will write the **frequency domain** equations using impedances and nodal analysis.

$$A: -\frac{(X - V_A)}{R_1} + \frac{(V_A - Y)}{Z_{C2}} + \frac{(V_A - V_B)}{R_2} = 0$$

$$B: -\frac{(V_A - V_B)}{R_2} + \frac{V_B}{Z_{C1}} = 0$$

Constraint:

$$V_B = V_C = \frac{R_3}{R_3 + R_4} \cdot Y = \frac{1}{K} \cdot Y$$



All variables are Fourier transformed with ω as their argument.
Hence, we use capital letters.

Writing the frequency domain equations in Maple:

Constraint:

$$V_B = \frac{1}{K} \cdot Y$$

Equations:

$$A: -\frac{(X - V_A)}{R_1} + \frac{(V_A - Y)}{Z_{C2}} + \frac{(V_A - V_B)}{R_2} = 0$$

$$B: -\frac{(V_A - V_B)}{R_2} + \frac{V_B}{Z_{C1}} = 0$$

Constraints

$$VB := \frac{1}{K} \cdot Y :$$

Equations

$$eqA1 := -\frac{(X - VA)}{R1} + \frac{(VA - Y)}{ZC2} + \frac{(VA - VB)}{R2} = 0 :$$

$$eqB1 := -\frac{(VA - VB)}{R2} + \frac{VB}{ZC1} = 0 :$$

Solving for a symbolic expression for Y :

Solve

```
solutions := simplify(solve( {eqA1, eqB1}, [VA, Y], symbolic = true) ) :  
assign(solutions)  
simplify(Y)
```

$$-\frac{K X Z C 1 \ Z C 2}{\left(\left(K-1\right) Z C 1-Z C 2-R 2\right) R 1-Z C 2\left(R 2+Z C 1\right)}$$

Impedances

Defining the impedances:

$$Z C 1:=\frac{1}{j \cdot \omega \cdot C 1} ; ; Z C 2:=\frac{1}{j \cdot \omega \cdot C 2} ; ;$$

Obtaining V_O expressed in terms of the components:

simplify(Y)

$$-\frac{K X}{-1+C 1 \ C 2 \ R 1 \ R 2 \ \omega^2+\left(\left(-I \ C 1+(I K-I) \ C 2\right) R 1-I \ C 1 \ R 2\right) \omega}$$

$$Y:=\omega \rightarrow \frac{K}{R 1 \cdot R 2 \cdot C 1 \cdot C 2\left(j \omega\right)^2+\left(j \omega\right)\left(R 1 \cdot C 1+R 2 \cdot C 1+(1-K) R 1 \cdot C 2\right)+1} \cdot X ;$$

Constructing a function for $Y(\omega)$:

Observation:

The gain K is part of the coefficient to $j\omega$. This means that the gain interferes with the cut-off frequency.

Changing the gain resistors will also change the cut-off frequency.

For this reason, the Sallen-Key filter is often used with $K = 1$. In such cases, we build the filter around a unity gain op-amp circuit.

Transfer function

$$H := \omega \rightarrow \frac{K}{(j\omega)^2 + (j\omega) \cdot \left(\frac{1}{R_2 \cdot C_2} + \frac{1}{R_1 \cdot C_2} + \frac{(1 - K)}{R_2 \cdot C_1} \right) + \frac{1}{R_1 \cdot R_2 \cdot C_1 \cdot C_2}} :$$

Choice of components

$$C_1 := 1795.2E-9 ; ; C_2 := 398.94E-9 ; ; R_1 := 3.9894E3 ; ; R_2 := 0.8865E3 ; ; R_3 := 1E3 ; ; \\ R_4 := 1E3 ; ;$$

$$K := 1 + \frac{R_4}{R_3} :$$

Alternative form:

$$H(\omega) = \frac{K}{(j\omega)^2 R_1 R_2 C_1 C_2 + j\omega (C_1 (R_1 + R_2) + R_1 C_2 (1 - K)) + 1}$$

Filter table

		Lowpass	Highpass	Bandpass
R_1	$k\Omega$	3.9894	1784.1	56.419
R_2	$k\Omega$	0.8865	892.06	34.117
R_3	$k\Omega$	1.0	1.0	149.72
R_4	$k\Omega$	1.0	1.0	1.0
R_5	$k\Omega$	–	–	1.0
C_1	nF	1795.2	1784.1	56.419
C_2	nF	398.94	3568.2	56.419
a_1		$2.83 \cdot 10^3$	$6.283 \cdot 10^{-1}$	$3.141 \cdot 10^1$
a_0		$3.95 \cdot 10^5$	$9.87 \cdot 10^{-2}$	$9.87 \cdot 10^4$
b_2		0	2	0
b_1		0	0	$6.283 \cdot 10^2$
b_0		$7.89 \cdot 10^5$	0	0

Coefficient values

$$a1 := \frac{1}{R2 \cdot C2} + \frac{1}{R1 \cdot C2} + \frac{(1 - K)}{R2 \cdot C1}$$

$$a1 := 2827.537902$$

$$a0 := \frac{1}{R1 \cdot R2 \cdot C1 \cdot C2}$$

$$a0 := 394814.6451$$

$$b0 := \frac{K}{R1 \cdot R2 \cdot C1 \cdot C2}$$

$$b0 := 789629.2901$$

The coefficients agree with what was obtained in the time domain.

Filter 10 – Lowpass filter (sol)

We readily obtain the differential equation from the frequency characteristic:

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{\frac{K}{R_1 R_2 C_1 C_2}}{(j\omega)^2 + \left(\frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} + \frac{1-K}{R_2 C_1} \right) (j\omega) + \frac{1}{R_1 R_2 C_1 C_2}}$$

$$\left((j\omega)^2 + \left(\frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} + \frac{1-K}{R_2 C_1} \right) (j\omega) + \frac{1}{R_1 R_2 C_1 C_2} \right) Y(\omega) = \frac{K}{R_1 R_2 C_1 C_2} X(\omega)$$

Inverse Fourier transforming each term:

$$(j\omega)^2 Y(\omega) + \left(\frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} + \frac{1-K}{R_2 C_1} \right) (j\omega) Y(\omega) + \frac{1}{R_1 R_2 C_1 C_2} Y(\omega) = \frac{K}{R_1 R_2 C_1 C_2} X(\omega)$$

$$\ddot{y}(t) + \left(\frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} + \frac{1-K}{R_2 C_1} \right) \dot{y}(t) + \frac{1}{R_1 R_2 C_1 C_2} y(t) = \frac{K}{R_1 R_2 C_1 C_2} x(t)$$

Before plotting the magnitude and phase of $H(\omega)$ it is helpful to have an idea about what to expect. We can get that from low and high frequency asymptotes.

Low frequency (LF) asymptote:

High frequency (HF) asymptote:

$$H(\omega) = \frac{K}{(j\omega)^2 R_1 R_2 C_1 C_2 + j\omega (C_1(R_1 + R_2) + R_1 C_2(1 - K)) + 1}$$

$$\lim_{\omega \rightarrow 0} H(\omega) = H(\omega_{LF}) = \frac{K}{1} = K$$

$$\lim_{\omega \rightarrow \infty} H(\omega) = H(\omega_{HF}) = \frac{K}{(j\omega)^2 R_1 R_2 C_1 C_2} = -\frac{K}{R_1 R_2 C_1 C_2} \cdot \frac{1}{\omega^2} \rightarrow 0$$

The asymptotes tell us that the filter has constant gain at low frequencies and that the gain goes to zero for high frequency. So, we have a low pass filter.

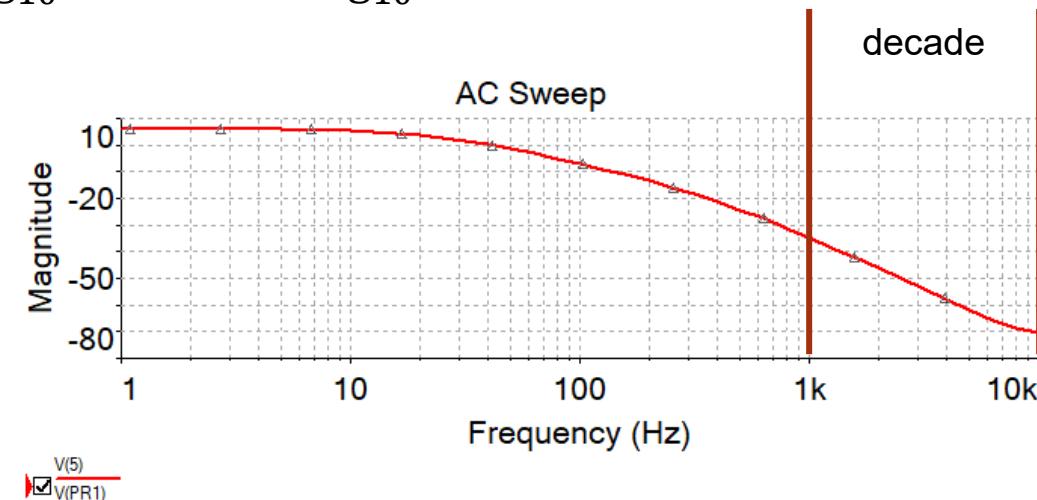
When interpreting the limit of an expression, we include only the term with dominant magnitude and consider the contributions from the other terms to be insignificant. So, for the denominator, we only include the 2nd order term for the high frequency asymptote, while to the low frequency asymptote the constant term will dominate.

Filter 10 – Lowpass filter (sol)

We can also predict the order of the filter. It is revealed by the slope of the declining asymptote, but we need to plot the amplitude spectrum in decibels. A first order filter has a slope of 20 dB per decade frequency change. A 2nd order filter a slope of 40 dB/decade, a 3rd order filter a slope of 60 dB/decade, and so on.

DC gain in decibels:

$$|H(\omega_{LF})|_{dB} = 20dB \cdot \log_{10} K = 20dB \cdot \log_{10} 2 = 6dB$$



HF slope:

$$\begin{aligned}
 |H(10 \cdot \omega_{HF})|_{dB} - |H(\omega_{HF})|_{dB} &= 20dB \cdot \log_{10} \frac{K}{R_1 R_2 C_1 C_2} \cdot \frac{1}{(10 \cdot \omega)^2} - 20dB \cdot \log_{10} \frac{K}{R_1 R_2 C_1 C_2} \cdot \frac{1}{(\omega)^2} \\
 &= 20dB \cdot \log_{10} \frac{(\omega)^2}{(10 \cdot \omega)^2} = 20dB \cdot \log_{10} \frac{1}{(10)^2} = 20dB \cdot \log_{10} 10^{-2} \\
 &= -2 \cdot 20dB \cdot \log_{10} 10 = -40dB/\text{decade}
 \end{aligned}$$

Phase characteristic:

$$H(\omega_{LF}) = \frac{K}{1} = K$$

Low frequency (LF) asymptote:

$$\angle H(\omega_{LF}) = \tan^{-1} \frac{0}{K} = 0^\circ$$

High frequency (HF) asymptote:

$$\angle H(\omega_{HF}) = \pm 180^\circ$$

Tricky cases of $\tan^{-1}()$:

$$z = x + jy$$

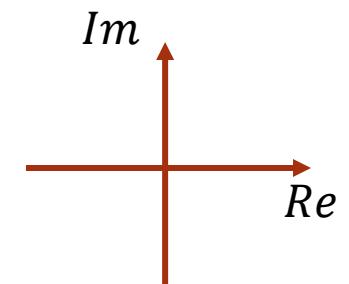
$$\theta = \tan^{-1} \frac{\text{Im}\{z\}}{\text{Re}\{z\}}$$

If $\text{Im}\{z\} = 0$, then if $\text{Re}\{z\} > 0 \Rightarrow \theta = 0$

If $\text{Im}\{z\} = 0$, then if $\text{Re}\{z\} < 0 \Rightarrow \theta = \pm 180^\circ$

If $\text{Re}\{z\} = 0$, then if $\text{Im}\{z\} > 0 \Rightarrow \theta = 90^\circ$

If $\text{Re}\{z\} = 0$, then if $\text{Im}\{z\} < 0 \Rightarrow \theta = -90^\circ$



A few commands to standardize plots.

Magnitude plots are in decibel with logarithmic frequency axis in Hz.

Phase plots are in degrees with logarithmic frequency axes in Hz.

Using display with *aligncolumns* will align the y-axes for better comparison of magnitude and phase plots.

There will be some annoying table borders. They can be removed using the interactive Table tools palette.

$$dB := \omega \rightarrow 20 \cdot \log10(|H(\omega)|) : \\ angle := \omega \rightarrow \text{argument}(H(\omega)) \cdot \frac{180}{\pi} :$$

Plotting solution

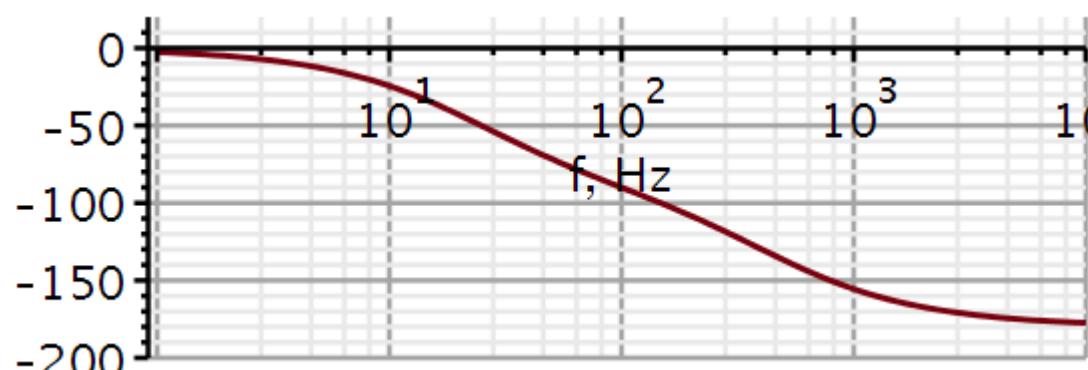
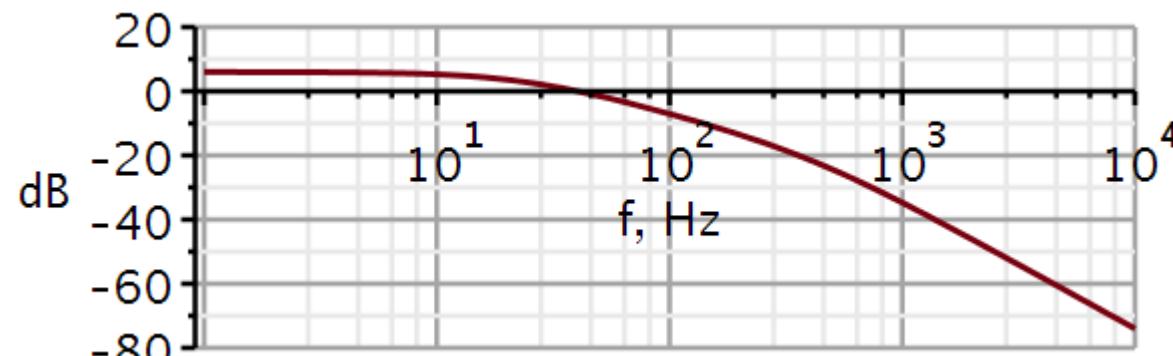
$$p1 := \text{semilogplot}(dB(2 \cdot \pi \cdot f), f = 1 .. 1E5, -80 .. 20, \text{thickness} = 3, \text{font} = [\text{Helvetica, roman, 18}], \text{axis}[2] = [\text{thickness} = 2.5], \text{axis}[1] = [\text{thickness} = 2.5], \text{labels} = ["f, Hz", "dB"], \text{labelfont} = [\text{Helvetica, 18}], \text{numpoints} = 100, \text{gridlines}, \text{size} = [600, 200]) :$$

$$p2 := \text{semilogplot}(angle(2 \cdot \pi \cdot f), f = 1 .. 1E5, -200 .. 200, \text{thickness} = 3, \text{font} = [\text{Helvetica, roman, 18}], \text{axis}[2] = [\text{thickness} = 2.5], \text{axis}[1] = [\text{thickness} = 2.5], \text{labels} = ["f, Hz", " "], \text{labelfont} = [\text{Helvetica, 18}], \text{numpoints} = 100, \text{gridlines}, \text{size} = [600, 200]) :$$

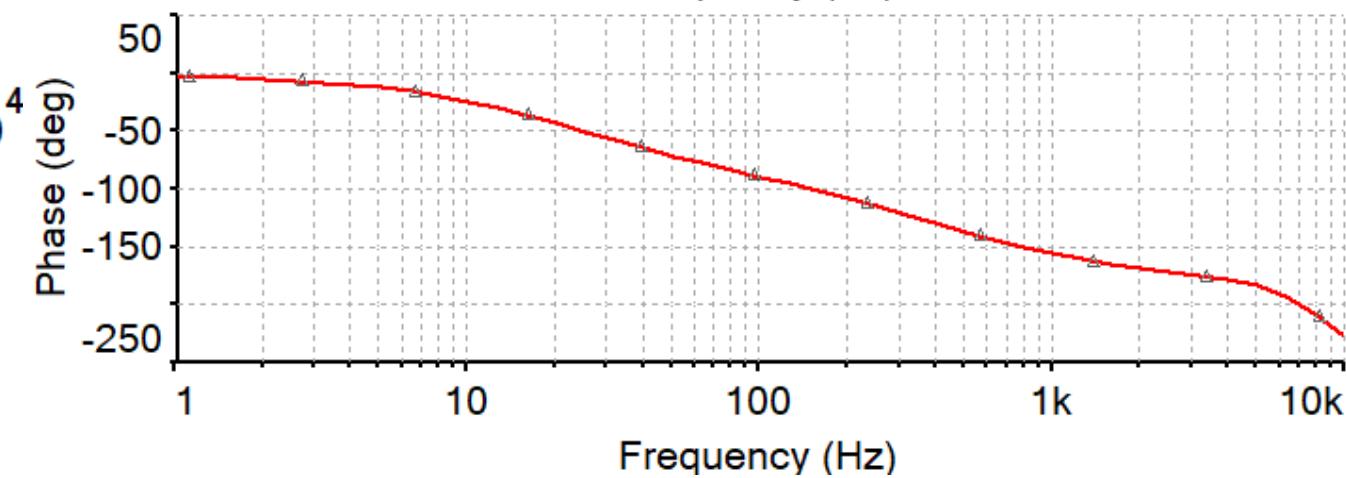
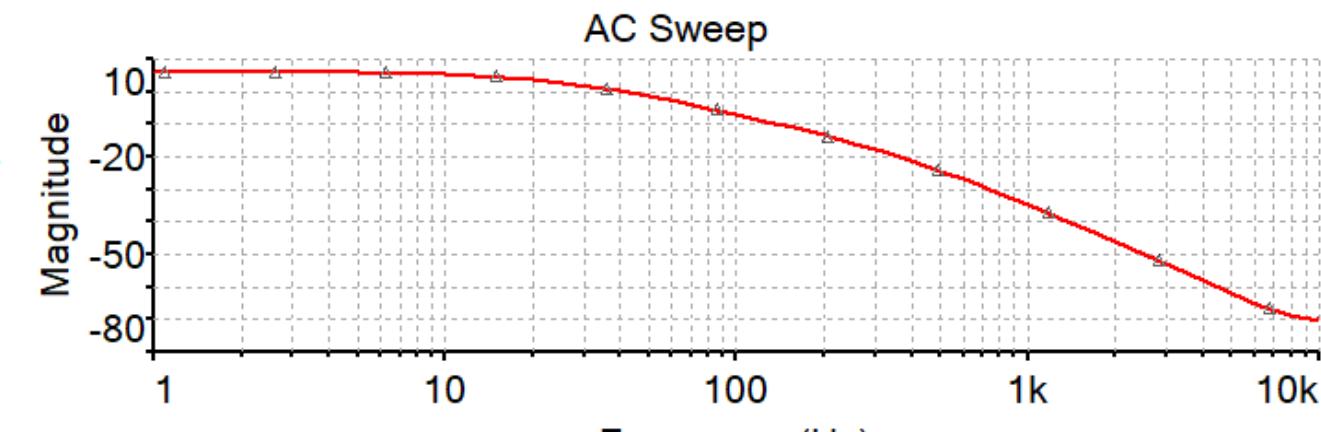
$$\text{plotstack} := \text{Array}(1 .. 2, 1 .. 1) : \\ \text{plotstack}[1, 1] := p1 : \\ \text{plotstack}[2, 1] := p2 : \\ \text{display}(\text{plotstack}, \text{aligncolumns} = [1])$$

Filter 10 – Lowpass filter (sol)

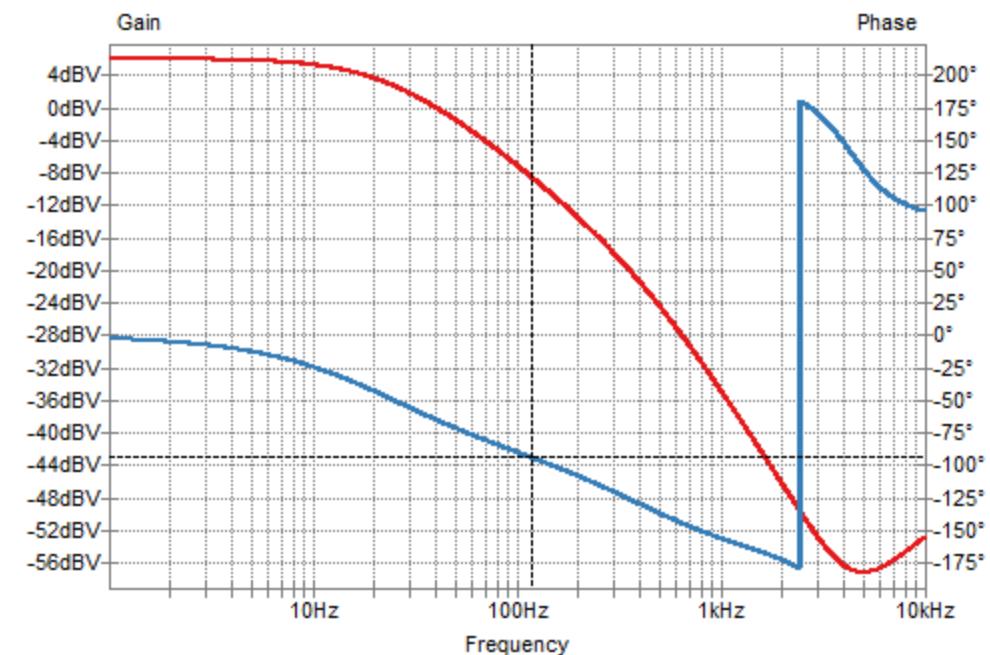
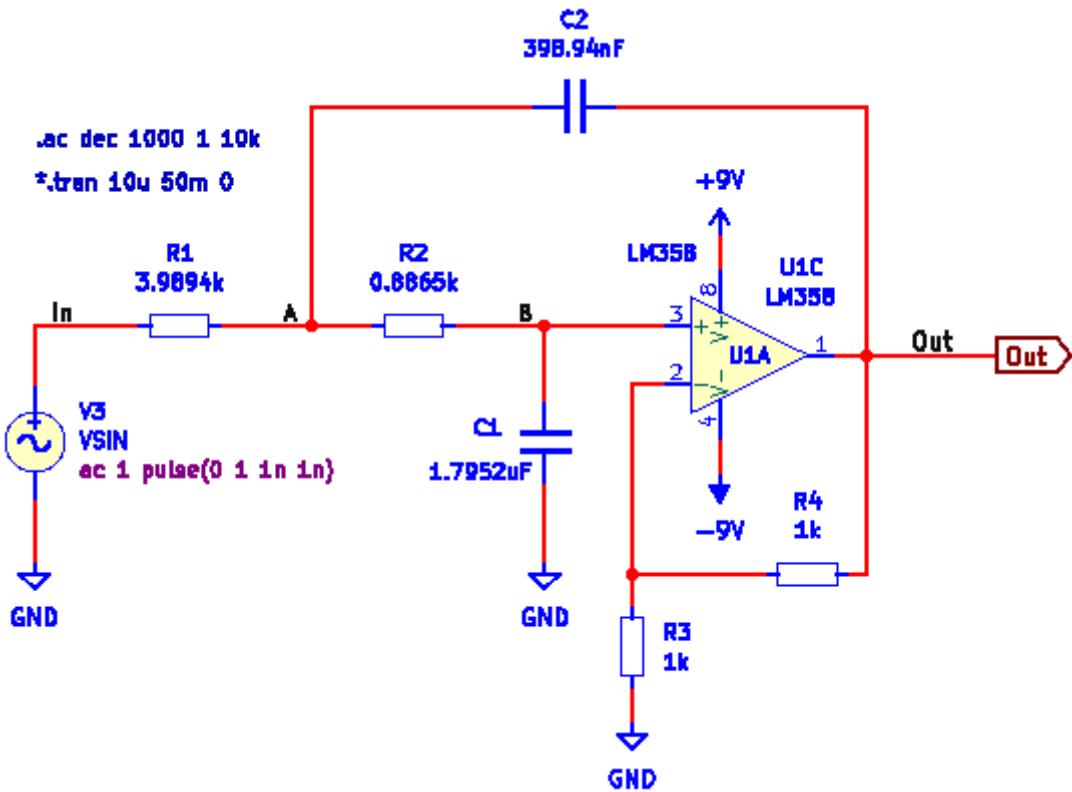
Maple



Multisim



Filter 10 – Lowpass filter (sol)



Filter 10 – Lowpass filter (sol)

We can derive a general expression for the 3-dB cut-off frequency for a 2nd order lowpass filter.

$$|H(j\omega_{3dB})| = \frac{|b_0|}{|(j\omega_{3dB})^2 + a_1(j\omega_{3dB}) + a_0|} = \frac{1}{\sqrt{2}} \cdot |H(j\omega_{LF})|$$

$$|H(j\omega_{3dB})| = \frac{|b_0|}{|a_0| \sqrt{\left(1 - \frac{\omega_{3dB}^2}{a_0}\right)^2 + \left(\frac{a_1}{a_0}\omega_{3dB}\right)^2}} = \frac{1}{\sqrt{2}} \cdot \frac{|b_0|}{|a_0|}$$

$$\left(1 - \frac{\omega_{3dB}^2}{a_0}\right)^2 + \left(\frac{a_1}{a_0}\omega_{3dB}\right)^2 - 2 = 0$$

$$a_0^2 - 2a_0\omega_{3dB}^2 + \omega_{3dB}^4 + a_1^2\omega_{3dB}^2 - 2a_0^2 = 0$$

$$\omega_{3dB}^4 + (a_1^2 - 2a_0)\omega_{3dB}^2 - a_0^2 = 0$$

Filter 10 – Lowpass filter (sol)

$$\omega_{3dB}^4 + (a_1^2 - 2a_0)\omega_{3dB}^2 - a_0^2 = 0$$

$$y = \omega^2$$

$$y^2 + (a_1^2 - 2a_0)y - a_0^2 = 0$$

$$y = \frac{-(a_1^2 - 2a_0) \pm \sqrt{(a_1^2 - 2a_0)^2 - 4(-a_0^2)}}{2}$$

$$y = \frac{-(a_1^2 - 2a_0) \pm \sqrt{a_1^4 - 4a_0a_1^2 + 8a_0^2}}{2}$$

Cut-off frequencies2nd order lowpass filter

$$H := \omega \rightarrow \frac{b0}{(j\cdot\omega)^2 + a1\cdot(j\cdot\omega) + a0} :$$

$$\text{solve}(y^2 + (a1^2 - 2\cdot a0)\cdot y - a0^2 = 0, y)$$

$$-\frac{a1^2}{2} + a0 + \frac{\sqrt{a1^4 - 4\ a0\ a1^2 + 8\ a0^2}}{2}, -\frac{a1^2}{2} + a0 - \frac{\sqrt{a1^4 - 4\ a0\ a1^2 + 8\ a0^2}}{2}$$

$$y1 := -\frac{a1^2}{2} + a0 + \frac{\sqrt{a1^4 - 4\ a0\ a1^2 + 8\ a0^2}}{2} :$$

$$y2 := -\frac{a1^2}{2} + a0 - \frac{\sqrt{a1^4 - 4\ a0\ a1^2 + 8\ a0^2}}{2} :$$

$$a0 := 1 ; a1 := \sqrt{2} ; b0 := 1 :$$

$$\text{evalf}(y1)$$

$$1.000000000$$

$$\text{evalf}(y2)$$

$$-1.000000000$$

$$x1 := \text{evalf}(\sqrt{y1})$$

$$x1 := 1.$$

$$a_0 = 1; a_1 = \sqrt{2}$$

Filter 10 – Lowpass filter (sol)

Same filter, but different components:

restart

$$y1 := -\frac{a1^2}{2} + a0 + \frac{\sqrt{a1^4 - 4 a0 a1^2 + 8 a0^2}}{2};$$

$$y2 := -\frac{a1^2}{2} + a0 - \frac{\sqrt{a1^4 - 4 a0 a1^2 + 8 a0^2}}{2};$$

$$C1 := 33 \cdot 10^{-9};; C2 := 68 \cdot 10^{-9};; R1 := 22000;; R2 := 22000;; K := 1;;$$

$$a1 := \frac{1}{R1 \cdot C2} + \frac{1}{R2 \cdot C2} + \frac{(1 - K)}{R2 \cdot C1}$$

$$a1 := \frac{250000}{187}$$

$$a0 := \frac{1}{R1 \cdot R2 \cdot C1 \cdot C2}$$

$$a0 := \frac{62500000000}{67881}$$

$$b0 := \frac{K}{R1 \cdot R2 \cdot C1 \cdot C2}$$

$$b0 := \frac{62500000000}{67881}$$

$$x1 := \text{evalf}(\sqrt{y1})$$

$$x1 := 973.7593838$$

$$x1$$

$$973.7593838$$

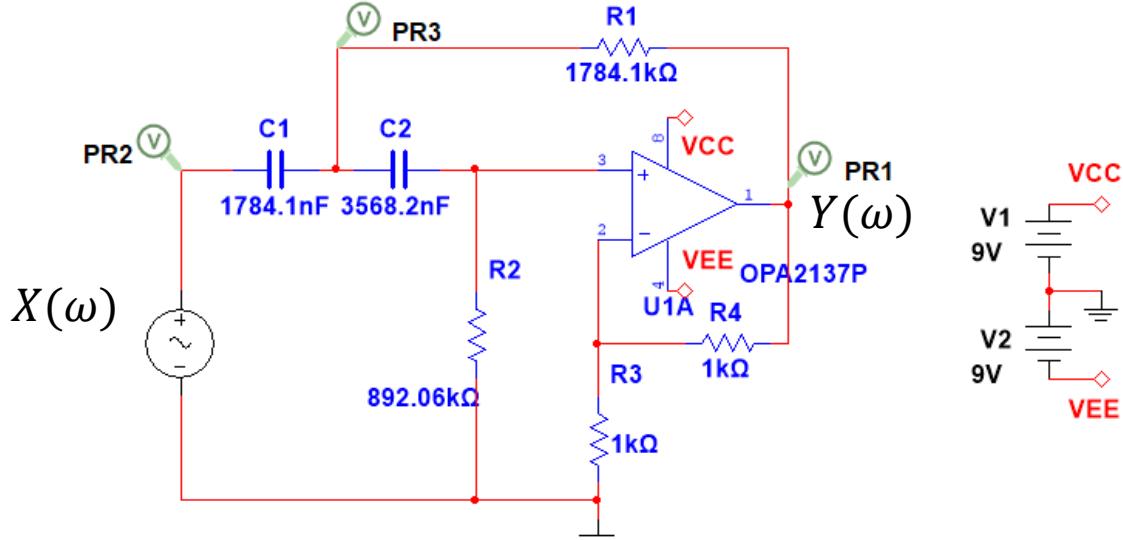
$$\frac{x1}{2 \cdot \pi}$$

$$154.9786193$$

Filter 11 – Highpass filter

A Sallen-Key filter is shown here. You are asked to analyse this circuit and:

- Derive the equation for the frequency characteristic $H(\omega)$. Use Maple/Python if possible. Verify the numerical values of the coefficients (a_1, a_0) with those calculated in Lecture 1.
- Derive the differential equation.
- Derive the equation for the amplitude characteristic $|H(\omega)|$ and determine Pass band gain and Stop band asymptote slope (dB/decade).
- Derive the equation for phase angle $\angle H(\omega)$ and predict phase angles at low and high frequencies.
- Plot the amplitude and phase characteristics in Maple/Python.
- Draw the circuit in KiCad/Spice, run an AC sweep and plot the amplitude and phase characteristics.
- Validate the mathematical analysis by comparing Maple/Python graphs with KiCad/Spice graphs.



C values
E6:
10
15
22
33
47
68

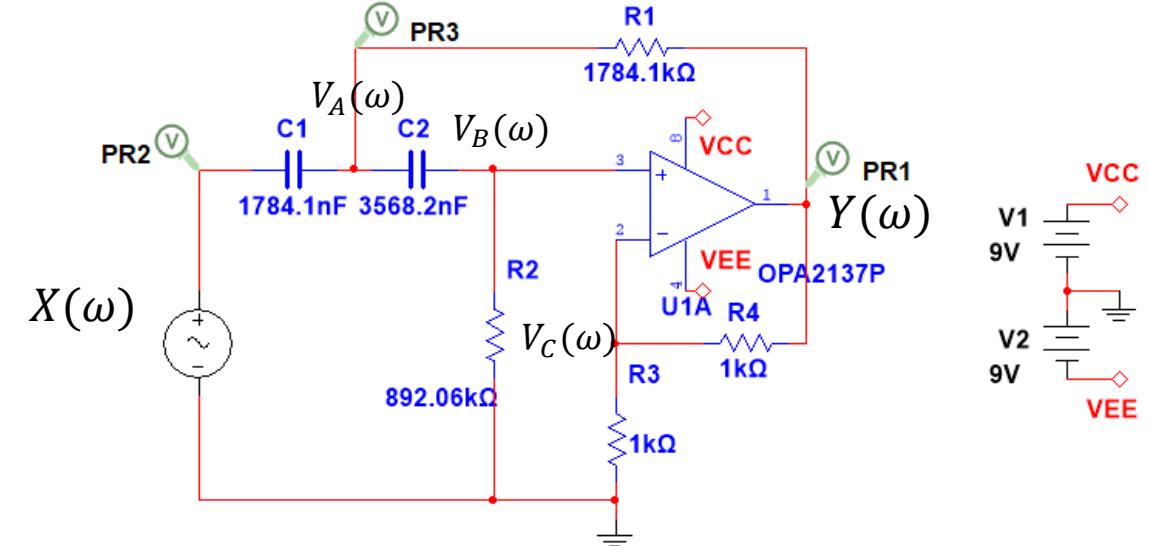
Filter 11 – Highpass filter (sol)

Nodal equations:

$$A: -\frac{X(\omega) - V_A(\omega)}{Z_{C_1}} + \frac{V_A(\omega) - Y(\omega)}{R_1} + \frac{V_A(\omega) - V_B(\omega)}{Z_{C_2}} = 0$$

$$B: -\frac{V_A(\omega) - V_B(\omega)}{Z_{C_2}} + \frac{V_B(\omega)}{R_2} = 0$$

$$C: V_C(\omega) = \frac{R_3}{R_3 + R_4} Y(\omega) = \frac{1}{K} Y(\omega)$$



$$\frac{V_C(\omega)}{Y(\omega)} = \frac{R_3}{R_3 + R_4}$$

Constraints:

$$V_B(\omega) = V_C(\omega) = \frac{1}{K} Y(\omega)$$

$$V_B(\omega) = V_C(\omega) = \frac{R_3}{R_3 + R_4} Y(\omega) = \frac{1}{K} Y(\omega)$$

$$Y(\omega) = \left(1 + \frac{R_4}{R_3}\right) V_B(\omega) = K V_B(\omega)$$

Filter 11 – Highpass filter (sol)

Nodal equations:

$$A: -\frac{X(\omega) - V_A(\omega)}{Z_{C_1}} + \frac{V_A(\omega) - Y(\omega)}{R_1} + \frac{V_A(\omega) - V_B(\omega)}{Z_{C_2}} = 0$$

$$B: -\frac{V_A(\omega) - V_B(\omega)}{Z_{C_2}} + \frac{V_B(\omega)}{R_2} = 0$$

$$C: V_C(\omega) = \frac{R_3}{R_3 + R_4} Y(\omega) = \frac{1}{K} Y(\omega)$$

Constraints:

$$V_B(\omega) = V_C(\omega) = \frac{1}{K} Y(\omega)$$

Constraints

$$VB := \frac{1}{K} \cdot Y;$$

Equations

$$eqA1 := -\frac{(X - VA)}{ZC1} + \frac{(VA - Y)}{R1} + \frac{(VA - VB)}{ZC2} = 0;$$

$$eqB1 := -\frac{(VA - VB)}{ZC2} + \frac{VB}{R2} = 0;$$

Solve

solutions := simplify(solve({eqA1, eqB1}, [VA, Y], symbolic = true)):
 assign(solutions)
 simplify(Y)

$$-\frac{K R1 R2 X}{(-R1 + (K - 1) R2 - ZC2) ZC1 - R1 (R2 + ZC2)}$$

Impedances

$$ZC1 := \frac{1}{j \cdot \omega \cdot C1}; \quad ZC2 := \frac{1}{j \cdot \omega \cdot C2};$$

simplify(Y)

$$\frac{K R1 R2 X \omega^2 C2 C1}{-1 + C1 C2 R1 R2 \omega^2 + ((-I R1 + (I K - I) R2) C2 - I C1 R1) \omega}$$

$$Y := \omega \rightarrow \frac{K R1 R2 \cdot (j \cdot \omega)^2 C1 C2}{C1 C2 R1 R2 (j \cdot \omega)^2 + j \omega \cdot (R1 \cdot (C1 + C2) + R2 \cdot C2 \cdot (1 - K)) + 1} \cdot X;$$

2. Reorganizing and inverse Fourier transforming yields the differential equation:

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{K(j\omega)^2}{(j\omega)^2 + j\omega \left(\frac{1}{R_2 C_1} + \frac{1}{R_2 C_2} + \frac{(1-K)}{R_1 C_1} \right) + \frac{1}{R_1 R_2 C_1 C_2}}$$

$$\left((j\omega)^2 + j\omega \left(\frac{1}{R_2 C_1} + \frac{1}{R_2 C_2} + \frac{(1-K)}{R_1 C_1} \right) + \frac{1}{R_1 R_2 C_1 C_2} \right) Y(\omega) = K(j\omega)^2 X(\omega)$$

$$(j\omega)^2 Y(\omega) + (j\omega) Y(\omega) \left(\frac{1}{R_2 C_1} + \frac{1}{R_2 C_2} + \frac{(1-K)}{R_1 C_1} \right) + \frac{1}{R_1 R_2 C_1 C_2} Y(\omega) = K(j\omega)^2 X(\omega)$$

$$\ddot{y}(t) + \left(\frac{1}{R_2 C_1} + \frac{1}{R_2 C_2} + \frac{(1-K)}{R_1 C_1} \right) \dot{y}(t) + \frac{1}{R_1 R_2 C_1 C_2} y(t) = K \ddot{x}(t)$$

Filter 11 – Highpass filter (sol)

Transfer functionRewriting $H(\omega)$ manually

$$H := \omega \rightarrow \frac{K \cdot (j \cdot \omega)^2}{(j \cdot \omega)^2 + j \cdot \omega \cdot \left(\frac{1}{R_2 \cdot C_1} + \frac{1}{R_2 \cdot C_2} + \frac{(1 - K)}{R_1 \cdot C_1} \right) + \frac{1}{R_1 \cdot R_2 \cdot C_1 \cdot C_2}} :$$

Choice of components

$$C_1 := 1784.1E-9 ; C_2 := 3568.2E-9 ; R_1 := 1784.1E3 ; R_2 := 892.06E3 ; R_3 := 1E3 ; \\ R_4 := 1E3 :$$

$$K := 1 + \frac{R_4}{R_3} :$$

Calculated parameters

$$a_1 := \frac{C_1 + C_2}{R_2 \cdot C_1 \cdot C_2} + \frac{(1 - K)}{R_1 \cdot C_1}$$

$$a_1 := 0.6283249517$$

$$a_0 := \frac{1}{R_1 \cdot R_2 \cdot C_1 \cdot C_2}$$

$$a_0 := 0.09870027408$$

$$b_2 = K$$

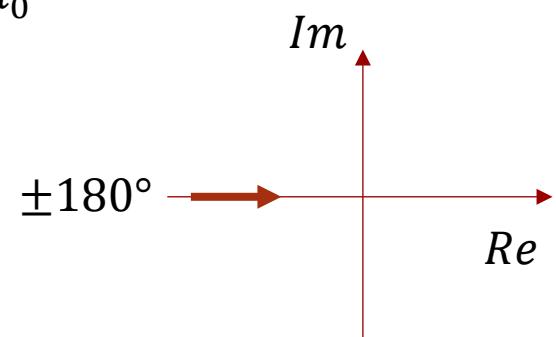
$$b_2 = 2.000000000$$

Coefficients agree with those calculated in Lecture 1.

3. Derive the equation for the amplitude characteristic $|H(\omega)|$ and determine Pass band gain and Stop band asymptote slope.

Low frequency asymptote

$$\lim_{\omega \rightarrow 0} H(\omega) = \lim_{\omega \rightarrow 0} \frac{(j\omega)^2 \cdot K}{a_0} = \lim_{\omega \rightarrow 0} \frac{-(\omega)^2 \cdot K}{a_0} = 0$$



High frequency asymptote

$$\lim_{\omega \rightarrow \infty} H(\omega) = \lim_{\omega \rightarrow \infty} \frac{(j\omega)^2 \cdot K}{(j\omega)^2} = K$$

The asymptotes reveal that the low frequencies are attenuated, and the high frequencies are amplified by K .

Filter 11 – Highpass filter (sol)

$$H(\omega) = \frac{K(j\omega)^2}{(j\omega)^2 + j\omega \left(\frac{1}{R_2 C_1} + \frac{1}{R_2 C_2} + \frac{(1-K)}{R_1 C_1} \right) + \frac{1}{R_1 R_2 C_1 C_2}}$$

The low-frequency asymptote is:

$$H_{LF}(\omega) = \frac{K(j\omega)^2}{\frac{1}{R_1 R_2 C_1 C_2}} = R_1 R_2 C_1 C_2 K (j\omega)^2$$

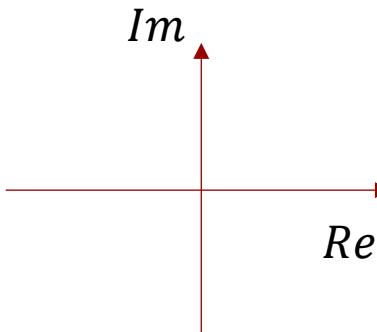
LF slope:

$$\begin{aligned} |H_{LF}(10 \cdot \omega)|_{dB} - |H_{LF}(\omega)|_{dB} &= 20\text{dB} \cdot \log_{10} R_1 R_2 C_1 C_2 K (10\omega)^2 - 20\text{dB} \cdot \log_{10} R_1 R_2 C_1 C_2 K (\omega)^2 \\ &= 20\text{dB} \cdot \log_{10} \frac{(10\omega)^2}{(\omega)^2} = 20\text{dB} \cdot \log_{10} 10^2 \\ &= 2 \cdot 20\text{dB} \cdot \log_{10} 10 = 40\text{dB/decade} \end{aligned}$$

$$\log A - \log B = \log \frac{A}{B}$$

Filter 11 – Highpass filter (sol)

4. Derive the equation for phase angle $\angle H(\omega)$ and predict phase angles at low and high frequencies.



$$H(\omega) = \frac{(j\omega)^2 \cdot K}{(j\omega)^2 + a_1(j\omega) + a_0}$$

$$\angle H(\omega) = \underbrace{\angle P(\omega)}_{\substack{\text{numerator} \\ \text{angle}}} - \underbrace{\angle Q(\omega)}_{\substack{\text{denominator} \\ \text{angle}}}$$

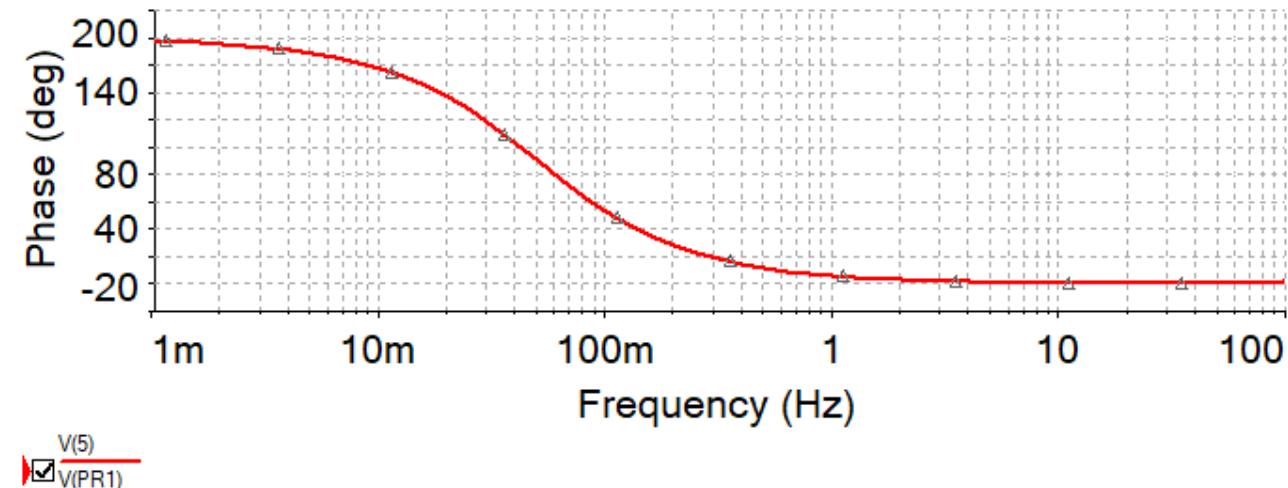
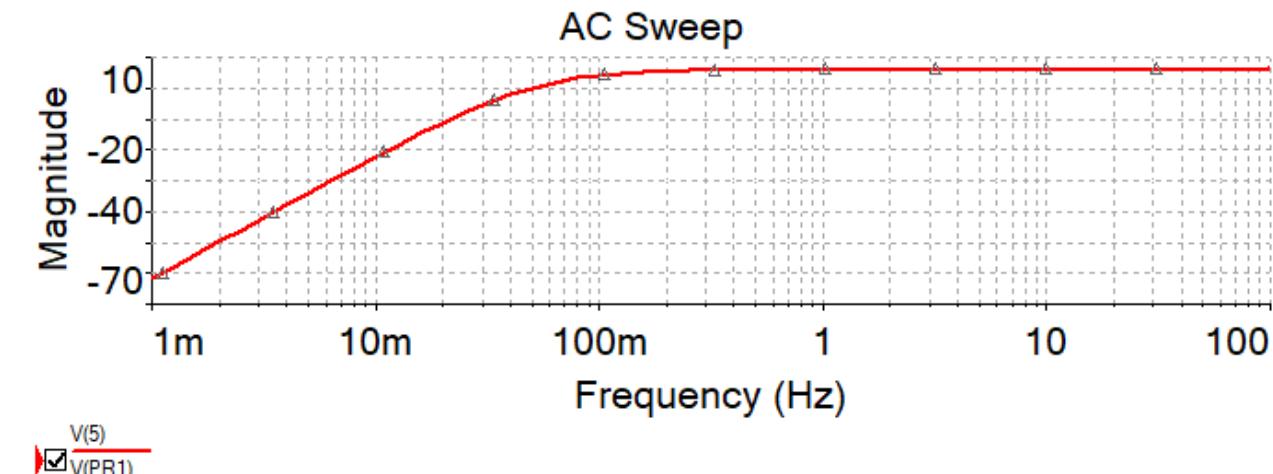
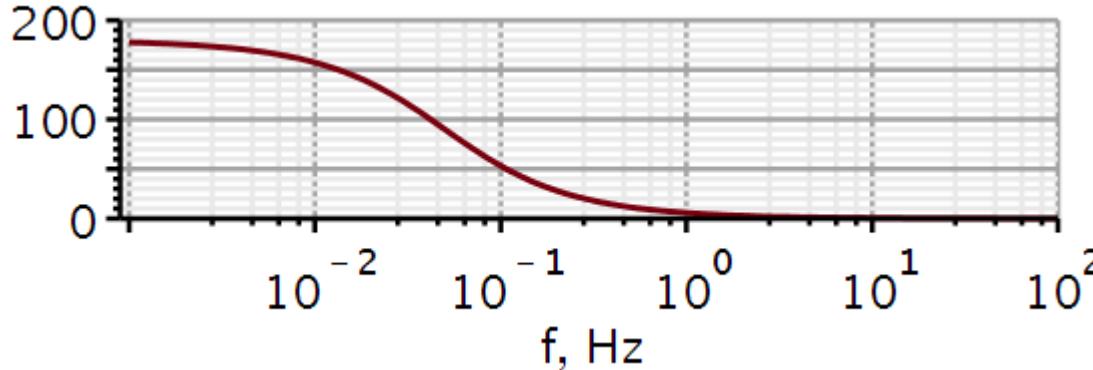
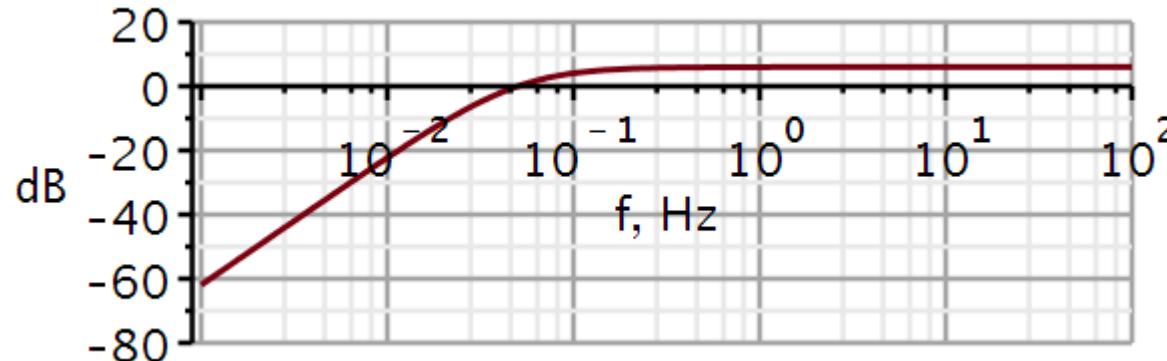
$$\angle H(\omega) = \angle(j\omega)^2 - \angle((j\omega)^2 + a_1(j\omega) + a_0)$$

$$\angle H(0) = 180^\circ - 0^\circ = 180^\circ$$

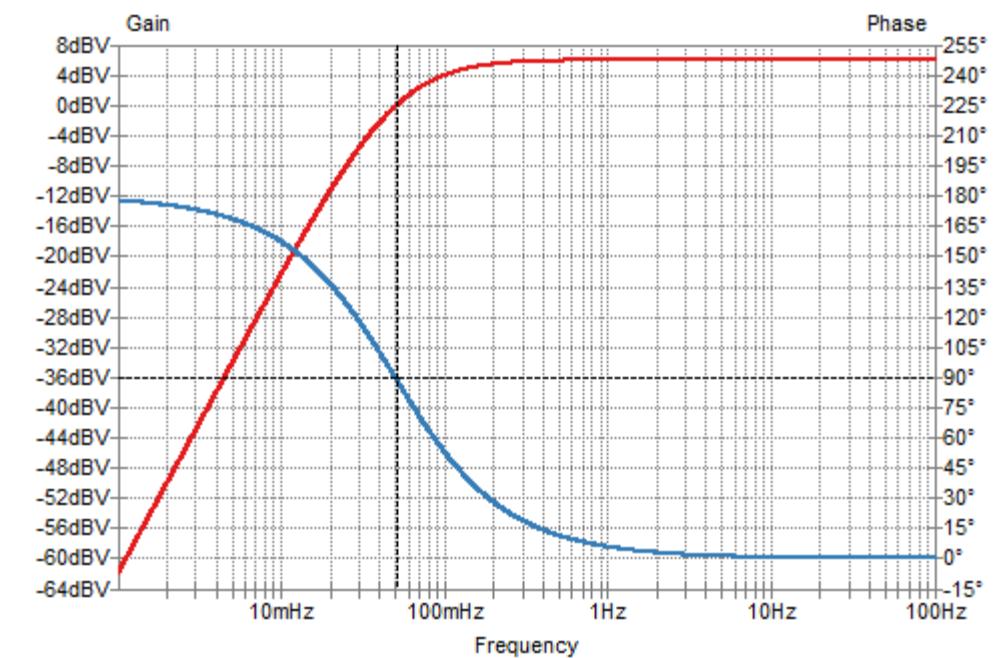
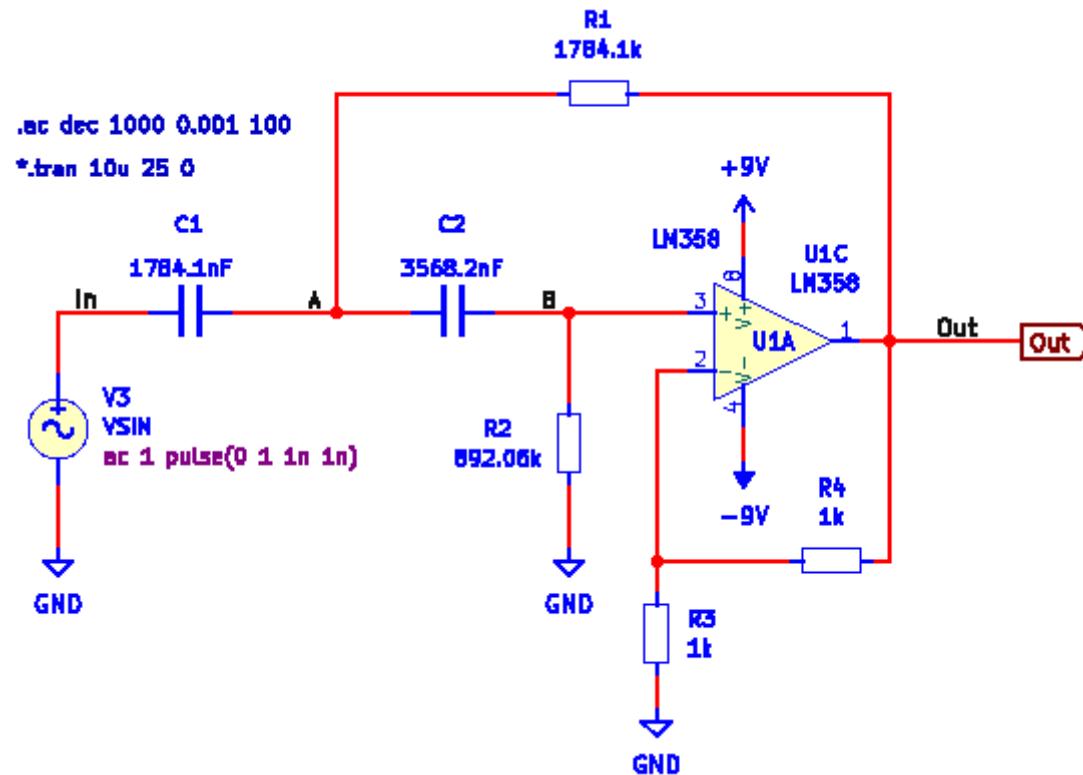
$$\angle H(\infty) = 180^\circ - 180^\circ = 0^\circ$$

Filter 11 – Highpass filter (sol)

5. - 7. Plot the amplitude and phase characteristics in Maple.



Filter 11 – Highpass filter (sol)



Filter 11 – Highpass filter (sol)

We can derive a general expression for the 3-dB cut-off frequency for a 2nd order highpass filter.

$$|H(j\omega_{3dB})| = \frac{|b_2| |(j\omega_{3dB})^2|}{|(j\omega_{3dB})^2 + a_1(j\omega_{3dB}) + a_0|} = \frac{1}{\sqrt{2}} \cdot |H(j\omega_{HF})|$$

$$|H(j\omega_{3dB})| = \frac{|b_2| |(j\omega_{3dB})^2|}{\sqrt{(a_0 - \omega_{3dB}^2)^2 + (a_1\omega_{3dB})^2}} = \frac{|b_2| |(j\omega_{3dB})^2|}{\sqrt{2} |(j\omega_{3dB})^2|}$$

$$(a_0 - \omega_{3dB}^2)^2 + (a_1\omega_{3dB})^2 = 2\omega_{3dB}^4$$

$$a_0^2 - 2a_0\omega_{3dB}^2 + \omega_{3dB}^4 + (a_1\omega_{3dB})^2 = 2\omega_{3dB}^4$$

$$-\omega_{3dB}^4 + (a_1^2 - 2a_0)\omega_{3dB}^2 + a_0^2 = 0$$

Filter 11 – Highpass filter (sol)

$$-\omega_{3dB}^4 + (a_1^2 - 2a_0)\omega_{3dB}^2 + a_0^2 = 0$$

$$y = \frac{-(a_1^2 - 2a_0) \pm \sqrt{(a_1^2 - 2a_0)^2 + 4a_0^2}}{-2}$$

$$x = \sqrt{y}$$

2nd order highpass*restart*

$$\text{solve}(-y^2 + (a1^2 - 2 \cdot a0) \cdot y + a0^2 = 0, y)$$

$$\frac{a1^2}{2} - a0 + \frac{\sqrt{a1^4 - 4 a0 a1^2 + 8 a0^2}}{2}, \frac{a1^2}{2} - a0 - \frac{\sqrt{a1^4 - 4 a0 a1^2 + 8 a0^2}}{2}$$

$$y1 := \frac{a1^2}{2} - a0 + \frac{\sqrt{a1^4 - 4 a0 a1^2 + 8 a0^2}}{2}:$$

$$y2 := \frac{a1^2}{2} - a0 - \frac{\sqrt{a1^4 - 4 a0 a1^2 + 8 a0^2}}{2}:$$

$$a0 := 1; a1 := \sqrt{2}; b2 := 1;$$

$$\text{evalf}(y1)$$

$$1.000000000$$

$$\text{evalf}(y2)$$

$$-1.000000000$$

$$x1 := \text{evalf}(\sqrt{y1})$$

$$x1 := 1.$$

Filter 11 – Highpass filter (sol)

Highpass filter, not normalized*restart*

$$\text{solve}(-y^2 + (al^2 - 2 \cdot a0) \cdot y + a0^2 = 0, y)$$

$$\frac{al^2}{2} - a0 + \frac{\sqrt{al^4 - 4 a0 al^2 + 8 a0^2}}{2}, \frac{al^2}{2} - a0 - \frac{\sqrt{al^4 - 4 a0 al^2 + 8 a0^2}}{2}$$

$$y1 := \frac{al^2}{2} - a0 + \frac{\sqrt{al^4 - 4 a0 al^2 + 8 a0^2}}{2};$$

$$y2 := \frac{al^2}{2} - a0 - \frac{\sqrt{al^4 - 4 a0 al^2 + 8 a0^2}}{2};$$

$$C1 := 10 \cdot 10^{-6}; C2 := 10 \cdot 10^{-6}; RI := 470000; R2 := 220000; K := 1;$$

$$al := \frac{1}{RI \cdot C1} + \frac{1}{RI \cdot C2} + \frac{(1 - K)}{R2 \cdot C1}$$

$$al := \frac{20}{47}$$

$$a0 := \frac{1}{RI \cdot R2 \cdot C1 \cdot C2}$$

$$a0 := \frac{50}{517}$$

$$b2 := K$$

$$b2 := 1$$

$$x1 := \text{evalf}(\sqrt{y1})$$

$$x1 := 0.3012233701$$

$$xI$$

$$0.3012233701$$

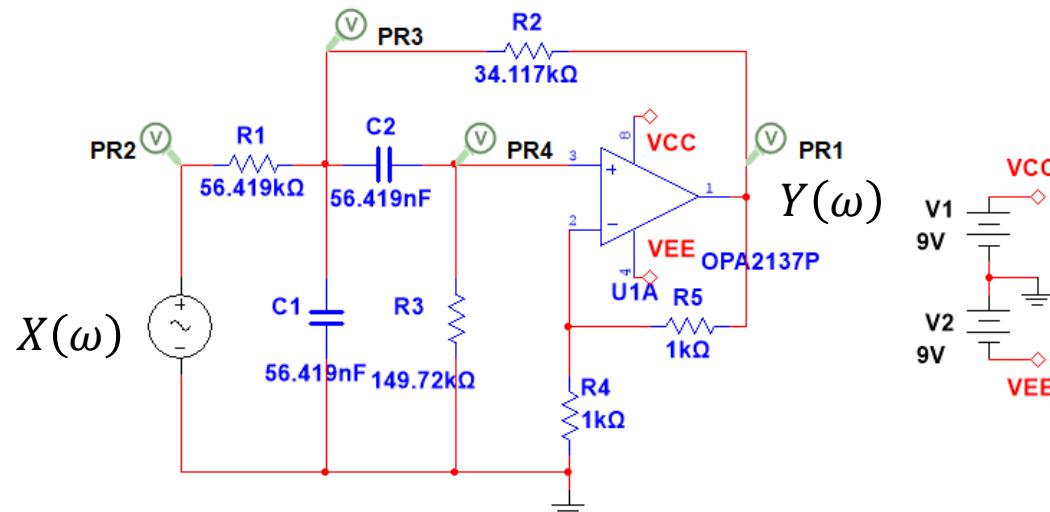
$$\frac{xI}{2 \cdot \pi}$$

$$0.04794118832$$

Filter 12 – Bandpass filter

A Sallen-Key filter is shown here. You are asked to analyse this circuit and:

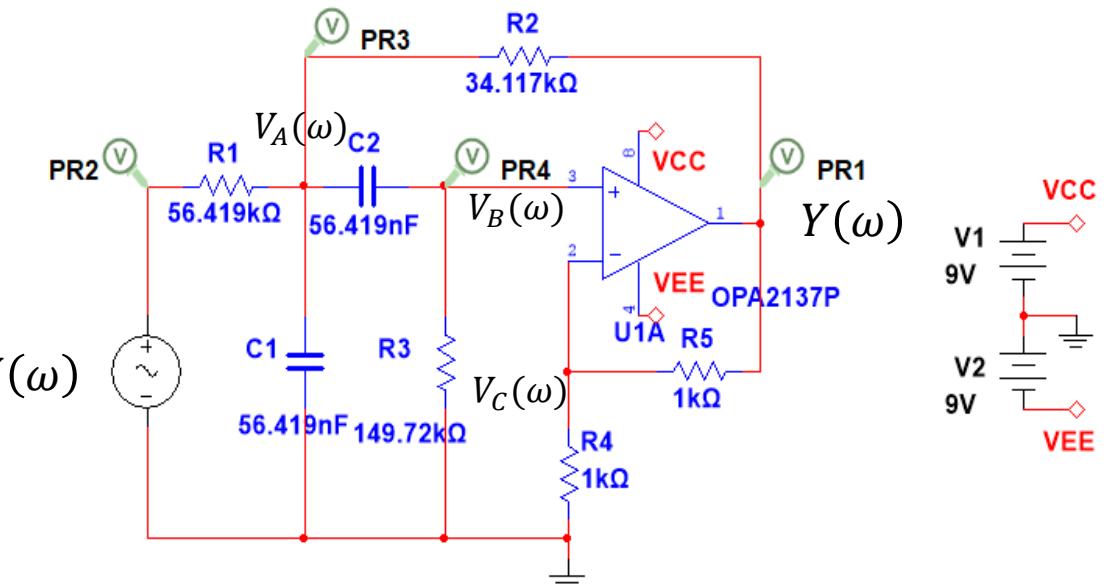
- Derive the equation for the frequency characteristic $H(\omega)$. Use Maple/Python if possible. Verify the numerical values of the coefficients (a_1, a_0) with those calculated in Lecture 1.
- Derive the differential equation.
- Derive the equation for the amplitude characteristic $|H(\omega)|$ and determine Pass band gain and Stop band asymptote slope (dB/decade).
- Derive the equation for phase angle $\angle H(\omega)$ and predict phase angles at low and high frequencies.
- Plot the amplitude and phase characteristics in Maple/Python.
- Draw the circuit in KiCad/Spice, run an AC sweep and plot the amplitude and phase characteristics.
- Validate the mathematical analysis by comparing Maple/Python graphs with KiCad/Spice graphs.



C values
E6:
10
15
22
33
47
68

Filter 12 – Bandpass filter (sol)

Setting up the nodal equations in the frequency domain:

**Constraints**

$$VB := \frac{1}{K} \cdot Y :$$

Equations

$$eqA1 := -\frac{(X - VA)}{R1} + \frac{(VA - Y)}{R2} + \frac{(VA - VB)}{ZC2} + \frac{VA}{ZC1} = 0 : \\ eqB1 := -\frac{(VA - VB)}{ZC2} + \frac{VB}{R3} = 0 :$$

Solve

```
solutions := simplify(solve( {eqA1, eqB1}, [VA, Y], symbolic = true) ) :
```

```
assign(solutions)
```

```
simplify(Y)
```

$$-\frac{K R_2 R_3 X ZC1}{((-R_2 + (K - 1) R_3 - ZC2) R_1 - R_2 (R_3 + ZC2)) ZC1 - R_1 R_2 (R_3 + ZC2)}$$

Impedances

$$ZC1 := \frac{1}{j \cdot \omega \cdot C1} ; ; ZC2 := \frac{1}{j \cdot \omega \cdot C2} ; ;$$

```
simplify(Y)
```

$$\frac{-I K R_2 R_3 X \omega C2}{C1 C2 R_1 R_2 R_3 \omega^2 + (((-I R_2 + (I K - I) R_3) R_1 - I R_2 R_3) C2 - I C1 R_1 R_2) \omega - R_1 - R_2}$$

After some manual rewriting, we get the equation in polynomial form:

$$H(\omega) = \frac{\frac{K}{R_1 C_1}(j\omega)}{(j\omega)^2 + j\omega \left(\frac{1}{R_1 C_1} + \frac{1}{R_3 C_1} + \frac{1}{R_3 C_2} + \frac{(1-K)}{R_2 C_1} \right) + \frac{R_1 + R_2}{R_1 R_2 R_3 C_1 C_2}}$$

2. Coefficients agree with those calculated in Lecture 1.

Transfer function

$$H := \omega \rightarrow \frac{\frac{K \cdot j \cdot \omega}{R1 \cdot C1}}{(j \cdot \omega)^2 + \left(\frac{1}{R1 \cdot C1} + \frac{1}{R3 \cdot C1} + \frac{1}{R3 \cdot C2} + \frac{(1 - K)}{R2 \cdot C1} \right) (j \cdot \omega) + \frac{(R1 + R2)}{R1 \cdot R2 \cdot R3 \cdot C1 \cdot C2}} :$$

Choice of components

$$C1 := 56.419E-9 ; ; C2 := 56.419E-9 ; ; R1 := 56.419E3 ; ; R2 := 34.117E3 ; ; R3 := 149.72E3 ; ; R4 := 1E3 ; ; R5 := 1E3 :$$

$$K := 1 + \frac{R5}{R4} :$$

Calculated parameters

$$al := \frac{1}{R1 \cdot C1} + \frac{1}{R3 \cdot C1} + \frac{1}{R3 \cdot C2} + \frac{(1 - K)}{R2 \cdot C1}$$

al := 31.4059709

$$a0 := \frac{(R1 + R2)}{R1 \cdot R2 \cdot R3 \cdot C1 \cdot C2}$$

a0 := 98694.85113

$$b1 := \frac{K}{R1 \cdot C1}$$

b1 := 628.3176031

3. Derive the equation for the amplitude characteristic $|H(\omega)|$ and determine Pass band gain and Stop band asymptote slope.

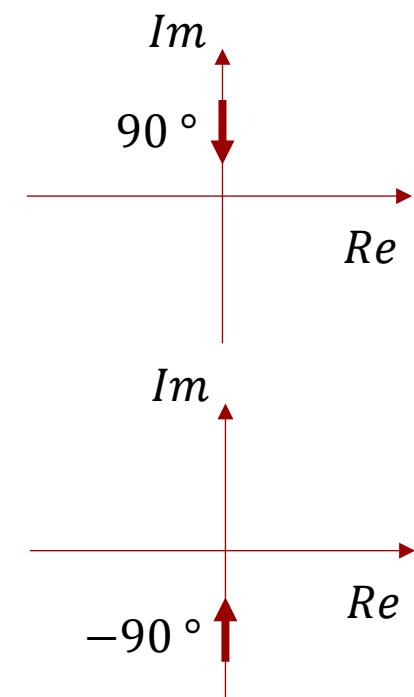
Low frequency asymptote

$$\lim_{\omega \rightarrow 0} H(\omega) = \lim_{\omega \rightarrow 0} \frac{j \omega \cdot b_1}{a_0} = 0$$

High frequency asymptote

$$\lim_{\omega \rightarrow \infty} H(\omega) = \lim_{\omega \rightarrow \infty} \frac{j \omega \cdot b_1}{(j \omega)^2} = \lim_{\omega \rightarrow \infty} \frac{b_1}{j \omega} = \lim_{\omega \rightarrow \infty} \frac{-j \cdot b_1}{\omega} = 0$$

$$H(\omega) = \frac{j \omega \cdot b_1}{(j \omega)^2 + a_1(j\omega) + a_0}$$



The asymptotes reveal that both the low and high frequencies are attenuated. It must be a bandpass filter.

Filter 12 – Bandpass filter (sol)

$$H(\omega) = \frac{\frac{K}{R_1 C_1}(j\omega)}{(j\omega)^2 + j\omega \left(\frac{1}{R_1 C_1} + \frac{1}{R_3 C_1} + \frac{1}{R_3 C_2} + \frac{(1-K)}{R_2 C_1} \right) + \frac{R_1 + R_2}{R_1 R_2 R_3 C_1 C_2}}$$

The low-frequency asymptote is:

$$H_{LF}(\omega) = \frac{\frac{K}{R_1 C_1}(j\omega)}{\frac{R_1 + R_2}{R_1 R_2 R_3 C_1 C_2}} = \frac{K R_2 R_3 C_2}{R_1 + R_2}(j\omega)$$

LF slope:

$$\begin{aligned} |H_{LF}(10 \cdot \omega)|_{dB} - |H_{LF}(\omega)|_{dB} &= 20dB \cdot \log_{10} \frac{K R_2 R_3 C_2}{R_1 + R_2}(10 \omega) - 20dB \cdot \log_{10} \frac{K R_2 R_3 C_2}{R_1 + R_2}(\omega) \\ &= 20dB \cdot \log_{10} \frac{(10 \omega)^1}{(\omega)^1} = 20dB \cdot \log_{10} 10^1 \\ &= 1 \cdot 20dB \cdot \log_{10} 10 = 20dB/\text{decade} \end{aligned}$$

$$\log A - \log B = \log \frac{A}{B}$$

Filter 12 – Bandpass filter (sol)

$$H(\omega) = \frac{\frac{K}{R_1 C_1}(j\omega)}{(j\omega)^2 + j\omega \left(\frac{1}{R_1 C_1} + \frac{1}{R_3 C_1} + \frac{1}{R_3 C_2} + \frac{(1-K)}{R_2 C_1} \right) + \frac{R_1 + R_2}{R_1 R_2 R_3 C_1 C_2}}$$

The high-frequency asymptote is:

$$H_{HF}(\omega) = \frac{\frac{K}{R_1 C_1}(j\omega)}{(j\omega)^2} = \frac{K}{R_1 C_1(j\omega)}$$

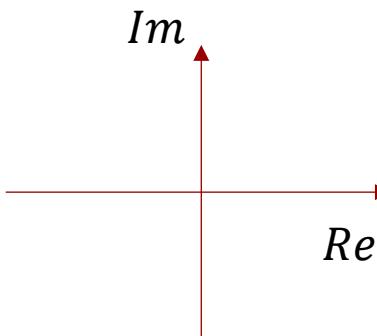
HF slope:

$$\begin{aligned} |H_{HF}(10 \cdot \omega)|_{dB} - |H_{HF}(\omega)|_{dB} &= 20 \text{dB} \cdot \log_{10} \frac{K}{R_1 C_1(10 \omega)} - 20 \text{dB} \cdot \log_{10} \frac{K}{R_1 C_1(\omega)} \\ &= 20 \text{dB} \cdot \log_{10} \frac{(\omega)^1}{(10 \omega)^1} = 20 \text{dB} \cdot \log_{10} \frac{1}{(10)^1} = 20 \text{dB} \cdot \log_{10} 10^{-1} \\ &= -1 \cdot 20 \text{dB} \cdot \log_{10} 10 = -20 \text{dB/decade} \end{aligned}$$

$$\log A - \log B = \log \frac{A}{B}$$

4. Derive the equation for phase angle $\angle H(\omega)$ and predict phase angles at low and high frequencies.

$$H(\omega) = \frac{(j\omega)b_1}{(j\omega)^2 + a_1(j\omega) + a_0}$$



$$\angle H(\omega) = \underbrace{\angle P(\omega)}_{\substack{\text{numerator} \\ \text{angle}}} - \underbrace{\angle Q(\omega)}_{\substack{\text{denominator} \\ \text{angle}}}$$

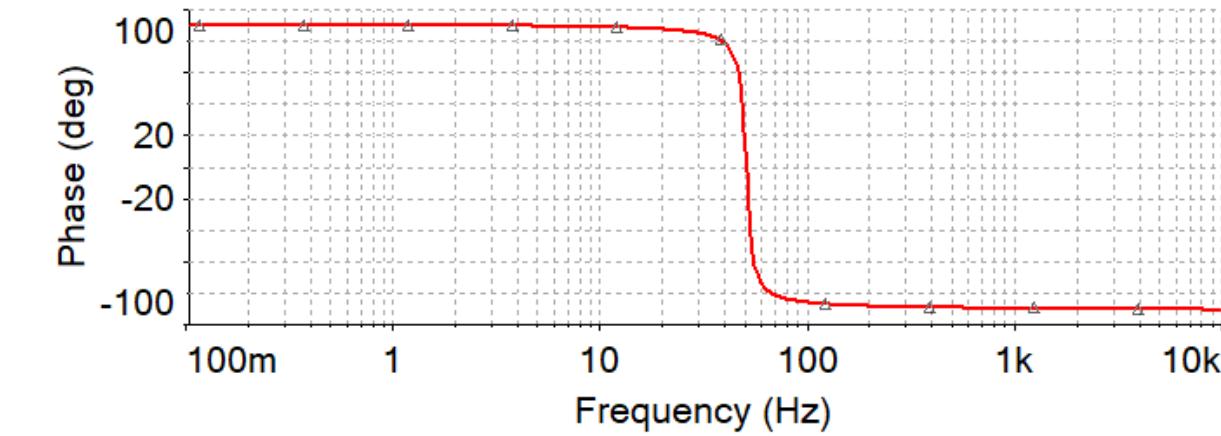
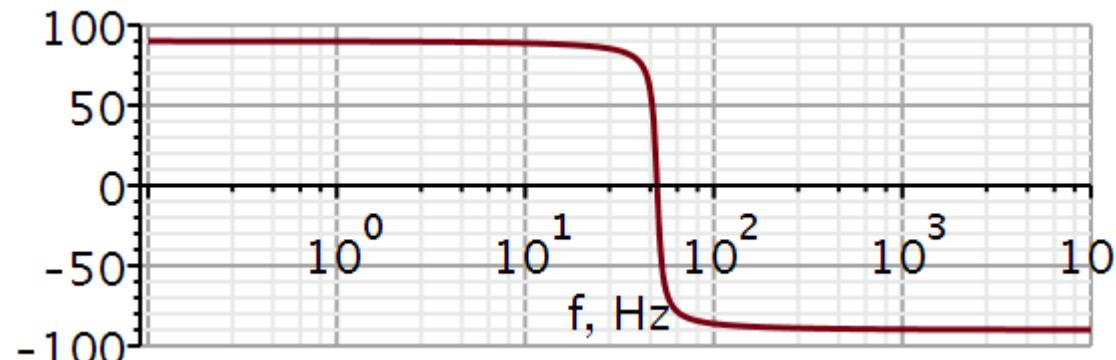
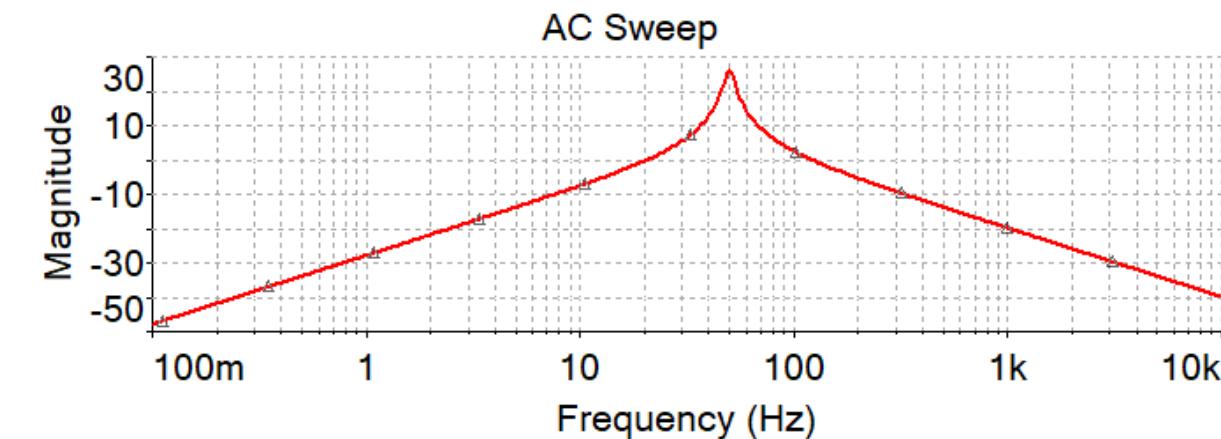
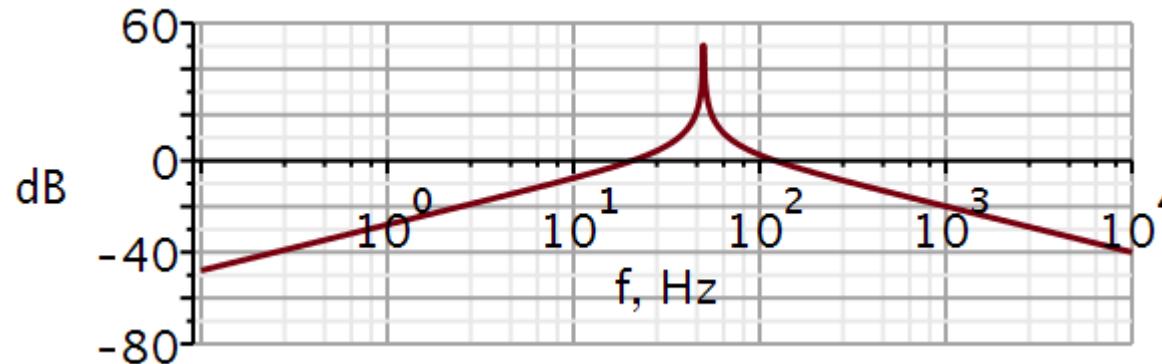
$$\angle H(\omega) = \angle(j\omega)b_1 - \angle((j\omega)^2 + a_1(j\omega) + a_0)$$

$$\angle H(0) = 90^\circ - 0^\circ = 90^\circ$$

$$\angle H(\infty) = 90^\circ - 180^\circ = -90^\circ$$

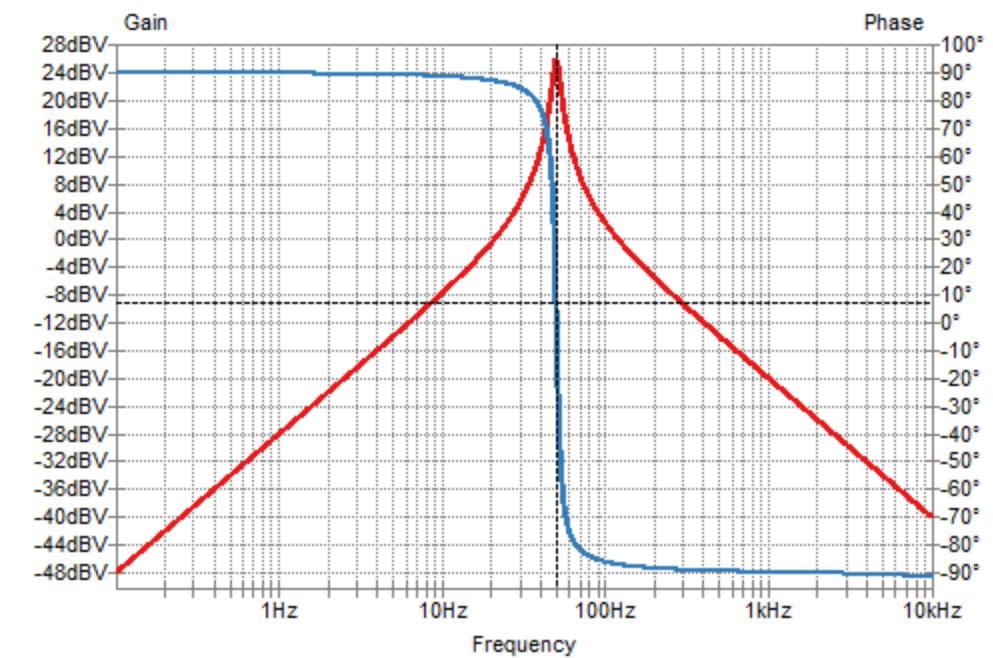
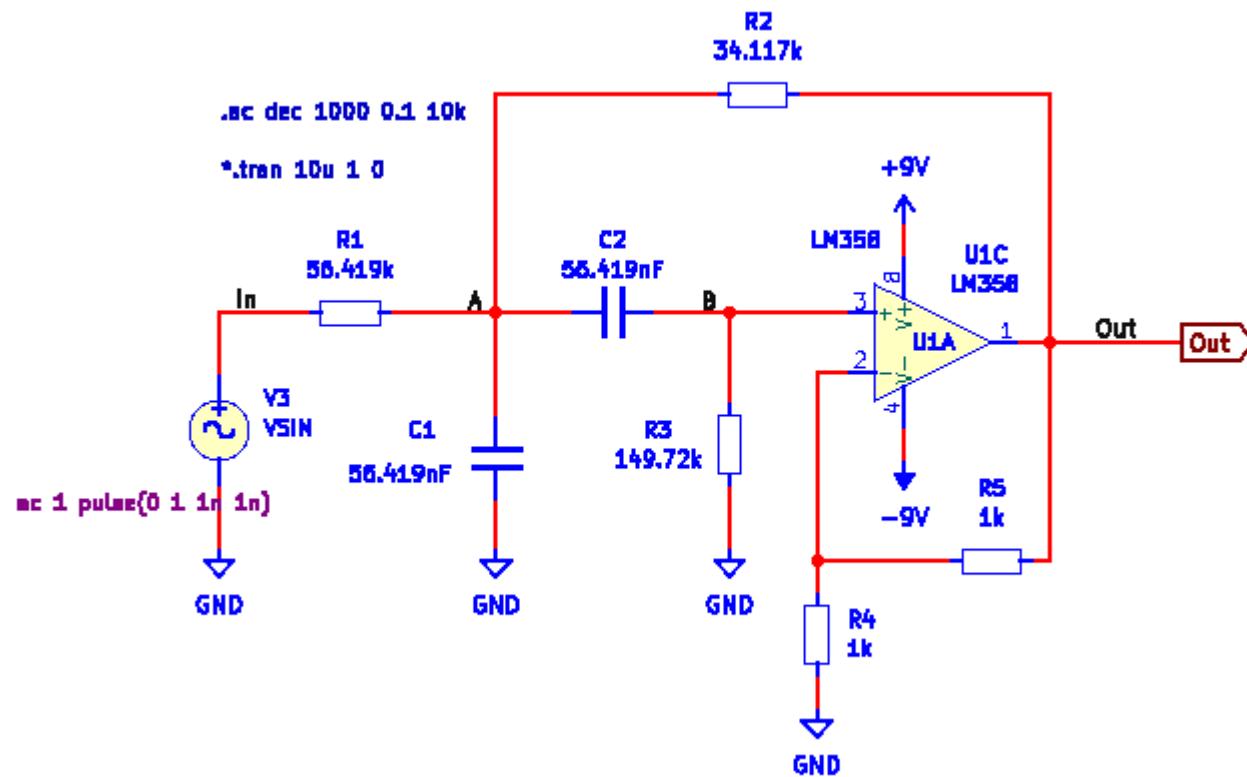
Filter 12 – Bandpass filter (sol)

5. – 7. Plot and compare.



See next slide

Filter 12 – Bandpass filter (sol)



Filter 12 – Bandpass filter (sol)

$$H := \omega \rightarrow \frac{\frac{K \cdot j \cdot \omega}{R1 \cdot C1}}{(j \cdot \omega)^2 + \left(\frac{1}{R1 \cdot C1} + \frac{1}{R3 \cdot C1} + \frac{1}{R3 \cdot C2} + \frac{(1 - K)}{R2 \cdot C1} \right) (j \cdot \omega) + \frac{(R1 + R2)}{R1 \cdot R2 \cdot R3 \cdot C1 \cdot C2}} :$$

$$dB := \omega \rightarrow 20 \cdot \log10(|H(\omega)|) :$$

$$angle := \omega \rightarrow \text{argument}(H(\omega)) \cdot \frac{180}{\pi} :$$

$$\theta := \omega \rightarrow 90 - \frac{180}{\pi} \arctan \left(\left(\frac{1}{R1 \cdot C1} + \frac{1}{R3 \cdot C1} + \frac{1}{R3 \cdot C2} + \frac{(1 - K)}{R2 \cdot C1} \right) \cdot \omega, \frac{(R1 + R2)}{R1 \cdot R2 \cdot R3 \cdot C1 \cdot C2} - \omega^2 \right) :$$

Imaginary argument

Real argument

So far, the phase function `angle()` has given satisfying results when plotting the phase. For unknown reasons `angle()` gives a very sharp phase jump from 90 to -90 degrees for the bandpass filter. Then I tried to use a function $\theta()$ shown above, it gives a smoother phase transition in good agreement with what KiCad shows.

`arctan()` can take two arguments, the imaginary part first, and the real part second. In this way it can figure out, in what quadrant the complex number is located.

22050 Continuous-Time Signals and Linear Systems

Kaj-Åge Henneberg

L07

Forward and Inverse Laplace Transformation Laplace Theorems

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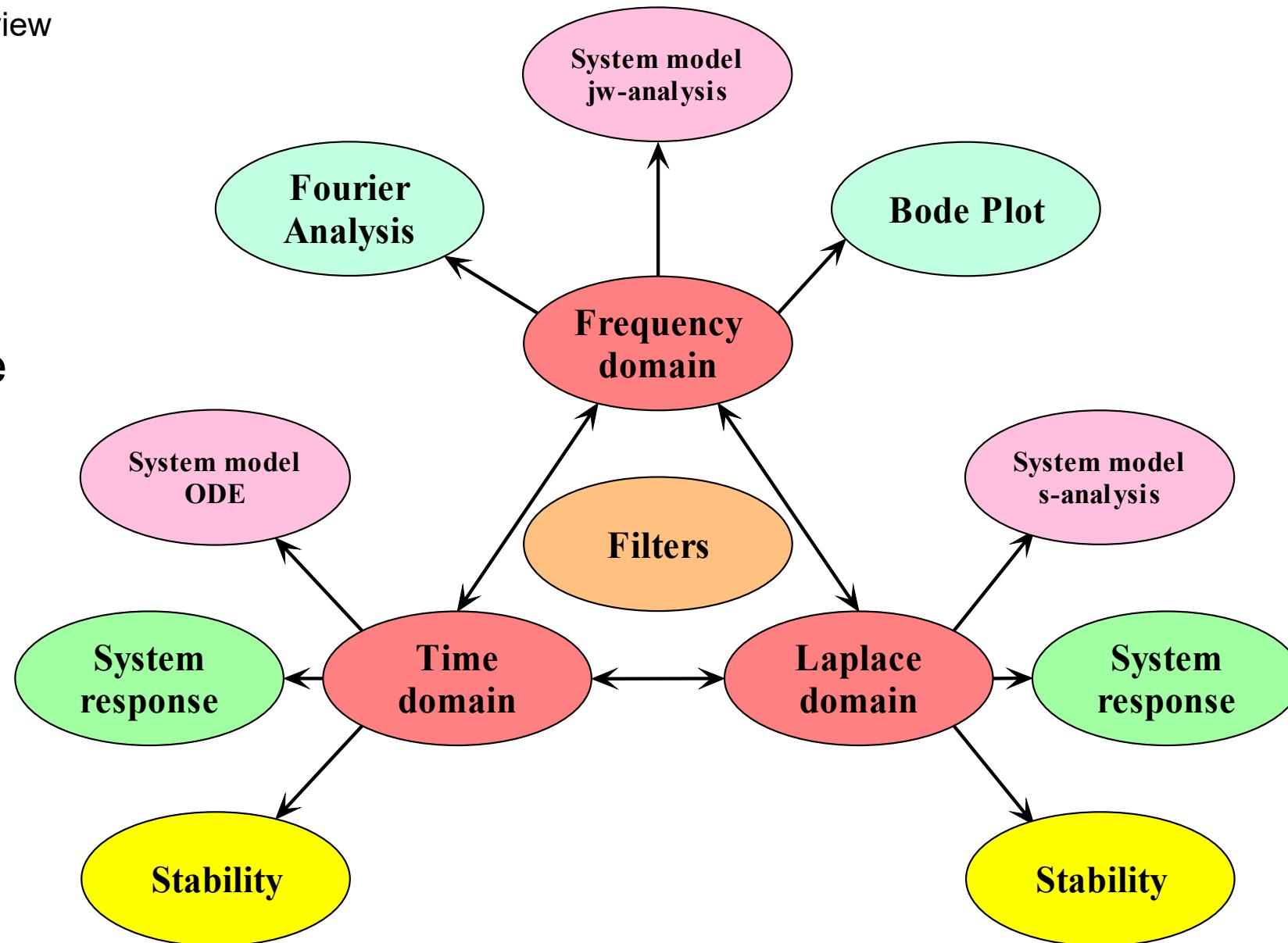
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Course overview

- System
 - Model
 - Response
 - Performance
- Domains
 - Time
 - Frequency
 - s-domain



6.1 Laplace Transform

6.2 Some properties of the Laplace Transform

The Fourier Transformation has two important limitations:

1. Convergence:
2. It cannot handle transient behaviors due to the abrupt onset of a signal. Systems cannot have initial conditions.

$$\int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt < \int_{-\infty}^{\infty} |x(t)| |e^{-j\omega t}| dt = \int_{-\infty}^{\infty} |x(t)| dt < \infty$$

Solution:

As far as convergence is concerned, we can modify the signal in order to ensure convergence of the Fourier integral. We do this by damping the signal adequately with a decaying exponential:

Then we can always find a value σ that makes $|\varphi(t)|$ integrable.

$$\varphi(t) = e^{-\sigma t} \cdot x(t)$$

Provided that: $x(t)$ is an exponential order signal

$$|x(t)| < M e^{\alpha t}, 0 < M, \alpha < \infty$$

Example

$$x(t) = e^{3t}$$

$$\varphi(t) = e^{-\sigma t} \cdot e^{3t} = e^{(3-\sigma)t}$$

$$3 - \sigma < 0 \Rightarrow \sigma > 3$$

Fourier Transform of exponentially damped signal

Forward transform:

What transform do we get if we multiply a time function by an exponentially decaying function?

The transform is a function of a complex-valued frequency variable. We call this the **forward Laplace transform**.

$$X(j\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\mathcal{F}\{e^{-\sigma t}x(t)\} = \underbrace{\int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t} dt}_{\text{Fourier transform}} = \underbrace{\int_{-\infty}^{\infty} x(t) e^{-(\sigma+j\omega)t} dt}_{\text{Laplace transform}}$$

$$= X(\sigma + j\omega)$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt \quad s \stackrel{\text{def}}{=} \sigma + j\omega$$

We started out with a Fourier transformation of a product $e^{-\sigma t}x(t)$. Then we reinterpret the expression as a new type of transformation of $x(t)$ where the pre-multiplication by $e^{-\sigma t}$ is an integral part of the transformation rather than a part of the signal being transformed. In doing so, we observe that the Laplace transform is equipped to handle exponentially rising signals, - provided that we choose σ such that product $e^{-\sigma t}x(t)$ has a Fourier transformation. This simply implies that the product $e^{-\sigma t}x(t)$ must be a decaying signal in time, i.e., that $e^{-\sigma t}$ falls at a faster rate than $x(t)$ increases. If $x(t)$ is an exponential order signal such a σ always exists.

Fourier Transform of exponentially damped signal

This is the forward Fourier transform of the function $x(t)e^{-\sigma t}$:

$$x(t)e^{-\sigma t} \leftrightarrow \int_{-\infty}^{\infty} x(t)e^{-\sigma t} e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t} dt = X(\sigma + j\omega)$$

We can then write the inverse Fourier transform:

$$x(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega$$

Multiplying on both sides with $e^{\sigma t}$:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{(\sigma+j\omega)t} d\omega$$

Variable substitution: $s \stackrel{\text{def}}{=} \sigma + j\omega \Rightarrow \frac{1}{j} ds = d\omega$

Inverse Laplace Transform:

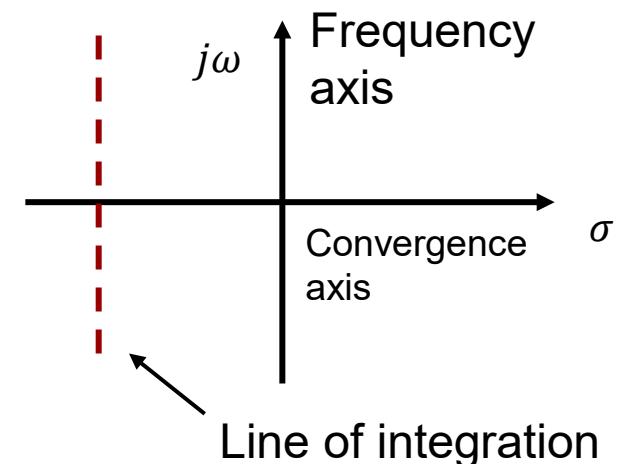
$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$$

s-plane

Complex frequency: $s \stackrel{\text{def}}{=} \sigma + j\omega$ $ds = j \cdot d\omega$

In the s -plane, we integrate from $\sigma - j\infty$ to $\sigma + j\infty$

Region of convergence: What should be the value of σ ?
It should have a value c such that $|x(t)e^{-ct}|$ is integrable

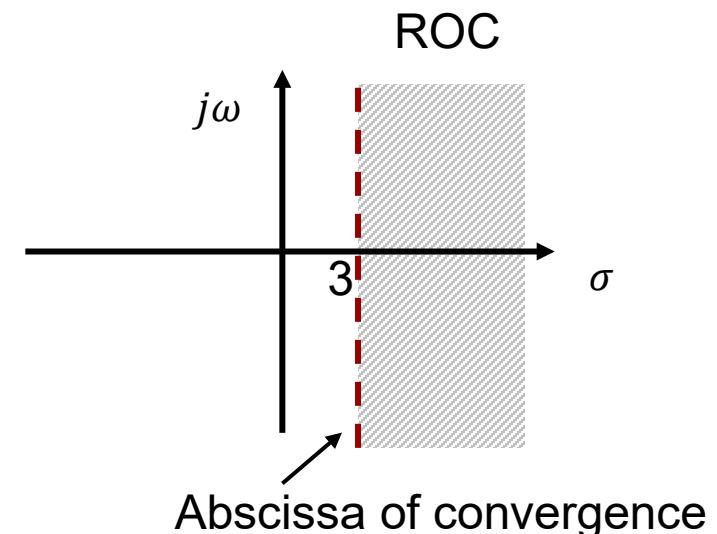


Example: $x(t) = e^{3t}$

We must now determine c , so that: $|e^{-ct} \cdot e^{3t}| = |e^{(3-c)t}|$

is integrable. This is the case when: $c > 3$

The region of convergence (ROC) for $\mathcal{L}\{e^{3t}\}$ is $\sigma > 3$.



Forward transformation:

$$\begin{aligned}\mathcal{F}\{e^{-\sigma t}x(t)\} &= \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-(\sigma+j\omega)t} dt \\ &= X(\sigma + j\omega)\end{aligned}$$

$$s \stackrel{\text{def}}{=} \sigma + j\omega$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$X(s) \stackrel{\text{def}}{=} \mathcal{L}\{x(t)\}$$

Transform pair:

$$X(s) \leftrightarrow x(t)$$

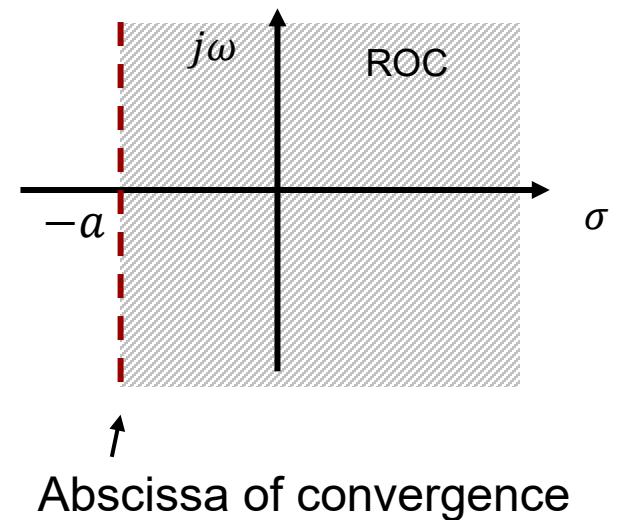
Example

$$x(t) = e^{-at}u(t) \quad a \in \mathbb{R}$$

$$X(s) = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt = -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^\infty = \frac{1}{s+a} - \lim_{t \rightarrow \infty} \frac{1}{s+a}e^{-(s+a)t}$$

$$\lim_{t \rightarrow \infty} e^{-(s+a)t} = \lim_{t \rightarrow \infty} e^{-\text{Re}(s+a)t - j\text{Im}(s+a)t} = \lim_{t \rightarrow \infty} \underbrace{e^{-\text{Re}(s+a)t}}_{\text{modulus}} e^{-j\omega t} = \begin{cases} 0 & \text{Re}(s+a) > 0 \\ \infty & \text{Re}(s+a) < 0 \end{cases}$$

$$X(s) = \frac{1}{s+a}, \text{Re}(s+a) > 0 \quad \text{Re}(s) > -a$$



Unfortunately, the bilateral transformation is not unique, as the following example will show:

$x(t)$ causal

$a \in \mathbb{R}$

$$x(t) = e^{-at}u(t)$$

$$X(s) = \frac{1}{s+a}, \operatorname{Re}\{s\} > -a$$

$x(t)$ anti-causal

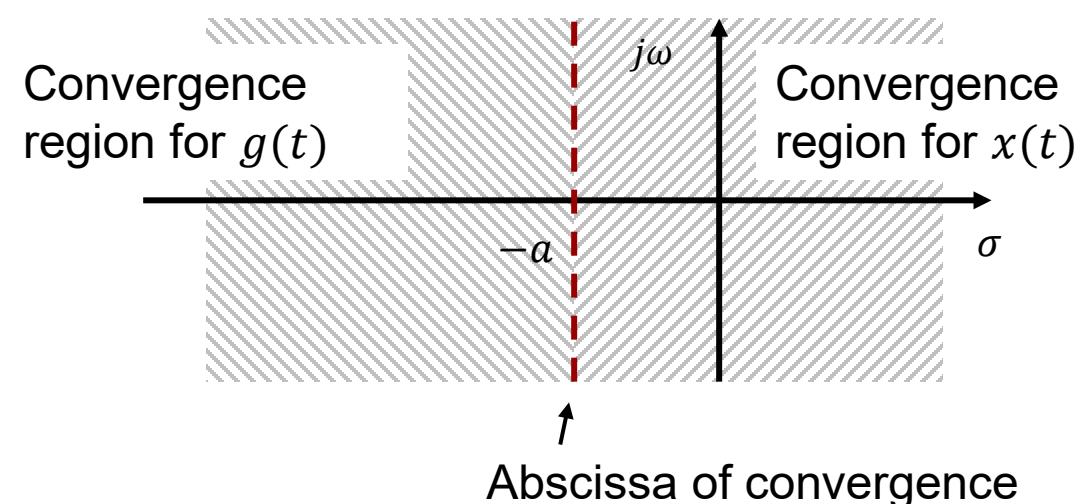
$a \in \mathbb{R}$

$$g(t) = -e^{-at}u(-t)$$

$$G(s) = \frac{1}{s+a}, \operatorname{Re}\{s\} < -a$$

The bilateral Laplace Transform is not unique.

In this example the two functions have the same Laplace transform. The only difference is the region of convergence.



All causal signals have convergence regions extending to the right from a vertical line c .

All anti-causal signals have convergence regions extending to the left from a vertical line c .

If we from hereon decide that we only consider causal signals, then the convergence region is given, and the Laplace transformation is unique.

The Laplace transformation of causal signals is called the **UNILATERAL Laplace transformation**.

The unilateral Laplace transform is by far the most used Laplace transform, and without other indications:

the term “Laplace transform” is always taken to mean “the unilateral Laplace transform”.

Remember this at the exam.

$$X(s) = \int_{0_-}^{\infty} x(t)e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s)e^{st} ds$$

The unilateral Laplace transform put a constraint on $x(t)$, not $X(s)$. Hence, the inverse Laplace transform is the same.

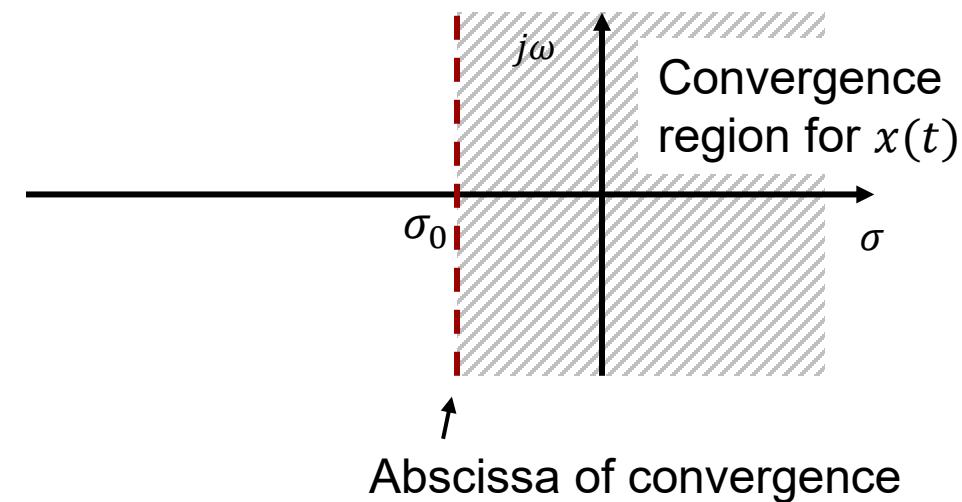
Existence of Unilateral Laplace Transformation

If $x(t)$ is an exponential order signal, i.e.,

If: $|x(t)| < M e^{\sigma_0 t}$ and $\sigma > \sigma_0$ then

$$\begin{aligned} X(s) &= \int_{0_-}^{\infty} x(t)e^{-st} dt = \int_{0_-}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t} dt \\ &< \int_{0_-}^{\infty} |[x(t)e^{-\sigma t}]e^{-j\omega t}| dt = \int_{0_-}^{\infty} |\textcolor{red}{x(t)}e^{-\sigma t}| dt < \int_{0_-}^{\infty} |\textcolor{red}{M} e^{\sigma_0 t} e^{-\sigma t}| dt < M \int_{0_-}^{\infty} |e^{-(\sigma-\sigma_0)t}| dt < \infty \end{aligned}$$

Abscissa of convergence: $\sigma = \sigma_0$



Example signals

$$x(t) = \delta(t)$$

$$X(s) = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = \int_{0^-}^{\infty} \delta(t) e^{-s\mathbf{0}} dt = \int_{0^-}^{\infty} \delta(t) dt = 1$$

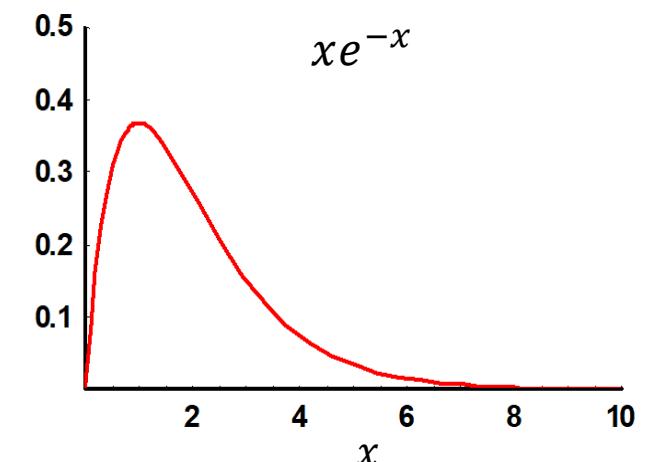
$$x(t) = u(t)$$

$$X(s) = \int_{0^-}^{\infty} u(t) e^{-st} dt = \int_{0^-}^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_{0^-}^{\infty} = \frac{1}{s}, \text{Re}(s) > 0$$

$$x(t) = tu(t)$$

$$\begin{aligned} X(s) &= \int_{0^-}^{\infty} tu(t) e^{-st} dt = \int_{0^-}^{\infty} te^{-st} dt = -\frac{1}{s^2} e^{-st} (1 + st) \Big|_{0^-}^{\infty} \\ &= \frac{1}{s^2} - \lim_{t \rightarrow \infty} \frac{(1 + st)e^{-st}}{s^2} \\ &= \frac{1}{s^2}, \text{Re}(s) > 0 \end{aligned}$$

The limit of
 $st e^{-st}$



Comparison with Fourier transform

$x(t)$	$X(\omega)$	$X(s)$
$\delta(t)$	1	1
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$	$\frac{1}{s}, \sigma > 0$
$tu(t)$	nonexistent	$\frac{1}{s^2}, \sigma > 0$

The step function is a special case. It has a Fourier transform, yet the region of convergence for the Laplace transform does not include the imaginary axis.

The class of signals with a Laplace transform is larger than that with a Fourier transform.

Inverse Laplace transform

For the inverse Laplace transform, there is no difference between the bilateral transform and the unilateral transform.

$$e^{-\sigma t} x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega$$

↔

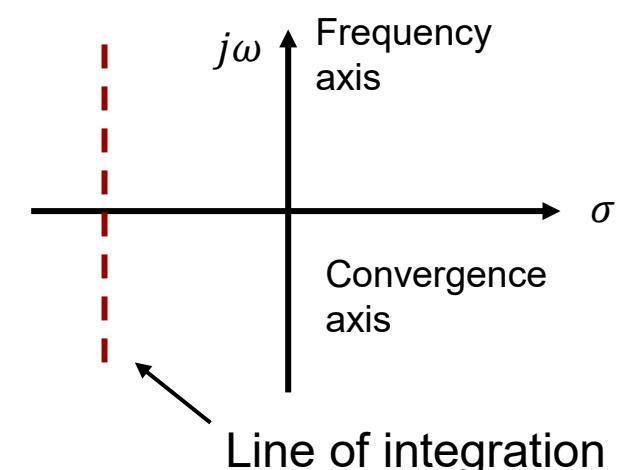
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{(\sigma+j\omega)t} d\omega$$

In the s-plane we integrate from $\sigma - j\infty$ to $\sigma + j\infty$

$$s \stackrel{\text{def}}{=} \sigma + j\omega$$

$$ds = j \cdot d\omega$$

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds$$



In general, the computation of this integral requires integration in the complex plane.

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s)e^{st} ds$$

To avoid this, the function $X(s)$ is factorized in simple factors and **partial fraction expansion** is performed.

$$\begin{aligned} X(s) &= \frac{P(s)}{Q(s)} = \frac{(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)} \\ &= \frac{A_1}{s + p_1} + \frac{A_2}{s + p_2} + \cdots + \frac{A_n}{s + p_n} \end{aligned}$$

Partial Fraction Expansion
(Lathi appendix B.5)

Using **tables of Laplace transforms**, we then inverse transform term by term:

$$x(t) = [A_1 e^{-p_1 t} + A_2 e^{-p_2 t} + \cdots + A_n e^{-p_n t}] u(t)$$

Next topic

Partial fraction expansion

Video 2

Let us consider the step response:

It will be shown in a later slide that convolution in the time domain corresponds to multiplying the transforms in the Laplace domain:

$$y(t) = h(t) * x(t) = Ae^{-at} * u(t)$$

$$Y(s) = H(s) \cdot X(s) = \frac{A}{s+a} \cdot \frac{1}{s} = \frac{A}{s(s+a)}$$

To obtain the solution in the time domain, we need to find the inverse Laplace transform of $Y(s)$. This is done by splitting up the expression into several fractions and then finding the inverse Laplace transform of each term.

$$y(t) = \mathcal{L}^{-1} \left[\frac{A}{s(s+a)} \right] = \mathcal{L}^{-1} \left[\frac{k_1}{s} + \frac{k_2}{s+a} \right] = \mathcal{L}^{-1} \left[\frac{k_1}{s} \right] + \mathcal{L}^{-1} \left[\frac{k_2}{s+a} \right]$$

The technique of splitting a fraction up into simpler fractions is called **Partial Fraction Expansion**.

Techniques of partial fraction expansions

To determine the unknown coefficient k_1 , we need to isolate it.

We do this by multiplying both sides of the equation with its denominator (s) and then setting $s = 0$. This removes the term with k_2 .

$$Y(s) = \frac{A}{s(s+a)} = \frac{k_1}{s} + \frac{k_2}{s+a}$$

$$s \cdot Y(s) \Big|_{s=0} = s \cdot \frac{A}{s(s+a)} \Big|_{s=0} = \textcolor{green}{s} \cdot \frac{k_1}{\textcolor{green}{s}} \Big|_{s=0} + \textcolor{red}{s} \cdot \frac{k_2}{\textcolor{red}{s+a}} \Big|_{s=0}$$

$$s \cdot Y(s) \Big|_{s=0} = \frac{A}{(0+a)} = k_1 + \textcolor{red}{0} \cdot \frac{k_2}{a}$$

$$k_1 = \frac{A}{a}$$

Techniques of partial fraction expansions

Next, we use the same approach to isolate k_2 :

$$\begin{aligned}(s + a) \cdot Y(s) \Big|_{s=-a} &= (s + a) \cdot \frac{A}{s(s + a)} \Big|_{s=-a} \\ &= (s + a) \cdot \frac{k_1}{s} \Big|_{s=-a} + (s + a) \cdot \frac{k_2}{s + a} \Big|_{s=-a}\end{aligned}$$

$$(s + a) \cdot Y(s) \Big|_{s=-a} = \frac{A}{s} \Big|_{s=-a} = 0 \cdot \frac{k_1}{s} \Big|_{s=-a} + k_2 \Rightarrow k_2 = -\frac{A}{a}$$

Inserting the two coefficients, we get:

$$Y(s) = \frac{A}{s(s + a)} = \frac{k_1}{s} + \frac{k_2}{s + a} = \frac{A}{a} \cdot \frac{1}{s} - \frac{A}{a} \cdot \frac{1}{s + a}$$

Looking up the inverse Laplace transforms of each term, we get:

$$\begin{aligned}y(t) &= \frac{A}{a} \cdot \mathcal{L}^{-1} \left[\frac{1}{s} \right] - \frac{A}{a} \cdot \mathcal{L}^{-1} \left[\frac{1}{s + a} \right] = \frac{A}{a} \cdot u(t) - \frac{A}{a} \cdot e^{-at} \cdot \underbrace{u(t)}_{\text{for causality}} \\ &= \frac{A}{a} (1 - e^{-at}) \cdot u(t)\end{aligned}$$

Here is a problem
with repeated
roots:

$$X(s) = \frac{A}{s(s+a)^2} = \frac{k_0}{s} + \frac{k_1}{(s+a)} + \frac{k_2}{(s+a)^2}$$

$$k_0 = s \frac{A}{s(s+a)^2} \Big|_{s=0} - s \frac{k_1}{(s+a)} \Big|_{s=0} - s \frac{k_2}{(s+a)^2} \Big|_{s=0} = \frac{A}{a^2}$$

$$k_2 = (s+a)^2 \frac{A}{s(s+a)^2} \Big|_{s=-a} - (s+a)^2 \frac{k_0}{s} \Big|_{s=-a} (s+a)^2 \frac{k_1}{(s+a)} \Big|_{s=-a} = -\frac{A}{a}$$

$$X(s) = \frac{A}{s(s+a)^2} = \frac{A}{a^2} \frac{1}{s} + \frac{k_1}{(s+a)} - \frac{A}{a} \frac{1}{(s+a)^2}$$

Techniques of partial fraction expansions

Set $s = 1$ and solve for k_1 :

$$X(s) = \frac{A}{s(s+a)^2} = \frac{A}{a^2} \frac{1}{s} + \frac{k_1}{(s+a)} - \frac{A}{a} \frac{1}{(s+a)^2}$$

$$X(1) = \frac{A}{1(1+a)^2} = \frac{A}{a^2} \frac{1}{1} + \frac{k_1}{(1+a)} - \frac{A}{a} \frac{1}{(1+a)^2}$$

$$X(1) = \frac{A}{(1+a)^2} = \frac{A}{a^2} + \frac{k_1}{(1+a)} - \frac{A}{a} \frac{1}{(1+a)^2}$$

$$\frac{k_1}{(1+a)} = \frac{A}{(1+a)^2} + \frac{A}{a} \frac{1}{(1+a)^2} - \frac{A}{a^2}$$

$$\begin{aligned} k_1 &= \frac{A}{(1+a)} + \frac{A}{a} \frac{1}{(1+a)} - \frac{A(1+a)}{a^2} = A \left(\frac{a^2}{a^2(1+a)} + \frac{a}{a^2(1+a)} - \frac{(1+a)^2}{a^2(1+a)} \right) \\ &= A \left(\frac{a^2 + a - (1+2a+a^2)}{a^2(1+a)} \right) = A \left(\frac{-1-a}{a^2(1+a)} \right) = -\frac{A}{a^2} \end{aligned}$$

$$X(s) = \frac{A}{s(s+a)^2} = \frac{A}{a^2} \left[\frac{1}{s} - \frac{1}{(s+a)} - \frac{a}{(s+a)^2} \right]$$

Inverse Laplace transformation of each term using a table of transforms:

$$x(t) = \frac{A}{a^2} [1 - e^{-at} - at \cdot e^{-at}] \cdot u(t)$$

Let us consider the step response of a second order lowpass filter. We will discover that the Laplace transform of the impulse response is:

We will often prefer to express it in terms of the **resonance frequency** ω_n and its **damping factor** ζ :

To obtain the solution in the time domain, we need to inverse Laplace transform this:

This requires partial fraction expansion.

$$y(t) = h(t) * u(t)$$

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_0}$$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \omega_n = \sqrt{a_0} \quad \zeta = \frac{a_1}{2\sqrt{a_0}}$$

$$Y(S) = H(s) \cdot \frac{1}{s}$$

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{k_0}{s} + \frac{k_1 s + k_2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{k_0}{s} + \frac{k_1 s + k_2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Solving for k_0 :

$$k_0 = s \cdot Y(s) \Big|_{s=0} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Big|_{s=0} = 1$$

Solving for k_1 :

$$s \cdot Y(s) \Big|_{s \rightarrow \infty} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Big|_{s \rightarrow \infty} = \frac{k_0 s}{s} \Big|_{s \rightarrow \infty} + \frac{k_1 s^2}{s^2} \Big|_{s \rightarrow \infty} = k_0 + k_1 = 0 \quad k_1 = -k_0 = -1$$

Solving for k_2 :

$$s = 1 \Rightarrow \frac{\omega_n^2}{1(1^2 + 2\zeta\omega_n 1 + \omega_n^2)} = \frac{1}{1} + \frac{-1 \cdot 1 + k_2}{1^2 + 2\zeta\omega_n 1 + \omega_n^2} \Rightarrow k_2 = -2\zeta\omega_n$$

Inserting results:

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Techniques of partial fraction expansions

Next, we need to use a table of transforms. This often requires switching to other symbols:

$$Y(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s} - \frac{As + B}{s^2 + 2as + c}$$

$$y(t) = [1 - \textcolor{red}{r} e^{-\textcolor{brown}{a}t} \cos(\textcolor{blue}{b}t + \theta)]u(t) \quad \text{Solution in table of transforms}$$

$$r = \sqrt{\frac{A^2c + B^2 - 2ABa}{c - a^2}} = \frac{1}{\sqrt{1 - \zeta^2}}$$

$$\theta = \tan^{-1}\left(\frac{Aa - B}{A\sqrt{c - a^2}}\right) = \tan^{-1}\left(\frac{-\zeta}{\sqrt{1 - \zeta^2}}\right)$$

$$b = \sqrt{c - a^2} = \omega_n \sqrt{1 - \zeta^2}$$

To obtain the solution using our notation, some work is required to rephrase the solution:

$$y(t) = \left[1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \cos\left(\omega_n \sqrt{1 - \zeta^2} t + \tan^{-1}\left[\frac{-\zeta}{\sqrt{1 - \zeta^2}}\right]\right) \right] u(t)$$

Geometric relations of poles

We would like to find the poles:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

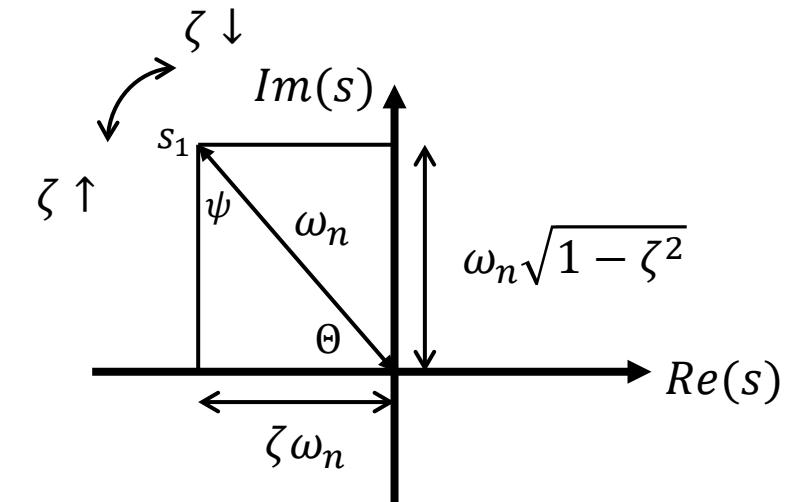
$$s_{1,2} = -\zeta\omega_n \pm j \cdot \omega_n\sqrt{1 - \zeta^2}$$

$$\omega_n = |s_1|$$

$$\cos \Theta = \frac{\zeta\omega_n}{\omega_n} = \zeta \Rightarrow \quad \Theta = \cos^{-1} \zeta$$

$$\tan \psi = \frac{\zeta\omega_n}{\omega_n\sqrt{1 - \zeta^2}} = \frac{\zeta}{\sqrt{1 - \zeta^2}} \Rightarrow \quad \psi = \tan^{-1} \frac{\zeta}{\sqrt{1 - \zeta^2}}$$

$$\psi = 90^\circ - \Theta \Rightarrow -\psi = \Theta - 90^\circ \quad -\psi = \tan^{-1} \frac{-\zeta}{\sqrt{1 - \zeta^2}} = \theta - 90^\circ = \cos^{-1} \zeta - 90^\circ$$



$$y(t) = \left[1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1 - \zeta^2}t + \cos^{-1} \zeta) \right] u(t)$$

Partial fraction expansion

We could also just use Maple:

$$H1 := \frac{1}{s \cdot (s + a)} : |$$

convert(H1, parfrac, s)

$$-\frac{1}{a(s + a)} + \frac{1}{as}$$

$$H2 := \frac{A}{s \cdot (s + a)^2} :$$

convert(H2, parfrac, s)

$$-\frac{A}{a(s + a)^2} - \frac{A}{a^2(s + a)} + \frac{A}{a^2s}$$

$$H3 := \frac{\omega_n^2}{s \cdot (s^2 + 2 \cdot \zeta \cdot \omega_n \cdot s + \omega_n^2)} :$$

convert(H3, parfrac, s)

$$\frac{-2\zeta\omega_n - s}{2\zeta\omega_n s + s^2 + \omega_n^2} + \frac{1}{s}$$

Using Maple can save you a lot of time.
 Which you then can use to figure out if the result is identical to alternative forms ☺

restart
with(inttrans) : ←

Inverse Laplace transformation

invlaplace(H1, s, t)

$$\frac{1 - e^{-at}}{a}$$

invlaplace(H2, s, t)

$$\frac{(-(at + 1)e^{-at} + 1)A}{a^2}$$

invlaplace(H3, s, t)

$$1 - \frac{1}{\omega_n(\zeta^2 - 1)} \left(e^{-t\zeta\omega_n} \left(\cosh \left(t \sqrt{\omega_n^2 (\zeta^2 - 1)} \right) \omega_n (\zeta^2 - 1) \right. \right. \\ \left. \left. + \sqrt{\omega_n^2 (\zeta^2 - 1)} \zeta \sinh \left(t \sqrt{\omega_n^2 (\zeta^2 - 1)} \right) \right) \right)$$

Laplace theorems

Video 3

Time shift:

$$\begin{aligned} \int_{0^-}^{\infty} x(t - t_0) u(t - t_0) e^{-st} dt &= \int_{-t_0}^{\infty} x(y) \color{green}{u(y)} e^{-s(y+t_0)} dy && y \stackrel{\text{def}}{=} t - t_0 \\ &= \int_0^{\infty} x(y) \color{blue}{e^{-s(y+t_0)}} dy \\ &= \color{blue}{e^{-st_0}} \int_0^{\infty} x(y) \color{blue}{e^{-sy}} dy \\ &= X(s) e^{-st_0} \end{aligned}$$

We can only delay the signal.

Time advancing may violate
the assumption of a causal
signal.

$$x(t - t_0) \leftrightarrow e^{-st_0} X(s), t_0 \geq 0$$

Frequency shift theorem:

$$\int_{0^-}^{\infty} [x(t)e^{s_0 t}] e^{-st} dt = \int_{0^-}^{\infty} x(t) e^{-(s-s_0)t} dt \\ = X(s - s_0)$$

$$x(t)e^{s_0 t} \leftrightarrow X(s - s_0)$$

Integration by parts:

$$\int f \cdot g' dx = f \cdot g - \int f' \cdot g dx$$

$$\begin{aligned} \int_{0_-}^{\infty} \frac{dx}{dt} e^{-st} dt &= x(t) e^{-st} \Big|_{0_-}^{\infty} - (-s) \int_{0_-}^{\infty} x(t) e^{-st} dt \\ &= -x(0_-) + sX(s) \end{aligned}$$

Differentiation theorem:

$$\frac{dx}{dt} \leftrightarrow sX(s) - x(0_-)$$

$$\frac{d^2x}{dt^2} \leftrightarrow s^2X(s) - sx(0_-) - \dot{x}(0_-)$$

Examples:

$$i_C(t) = C \frac{dv_C}{dt} \quad I_C(s) = sCV_C(s) - Cv_C(0_-)$$

$$v_L(t) = L \frac{di_L}{dt} \quad V_L(s) = sLI_L(s) - Li_L(0_-)$$

Consider:

$$g(t) \stackrel{\text{def}}{=} \int_{0_-}^t x(\tau)d\tau \quad \text{so that} \quad \frac{dg}{dt} = x(t) \quad \text{and} \quad g(0_-) = 0$$

Transform pairs:

$$g(t) \leftrightarrow G(s) \quad \text{and} \quad x(t) \leftrightarrow X(s)$$

$$x(t) = \frac{dg}{dt} \leftrightarrow X(s) = sG(s) - g(0_-) = sG(s)$$

$$G(s) = \frac{X(s)}{s}$$

Integration theorem 1:

$$\int_{0_-}^t x(\tau)d\tau \leftrightarrow \frac{X(s)}{s}$$

Consider:

$$g(t) \stackrel{\text{def}}{=} \int_{-\infty}^t x(\tau)d\tau = \int_{-\infty}^{0_-} x(\tau)d\tau + \int_{0_-}^t x(\tau)d\tau = g(0_-) + \int_{0_-}^t x(\tau)d\tau$$

$$x(t) = \frac{dg}{dt} \leftrightarrow X(s) = sG(s) - g(0_-)$$

$$G(s) = \frac{X(s)}{s} + \frac{\int_{-\infty}^{0_-} x(\tau)d\tau}{s} \quad \Leftrightarrow \quad g(t) \leftrightarrow \frac{X(s)}{s} + \frac{\int_{-\infty}^{0_-} x(\tau)d\tau}{s}$$

Integration theorem 2:

$$\int_{-\infty}^t x(\tau)d\tau \leftrightarrow \frac{X(s)}{s} + \frac{\int_{-\infty}^{0_-} x(\tau)d\tau}{s}$$

Example:

$$v_C(t) = \frac{1}{C} \int_{-\infty}^t i_C(\tau)d\tau \leftrightarrow V_C(s) = \frac{I_C(s)}{sC} + \frac{v_C(0_-)}{s}$$

Time scaling:

$$y \stackrel{\text{def}}{=} at \Rightarrow dt = dy/a$$

$$\begin{aligned} \int_{0^-}^{\infty} x(at)e^{-st}dt &= \frac{1}{a} \int_{0^-}^{\infty} x(y)e^{-sy/a}dy \\ &= \frac{1}{a} X\left(\frac{s}{a}\right) \end{aligned}$$

Time inversion is not permissible
as this would turn a causal signal
into an anti-causal.

$$x(at) \leftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right), a > 0$$

Convolution theorem:

$$x_1(t) * x_2(t) \leftrightarrow X_1(s) \cdot X_2(s)$$

$$\begin{aligned} x_1(t) * x_2(t) &\leftrightarrow \int_{0^-}^{\infty} \int_{0^-}^{\infty} x_1(\tau) x_2(t - \tau) d\tau e^{-st} dt = \int_{0^-}^{\infty} x_1(\tau) \int_{0^-}^{\infty} x_2(t - \tau) e^{-st} dt d\tau \\ &= \int_{0^-}^{\infty} x_1(\tau) X_2(s) e^{-s\tau} d\tau = X_2(s) \cdot \int_{0^-}^{\infty} x_1(\tau) e^{-s\tau} d\tau \\ &= X_1(s) \cdot X_2(s) \end{aligned}$$

Example:

$$\begin{array}{rcl} y_{zs}(t) &=& h(t) * x(t) \\ \uparrow_3 && \downarrow_1 \quad \downarrow_2 \\ Y_{zs}(s) &=& H(s) \cdot X(s) \end{array}$$

Laplace Theorems – Frequency domain convolution

Let us find the Laplace transform of a product of two signals:

$$y(t) = h(t)x(t) \quad y(t) \leftrightarrow Y(s) \quad h(t) \leftrightarrow H(s) \quad x(t) \leftrightarrow X(s)$$

$$\begin{aligned} \mathcal{L}\{y(t)\} &= \int_0^\infty y(t)e^{-st}dt = \int_0^\infty h(t)x(t)e^{-st}dt \\ &= \int_0^\infty h(t) \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(p)e^{pt}dp e^{-st}dt = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(p) \int_0^\infty h(t)e^{pt}e^{-st}dt dp \\ &= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(p) \int_0^\infty h(t)e^{-(s-p)t}dt dp = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(p)H(s-p)dp \\ &= \frac{1}{2\pi j} [X(s) * H(s)] \end{aligned}$$

$$h(t)x(t) \leftrightarrow \frac{1}{2\pi j} [X(s) * H(s)]$$

Initial Value Theorem

$$x(0_+) = \lim_{s \rightarrow \infty} s X(s)$$

If the limit on the right-hand side exists.

Proof:

$$\begin{aligned}
 sX(s) - \cancel{x(0_-)} &= \int_{0_-}^{\infty} \frac{dx}{dt} e^{-st} dt = \int_{0_-}^{0_+} \frac{dx}{dt} e^{-st} dt + \int_{0_+}^{\infty} \frac{dx}{dt} e^{-st} dt \\
 &= x(t) \Big|_{0_-}^{0_+} + \int_{0_+}^{\infty} \frac{dx}{dt} e^{-st} dt = x(0_+) - \cancel{x(0_-)} + \int_{0_+}^{\infty} \frac{dx}{dt} e^{-st} dt \\
 \lim_{s \rightarrow \infty} s X(s) &= x(0_+) + \lim_{s \rightarrow \infty} \int_{0_+}^{\infty} \frac{dx}{dt} e^{-st} dt = x(0_+) + \int_{0_+}^{\infty} \frac{dx}{dt} \left(\lim_{s \rightarrow \infty} e^{-st} \right) dt \\
 &= x(0_+)
 \end{aligned}$$

Final Value Theorem

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} s X(s)$$

Provided that $sX(s)$ only has poles in the left half plane.

Proof:

$$\begin{aligned}\lim_{s \rightarrow 0} [sX(s) - x(0_-)] &= \lim_{s \rightarrow 0} \int_{0_-}^{\infty} \frac{dx}{dt} e^{-st} dt &= \int_{0_-}^{\infty} \frac{dx}{dt} \lim_{s \rightarrow 0} e^{-st} dt = \int_{0_-}^{\infty} \frac{dx}{dt} \mathbf{1} dt \\ &= x(t) \Big|_{0_-}^{\infty} = \lim_{t \rightarrow \infty} x(t) - x(0_-)\end{aligned}$$

$$\text{Example: } Y_{zs}(s) = \frac{2s}{(s + 2)^2}$$

$$\text{Initial value: } y_{zs}(0_+) = \lim_{s \rightarrow \infty} s Y_{zs}(s) = \lim_{s \rightarrow \infty} s \frac{2s}{(s + 2)^2} = 2$$

$$\text{Final value: } \lim_{t \rightarrow \infty} y_{zs}(t) = \lim_{s \rightarrow 0} s Y_{zs}(s) = \lim_{s \rightarrow 0} s \frac{2s}{(s + 2)^2} = 0$$

$$\text{From a Laplace transform table: } y_{zs}(t) = (2 - 4t)e^{-2t}u(t)$$

This agrees with our predictions.

So, before we do the inverse Laplace transform, we can quickly get the initial value and the final value of the response. A very valuable method to check for mistakes.

Problems

Problem 6.1 (6.1-1) Laplace transform of various functions

By direct integration [Eq. (6.8b)] find the Laplace transforms and the region of convergence of the following functions.

Use Maple to help with integrations.
The regions of convergence you will have to work out yourself.

$$(b) t e^{-t} u(t)$$

$$(c) t \cos \omega_0 t \ u(t)$$
 Hint: Substitute Euler for cos

$$(d) (e^{2t} - 2e^{-t}) u(t)$$

$$(a) u(t) - u(t - 1)$$

Problem 6.3 (6.1-4): Inverse Laplace transform and partial fraction expansion

Find the inverse Laplace transform of these functions.

$$(a) \frac{2s + 5}{s^2 + 5s + 6}$$

$$(b) \frac{3s + 5}{s^2 + 4s + 13}$$

$$(c) \frac{(s + 1)^2}{s^2 - s - 6}$$

$$(d) \frac{5}{s^2(s + 2)}$$

$$(e) \frac{2s + 1}{(s + 1)(s^2 + 2s + 2)}$$

Use Maple to test if partial fraction expansion is possible. If it is, use Maple to obtain the inverse Laplace transform using the results of the partial fraction expansion. Check each result by comparing with table 6.1.

Problem 6.3 a Problem 6.1.4 in newest book

$$\text{convert}\left(\frac{2 \cdot s + 5}{s^2 + 5 \cdot s + 6}, \text{parfrac}, s\right)$$

$$\frac{1}{s + 2} + \frac{1}{s + 3}$$

$$\text{invlaplace}\left(\frac{1}{s + 2} + \frac{1}{s + 3}, s, t\right)$$

$$e^{-2t} + e^{-3t}$$

Entry 5 in table 6.1

Solutions

Problem 6.1 (6.1-1) Laplace transform of various functions

By direct integration [Eq. (6.8b)] find the Laplace transforms and the region of convergence of the following functions.

Use Maple to help with integrations.
The regions of convergence you will have to work out yourself.

$$(b) t e^{-t} u(t)$$

$$(c) t \cos \omega_0 t \ u(t)$$
 Hint: Substitute Euler for cos

$$(d) (e^{2t} - 2e^{-t}) u(t)$$

$$(a) u(t) - u(t - 1)$$

Problem 6.1 (6.1-1) Laplace transform of various functions (sol)

By direct integration [Eq. (6.8b)] find the Laplace transforms and the region of convergence of the following functions.

$$(b) t e^{-t} u(t)$$

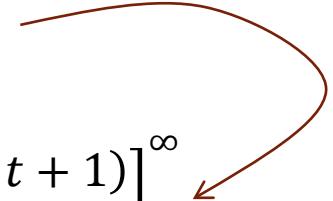
$$e^{-(s+1)t} = e^{-(\sigma+j\omega+1)t} = e^{-(\sigma+1)t} e^{-j\omega t}$$

$$|e^{-(s+1)t}| = |e^{-(\sigma+j\omega+1)t}| = |e^{-(\sigma+1)t}| \underbrace{|e^{-j\omega t}|}_{=1}$$

$$\lim_{t \rightarrow \infty} |e^{-(s+1)t}| = \lim_{t \rightarrow \infty} |e^{-(\sigma+1)t}| = \begin{cases} 0 & \text{if } \sigma + 1 > 0 \\ \infty & \text{if } \sigma + 1 < 0 \end{cases}$$

$$\begin{aligned} \int_{0-}^{\infty} t e^{-t} u(t) e^{-st} dt &= \int_{0-}^{\infty} t e^{-(s+1)t} dt \\ &= \left[-\frac{e^{-(s+1)t} (ts + t + 1)}{(s+1)^2} \right]_{0-}^{\infty} \\ &= \frac{1}{(s+1)^2} - \lim_{t \rightarrow \infty} \frac{e^{-(s+1)t} (ts + t + 1)}{(s+1)^2} \\ &= \frac{1}{(s+1)^2}, \quad \text{Re}\{s\} > -1 \end{aligned}$$

Use Maple to find
the integrand



Region of
convergence

Problem 6.1 (6.1-1) Laplace transform of various functions (sol)

Problem 6.1 b

Find the integrand

(b) $t e^{-t} u(t)$

$$\int t \cdot e^{-(s+1) \cdot t} dt - \frac{e^{-(s+1) t} (s t + t + 1)}{s^2 + 2 s + 1}$$

$$\int_0^\infty t \cdot e^{-(s+1) \cdot t} dt \\ \lim_{t \rightarrow \infty} \left(-\frac{e^{-(s+1) t} s t + t e^{-(s+1) t} + e^{-(s+1) t} - 1}{s^2 + 2 s + 1} \right)$$

Inverse Laplace transform

$laplace(t \cdot e^{-t} \cdot \text{Heaviside}(t), t, s)$

$$\frac{1}{(s+1)^2}$$

Problem 6.1 (6.1-1) Laplace transform of various functions (sol)

By direct integration [Eq. (6.8b)] find the Laplace transforms and the region of convergence of the following functions.

$$(c) t \cos \omega_0 t \ u(t)$$

$$(c) t \cos \omega_0 t \ u(t) = t \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t}) u(t)$$

$$\int_{0_-}^{\infty} t \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t}) e^{-st} dt = \int_{0_-}^{\infty} t \frac{1}{2} (e^{(j\omega_0 - s)t} + e^{-(j\omega_0 + s)t}) dt$$

$$\int t e^{at} dt = \frac{(at - 1)e^{at}}{a^2}$$

Problem 6.1 (6.1-1) Laplace transform of various functions (sol)

(c) continued

$$\int_{0_-}^{\infty} t \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t}) e^{-st} dt = \left[\frac{(j\omega_0 - s)t - 1}{2(j\omega_0 - s)^2} e^{(j\omega_0 - s)t} \right]_0^{\infty} + \left[\frac{-(j\omega_0 + s)t - 1}{2(j\omega_0 + s)^2} e^{-(j\omega_0 + s)t} \right]_0^{\infty}$$

$$\left[\frac{(j\omega_0 - s)t - 1}{2(j\omega_0 - s)^2} e^{(j\omega_0 - s)t} \right]_0^{\infty} = \frac{1}{2(j\omega_0 - s)^2}, \quad Re\{j\omega_0 - s\} < 0 \Rightarrow Re\{s\} > 0$$

$$\left[\frac{-(j\omega_0 + s)t - 1}{2(j\omega_0 + s)^2} e^{-(j\omega_0 + s)t} \right]_0^{\infty} = \frac{1}{2(j\omega_0 + s)^2}, \quad Re\{j\omega_0 + s\} > 0 \Rightarrow Re\{s\} > 0$$

$$\int_{0_-}^{\infty} t \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t}) e^{-st} dt = \frac{1}{2(j\omega_0 - s)^2} + \frac{1}{2(j\omega_0 + s)^2} = - \frac{s^2 - \omega_0^2}{(\omega_0^2 + 2j\omega_0 s - s^2)(-\omega_0^2 + 2j\omega_0 s + s^2)}$$

$$= - \frac{s^2 - \omega_0^2}{-s^4 - 2\omega_0^2 s^2 - \omega_0^4} = \frac{s^2 - \omega_0^2}{(s^2 + \omega_0^2)^2}, \quad Re\{s\} > 0$$

$$\int t e^{at} dt = \frac{(at - 1)e^{at}}{a^2}$$

Problem 6.1 (6.1-1) Laplace transform of various functions (sol)

Problem 6.1 c

Find the integrand

(c) $t \cos \omega_0 t \ u(t)$

$$\int t \cdot \frac{1}{2} \cdot e^{a \cdot t} dt$$

$$\frac{(at - 1) e^{at}}{2a^2}$$

$$\text{normal} \left(\frac{1}{\text{expand}(2 \cdot (j \cdot \omega_0 - s)^2)} + \frac{1}{\text{expand}(2 \cdot (j \cdot \omega_0 + s)^2)} \right)$$

$$- \frac{s^2 - \omega_0^2}{(2 I \omega_0 s - s^2 + \omega_0^2)(-\omega_0^2 + 2 I \omega_0 s + s^2)}$$

$$- \frac{s^2 - \omega_0^2}{\text{expand}((\omega_0^2 + 2 I s \omega_0 - s^2)(-\omega_0^2 + 2 I s \omega_0 + s^2))}$$

$$- \frac{s^2 - \omega_0^2}{-s^4 - 2 \omega_0^2 s^2 - \omega_0^4}$$

Inverse Laplace transform

$$\text{laplace}(t \cdot \cos(\omega_0 \cdot t) \cdot \text{Heaviside}(t), t, s)$$

$$\frac{s^2 - \omega_0^2}{(s^2 + \omega_0^2)^2}$$

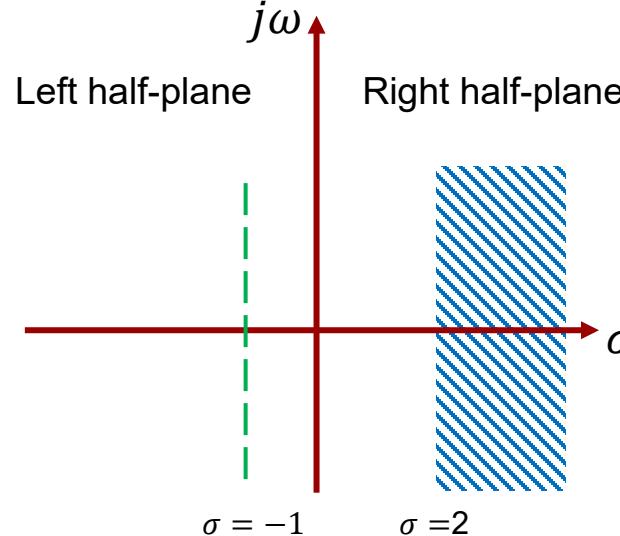
Problem 6.1 (6.1-1) Laplace transform of various functions (sol)

By direct integration [Eq. (6.8b)] find the Laplace transforms and the region of convergence of the following functions.

$$(d) (e^{2t} - 2e^{-t}) u(t)$$

$$\int_{0_-}^{\infty} (e^{2t} - 2e^{-t}) e^{-st} dt = \int_{0_-}^{\infty} e^{(2-s)t} dt - 2 \int_{0_-}^{\infty} e^{-(s+1)t} dt$$

$$\int_{0_-}^{\infty} e^{(2-s)t} dt = \left[-\frac{e^{-(s-2)t}}{s-2} \right]_0^{\infty} = \frac{1}{s-2} - \lim_{t \rightarrow \infty} \frac{e^{-(s-2)t}}{s-2} = \frac{1}{s-2}, \operatorname{Re}\{s\} > 2$$



$$\int_{0_-}^{\infty} e^{-(s+1)t} dt = \left[-\frac{e^{-(s+1)t}}{s+1} \right]_0^{\infty} = \frac{1}{s+1} - \lim_{t \rightarrow \infty} \frac{e^{-(s+1)t}}{s+1} = \frac{1}{s+1}, \operatorname{Re}\{s\} > -1$$

$$\int_{0_-}^{\infty} (e^{2t} - 2e^{-t}) e^{-st} dt = \frac{1}{s-2} - \frac{2}{s+1}, \operatorname{Re}\{s\} > 2$$

Problem 6.1 (6.1-1) Laplace transform of various functions (sol)

Problem 6.1 d

Find the integrand

(d) $(e^{2t} - 2e^{-t}) u(t)$

$$\int e^{(2-s)\cdot t} dt$$

$$-\frac{e^{(-2+s)t}}{-2+s}$$

$$\int e^{-(s+1)\cdot t} dt$$

$$-\frac{e^{-(s+1)t}}{s+1}$$

$$\text{laplace}\left((e^{2\cdot t} - 2\cdot e^{-t}) \cdot \text{Heaviside}(t), t, s\right)$$

$$-\frac{-5+s}{(-2+s)(s+1)}$$

$$\text{convert}\left(-\frac{-5+s}{(-2+s)(s+1)}, \text{parfrac}, s\right)$$

$$\frac{1}{-2+s} - \frac{2}{s+1}$$

Problem 6.1 (6.1-1) Laplace transform of various functions (sol)

By direct integration [Eq. (6.8b)] find the Laplace transforms and the region of convergence of the following functions.

$$(a) u(t) - u(t - 1)$$

$$\begin{aligned}\int_{0_-}^{\infty} (u(t) - u(t - 1)) e^{-st} dt &= \int_{0_-}^1 e^{-st} dt \\ &= \left[-\frac{1}{s} e^{-st} \right]_0^1 = -\frac{1}{s} e^{-s} + \frac{1}{s} \\ &= \frac{1}{s} (1 - e^{-s})\end{aligned}$$

There are no convergence conditions. The region of convergence is the entire s -plane.

Problem 6.1 a

Find the integrand

(a) $u(t) - u(t - 1)$

$$\int_0^{\infty} (v(t) - v(t - 1) \cdot e^{-s \cdot t}) dt$$

$$\lim_{t \rightarrow \infty} \left(\frac{e^{-s \cdot t} \text{Heaviside}(t - 1)}{s} - \frac{\text{Heaviside}(t - 1) e^{-s}}{s} + t \text{Heaviside}(t) \right)$$

$$\text{laplace}(v(t) - v(t - 1), t, s)$$

$$\frac{1 - e^{-s}}{s}$$

Problem 6.3 (6.1-4): Inverse Laplace transform and partial fraction expansion

Find the inverse Laplace transform of these functions.

$$(a) \frac{2s + 5}{s^2 + 5s + 6}$$

$$(b) \frac{3s + 5}{s^2 + 4s + 13}$$

$$(c) \frac{(s + 1)^2}{s^2 - s - 6}$$

$$(d) \frac{5}{s^2(s + 2)}$$

$$(e) \frac{2s + 1}{(s + 1)(s^2 + 2s + 2)}$$

Use Maple to test if partial fraction expansion is possible. If it is, use Maple to obtain the inverse Laplace transform using the results of the partial fraction expansion. Check each result by comparing with table 6.1.

Problem 6.3 a

Problem 6.1.4 in latest book

$$\text{convert}\left(\frac{2 \cdot s + 5}{s^2 + 5 \cdot s + 6}, \text{parfrac}, s\right)$$

$$\frac{1}{s+2} + \frac{1}{s+3}$$

$$\text{invlaplace}\left(\frac{1}{s+2} + \frac{1}{s+3}, s, t\right)$$

$$e^{-2t} + e^{-3t}$$

Entry 5 in table 6.1

Find the inverse Laplace transform of these functions.

$$(b) \frac{3s + 5}{s^2 + 4s + 13}$$

Problem 6.3 b Problem 6.1.4 in latest book

$$\text{convert}\left(\frac{3 \cdot s + 5}{s^2 + 4 \cdot s + 13}, \text{parfrac}, s\right)$$

$$\frac{3 s + 5}{s^2 + 4 s + 13}$$

$$\text{invlaplace}\left(\frac{3 s + 5}{s^2 + 4 s + 13}, s, t\right)$$

$$\frac{e^{-2 t} (9 \cos(3 t) - \sin(3 t))}{3}$$

Entry 10d in table 6.1

Problem 6.3 (6.1-3): Inverse Laplace transform and partial fraction expansion (sol)

Find the inverse Laplace transform of these functions.

$$(c) \frac{(s+1)^2}{s^2 - s - 6}$$

Problem 6.3 c

$$\text{convert}\left(\frac{(s+1)^2}{s^2 - s - 6}, \text{parfrac}, s\right)$$

$$1 + \frac{16}{5(s-3)} - \frac{1}{5(s+2)}$$

$$\text{invlaplace}\left(1 + \frac{16}{5(s-3)} - \frac{1}{5(s+2)}, s, t\right)$$

$$\text{Dirac}(t) + \frac{16 e^{3t}}{5} - \frac{e^{-2t}}{5}$$

Problem 6.1.4 in latest book

Entry 1+5 in table 6.1

Problem 6.3 (6.1-3): Inverse Laplace transform and partial fraction expansion (sol)

Find the inverse Laplace transform of these functions.

$$(d) \frac{5}{s^2(s+2)}$$

Problem 6.3 d

$$\text{convert}\left(\frac{5}{s^2 \cdot (s+2)}, \text{parfrac}, s\right)$$

$$\frac{5}{4(s+2)} + \frac{5}{2s^2} - \frac{5}{4s}$$

$$\text{invlaplace}\left(\frac{5}{4(s+2)} + \frac{5}{2s^2} - \frac{5}{4s}, s, t\right)$$

$$\frac{5e^{-2t}}{4} + \frac{5t}{2} - \frac{5}{4}$$

$\cdot u(t)$ implicit

Entry 2+3+5 in table 6.1

Find the inverse Laplace transform of these functions.

$$(e) \frac{2s + 1}{(s + 1)(s^2 + 2s + 2)}$$

Problem 6.3 e Problem 6.1.4 in latest book

$$\text{convert}\left(\frac{2 \cdot s + 1}{(s + 1) \cdot (s^2 + 2 \cdot s + 2)}, \text{parfrac}, s\right)$$
$$\frac{s + 3}{s^2 + 2 s + 2} - \frac{1}{s + 1}$$

$$\text{invlaplace}\left(\frac{s + 3}{s^2 + 2 s + 2} - \frac{1}{s + 1}, s, t\right)$$
$$e^{-t} (\cos(t) + 2 \sin(t) - 1)$$

Entry 5 + 10d in table 6.1

22050 Continuous-Time Signals and Linear Systems

Kaj-Åge Henneberg

L08

Solving differential equation and analyzing electric
circuits using Laplace Theorems

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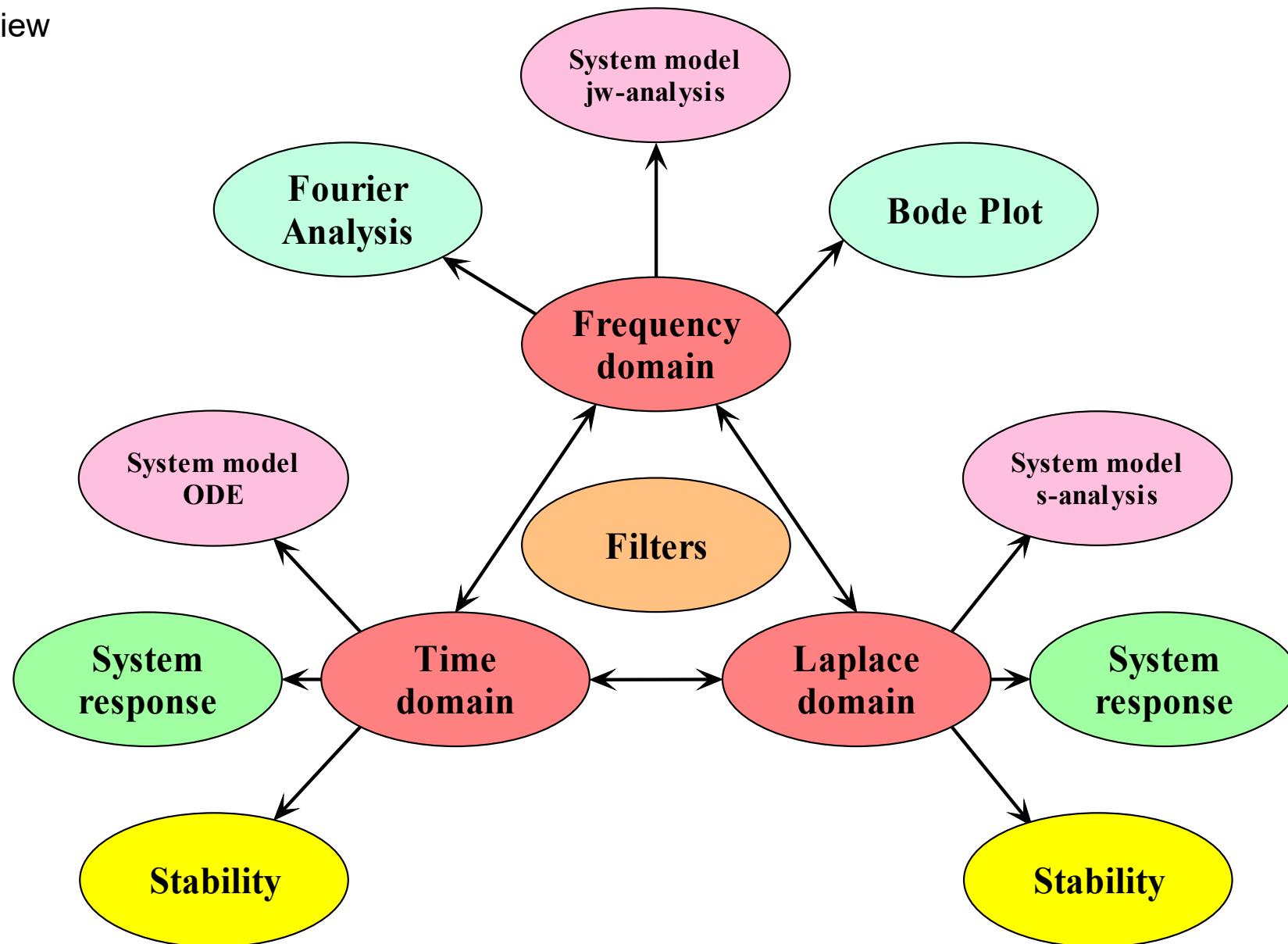
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Course overview



6.3 Solutions to differential equation using Laplace transform

6.4 Analysis of electrical circuits using the Laplace transform

Time scaling:

$$x(at) \leftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right), a > 0$$

Time shift:

$$x(t - t_0) \leftrightarrow e^{-st_0} X(s), t_0 \geq 0$$

Frequency shift theorem:

$$x(t)e^{s_0 t} \leftrightarrow X(s - s_0)$$

Convolution theorem:

$$x_1(t) * x_2(t) \leftrightarrow X_1(s) \cdot X_2(s)$$

Differentiation theorem:

$$\frac{dx}{dt} \leftrightarrow sX(s) - x(0_-)$$

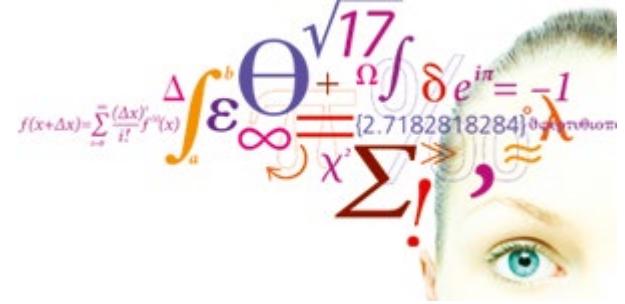
$$\frac{d^2x}{dt^2} \leftrightarrow s^2X(s) - sx(0_-) - \dot{x}(0_-)$$

Integration theorem:

$$\int_{0_-}^t x(\tau)d\tau \leftrightarrow \frac{X(s)}{s}$$

$$\int_{-\infty}^t x(\tau)d\tau \leftrightarrow \frac{X(s)}{s} + \frac{\int_{-\infty}^{0_-} x(\tau)d\tau}{s}$$

Next topic



Solving differential equations using Laplace transformation

Video 1

Solve for the zero-input response and the zero-state response and add them together to get the total solution.

$$\ddot{y} + 4\dot{y} + 4y = 2\ddot{x}, \quad y(0_-) = 1, \dot{y}(0_-) = -1$$

$$x(t) = u(t)$$

Transform pairs:

$$y(t) \leftrightarrow Y(s) \quad \text{and} \quad \dot{y}(t) \leftrightarrow sY(s) - y(0_-)$$

$$\ddot{y}(t) \leftrightarrow s^2Y(s) - sy(0_-) - \dot{y}(0_-)$$

$$\ddot{x}(t) \leftrightarrow s^2X(s) \quad \text{and} \quad \dot{x}(0_-) = \ddot{x}(0_-) = 0 : \text{causal signal}$$

$$[s^2 + 4s + 4]Y(s) = 2s^2X(s) + \underbrace{sy(0_-) + \dot{y}(0_-) + 4y(0_-)}_{\text{initial condition terms}}$$

$$Y(s) = \frac{2s^2}{s^2 + 4s + 4} \cdot X(s) + \frac{sy(0_-) + \dot{y}(0_-) + 4y(0_-)}{s^2 + 4s + 4}$$

$$Y(s) = \underbrace{\frac{P(s)}{Q(s)} \cdot X(s)}_{\text{zero-state}} + \underbrace{\frac{I(s)}{Q(s)}}_{\text{zero-input}}$$

Zero-input response using Laplace Transformation

Before going any further we will use the initial and final value theorems to predict $y_{zi}(0_+)$ and $y_{zi}(t \rightarrow \infty)$.

$$Y_{zi}(s) = \frac{I(s)}{Q(s)} = \frac{sy(0_-) + \dot{y}(0_-) + 4y(0_-)}{s^2 + 4s + 4} = \frac{s + 4 - 1}{s^2 + 4s + 4} = \frac{s + 3}{(s + 2)^2}$$

$$y_{zi}(0_+) = \lim_{s \rightarrow \infty} s Y_{zi}(s) = \lim_{s \rightarrow \infty} s \frac{s + 3}{(s + 2)^2} = \frac{s^2}{s^2} = 1$$

$$\lim_{t \rightarrow \infty} y_{zi}(t) = \lim_{s \rightarrow 0} s Y_{zi}(s) = \lim_{s \rightarrow 0} s \frac{s + 3}{(s + 2)^2} = \frac{0}{4} = 0$$

Zero-input response using
Laplace Transformation

$$Y_{zi}(s) = \frac{s+3}{(s+2)^2} = \frac{k_1}{(s+2)^2} + \frac{k_2}{s+2}$$

Partial fraction expansion

$$k_1 = (s+2)^2 \frac{(s+3)}{(s+2)^2} \Big|_{s=-2} - (s+2)^2 \frac{k_2}{s+2} \Big|_{s=-2} = (s+3) \Big|_{s=-2} = 1$$

Pick an arbitrary value for s
not used before.

$$s = 0 \quad \frac{s+3}{(s+2)^2} = \frac{1}{(s+2)^2} + \frac{k_2}{s+2} \quad \frac{3}{4} = \frac{1}{4} + \frac{k_2}{2} \Rightarrow k_2 = 1$$

$$Y_{zi}(s) = \frac{s+3}{(s+2)^2} = \frac{1}{(s+2)^2} + \frac{1}{s+2}$$

Zero-input response: $y_{zi}(t) = te^{-2t}u(t) + e^{-2t}u(t) = (1+t)e^{-2t}u(t)$

Checking with our predictions:

$$y_{zi}(0_+) = 1 \qquad \lim_{t \rightarrow \infty} y_{zi}(t) = 0$$

Zero-state response using
Laplace Transformation:

Before going any further we
will use the initial and final
value theorems to predict
 $y_{zs}(0_+)$ and $y_{zs}(t \rightarrow \infty)$.

$$Y_{zs}(s) = \frac{P(s)}{Q(s)} \cdot X(s) = H(s)X(s) \quad x(t) = u(t) \leftrightarrow X(s) = \frac{1}{s}$$

$$Y_{zs}(s) = \frac{2s^2}{(s + 2)^2} \cdot \frac{1}{s} = \frac{2s}{(s + 2)^2}$$

$$y_{zs}(0_+) = \lim_{s \rightarrow \infty} s Y_{zs}(s) = \lim_{s \rightarrow \infty} s \frac{2s}{(s + 2)^2} = \frac{2s^2}{s^2} = 2$$

$$\lim_{t \rightarrow \infty} y_{zs}(t) = \lim_{s \rightarrow 0} s Y_{zs}(s) = \lim_{s \rightarrow 0} s \frac{2s}{(s + 2)^2} = \frac{0}{4} = 0$$

Partial fraction expansion

$$Y_{zs}(s) = \frac{2s^2}{(s+2)^2} \cdot \frac{1}{s} = \frac{2s}{(s+2)^2} = \frac{k_1}{(s+2)^2} + \frac{k_2}{s+2}$$

$$k_1 = 2s \Big|_{s=-2} = -4 \quad k_2: s \stackrel{\text{def}}{=} 0 \Rightarrow 0 = \frac{-4}{4} + \frac{k_2}{2} \Rightarrow k_2 = 2$$

$$Y_{zs}(s) = \frac{2s}{(s+2)^2} = \frac{-4}{(s+2)^2} + \frac{2}{s+2}$$

$$y_{zs}(t) = -4te^{-2t}u(t) + 2e^{-2t}u(t)$$

Zero-state response:

$$y_{zs}(t) = (2 - 4t)e^{-2t}u(t)$$

Checking with our predictions:

$$y_{zs}(0_+) = 2 \quad \lim_{t \rightarrow \infty} y_{zs}(t) = 0$$

Zero-input response:

$$y_{zi}(t) = te^{-2t}u(t) + e^{-2t}u(t) = (1+t)e^{-2t}u(t)$$

Zero-state response:

$$y_{zs}(t) = (2 - 4t)e^{-2t}u(t)$$

Using the "Signals and Linear Systems approach", we obtain separate expressions for the response to the initial conditions and for the response to the input signal. This way we can state precisely how the input signal changes the output signal.

$$y(t) = \underbrace{y_{zi}(t)}_{\text{zero-input}} + \underbrace{y_{zs}(t)}_{\text{zero-state}} = \underbrace{(1+t)e^{-2t}u(t)}_{\text{zero-input}} + \underbrace{(2-4t)e^{-2t}u(t)}_{\text{zero-state}}$$

If we had used the classical approach where the initial conditions are given at 0_+ , then we would not be able to distinguish the two responses.

$$y(t) = (3 - 3t)e^{-2t}u(t)$$

Impulse Response using Laplace Transformation

Find the impulse response $h(t)$ using Laplace transform.

$$Y_{zs}(s) = \frac{P(s)}{Q(s)} \cdot X(s) = H(s)X(s)$$

$$x(t) = \delta(t) \leftrightarrow X(s) = 1$$

$$Y_{zs}(s) = H(s) = \frac{P(s)}{Q(s)}$$

Example:

$$H(s) = \frac{2s^2}{(s+2)^2}$$

Polynomial division

$$\begin{array}{r} 2 \\ (s+2)^2 \quad \boxed{2s^2} \\ \hline 2s^2 + 8s + 8 \\ \hline -8s - 8 \end{array}$$

$$\frac{2s^2}{(s+2)^2} = 2 - 8 \frac{s+1}{(s+2)^2}$$

$$H(s) = \frac{2s^2}{(s+2)^2} = 2 - 8 \frac{s+1}{(s+2)^2}$$

Impulse Response using Laplace Transformation

Predicting initial and final values:

$$H(s) = \frac{2s^2}{(s+2)^2} = 2 - 8 \frac{s+1}{(s+2)^2}$$

$$h(0_+) = \lim_{s \rightarrow \infty} s H(s) = \cancel{\lim_{s \rightarrow \infty} 2s} - \lim_{s \rightarrow \infty} 8s \frac{s+1}{(s+2)^2} = -8 \lim_{s \rightarrow \infty} \frac{s^2}{s^2} = -8$$

If the limit on the right-hand side exists.

$$\lim_{t \rightarrow \infty} h(t) = \lim_{s \rightarrow 0} s H(s) = \lim_{s \rightarrow 0} 2s - \lim_{s \rightarrow 0} 8s \frac{s+1}{(s+2)^2} = 0$$

$$\begin{aligned} H(s) &= \frac{2s^2}{(s+2)^2} = 2 - 8 \frac{s+1}{(s+2)^2} \\ &= 2 + \frac{8}{(s+2)^2} - \frac{8}{s+2} \end{aligned}$$

Partial fraction expansion:

Looking up the inverse transforms in table 6.1:

$$h(t) = 2\delta(t) + 8te^{-2t}u(t) - 8e^{-2t}u(t)$$

$$h(t) = 2\delta(t) + 8(t-1)e^{-2t}u(t)$$

Checking with our predictions:

$$h(0_+) = -8 \quad \lim_{t \rightarrow \infty} h(t) = 0$$

Observe:

The power of s in the numerator is equal to the power of s in the denominator. A clear sign that this is a **highpass** filter. And we got the $\delta(t)$ function we expected. The initial value theorem predicts the value at 0_+ . At $t = 0_+$ the impulse has vanished.

Solve for the zero-input response and the zero-state response and add them together to get the total solution.

$$\ddot{y} + 4\dot{y} + 3y = 2\dot{x} + x, \quad y(0_-) = 1, \dot{y}(0_-) = 2$$

$$x(t) = u(t)$$

Transform pairs:

$$y(t) \leftrightarrow Y(s) \text{ and } \dot{y}(t) \leftrightarrow sY(s) - y(0_-)$$

$$\ddot{y}(t) \leftrightarrow s^2Y(s) - sy(0_-) - \dot{y}(0_-)$$

$$\dot{x}(t) \leftrightarrow sX(s) \text{ and } \dot{x}(0_-) = \ddot{x}(0_-) = 0 : \text{causal signal}$$

What is the Laplace transformation of the differential equation?

$$[s^2 + 4s + 3]Y(s) = (2s + 1)X(s) + sy(0_-) + \dot{y}(0_-) + 4y(0_-)$$

$$Y(s) = \frac{2s + 1}{s^2 + 4s + 3} \cdot X(s) + \frac{sy(0_-) + \dot{y}(0_-) + 4y(0_-)}{s^2 + 4s + 3}$$

$$Q(D)y(t) = P(D)x(t)$$

$$Y(s) = \underbrace{\frac{P(s)}{Q(s)} \cdot X(s)}_{\text{zero-state}} + \underbrace{\frac{I(s)}{Q(s)}}_{\text{zero-input}}$$

Before going any further we will use the initial and final value theorems to predict $y_{zi}(0_+)$ and $y_{zi}(t \rightarrow \infty)$.

$$Y_{zi}(s) = \frac{I(s)}{Q(s)} = \frac{s + 6}{s^2 + 4s + 3}$$

What is the initial value?

$$y_{zi}(0_+) = \lim_{s \rightarrow \infty} s Y_{zi}(s) = \lim_{s \rightarrow \infty} s \frac{s + 6}{s^2 + 4s + 3} = \frac{s^2}{s^2} = 1$$

What is the final value?

$$\lim_{t \rightarrow \infty} y_{zi}(t) = \lim_{s \rightarrow 0} s Y_{zi}(s) = \lim_{s \rightarrow 0} s \frac{s + 6}{s^2 + 4s + 3} = \frac{0}{3} = 0$$

Zero-input response:

What do we do next?

$$Y_{zi}(s) = \frac{I(s)}{Q(s)} = \frac{s+6}{s^2 + 4s + 3} = \frac{s+6}{(s+1)(s+3)} = \frac{k_1}{s+1} + \frac{k_2}{s+3}$$

What do we do next?

$$k_1 = \left. \frac{(s+6)}{(s+3)} \right|_{s=-1} = \frac{5}{2} \quad k_2 = \left. \frac{(s+6)}{(s+1)} \right|_{s=-3} = -\frac{3}{2}$$

What do we do next?

$$Y_{zi}(s) = \frac{1}{2} \left(\frac{5}{s+1} - \frac{3}{s+3} \right)$$

What do we do next?

$$y_{zi}(t) = \left(\frac{5}{2} e^{-t} - \frac{3}{2} e^{-3t} \right) u(t)$$

Zero-state response:

$$Y_{zs}(s) = \frac{P(s)}{Q(s)} \cdot X(s) = H(s)X(s) \quad x(t) = u(t) \leftrightarrow X(s) = \frac{1}{s}$$

What do we do next?

$$Y_{zs}(s) = \frac{2s+1}{s^2+4s+3} \cdot \frac{1}{s} = \frac{2s+1}{s(s+1)(s+3)} = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+3}$$

What do we do next?

$$k_1 = \left. \frac{2s+1}{(s+1)(s+3)} \right|_{s=0} = \frac{1}{3} \quad k_2 = \left. \frac{2s+1}{s(s+3)} \right|_{s=-1} = \frac{1}{2} \quad k_3 = \left. \frac{2s+1}{s(s+1)} \right|_{s=-3} = \frac{-5}{6}$$

What do we do next?

$$Y_{zs}(s) = \frac{1/3}{s} + \frac{1/2}{s+1} - \frac{5/6}{s+3}$$

What do we do next?

$$y_{zs}(t) = \left(\frac{1}{3} + \frac{1}{2}e^{-t} - \frac{5}{6}e^{-3t} \right) u(t)$$

Total solution:

$$\begin{aligned}y(t) &= y_{zi}(t) + y_{zs}(t) \\&= \left(\frac{5}{2}e^{-t} - \frac{3}{2}e^{-3t}\right)u(t) + \left(\frac{1}{3} + \frac{1}{2}e^{-t} - \frac{5}{6}e^{-3t}\right)u(t) \\&= \left(\frac{1}{3} + 3e^{-t} - \frac{7}{3}e^{-3t}\right)u(t)\end{aligned}$$

By Laplace transformation of differential equations, we have shown:

$$Q(D) y_{zs}(t) = P(D) x(t)$$

$$Q(s) Y_{zs}(s) = P(s) X(s) + I(s)$$

$$Y_{zs}(s) = \frac{P(s)}{Q(s)} \cdot X(s)$$

We define the **transfer function** $H(s)$ as the zero-state response divided by the input signal:

The roots of the characteristic equation are the poles of the transfer function (only if common factors have not been cancelled out):

The transfer function is the Laplace transform of the impulse response:

$$H(s) \stackrel{\text{def}}{=} \frac{Y_{zs}(s)}{X(s)}$$

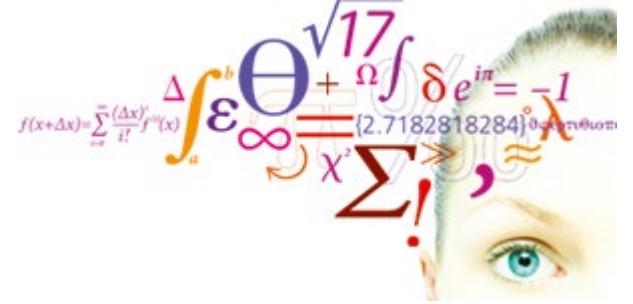
$$H(s) = \frac{P(s)}{Q(s)}$$

$$y_{zs}(t) = h(t) * x(t)$$

$$Y_{zs}(s) = H(s) \cdot X(s)$$

$$h(t) \leftrightarrow H(s)$$

Next topic



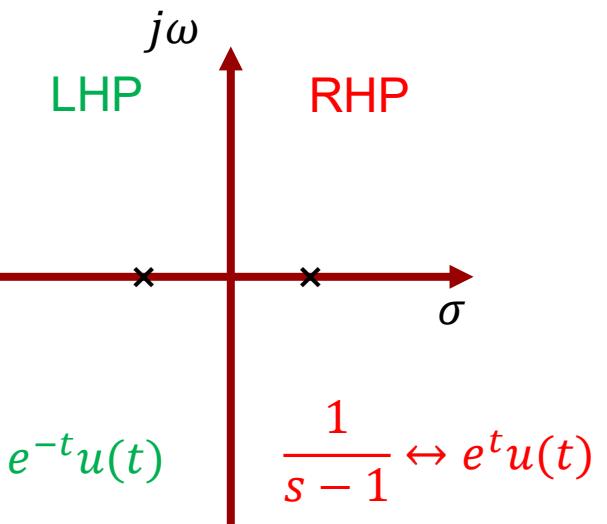
Stability analysis using Laplace transformation

We can now state the conditions of stability in terms of the poles of the transfer function.

1. An LTIC system is **asymptotically stable** if and only if all the poles of the transfer function $H(s)$ are in the left-half-plane (**LHP**). Poles in the LHP may be repeated or unrepeated.
2. An LTIC system is **unstable** if and only if either one or both of the following conditions are met: (i) at least one pole of $H(s)$ is in the **RHP**, (ii) there are repeated poles of $H(s)$ on the imaginary axis.
3. An LTIC system is **marginally stable** if and only if there are no poles of $H(s)$ in the RHP, and there are some *unrepeated* poles on the imaginary axis.

Direct quote from Lathi and Green, p. 555.

$$\frac{1}{s + a} \leftrightarrow e^{-at}u(t)$$



Observe:

A zero may cancel a *pole* in the transfer function.

However, zeros have no impact on the zero-input response. A pole in the RHP will thus give an unbounded response to a non-zero initial state, even if there is a superimposed zero.

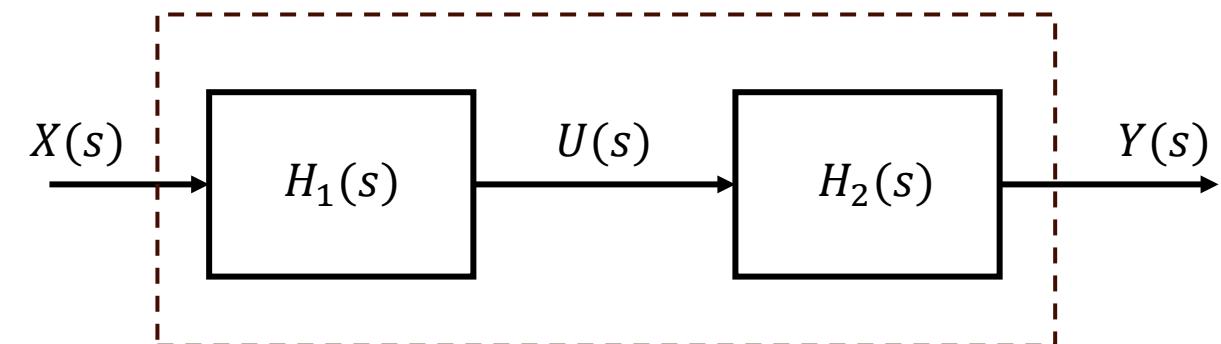
$$Y(s) = \frac{P(s)}{Q(s)} X(s) + \frac{I(s)}{Q(s)}$$

$$Y(s) = \underbrace{\frac{(s+1)(s-2)}{(s+2)(s-2)} X(s)}_{\substack{\text{zero-state} \\ \text{response}}} + \underbrace{\frac{I(s)}{(s+2)(s-2)}}_{\substack{\text{zero input} \\ \text{response}}}$$

$$Y(s) = \underbrace{\frac{(s+1)}{(s+2)} X(s)}_{\text{stable}} + \underbrace{\frac{I(s)}{(s+2)(s-2)}}_{\text{unstable}}$$

Stability

Here are shown a system, which, seen from the outside, appears to be one system, but in fact is composed of two subsystems in series.



$$U(s) = \frac{P_1(s)}{Q_1(s)} X(s) + \frac{I_1(s)}{Q_1(s)}$$

$$Y(s) = \frac{P_2(s)}{Q_2(s)} U(s) + \frac{I_2(s)}{Q_2(s)}$$

$$Y(s) = \frac{P_2(s)}{Q_2(s)} \left(\frac{P_1(s)}{Q_1(s)} X(s) + \frac{I_1(s)}{Q_1(s)} \right) + \frac{I_2(s)}{Q_2(s)}$$

$$Y(s) = \left(\frac{P_2(s)}{Q_2(s)} \frac{P_1(s)}{Q_1(s)} X(s) + \frac{P_2(s)}{Q_2(s)} \frac{I_1(s)}{Q_1(s)} \right) + \frac{I_2(s)}{Q_2(s)}$$

If we monitor the input and output signals, we may come to the conclusion that the signals are bounded in amplitude and hence that the system is stable in the Bounded Input - Bounded Output (BIBO) sense.

Let us look at an example.

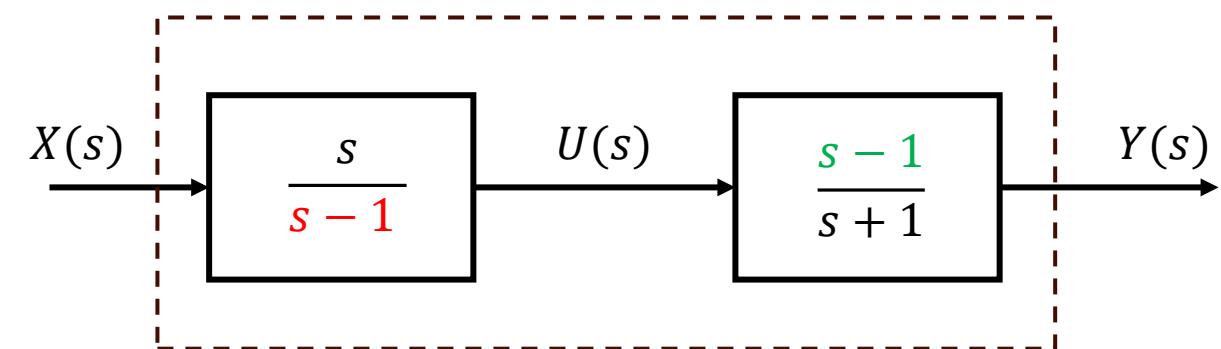
Stability by compensation

Transfer function $H_1(s)$ has a pole ($s - 1$) in the right-half plane. Hence subsystem 1 is unstable. But what about the composite system?

Subsystem 2 has a zero ($s - 1$) that attempts to cancel the troublesome pole of subsystem 1.

Doing the math, we end up with an equation which features only a pole ($s + 1$) in the left half plane. The impulse response of the composite system is $e^{-t}u(t)$, hence the composite system appears to be asymptotic stable (only poles in the half-plane).

However, things are much worse than they appear.



$$U(s) = \frac{s}{s-1} X(s) + \frac{I_1(s)}{s-1}$$

$$Y(s) = \frac{s-1}{s+1} U(s) + \frac{I_2(s)}{s+1}$$

$$Y(s) = \frac{s-1}{s+1} \left(\frac{s}{s-1} X(s) + \frac{I_1(s)}{s-1} \right) + \frac{I_2(s)}{s+1}$$

$$Y(s) = \left(\cancel{\frac{s-1}{s+1}} \frac{s}{\cancel{s-1}} X(s) + \cancel{\frac{s-1}{s+1}} \frac{I_1(s)}{\cancel{s-1}} \right) + \frac{I_2(s)}{s+1}$$

$$Y(s) = \left(\frac{s}{s+1} X(s) + \frac{I_1(s)}{s+1} \right) + \frac{I_2(s)}{s+1}$$

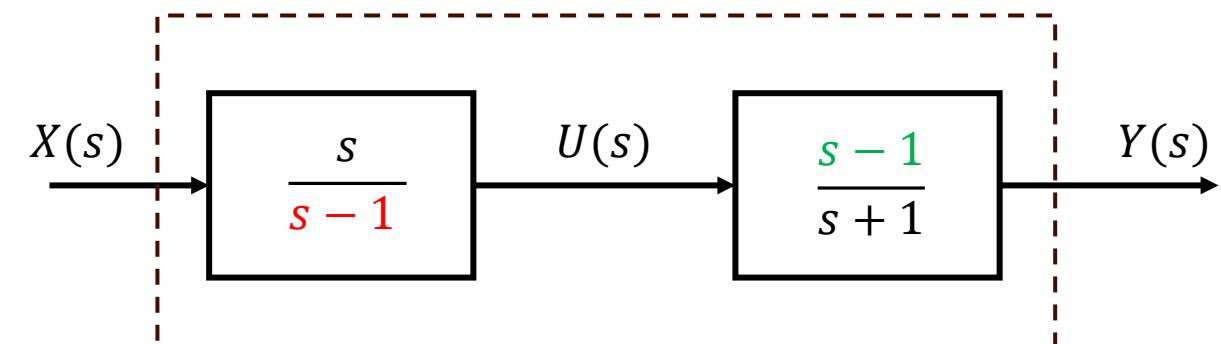
Stability by compensation

Transfer function $H_1(s)$ has a pole ($s - 1$) in the right-half plane. Hence subsystem 1 has an impulse response $e^t u(t)$. This behavior is independent of subsystem 2. Subsystem 2 cannot prevent the internal signal $u(t)$ from growing exponentially.

We notice also that $u(t)$ will grow exponentially whether stimulated by the input signal or simply from the presence of internal energy stored in subsystem 1.

In an electronic system, subsystem 1 would enter into a mode of saturation, thus clipping $u(t)$ at the supply voltage. Subsystem 2 cannot rectify this, but it will give the erroneous impression of a bounded output.

If subsystem 1 were a mechanical system, it would likely self-destruct.

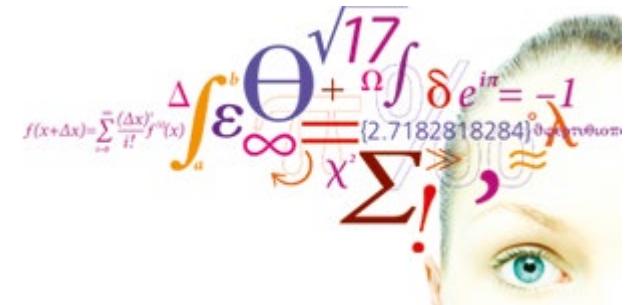


$$U(s) = \frac{s}{s - 1} X(s) + \frac{I_1(s)}{s - 1}$$

If a system $H_1(s)$ has an undesired performance, adding a compensating system $H_2(s)$ may in many cases improve performance.

This does not apply for an unstable system.

Next topic



Analyzing electric circuits using Laplace transformation

Video 2

The capacitor:

The voltage drop over a capacitor is the sum of the initial voltage and the voltage due to the charging current.

$$\int_{-\infty}^t x(\tau) d\tau \leftrightarrow \frac{X(s)}{s} + \frac{\int_{-\infty}^{0_-} x(\tau) d\tau}{s}$$

$$v_C(t) = \frac{1}{C} \int_{0_-}^t i_C(\tau) d\tau + v_C(0_-)$$

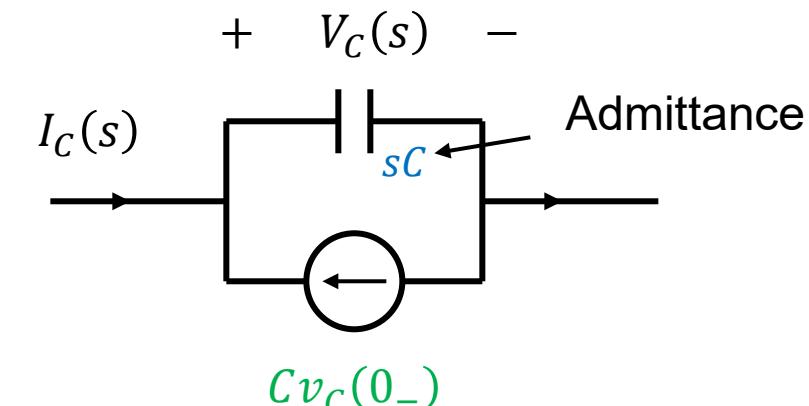
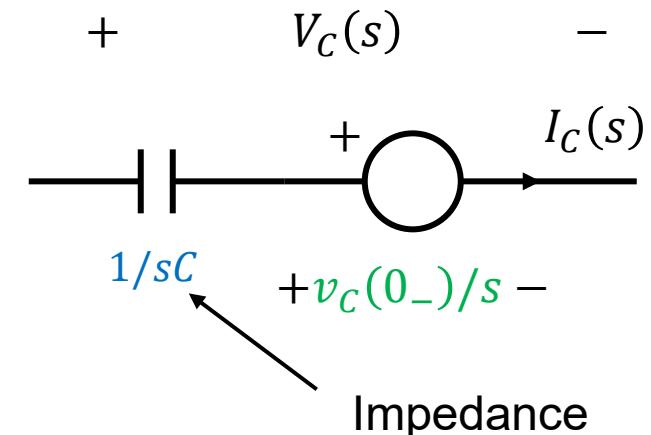
$$V_C(s) = \frac{1}{sC} I_C(s) + \frac{v_C(0_-)}{s}$$

Solving for the total current, we get a sum of two terms.

$$i_C(t) = C \frac{dv_C}{dt}$$

$$I_C(s) = sCV_C(s) - Cv_C(0_-)$$

We can think of the current source (green term) as a discharge current (or leakage current) reflecting the initial charged state of the capacitor. The blue term is the charging current needed to change the voltage at a given rate, if there were no initial charged state.



The inductor:

The voltage drop over an inductor is proportional to the rate of change in current.

$$v_L(t) = L \frac{di_L}{dt}$$

$$V_L(s) = sLI_L(s) - Li_L(0_-)$$

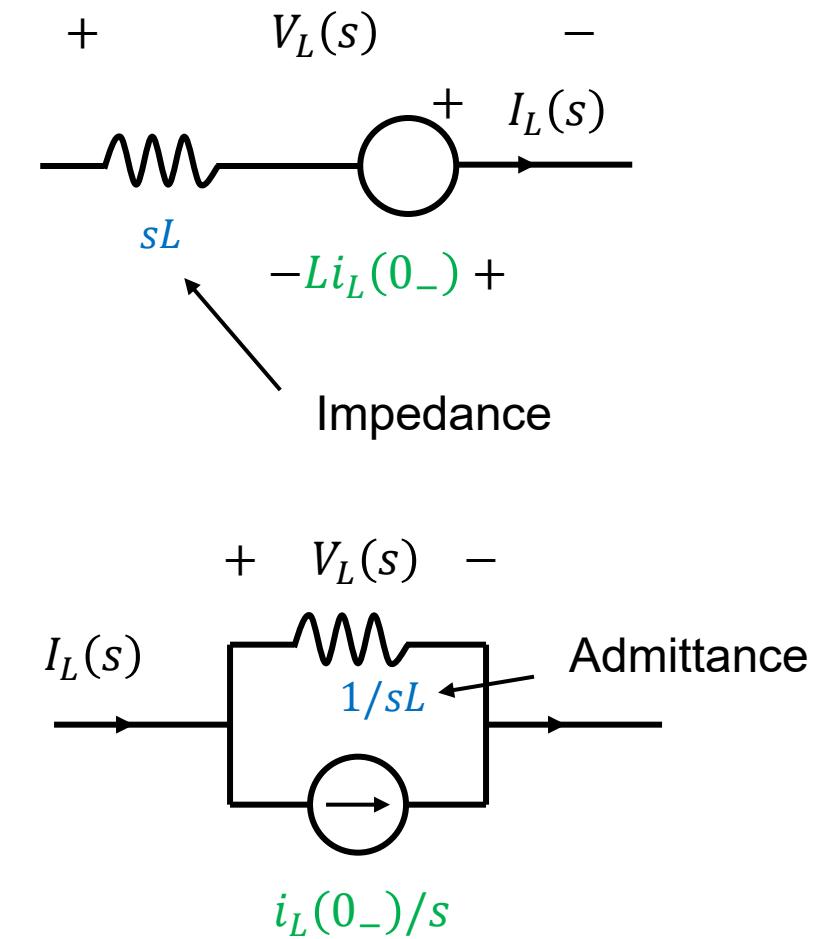
$$i_L(t) = \frac{1}{L} \int_{0_-}^t v_L(\tau) d\tau + i_L(0_-)$$

$$I_L(s) = \frac{1}{sL} V_L(s) + \frac{i_L(0_-)}{s}$$

Solving for the total current, we get a sum of two terms.

Hence, we obtain a model with two components in parallel.

The source (green) term represents the initial magnetized state of the inductor. An increase in current increases magnetic energy. A release of stored magnetic energy induces a current. The green source represents the release of an initial magnetic energy stored in the inductor.



For both the capacitor and the inductor we have two models (parallel or serial) to choose from. They are equally valid, but one may be more convenient to use than the other.

Nodal analysis: Here we add currents together. In this case the models with parallel components may be preferred because the two branches in the model are both currents.

$$I_C(s) = sC V_C(s) - Cv_C(0_-)$$

$$I_L(s) = \frac{1}{sL} V_L(s) + \frac{i_L(0_-)}{s}$$

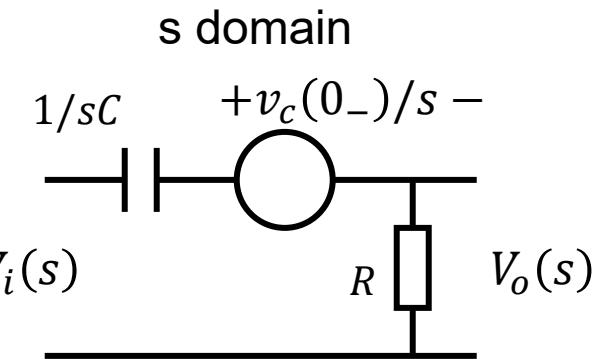
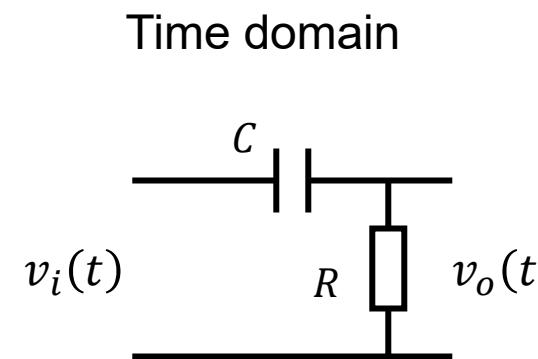
Mesh analysis: Here we add up voltage change in a loop. In this case the series model models may be preferred, because it adds voltages.

$$V_C(s) = \frac{1}{sC} I_C(s) + \frac{v_C(0_-)}{s}$$

$$V_L(s) = sL I_L(s) - Li_L(0_-)$$

Circuit Example 1 (using a series model in a nodal analysis, not the smartest way)

Laplace transform this circuit:



Using Kirchhoff's current law and Ohm's law for impedances in the $s -$ domain:

$$\frac{V_i(s) - \left(\frac{v_c(0_-)}{s} + V_o(s) \right)}{1/sC} = \frac{V_o(s)}{R}$$

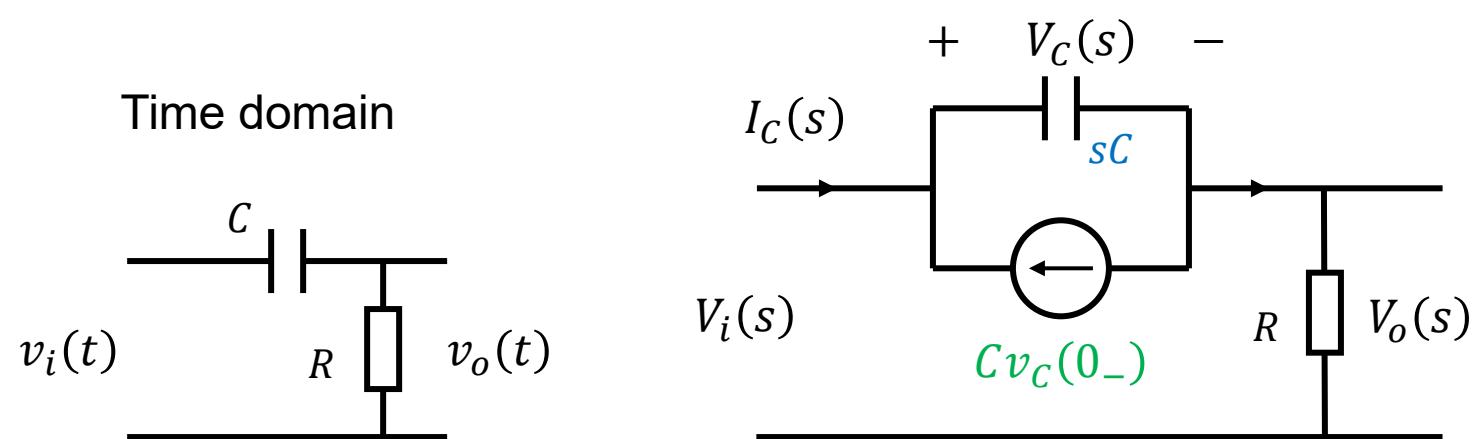
The series model adds a voltage term.

Simplifying and solving for $V_o(s)$:

$$sRCV_i(s) - RCv_c(0_-) = (sRC + 1)V_o(s)$$

We obtain a first order highpass filter.

$$V_o(s) = \underbrace{\frac{s}{s + 1/RC} V_i(s)}_{\text{zero-state}} - \underbrace{\frac{v_c(0_-)}{s + 1/RC}}_{\text{zero-input}}$$

Circuit Example 1 (using a *parallel* model in a nodal analysis, the smartest way)

Using Kirchhoff's current law and Ohm's law for impedances in the s – domain:

$$I_C(s) = sCV_C(s) - Cv_C(0_-)$$

$$sCV_C(s) - Cv_C(0_-) = \frac{V_o(s)}{R}$$

$$sRC(V_i(s) - V_o(s)) - RCv_c(0_-) = V_o(s)$$

$$(sRC + 1)V_o(s) = sRCV_i(s) - RCv_c(0_-)$$

Simplifying and solving for $V_o(s)$:

We obtain a first order highpass filter.

$$V_o(s) = \underbrace{\frac{s}{s + 1/RC} V_i(s)}_{\text{zero-state}} - \underbrace{\frac{v_c(0_-)}{s + 1/RC}}_{\text{zero-input}}$$

Circuit Example 1 continued

Setting $V_i(s) = 2/s$
and $v_C(0_-) = 1$ V:

Before computing the inverse Laplace transforms of the two terms, we would like to use the **initial and final value theorems** to predict the initial and final values of the two terms in the time domain:

How is it possible that the output dies away in response to a step input?

$$V_o(s) = \underbrace{\frac{s}{s + 1/RC} \cdot \frac{2}{s}}_{\text{zero-state}} - \underbrace{\frac{1}{s + 1/RC}}_{\text{zero-input}}$$

Initial values:

$$v_{o,ZI}(0_+) = \lim_{s \rightarrow \infty} s \cdot \frac{-1}{s + 1/RC} = -1$$

$$v_{o,ZS}(0_+) = \lim_{s \rightarrow \infty} s \cdot \frac{2}{s + 1/RC} = 2$$

Final values:

$$\lim_{t \rightarrow \infty} v_{o,ZI}(t) = \lim_{s \rightarrow 0} s \cdot \frac{-1}{s + 1/RC} = 0$$

$$\lim_{t \rightarrow \infty} v_{o,ZS}(t) = \lim_{s \rightarrow 0} s \cdot \frac{2}{s + 1/RC} = 0$$

Circuit Example 1 continued

The results agree with the results of the initial and final value theorems.

$$V_o(s) = \underbrace{\frac{2}{s + 1/RC}}_{\text{zero-state}} - \underbrace{\frac{1}{s + 1/RC}}_{\text{zero-input}}$$

$$v_o(t) = \underbrace{2e^{-t/RC}u(t)}_{\text{zero-state}} - \underbrace{e^{-t/RC}u(t)}_{\text{zero-input}}$$

Initial values:

$$v_{o,ZI}(0_+) = \lim_{s \rightarrow \infty} s \cdot \frac{-1}{s + 1/RC} = -1$$

$$v_{o,ZS}(0_+) = \lim_{s \rightarrow \infty} s \cdot \frac{2}{s + 1/RC} = 2$$

Final values:

$$\lim_{t \rightarrow \infty} v_{o,ZI}(t) = \lim_{s \rightarrow 0} s \cdot \frac{-1}{s + 1/RC} = 0$$

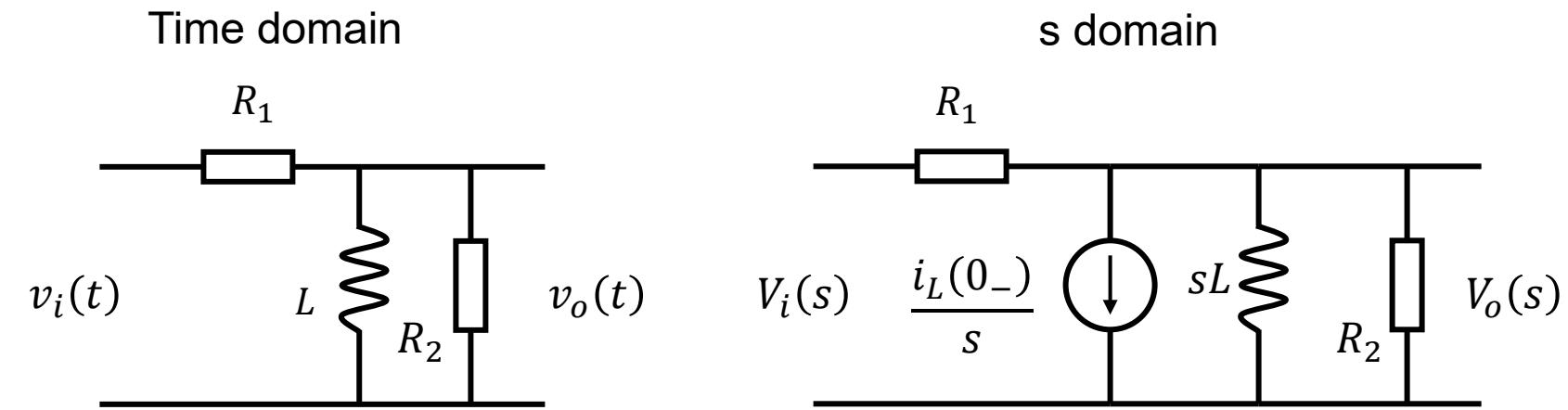
$$\lim_{t \rightarrow \infty} v_{o,ZS}(t) = \lim_{s \rightarrow 0} s \cdot \frac{2}{s + 1/RC} = 0$$

How is it possible that the output dies away in response to a step input?

What kind of circuit is this?

Circuit Example 2 (parallel model in nodal analysis, the smart thing to do)

Laplace transform
this circuit:



Here we only derive the input-output expression in the s domain.

$$\frac{V_i(s) - V_o(s)}{R_1} = \frac{V_o(s)}{sL} + \frac{i_L(0_-)}{s} + \frac{V_o(s)}{R_2}$$

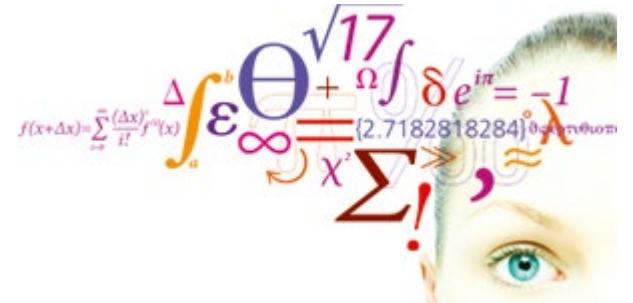
The parallel model gives us a simple term to add.

If we were asked to derive the response to a specific input signal, we would proceed with partial fraction expansion and inverse Laplace transform of the resulting terms.

$$V_o(s) = \frac{sR_1R_2L}{s(R_1 + R_2)L + R_1R_2} \cdot \frac{V_i(s)}{R_1} - \frac{R_1R_2L \cdot i_L(0_-)}{s(R_1 + R_2)L + R_1R_2}$$

$$V_o(s) = \underbrace{\frac{1}{R_1} \cdot \frac{sR_p}{s + R_p/L} \cdot V_i(s)}_{\text{zero-state}} - \underbrace{\frac{R_p \cdot i_L(0_-)}{s + \frac{R_p}{L}}}_{\text{zero-input}}; \quad R_p \stackrel{\text{def}}{=} \frac{R_1R_2}{R_1 + R_2}$$

Next topic



Another example, Notch filter

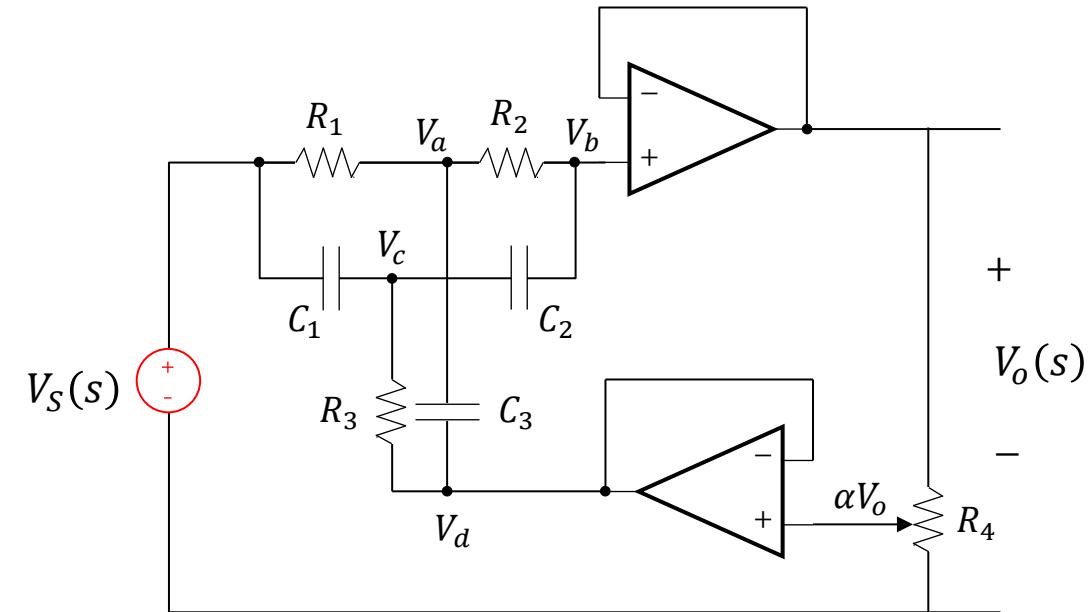
Twin-T notch filter

Deriving transfer function using the current superposition method.

The left-hand side expresses the partial current leaving the node, when all neighboring nodes are 0V. Because all neighboring nodes are at the same voltage, all components to the node are virtually in parallel and their parallel impedance is called the **shunt impedance**.

The right-hand side expresses the partial current entering the node, when the node voltage is zero.

We intend to use a CAS tool to solve the equations, hence we leave them in their most basic form, without inserting constraints.



$$a: \frac{V_a}{Z_a} = \frac{V_S}{R_1} + \frac{V_b}{R_2} + \frac{V_d}{Z_{C_3}}$$

$$\frac{1}{Z_a} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{Z_{C_3}}$$

$$b: \frac{V_b}{Z_b} = \frac{V_a}{R_2} + \frac{V_c}{Z_{C_2}}$$

$$\frac{1}{Z_b} = \frac{1}{R_2} + \frac{1}{Z_{C_2}}$$

$$c: \frac{V_c}{Z_c} = \frac{V_S}{Z_{C_1}} + \frac{V_b}{Z_{C_2}} + \frac{V_d}{R_3}$$

$$\frac{1}{Z_c} = \frac{1}{Z_{C_1}} + \frac{1}{Z_{C_2}} + \frac{1}{R_3}$$

Constraints: $V_b = V_o$ $V_d = \alpha V_o$ $Z_{C_x} = \frac{1}{sC_x}$

Twin-T notch filter

Solving for the transfer function: $H(s) = \frac{V_o(s)}{V_s(s)}$

$$H(s) = \frac{C_1 C_2 C_3 R_1 R_2 R_3 s^3 + ((R_1 + R_2) R_3 C_1 C_2) s^2 + R_3 (C_1 + C_2) s + 1}{C_1 C_2 C_3 R_1 R_2 R_3 s^3 + ((R_1 + R_2) R_3 C_1 C_2 + (1 - \alpha) R_1 R_3 (C_1 + C_2) C_3 + (1 - \alpha) R_1 R_2 C_2 C_3) s^2 + (R_3 (C_1 + C_2) + (1 - \alpha) (R_1 + R_2) C_2 + (1 - \alpha) R_1 C_3) s + 1}$$

Motivated to simplify the design of the Twin-T notch filter, we will make some constraints on the component values:

$$\begin{aligned} R_1 &= R_2 = 2R_3 = 2R \\ C_1 &= C_2 = \frac{C_3}{2} = \frac{C}{2} \end{aligned}$$

This reduces the expression significantly:

$$H(s) = \frac{(RC)^3 s^3 + (RC)^2 s^2 + (RC)s + 1}{(RC)^3 s^3 + (5 - 4\alpha)(RC)^2 s^2 + (5 - 4\alpha)(RC)s + 1}$$

We can seek common factors that cancel out:

$$H(s) = \frac{(RCs + 1)((RC)^2 s^2 + 1)}{(RCs + 1)((RC)^2 s^2 + 4(1 - \alpha)(RC)s + 1)}$$

Twin-T notch filter

We have obtained the transfer function of a **bandstop** filter.

It will have a notch at ω_0 .

A notch filter is required to have a narrow notch. The narrowness of the notch can be controlled by a parameter Q referred to as the **Quality factor** (DK: godhedsfaktor).

The parameter α represents a voltage divider in the form of a potentiometer (R_4). Hence the Q factor can be adjusted by turning R_4 .

$$H_{BS}(s) = \frac{((RC)^2 s^2 + 1)}{((RC)^2 s^2 + 4(1 - \alpha)(RC)s + 1)}$$

$$\omega_0 \stackrel{\text{def}}{=} \frac{1}{RC}$$

$$H_{BS}(j\omega) = \frac{\left(\frac{j\omega}{\omega_0}\right)^2 + 1}{\left(\frac{j\omega}{\omega_0}\right)^2 + \left(\frac{j\omega}{\omega_0}\right)4(1 - \alpha) + 1}$$

$$H_{BS}(j\omega) = \frac{\left(\frac{j\omega}{\omega_0}\right)^2 + 1}{\left(\frac{j\omega}{\omega_0}\right)^2 + \frac{j\omega}{\omega_0} \frac{1}{Q} + 1}$$

$$Q = \frac{1}{4(1 - \alpha)} \Rightarrow \alpha = 1 - \frac{1}{4Q}$$

Full feedback: $\alpha = 1 \Rightarrow Q = \infty$

0 feedback: $\alpha = 0 \Rightarrow Q = 0.25$ $Q = 10 \Rightarrow \alpha = 0.975$

Twin-T notch filter

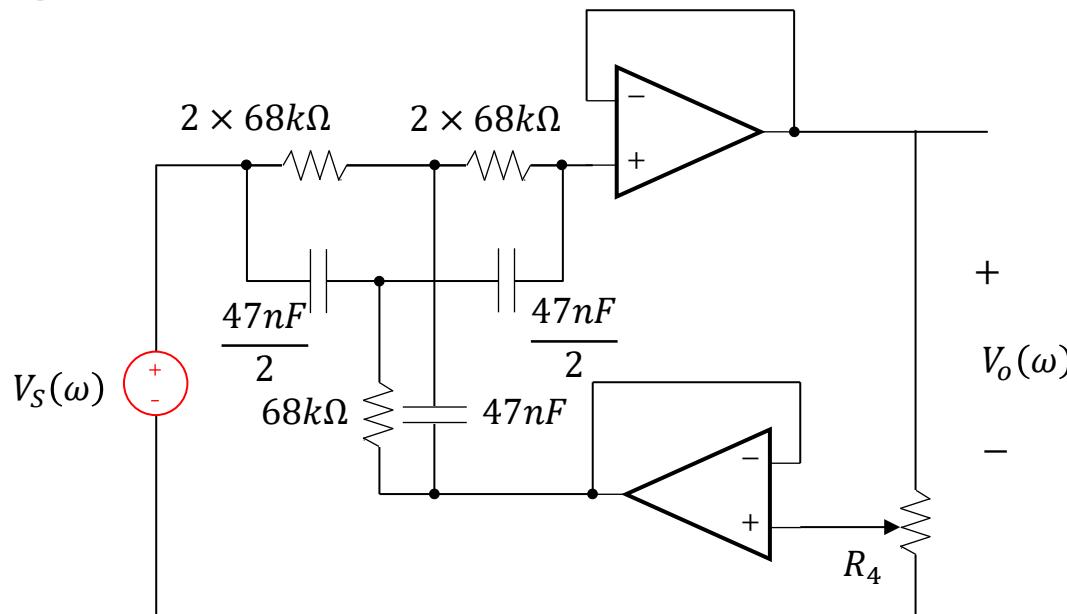
$$\omega_0 = \frac{1}{RC} \quad R = \frac{1}{2\pi 50 \times C}$$

E3	E6	E12	E24
10	10	10	10
		11	
		12	12
		13	
15	15	15	
		16	
		18	18
		20	
22	22	22	22
		24	
		27	27
		30	
33	33	33	
		36	
		39	39
		43	
47	47	47	47
		51	
		56	56
		62	
68	68	68	
		75	
		82	82
		91	

C nF	R kOhm	R1 kOhm	R2 kOhm	R3 kOhm	C1 nF	C2 nF	C3 nF
10,00	318,3	636,6	636,6	318,3	5,0	5,0	10,0
12,00	265,3	530,5	530,5	265,3	6,0	6,0	12,0
15,00	212,2	424,4	424,4	212,2	7,5	7,5	15,0
18,00	176,8	353,7	353,7	176,8	9,0	9,0	18,0
22,00	144,7	289,4	289,4	144,7	11,0	11,0	22,0
27,00	117,9	235,8	235,8	117,9	13,5	13,5	27,0
33,00	96,5	192,9	192,9	96,5	16,5	16,5	33,0
39,00	81,6	163,2	163,2	81,6	19,5	19,5	39,0
47,00	67,7	135,5	135,5	67,7	23,5	23,5	47,0
56,00	56,8	113,7	113,7	56,8	28,0	28,0	56,0
68,00	46,8	93,6	93,6	46,8	34,0	34,0	68,0
82,00	38,8	77,6	77,6	38,8	41,0	41,0	82,0

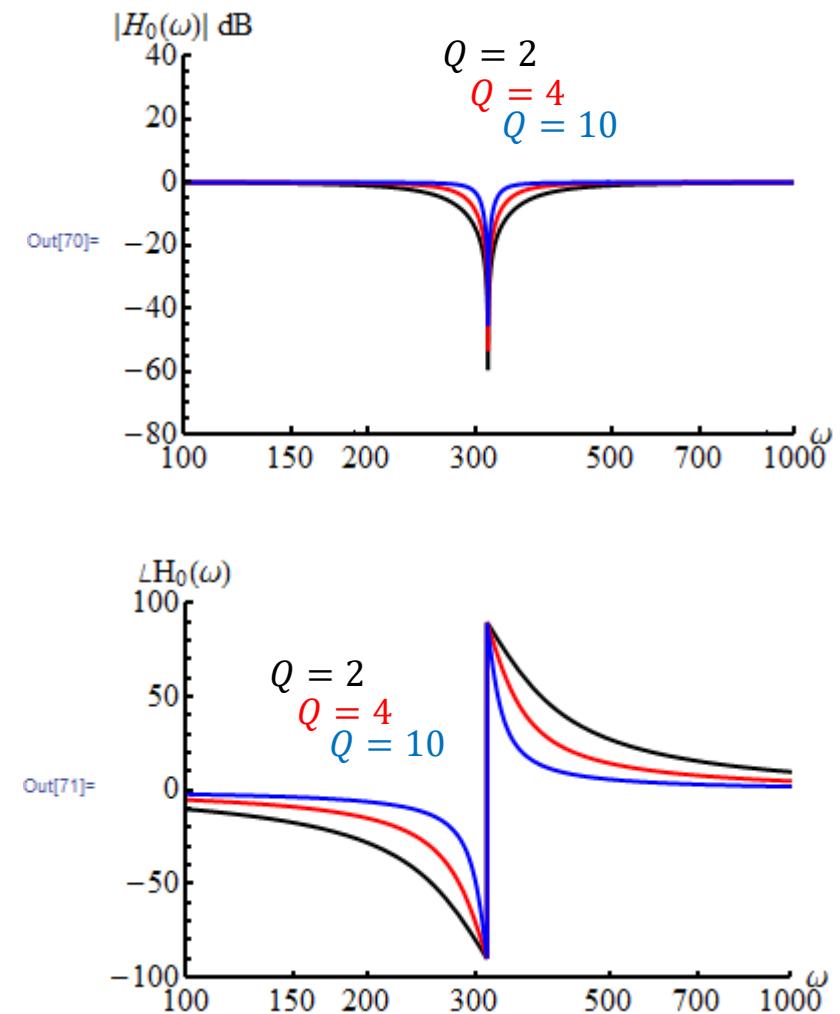
Choosing: $C = 47nF \Rightarrow R = 68k\Omega$

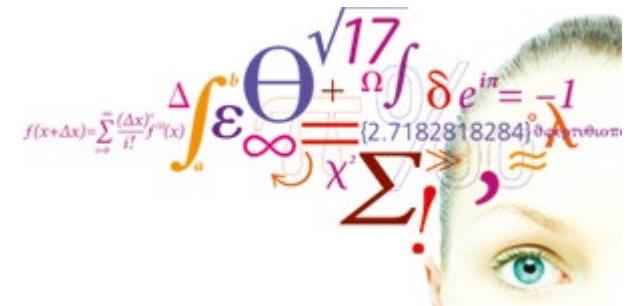
Twin-T notch filter



The Twin-T notch filter is a decent notch filter. However, placing the notch precisely at the desired frequency requires very precise component values, and the narrower the notch, the more precise they need to be. An attenuation in the range 30 – 40 dB is a realistic goal.

We will return to this filter later and then look into the relationship between the pole/zero positions and the behavior of the amplitude and phase spectra.





Problems

Filter 12: Lowpass filter – Question 1

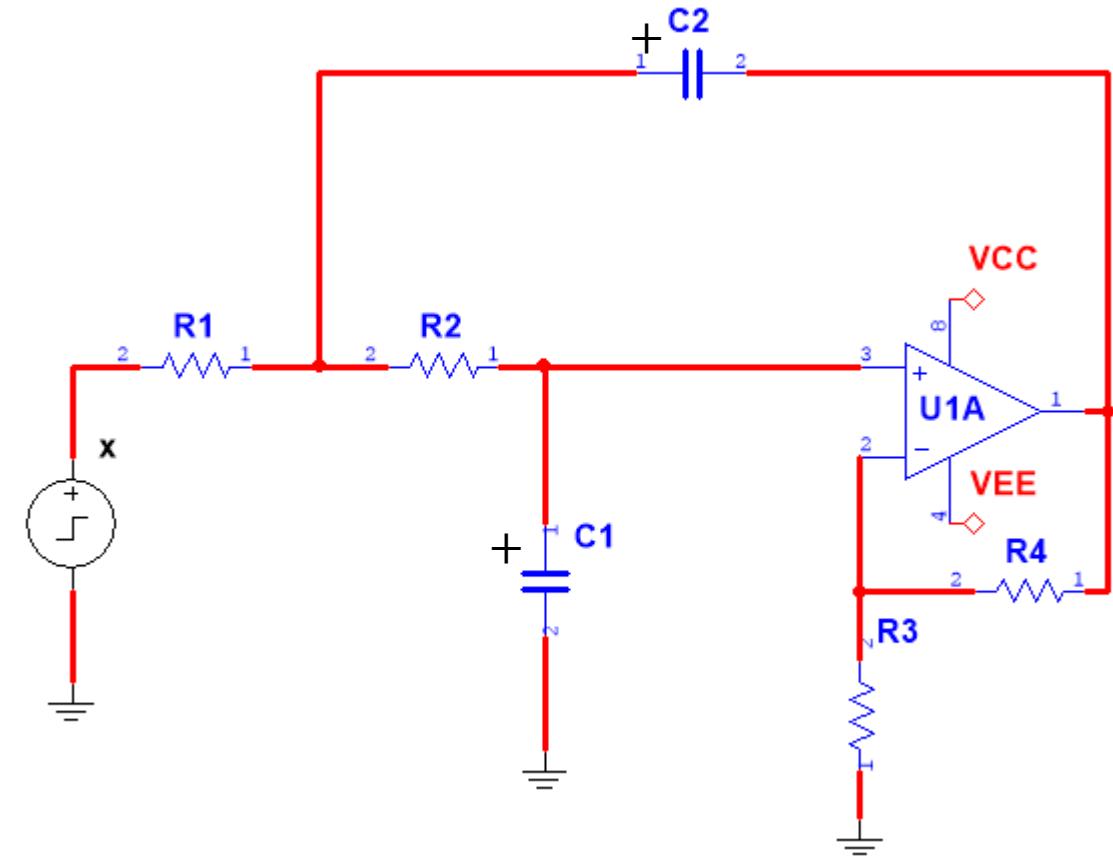
Question 1.

The circuit diagram is in a form suitable for time-domain analysis. We want to redraw the circuit, so it is suitable for s-domain analysis.

Laplace transform the frequency dependent components and the time-varying functions. Include the initial value generators for the filter capacitors.

$$v_{C_1}(0_-) = 1V, v_{C_2}(0_-) = -1V.$$

A + sign indicates the positive side of the capacitor with respect to their voltage at $t = 0_-$. It is not an electrolytic capacitor.



Question 2.

Derive the system response: $Y(s) = \frac{P(s)}{Q(s)}X(s) + \frac{I(s)}{Q(s)}$

Set up nodal equations and eliminate internal variables.

Answer:

$$Y(s) = \frac{b_0}{s^2 + a_1s + a_0}X(s) + \frac{\frac{K}{R_2C_1}v_{C_2}(0_-) + K\left(s + \frac{1}{R_2C_2} + \frac{1}{R_1C_2}\right)v_{C_1}(0_-)}{s^2 + a_1s + a_0}$$

Question 3

Solve for the zero-input response using partial fraction expansion and Laplace transform. Use Maple/Python.

Question 4

Solve for the zero-state response to the input $x(t) = 2u(t)$, using partial fraction expansion and Laplace transform. Use Maple/Python.

Answer: $y_{zs}(t) = (4 + 0.23 e^{-2680.2 t} - 4.23 e^{-147.25 t})u(t)$

Laplace transform the differential equation for the Sallen-Key highpass filter and derive $H(s)$. Look up the solutions to the problems on the first day of the course.

$$\ddot{y}(t) + \left(\frac{1}{R_2 C_2} + \frac{1}{R_2 C_1} + \frac{1-K}{R_1 C_1} \right) \dot{y}(t) + \frac{1}{R_1 R_2 C_1 C_2} y = K \ddot{x}(t)$$

Answer:

$$H(s) = \frac{b_2 s^2}{s^2 + a_1 s + a_0}$$
$$H(s) = \frac{b_2 s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Filter 13: Highpass filter - Question 2

Calculate the impulse response $h(t)$ by inverse Laplace transformation of $H(s)$.

We remember that this filter is supposed to have a double root. Due to numerical rounding errors, Maple/Python may not produce a result, that indicates this.

Set the following parameter values and make some common sense rewriting of the results.

Adjusted coefficients

$$\begin{aligned}f_0 &:= \frac{1}{20} : \omega_n := 2 \cdot \pi \cdot f_0 : \\ \varsigma &:= 1 :\end{aligned}$$

$$\zeta = 1 \Rightarrow H(s) = \frac{b_2 s^2}{(s + \omega_n)^2}$$

Use this form for the inverse Laplace transform

Answer: $h(t) = 2\delta(t) - 0.6283(2 - 0.31415 t)e^{-0.31415 t}u(t)$

$$h(t) = 2\delta(t) - \frac{2\pi}{10} \left(2 - \frac{\pi}{10}t\right) e^{-\frac{\pi}{10}t}$$

Using precise coefficients

Filter 13: Highpass filter - Question 3

Calculate the zero-state response to $x(t) = 2 u(t)$ by calculating the inverse Laplace transform of $Y(s) = H(s)X(s)$.

Answer: $y_{zs}(t) = 4(1 - 0.31415 t)e^{-0.31415 t} u(t)$

Filter 14: Bandpass filter - Question 1

Laplace transform the differential equation for the Sallen-Key bandpass filter and derive $H(s)$. Look up the solutions to the problems on the first day of the course.

$$\ddot{y} + \left(\frac{1}{R_1 C_1} + \frac{1}{R_3 C_2} + \frac{1}{R_3 C_1} + \frac{1-K}{R_2 C_1} \right) \dot{y} + \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{1}{R_3 C_1 C_2} y = K \frac{\dot{x}}{R_1 C_1}$$

Answer: $H(s) = \frac{b_1 s}{s^2 + a_1 s + a_0}$

Filter 14: Bandpass filter - Question 2

Calculate the impulse response $h(t)$ by inverse Laplace transformation of $H(s)$.

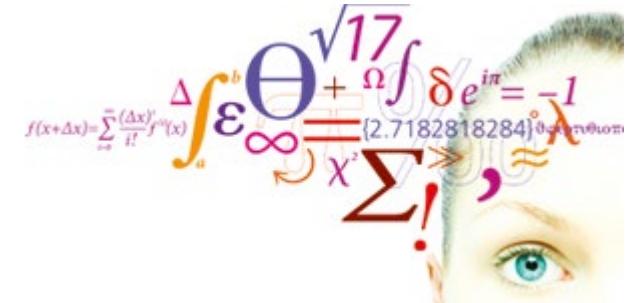
Answer:
$$h(t) = 629.1e^{-15.7t} \cos(314t + 0.05) u(t)$$

$$h(t) = 628.3 e^{-15.7 t} \cos(314 t) u(t) - 31.4 e^{-15.7 t} \sin(314 t) u(t)$$

Filter 14: Bandpass filter - Question 3

Calculate the zero-state response to $x(t) = 2 u(t)$ by calculating the inverse Laplace transform of $Y(s) = H(s)X(s)$.

Answer: $y_{zs}(t) = 4e^{-15.7 t} \sin(313.76 t) u(t) = 4e^{-15.7 t} \cos\left(313.76 t + \frac{\pi}{2}\right) u(t)$



Solutions

Filter 12: Lowpass filter – Question 1

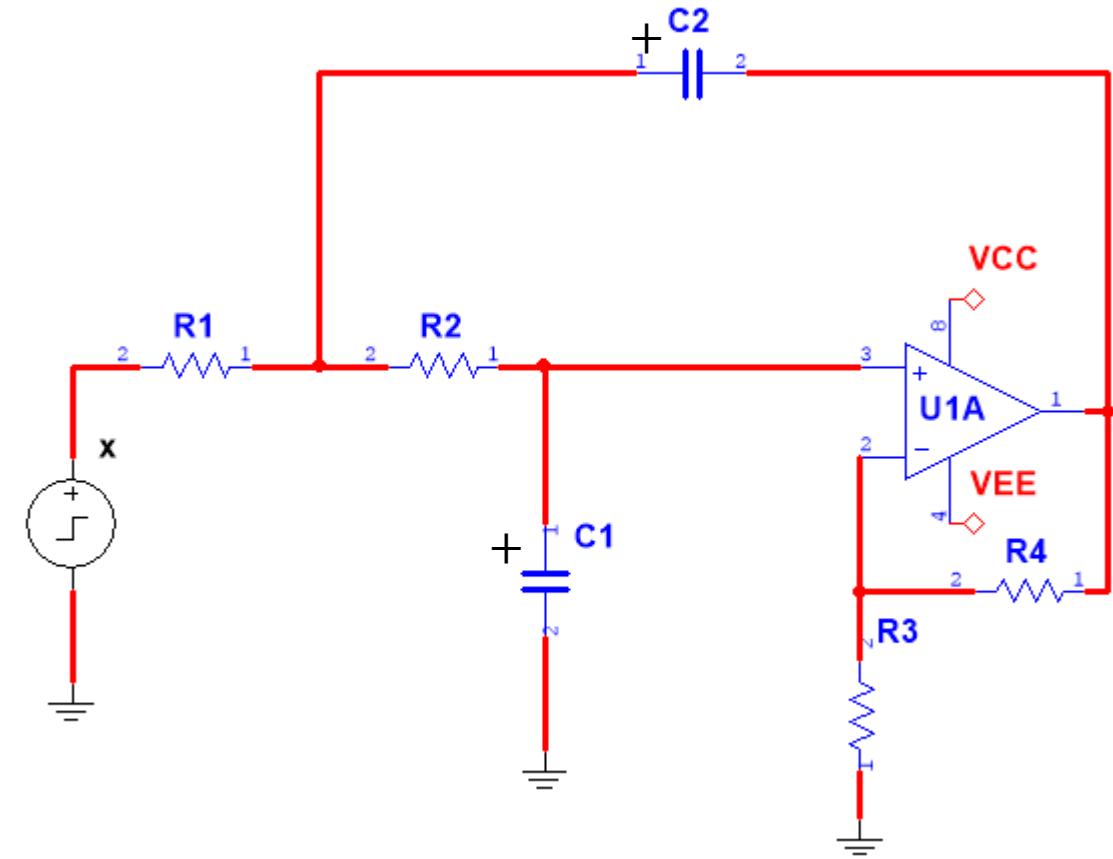
Question 1.

The circuit diagram is in a form suitable for time-domain analysis. We want to redraw the circuit, so it is suitable for s-domain analysis.

Laplace transform the frequency dependent components and the time-varying functions. Include the initial value generators for the filter capacitors.

$$v_{C_1}(0_-) = 1V, v_{C_2}(0_-) = -1V.$$

A + sign indicates the positive side of the capacitor with respect to their voltage at $t = 0_-$. It is not an electrolytic capacitor.



Filter 12: Lowpass filter – Question 1 (sol)

Question 1.

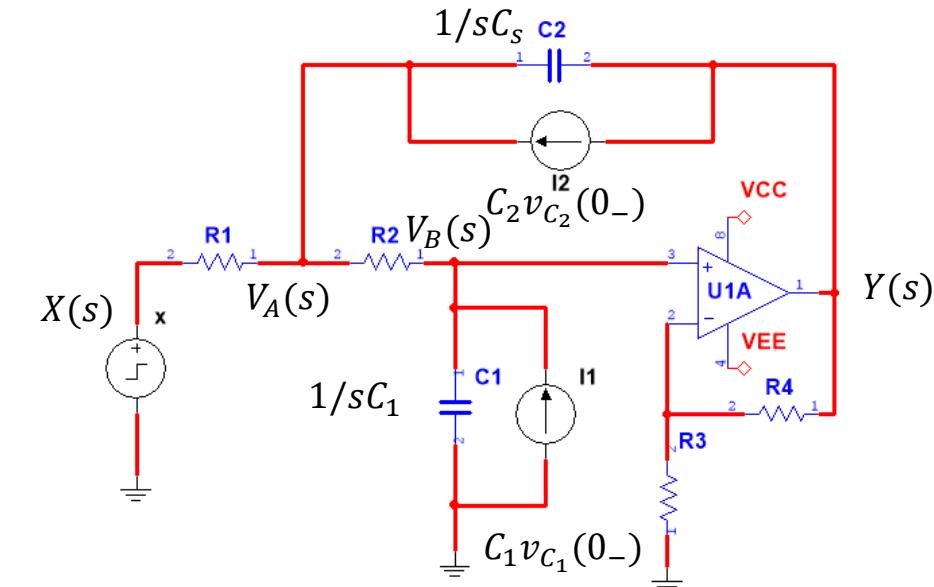
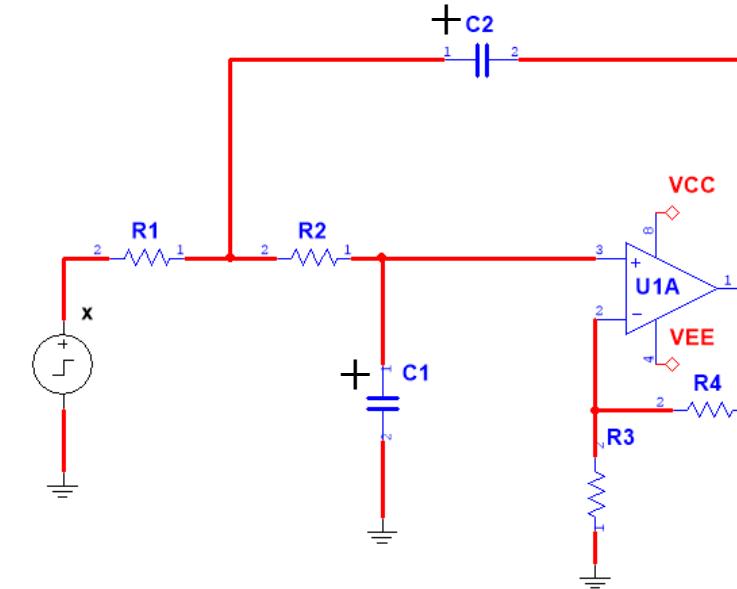
The circuit diagram is in a form suitable for time-domain analysis. We want to redraw the circuit, so it is suitable for s-domain analysis.

Laplace transform the frequency dependent components and the time-varying functions.
Include the initial value generators for the filter capacitors.

$$V_{C_1} = 1V, V_{C_2} = -1V.$$

I have chosen the **parallel component model**, because it is a sum of currents. This is most practical when we are writing nodal equations, where we sum up currents.

Series models are more convenient when we are using mesh analysis.



Question 2.

Derive the system response: $Y(s) = \frac{P(s)}{Q(s)}X(s) + \frac{I(s)}{Q(s)}$

Set up nodal equations and eliminate internal variables.

Answer:

$$Y(s) = \underbrace{\frac{b_0}{s^2 + a_1s + a_0}X(s)}_{\text{zero-state}} + \underbrace{\frac{\frac{K}{R_2C_1}v_{C_2}(0_-) + K\left(s + \frac{1}{R_2C_2} + \frac{1}{R_1C_2}\right)v_{C_1}(0_-)}{s^2 + a_1s + a_0}}_{\text{zero-input}}$$

Filter 12: Lowpass filter – Question 2 (sol)

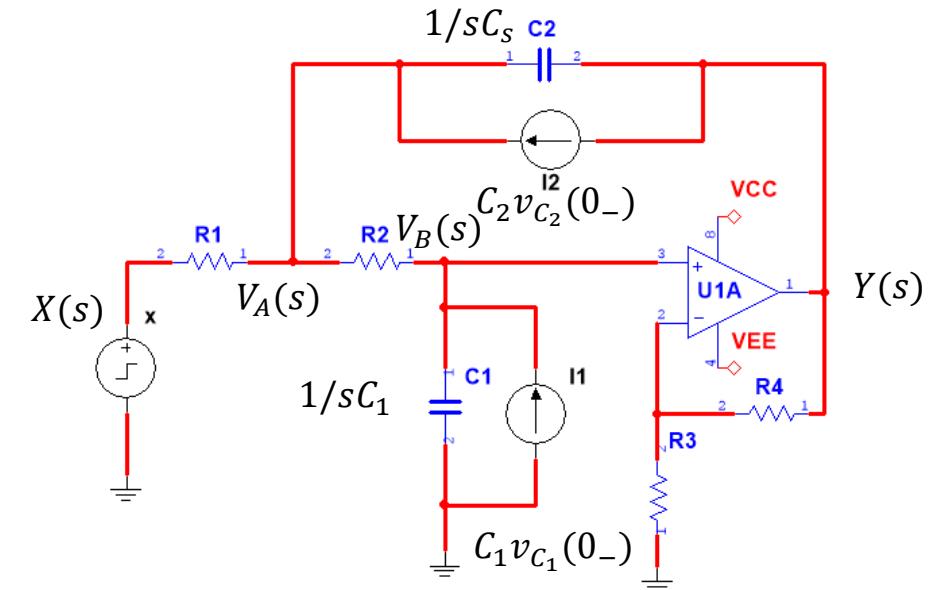
$$A1: -\frac{X(s) - V_A(s)}{R_1} + s C_2 (V_A(s) - Y(s)) - C_2 v_{C_2}(0_-) + \frac{V_A(s) - V_B(s)}{R_2} = 0$$

$$B1: -\frac{V_A(s) - V_B(s)}{R_2} + s C_1 V_B(s) - C_1 v_{C_1}(0_-) = 0$$

Constraint:

$$V_B(s) = \frac{1}{K} Y(s)$$

$V_B(s)$ is not an unknown. We should not solve for $V_B(s)$.



Filter 12: Lowpass filter – Question 2 (sol)

Setting up the nodal equations, remembering to add the initial conditions:

$$A1: -\frac{X(s) - V_A(s)}{R_1} + s C_2 (V_A(s) - Y(s)) \\ -C_2 v_{C_2}(0_-) + \frac{V_A(s) - V_B(s)}{R_2} = 0$$

$$B1: -\frac{V_A(s) - V_B(s)}{R_2} + s C_1 V_B(s) - C_1 v_{C_1}(0_-) = 0$$

Insert the impedances in the Laplace domain:

Constraints

$$VB := \frac{1}{K} \cdot Y :$$

Equations

$$eqA1 := -\frac{(X - VA)}{R1} + \frac{(VA - Y)}{ZC2} - C2 v_{C20} + \frac{(VA - VB)}{R2} = 0 :$$

$$eqB1 := -\frac{(VA - VB)}{R2} + \frac{VB}{ZC1} - C1 v_{C10} = 0 :$$

Solve

$$solutions := simplify(solve(\{eqA1, eqB1\}, [VA, Y], symbolic = true)) : \\ assign(solutions) \\ simplify(Y) \\ - \frac{K(((C1 v_{C10} + C2 v_{C20}) R1 + C1 v_{C10} R2 + X) ZC2 + C1 R1 R2 v_{C10}) ZC1}{(-R1 - R2 - ZC1) ZC2 + R1 ((K - 1) ZC1 - R2)}$$

Impedances

$$ZC1 := \frac{1}{s \cdot C1} ; ; ZC2 := \frac{1}{s \cdot C2} ; ;$$

$$simplify(Y)$$

$$- \frac{((v_{C10} (C2 R2 s + 1) C1 + C2 v_{C20}) R1 + C1 v_{C10} R2 + X) K}{s ((-C2 R2 s - 1) C1 + C2 (K - 1)) R1 - R2 s C1 - 1}$$

Filter 12: Lowpass filter – Question 2 (sol)

Using various Maple tools to manipulate the expression:

numer()
denom()
sort()
collect()

$$\begin{aligned} & \text{simplify}(Y) \\ &= \frac{((vC10(C2R2s + 1)C1 + C2vC20)R1 + C1vC10R2 + X)K}{((-C2R2s - 1)C1 + C2(K - 1))sR1 - R2sC1 - 1} \\ & \text{expl} := \frac{\text{numer}(Y)}{\text{denom}(Y)} \\ & \text{expl} := \frac{K(C1R1R2vC10sC2 + C1R1vC10 + C1vC10R2 + C2R1vC20 + X)}{C1C2R1R2s^2 - C2KR1s + R1sC1 + R2sC1 + R1sC2 + 1} \\ & \text{num1} := \text{sort}(\text{numer}(\text{expl}), s) \\ & \text{num1} := (C1R1R2vC10C2s + C1R1vC10 + C1vC10R2 + C2R1vC20 + X)K \\ & \text{den1} := \text{sort}(\text{collect}(\text{denom}(\text{expl}), s)) \\ & \text{den1} := C1C2R1R2s^2 + (-C2KR1 + C1R1 + C1R2 + C2R1)s + 1 \end{aligned}$$

Zero-state response:

$$Y_{zs} := s \rightarrow \frac{K}{R1 \cdot R2 \cdot C1 \cdot C2 \cdot s^2 + (R1 \cdot C1 + R2 \cdot C1 + (1 - K) \cdot R1 \cdot C2) \cdot s + 1} \cdot X;$$

Zero-input response:

$$Y_{zi} := s \rightarrow \frac{C1 \cdot R1 \cdot R2 \cdot vC10 \cdot C2 \cdot s + C1 \cdot R1 \cdot vC10 + C1 \cdot vC10 \cdot R2 + C2 \cdot R1 \cdot vC20}{R1 \cdot R2 \cdot C1 \cdot C2 \cdot s^2 + (R1 \cdot C1 + R2 \cdot C1 + (1 - K) \cdot R1 \cdot C2) \cdot s + 1} \cdot K;$$

$$Y_{zi2} := s \rightarrow \frac{K \left(s + \frac{1}{R2 \cdot C2} + \frac{1}{R1 \cdot C2} \right) \cdot vC10 + \frac{K}{R2 \cdot C1} \cdot vC20}{s^2 + \left(\frac{1}{R2 \cdot C2} + \frac{1}{R1 \cdot C2} + \frac{(1 - K)}{R2 \cdot C1} \right) \cdot s + \frac{1}{R1 \cdot R2 \cdot C1 \cdot C2}};$$

Question 3

Solve for the zero-input response using partial fraction expansion and Laplace transform. Use Maple.

Solve for the zero-input response using partial fraction expansion and Laplace transform. Use Maple.

Zero input response

$$\text{convert}\left(\frac{\frac{K \cdot \left(s + \frac{1}{R2 \cdot C2} + \frac{1}{R1 \cdot C2}\right) \cdot vC10 + \frac{K}{R2 \cdot C1} \cdot vC20}{s^2 + \left(\frac{1}{R2 \cdot C2} + \frac{1}{R1 \cdot C2} + \frac{(1 - K)}{R2 \cdot C1}\right) \cdot s + \frac{1}{R1 \cdot R2 \cdot C1 \cdot C2}}}{\frac{2.116313058}{s + 147.3061597} - \frac{0.1163130577}{s + 2680.231742}}, \text{parfrac}, s\right)$$

$$\text{invlaplace}\left(\frac{2.116313058}{s + 147.3061597} - \frac{0.1163130577}{s + 2680.231742}, s, t\right)$$
$$2.116313058 e^{-147.3061597 t} - 0.1163130577 e^{-2680.231742 t}$$

Question 4

Solve for the zero-state response to the input $x(t) = 2u(t)$, using partial fraction expansion and Laplace transform. Use Maple.

Answer: $y_{zs}(t) = (4 + 0.23 e^{-2680.2 t} - 4.23 e^{-147.25 t})u(t)$

Filter 12: Lowpass filter – Question 4 (sol)

Solve for the zero-state response to the input $x(t) = 2u(t)$,
using partial fraction expansion and Laplace transform. Use
Maple.

Answer: $y_{zs}(t) = (4 + 0.23 e^{-2680.2 t} - 4.23 e^{-147.25 t})u(t)$

Zero-state response

$$Y_{zs}(s) = H(s)X(s)$$

$$Y_{zs}(s) = H(s) \frac{2}{s}$$

convert $\left(\frac{\frac{K}{R1 \cdot R2 \cdot C1 \cdot C2}}{s^2 + \left(\frac{1}{R2 \cdot C2} + \frac{1}{R1 \cdot C2} + \frac{(1-K)}{R2 \cdot C1} \right) \cdot s + \frac{1}{R1 \cdot R2 \cdot C1 \cdot C2}} \cdot \frac{2}{s}, \text{parfrac, s} \right)$

$$\frac{4.000000000}{s} - \frac{4.232626116}{s + 147.3061597} + \frac{0.2326261165}{s + 2680.231742}$$

invlaplace $\left(\frac{4.000000000}{s} - \frac{4.232626116}{s + 147.3061597} + \frac{0.2326261165}{s + 2680.231742}, s, t \right)$

$$4. - 4.232626116 e^{-147.3061597 t} + 0.2326261165 e^{-2680.231742 t}$$

Filter 13: Highpass filter - Question 1

Laplace transform the differential equation for the Sallen-Key highpass filter and derive $H(s)$. Look up the solutions to the problems on the first day of the course.

$$\ddot{y}(t) + \left(\frac{1}{R_2 C_2} + \frac{1}{R_2 C_1} + \frac{1-K}{R_1 C_1} \right) \dot{y}(t) + \frac{1}{R_1 R_2 C_1 C_2} y = K \ddot{x}(t)$$

Answer:

$$H(s) = \frac{b_2 s^2}{s^2 + a_1 s + a_0}$$
$$H(s) = \frac{b_2 s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Laplace transform the differential equation for the Sallen-Key highpass filter and derive $H(s)$. Look up the solutions to the problems on the first day of the course.

We are only asked to find $H(s)$.
 We do not need to include the initial values because they do not multiply with $H(s)$.

Answer:

$$\ddot{y}(t) + \left(\frac{1}{R_2 C_2} + \frac{1}{R_2 C_1} + \frac{1-K}{R_1 C_1} \right) \dot{y}(t) + \frac{1}{R_1 R_2 C_1 C_2} y = K \ddot{x}(t)$$

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_2 \ddot{x}$$

$$s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = b_2 s^2 X(s)$$

$$(s^2 + a_1 s + a_0) Y(s) = b_2 s^2 X(s)$$

$$H(s) = \frac{Y_{zs}(s)}{X(s)} = \frac{b_2 s^2}{s^2 + a_1 s + a_0}$$

$$H(s) = \frac{b_2 s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Filter 13: Highpass filter - Question 2

Calculate the impulse response $h(t)$ by inverse Laplace transformation of $H(s)$.

When $\zeta = 1$, $s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + \omega_n)^2$

Only in this form will Maple give us an inverse Laplace transform of the form $t e^{-a t}$. This is the form we associate with a repeated root.

Answer: $h(t) = 2\delta(t) - 0.6283(2 - 0.31415 t)e^{-0.31415 t}u(t)$

$$h(t) = 2\delta(t) - \frac{2\pi}{10} \left(2 - \frac{\pi}{10}t\right) e^{-\frac{\pi}{10}t}$$

Using precise coefficients

We remember that this filter is supposed to have a double root. Due to numerical rounding errors, Maple will not produce a result, that indicates this.

Set the following parameter values and make some common sense rewriting of the results.

Adjusted coefficients

$$\begin{aligned} f0 &:= \frac{1}{20} : \omega_n := 2 \cdot \pi \cdot f0 : \\ \zeta &:= 1 : \end{aligned}$$

$$\zeta = 1 \Rightarrow H(s) = \frac{b_2 s^2}{(s + \omega_n)^2}$$

Use this form for the inverse Laplace transform

Calculate the impulse response $h(t)$ by inverse Laplace transformation of $H(s)$.

Answer: $h(t) = 2\delta(t) - 0.6283(2 - 0.31415 t)e^{-0.31415 t}u(t)$

We remember that this filter is supposed to have a double root. Due to numerical rounding errors, Maple will not produce a result, that indicates this.

Set the following parameter values and make some common sense rewriting of the results.

Adjusted coefficients

$$f_0 := \frac{1}{20} : \omega_n := 2 \cdot \pi f_0 :$$

$$\varsigma := 1 :$$

$$\varsigma = 1 \Rightarrow H(s) = \frac{b_2 s^2}{(s + \omega_n)^2}$$

Use this form for the inverse Laplace transform

Impulse response

$$\text{invlaplace}\left(\frac{K \cdot s^2}{(s + \omega_n)^2}, s, t\right)$$

$$2. \text{Dirac}(t) + 0.06283185308 e^{-0.3141592654 t} (3.141592654 t - 20.)$$

Filter 13: Highpass filter - Question 3

Calculate the zero-state response to $x(t) = 2 u(t)$ by calculating the inverse Laplace transform of $Y(s) = H(s)X(s)$.

Answer: $y_{zs}(t) = 4(1 - 0.31415 t)e^{-0.31415 t} u(t)$

Calculate the zero-state response to $x(t) = 2 u(t)$ by calculating the inverse Laplace transform of $Y(s) = H(s)X(s)$.

Answer: $y_{zs}(t) = 4(1 - 0.31415 t)e^{-0.31415 t} u(t)$

In this case it is better **not** to do a partial fraction expansion before the inverse Laplace transform.

Zero-state response

$$\text{invlaplace}\left(\frac{K \cdot s^2}{(s + \omega n)^2} \cdot \frac{2}{s}, s, t\right)$$
$$-0.4000000000 e^{-0.3141592654 t} (3.141592654 t - 10.)$$

Filter 14: Bandpass filter - Question 1

Laplace transform the differential equation for the Sallen-Key bandpass filter and derive $H(s)$. Look up the solutions to the problems on the first day of the course.

$$\ddot{y} + \left(\frac{1}{R_1 C_1} + \frac{1}{R_3 C_2} + \frac{1}{R_3 C_1} + \frac{1-K}{R_2 C_1} \right) \dot{y} + \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{1}{R_3 C_1 C_2} y = K \frac{\dot{x}}{R_1 C_1}$$

Answer: $H(s) = \frac{b_1 s}{s^2 + a_1 s + a_0}$

Laplace transform the differential equation for the Sallen-Key bandpass filter and derive $H(s)$. Look up the solutions to the problems on the first day of the course.

$$\ddot{y} + \left(\frac{1}{R_1 C_1} + \frac{1}{R_3 C_2} + \frac{1}{R_3 C_1} + \frac{1-K}{R_2 C_1} \right) \dot{y} + \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{1}{R_3 C_1 C_2} y = K \frac{\dot{x}}{R_1 C_1}$$

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{x}$$

$$s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = b_1 s X(s)$$

Answer:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_1 s}{s^2 + a_1 s + a_0}$$

Filter 14: Bandpass filter - Question 2

Calculate the impulse response $h(t)$ by inverse Laplace transformation of $H(s)$.

Answer:
$$h(t) = 629.1e^{-15.7t} \cos(314t + 0.05) u(t)$$

$$h(t) = 628.3 e^{-15.7 t} \cos(314 t) u(t) - 31.4 e^{-15.7 t} \sin(314 t) u(t)$$

Calculate the impulse response $h(t)$ by inverse Laplace transformation of $H(s)$.

Answer: $h(t) = 629.1e^{-15.7t} \cos(314t + 0.05) u(t)$

$$h(t) = 628.3 e^{-15.7 t} \cos(314 t) u(t) - 31.4 e^{-15.7 t} \sin(314 t) u(t)$$

Impulse response

$$\begin{aligned} & \text{simplify}\left(\text{invlaplace}\left(\frac{b1 \cdot s}{s^2 + a1 \cdot s + a0}, s, t\right)\right) \\ & (628.3176032 \cos(313.7646688 t) \\ & - 31.44542125 \sin(313.7646688 t)) e^{-15.70298545 t} \end{aligned}$$

Filter 14: Bandpass filter - Question 3

Calculate the zero-state response to $x(t) = 2 u(t)$ by calculating the inverse Laplace transform of $Y(s) = H(s)X(s)$.

Answer: $y_{zs}(t) = 4e^{-15.7 t} \sin(313.76 t) u(t) = 4e^{-15.7 t} \cos\left(313.76 t + \frac{\pi}{2}\right) u(t)$

Calculate the zero-state response to $x(t) = 2 u(t)$ by calculating the inverse Laplace transform of $Y(s) = H(s)X(s)$.

Answer: $y_{zs}(t) = 4e^{-15.7 t} \sin(313.76 t) u(t) = 4e^{-15.7 t} \cos\left(313.76 t + \frac{\pi}{2}\right) u(t)$

Zero-state response

$$\text{invlaplace}\left(\frac{b1 \cdot s}{s^2 + a1 \cdot s + a0} \cdot \frac{2}{s}, s, t\right)$$
$$4.005024565 e^{-15.70298545 t} \sin(313.7646688 t)$$

$$Y_{zs}(s) = H(s) \cdot X(s)$$

When we calculated the step response, we first Laplace transformed the step response: $X(s) = \frac{2}{s}$
Why didn't we do a similar thing when calculating the impulse response?

The Laplace transforms of $\delta(t)$ is $X(s) = 1$.

$$Y_{zs}(s) = H(s) \cdot X(s) = H(s) \cdot 1 = H(s)$$

$$y_{zs}(t) = \mathcal{L}^{-1}\{Y_{zs}(s)\} = \mathcal{L}^{-1}\{H(s)\} = h(t)$$

So, in an implicit way, we did do the same thing. We just didn't see it.

22050 Continuous-Time Signals and Linear Systems

Kaj-Åge Henneberg

L09

Quantitative features of step response
Bode plot 1: Constants and linear factors

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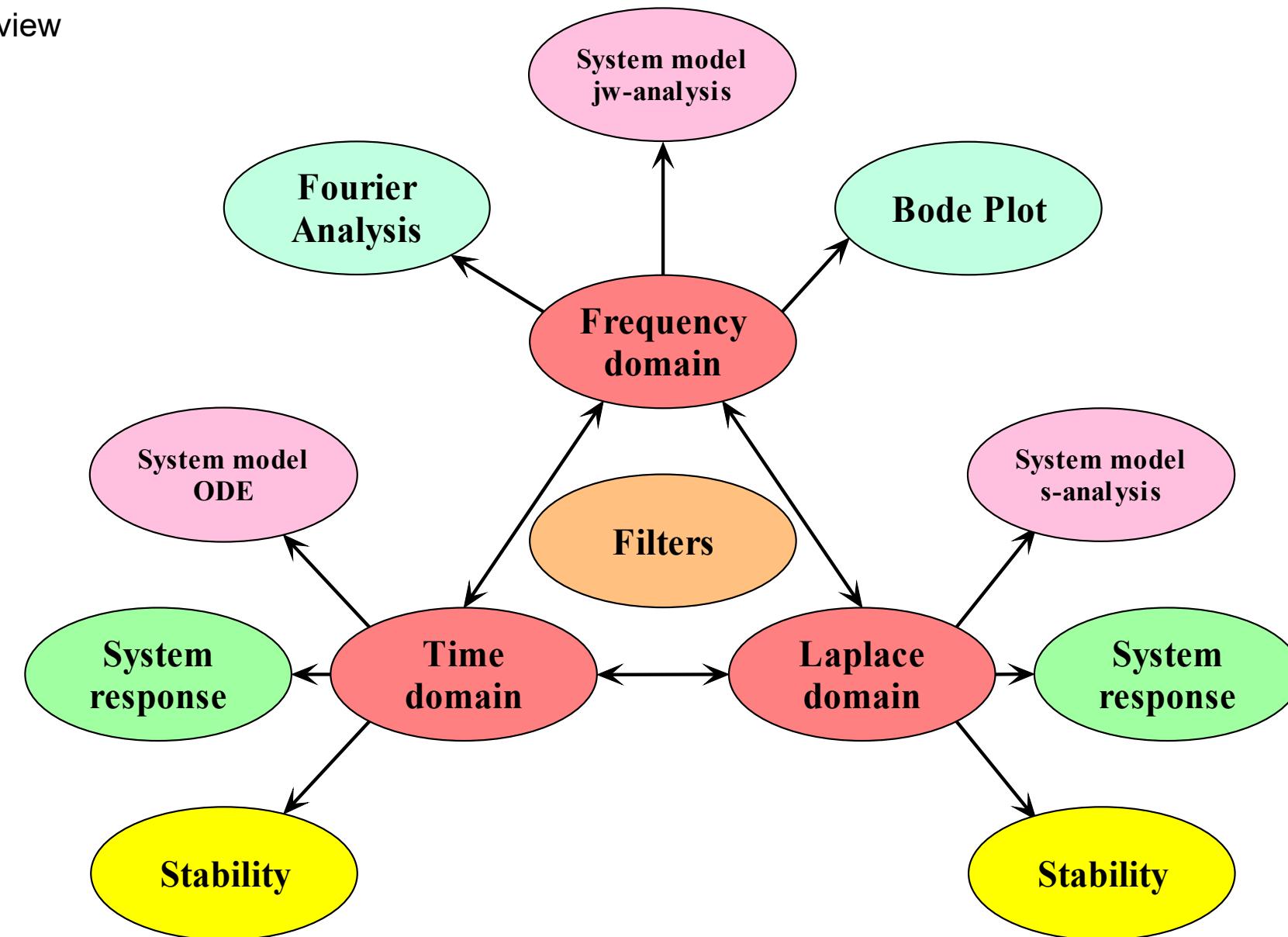
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- In course assignments, the reader is permitted to write the same equations and draw the same diagrams, but should do this using their own tools.



Control systems with feedback

6.7.1 (not lectured)

Analysis of second order system

6.7.2

Bode plot 1: Constants and linear factors

7.2

Lathi 6.7.2

Analysis of damped 2nd order system

Video 2

Damped Systems – types of roots

We start by identifying what influences the position of poles in the transfer function.

We solve for the poles by finding the roots of the denominator of the transfer function.

Based on the **discriminant**, we have three outcomes:

1. Distinct and real roots.
2. Repeated roots
3. Complex conjugated root.

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_0} = \frac{b_0}{(s + p_1)(s + p_2)}$$

$$p_1 = \frac{a_1}{2} + \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0}$$

$$p_2 = \frac{a_1}{2} - \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0}$$

Poles:

$$s_1 = -p_1 = -\frac{a_1}{2} - \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0}$$

$$s_2 = -p_2 = -\frac{a_1}{2} + \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0}$$

Discriminant:

$$D = \left(\frac{a_1}{2}\right)^2 - a_0 \begin{cases} > 0 & \text{Overdamped} \\ = 0 & \text{Critically damped} \\ < 0 & \text{Underdamped} \end{cases}$$

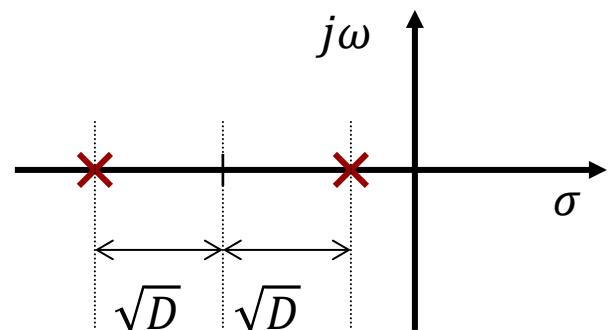
Damped Systems – types of roots

Overdamped:

$$D > 0$$

$$\begin{aligned} H(s) &= \frac{C}{(s + p_1)(s + p_2)} \\ &= \frac{A_1}{s + p_1} + \frac{A_2}{s + p_2} \end{aligned}$$

$$h(t) = [A_1 e^{-p_1 t} + A_2 e^{-p_2 t}] u(t)$$



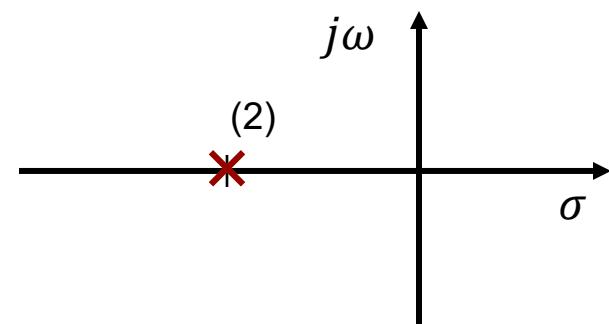
Critically damped:

$$D = 0$$

$$H(s) = \frac{A}{(s + p)^2}$$

$$p = \frac{a_1}{2}$$

$$h(t) = A[t e^{-pt}] u(t)$$



Damped Systems – types of roots

Underdamped: $D < 0$

$$H(s) = \frac{b_0}{\underbrace{s^2 + a_1 s + a_0}_{\text{Mathematical form}}} = \frac{b_0}{a_0} \cdot \frac{\omega_n^2}{\underbrace{s^2 + 2\zeta\omega_n s + \omega_n^2}_{\text{Physical form}}}$$

Natural (undamped) frequency: $\omega_n = \sqrt{a_0}$

Damping ratio: $\zeta = \frac{a_1}{2\sqrt{a_0}}$

For underdamped systems, it is very useful to express the roots in terms of physically meaningful systems parameters, namely:

$$D = \left(\frac{a_1}{2}\right)^2 - a_0 = (\zeta\omega_n)^2 - \omega_n^2 = (\zeta^2 - 1)\omega_n^2$$

ζ : damping ratio

ω_n : the natural (undamped resonance frequency).

$$D \begin{cases} > 0 & \text{if } \zeta > 1: \\ = 0 & \text{if } \zeta = 1: \\ < 0 & \text{if } 0 < \zeta < 1: \end{cases} \begin{array}{ll} \text{overdamped} \\ \text{critically damped} \\ \text{underdamped} \end{array}$$

Damped Systems – pole positions expressed in terms of new systems parameters

Poles:

$$s_{1,2} = -p_{1,2} = -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0} = -\zeta\omega_n \pm \sqrt{(\zeta^2 - 1)\omega_n^2}$$

$$= -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} = -\zeta\omega_n \pm j\omega_d$$

Damped frequency: ω_d **Observation:**

The resonance frequency is influenced by the damping factor. The largest resonance frequency is for the completely undamped system ($\zeta = 0$). As $\zeta \rightarrow 1$ the damped frequency $\omega_d \rightarrow 0$.

Hence for critically damped and overdamped systems, there are no oscillations.

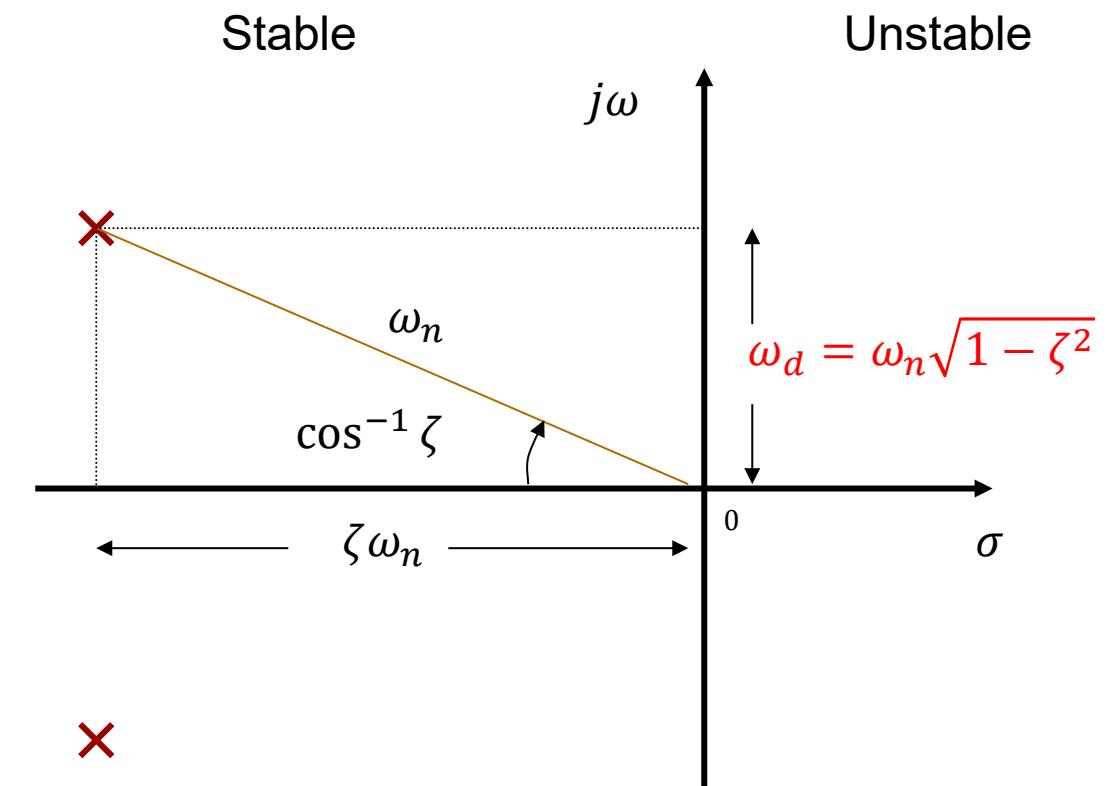
$$\zeta\omega_n = \omega_n \cos(\theta)$$

$$\zeta = \cos(\theta)$$

$$\zeta = 0 \Leftrightarrow \theta = 90^\circ$$

$$\zeta = \frac{1}{\sqrt{2}} \Leftrightarrow \theta = 45^\circ$$

$$\zeta = 1 \Leftrightarrow \theta = 0^\circ$$



Frequency Response – influence of damping factor

Expressing the transfer function $H(s)$ in terms of ζ and ω_n , we can now study the influence of these parameters on the **frequency response**.

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

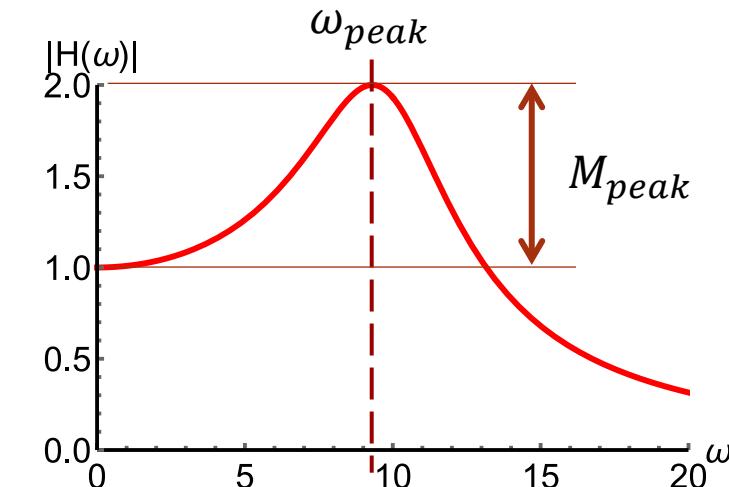
For very underdamped systems the amplitude spectrum features a local peak close to the 3dB cut-off frequency.

The frequency and the magnitude of the peak can be found by solving:

Observation:

$\omega_{peak} \rightarrow \omega_n$ when $\zeta \rightarrow 0$

$\omega_{peak} \rightarrow 0$ when $\zeta \rightarrow 1/\sqrt{2}$



$$|H(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$

$$\text{If } 0 < \zeta < 1/\sqrt{2}: \quad \omega_{peak} = \omega_n \sqrt{1 - 2\zeta^2}$$

$$M_{peak} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

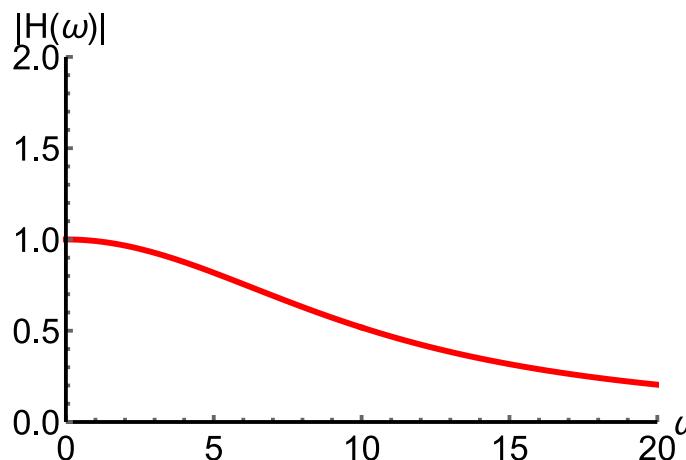
$$\zeta \rightarrow 1/\sqrt{2} \Rightarrow \omega_{peak} \rightarrow 0$$

Not in Lathi

Frequency Response – influence of damping factor

No peak, but roles off too early.

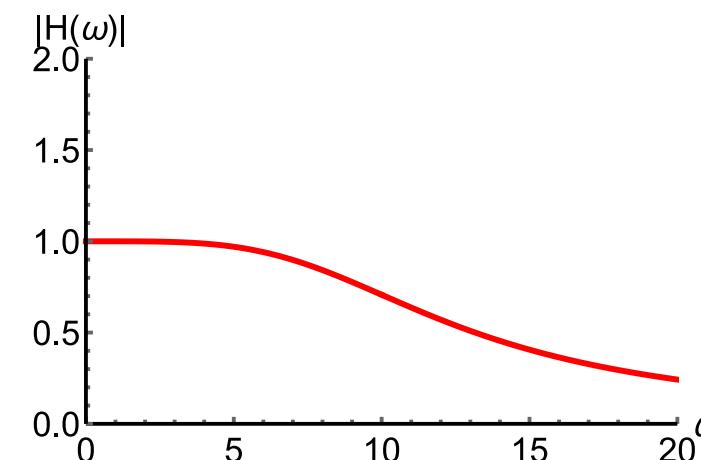
$$\zeta = 0.966 \quad \omega_n = 10$$
$$\theta = 15^\circ$$



Close to critically damped

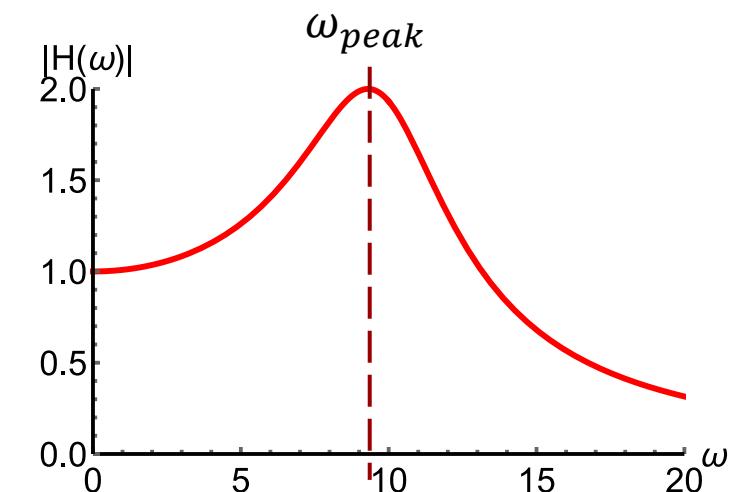
No peak, does not role off too early.

$$\zeta = \frac{1}{\sqrt{2}} \approx 0.707 \quad \omega_n = 10$$
$$\theta = 45^\circ$$



Resonance peak

$$\zeta = 0.259 \quad \omega_n = 10$$
$$\theta = 75^\circ$$



Close to undamped

Cut-off frequency dependence on damping factor

We would also like to know how the cut-off frequency ω_{3dB} varies as a function of damping ratio ζ .

We can get the modulus of the frequency from the transfer function:

We solve for that frequency ω_{3dB} where the pass-band gain has been damped by a factor $\sqrt{2}$:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

$$|H(j\omega_{3dB})| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega_{3dB}^2)^2 + (2\zeta\omega_n\omega_{3dB})^2}} = \frac{1}{\sqrt{2}}$$

$$\omega_{3dB} = \sqrt{-2\zeta^2\omega_n^2 + \omega_n^2 + \sqrt{2} \sqrt{\omega_n^4(2\zeta^4 - 2\zeta^2 + 1)}}$$

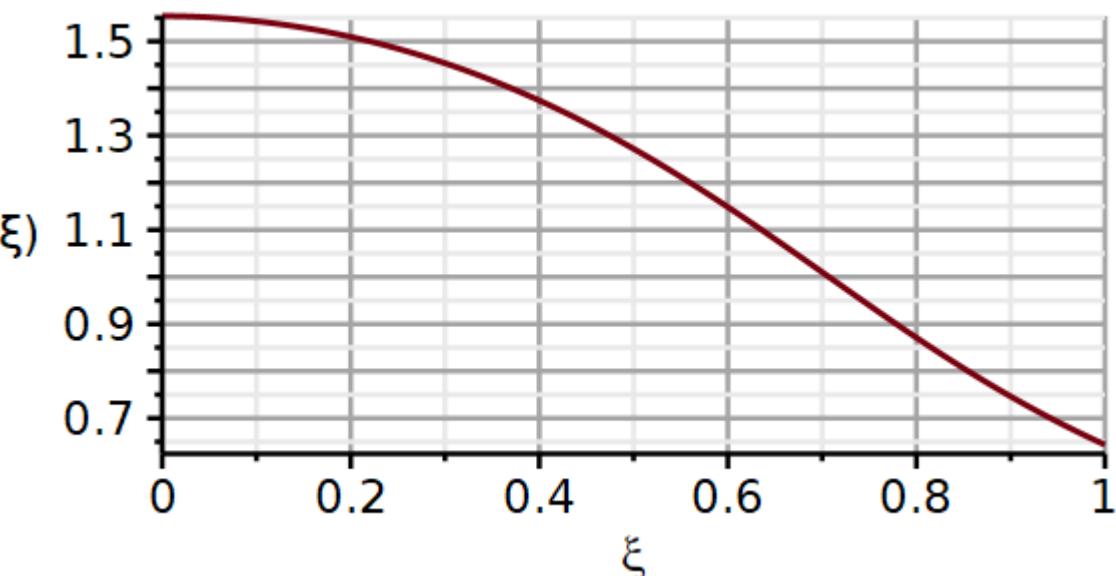
$$\omega_{3dB} = \omega_n \sqrt{-2\zeta^2 + 1 + \sqrt{2} \sqrt{(2\zeta^4 - 2\zeta^2 + 1)}}$$

$$\frac{\omega_{3dB}}{\omega_n} = \sqrt{1 - 2\zeta^2 + \sqrt{(4\zeta^4 - 4\zeta^2 + 2)}}$$

Cut-off frequency dependence on damping factor

$$\frac{\omega_{3dB}}{\omega_n} = \sqrt{1 - 2\xi^2 + \sqrt{(4\xi^4 - 4\xi^2 + 2)}}$$

$$\frac{\omega_{3dB}}{\omega_n}$$



Observation:

The analysis tells us that the 3dB cut-off frequency for a 2nd order lowpass filter will increase if the damping factor is reduced while the undamped resonance frequency ω_n is kept unchanged.

$$\text{evalf}(g(1), 3)$$

0.640

$$\text{evalf}\left(g\left(\frac{1}{\sqrt{2}}\right), 3\right)$$

1.

$$\text{evalf}(g(0), 3)$$

1.55

Cut-off frequency dependence on damping factor

We can also find out how ω_{3dB} changes relative to ω_d

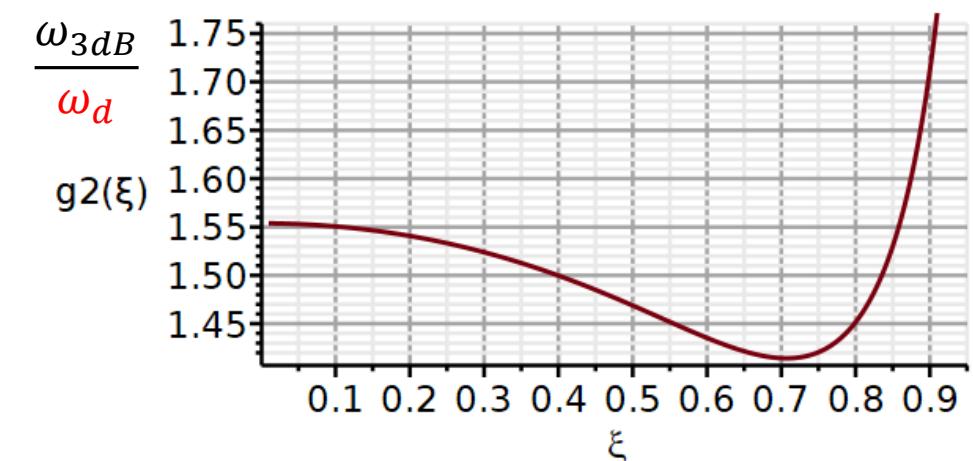
$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$\frac{\omega_{3dB}}{\omega_n} = \sqrt{1 - 2\zeta^2 + \sqrt{(4\zeta^4 - 4\zeta^2 + 2)}}$$

$$\frac{\omega_{3dB}}{\omega_n \sqrt{1 - \zeta^2}} = \frac{\sqrt{1 - 2\zeta^2 + \sqrt{(4\zeta^4 - 4\zeta^2 + 2)}}}{\sqrt{1 - \zeta^2}}$$

$$\frac{\omega_{3dB}}{\omega_d} = \frac{\sqrt{1 - 2\zeta^2 + \sqrt{(4\zeta^4 - 4\zeta^2 + 2)}}}{\sqrt{1 - \zeta^2}}$$

$$\frac{\omega_{3dB}}{\omega_d} = \sqrt{1 - \frac{\zeta^2}{(1 - \zeta^2)} + \frac{\sqrt{(4\zeta^4 - 4\zeta^2 + 2)}}{(1 - \zeta^2)}}$$



$$\text{evalf}\left(g2\left(\frac{1}{\sqrt{2}}\right), 3\right)$$

1.41

$$\text{evalf}(g2(0), 3)$$

1.55

Cut-off frequency dependence on damping factor

The position of ω_{peak} , ω_d , ω_n , ω_{3dB}

$$\omega_n := 10 : \zeta := 0.1 ;;$$

$$\omega_{3dB} := (\zeta, \omega_n) \rightarrow \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} ;$$

$$\omega_d := (\zeta, \omega_n) \rightarrow \omega_n \cdot \sqrt{1 - \zeta^2} ;$$

$$H0 := \omega \rightarrow \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} ;$$

$$Peak := [\omega_p, 20 \cdot \log_{10}(H_p(\zeta))]$$

$$Peak := [9.8995, 14.023]$$

$$Hod := [\omega_d(\zeta, \omega_n), dB(H0, \omega_d(\zeta, \omega_n))]$$

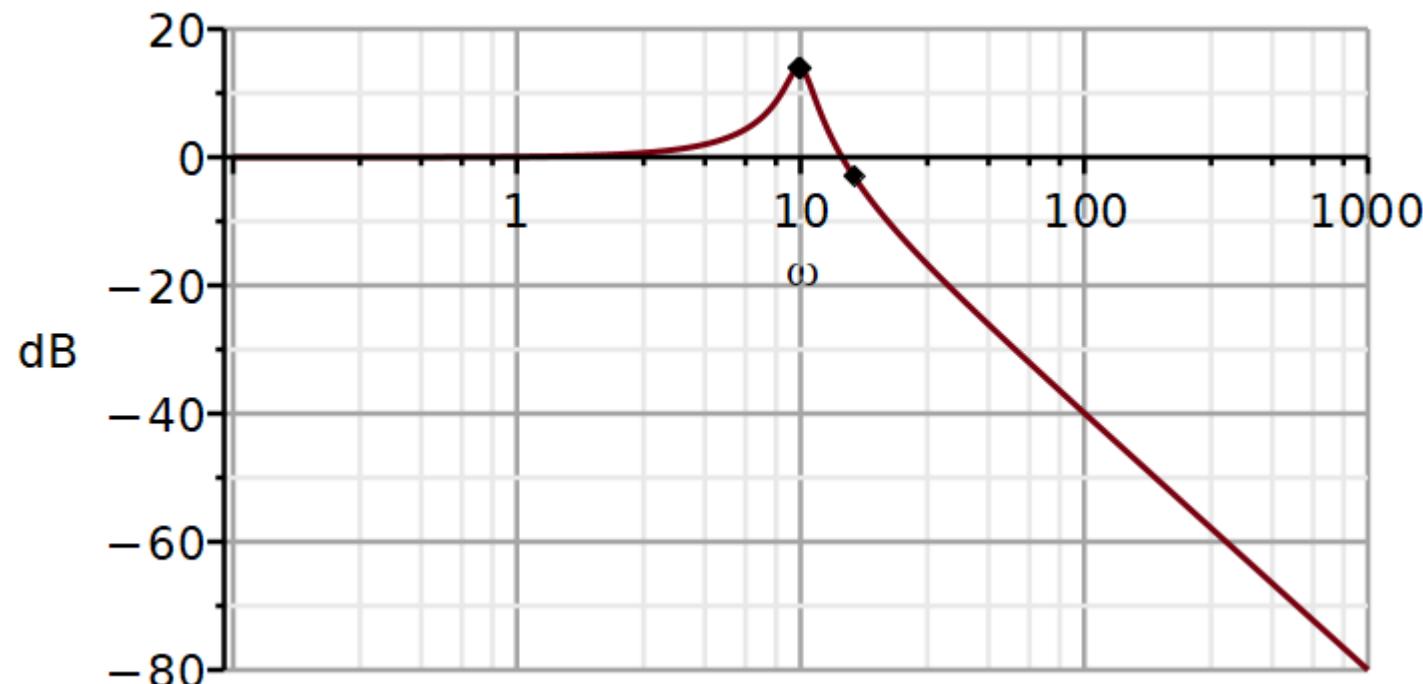
$$Hod := [9.9499, 14.012]$$

$$Hn := [\omega_n, dB(H0, \omega_n)];$$

$$Hn := [10, 13.979]$$

$$H3dB := [\omega_{3dB}(\zeta, \omega_n), dB(H0, \omega_{3dB}(\zeta, \omega_n))]$$

$$H3dB := [15.428, -3.0108]$$



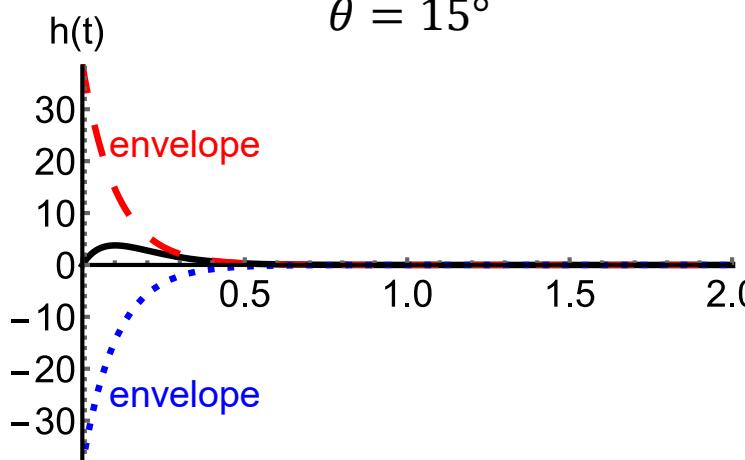
Time domain: Impulse response – influence of damping factor

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n}{\sqrt{1 - \zeta^2}} \cdot \frac{\omega_n \sqrt{1 - \zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

Table 6.1-9b: $h(t) = \underbrace{\frac{\omega_n}{\sqrt{1 - \zeta^2}} \cdot e^{-\zeta\omega_n t}}_{\text{envelope}} \sin \left[(\omega_n \sqrt{1 - \zeta^2}) t \right] u(t), \tau = \frac{1}{\zeta\omega_n}$

$$\zeta = 0.966$$

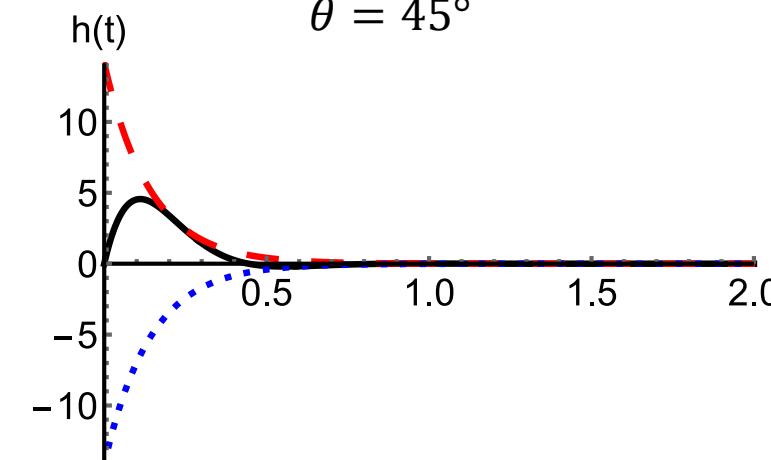
$$\theta = 15^\circ$$



Close to critically damped

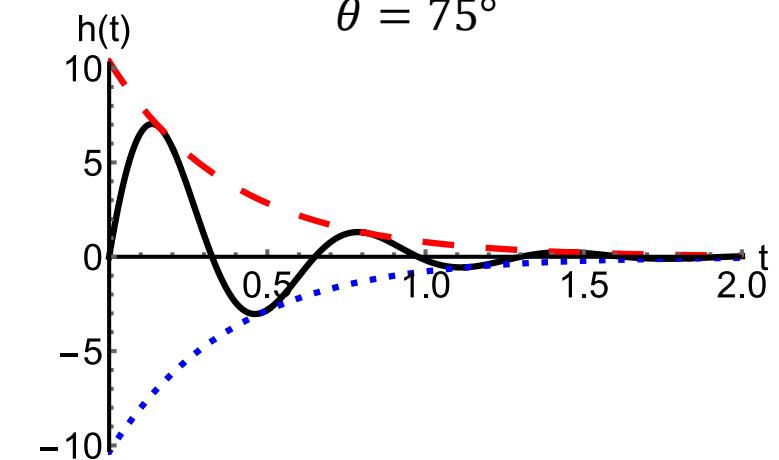
$$\zeta = 0.707$$

$$\theta = 45^\circ$$



$$\zeta = 0.259$$

$$\theta = 75^\circ$$



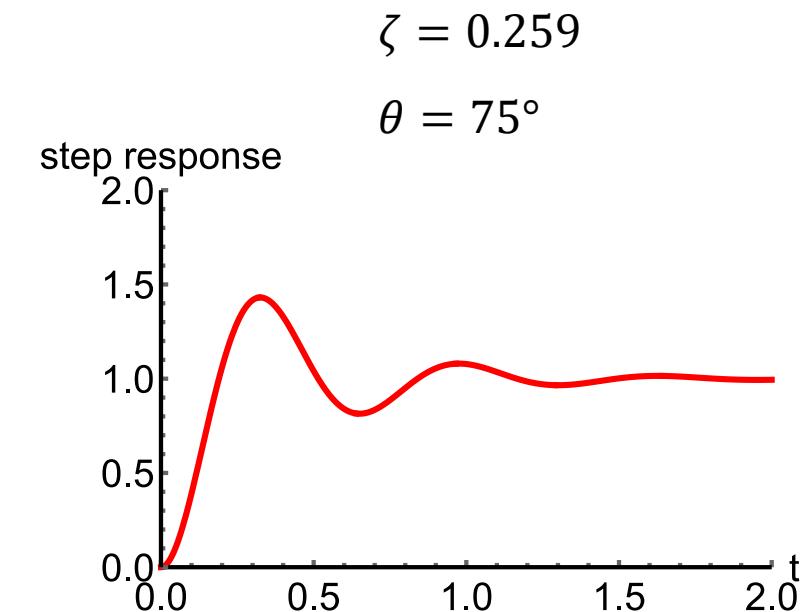
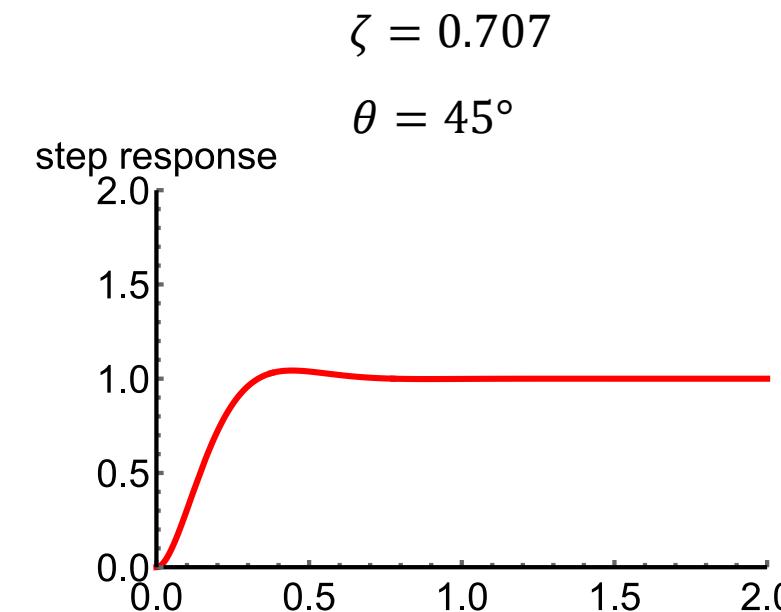
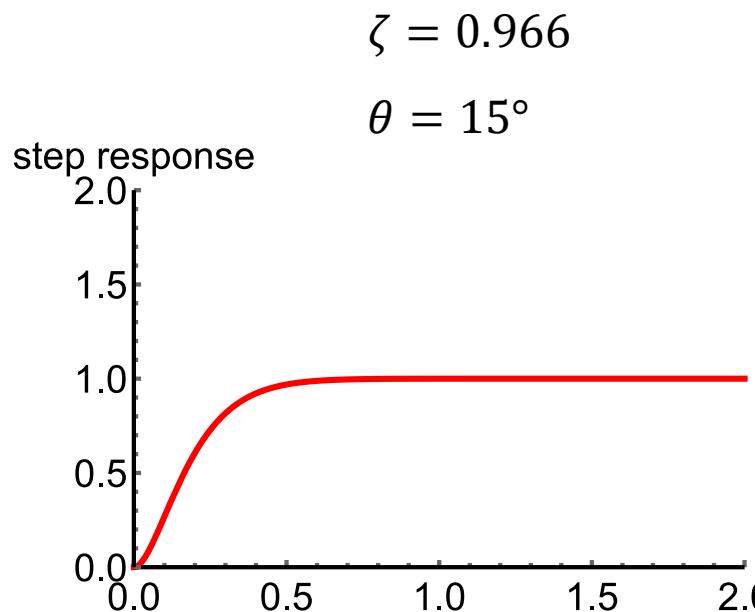
Close to undamped

Time domain: Step response – influence of damping factor

$$Y_{step}(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Table 6.1-10c:

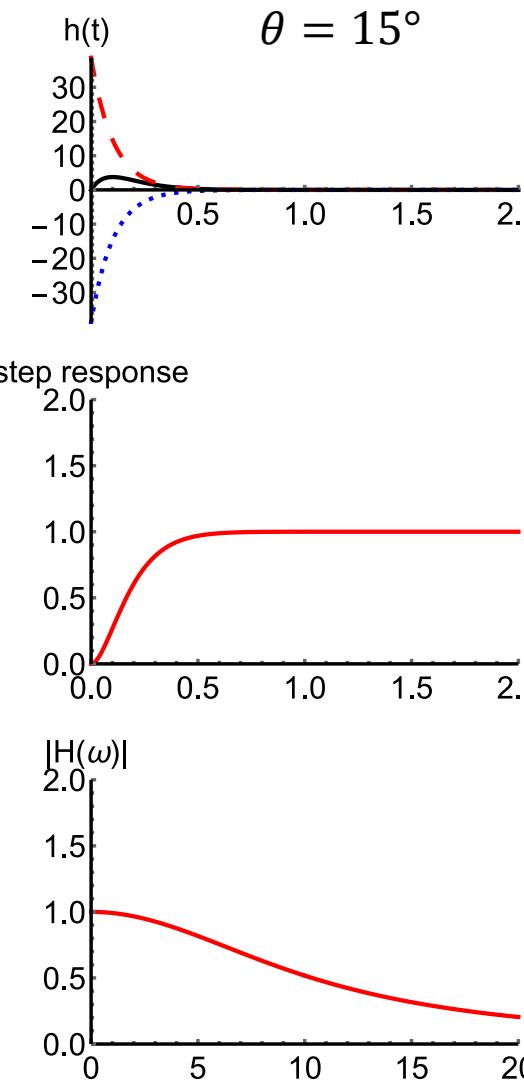
$$y_{step}(t) = \left(1 - \frac{1}{\sqrt{1-\zeta^2}} \cdot e^{-\zeta\omega_n t} \sin \left[(\omega_n \sqrt{1-\zeta^2}) t + \cos^{-1}(\zeta) \right] \right) u(t)$$



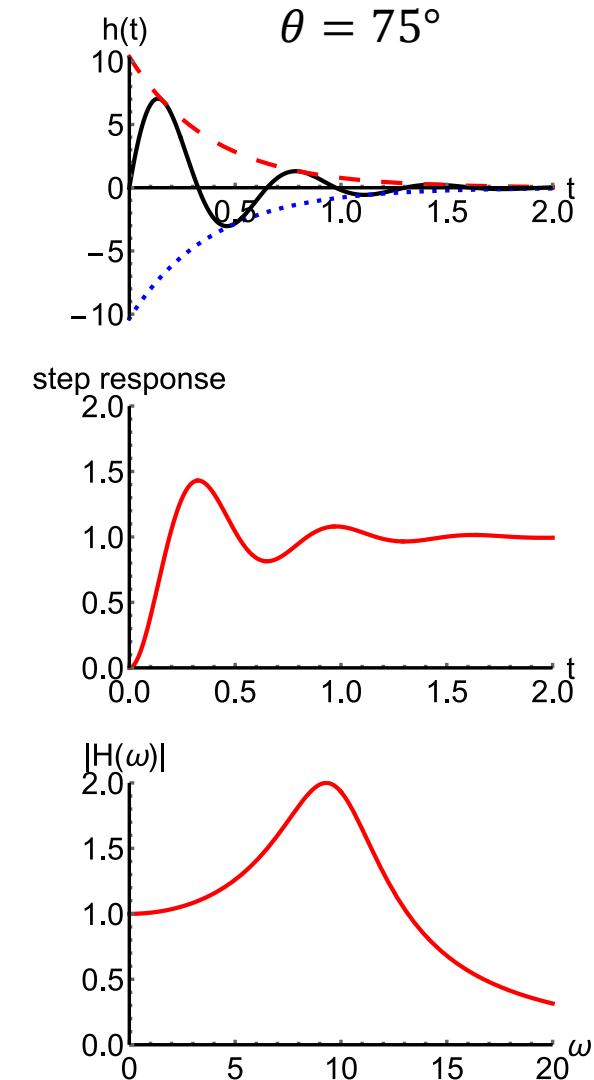
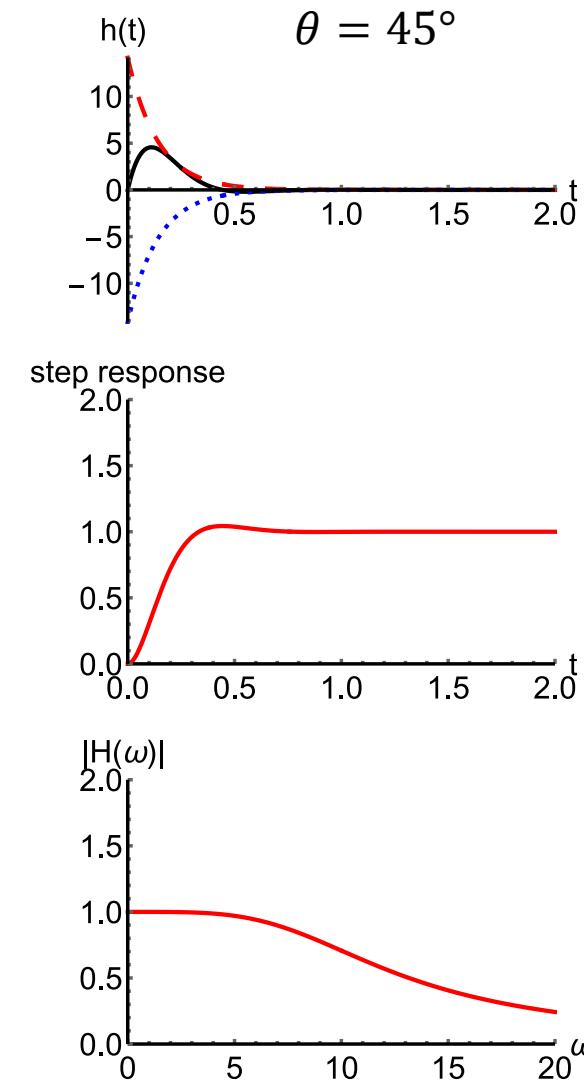
Close to critically damped

Close to undamped

Response for various degrees of damping



Close to critically damped



Close to undamped

[Next topic](#)

Step response features

Video 2

We have seen on the previous slides, that different system parameters yield different system responses.

Can we flip this around and determine system parameters from systems response curves?

If we could, then we would be able to conduct an experiment with an unknown system, apply a step input, determine the system's response and from that write the differential equation for the system.

To do so, we need to define quantitative features of the system response. With these definitions in place, we may be able to determine what values the system parameters need to have to meet system performance criteria expressed in terms of quantitative time domain response features.

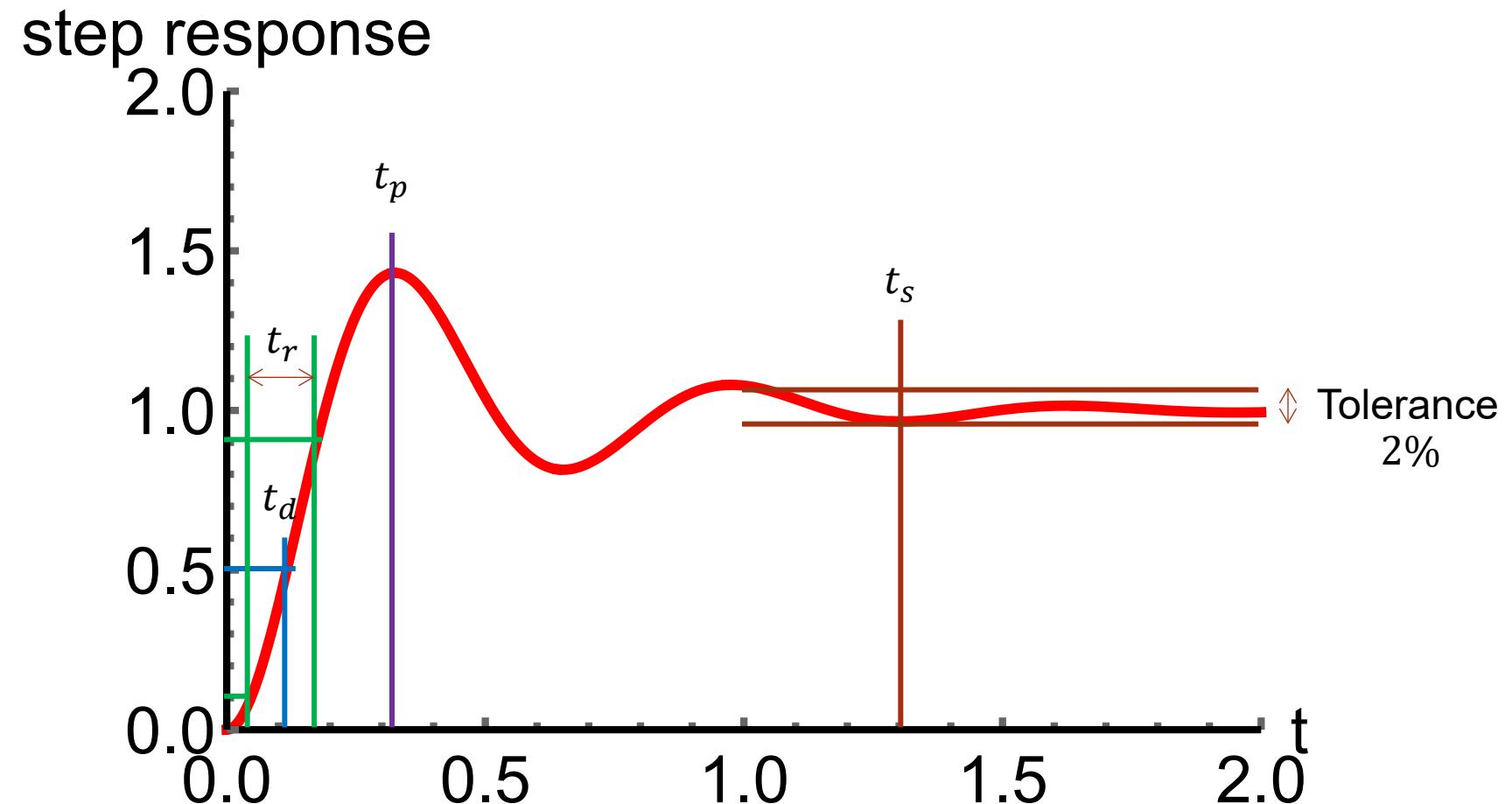
One set of response features are the **response times** measurable on the step response.

Rise time (10 to 90%): t_r

Delay time(time to 50%: t_d

Time to peak: t_p

2% settling time: t_s



Response times derived from the analytical expression for the step response.

$$y_{step}(t) = \left(1 - \frac{1}{\sqrt{1-\zeta^2}} \cdot e^{-\zeta\omega_n t} \sin \left[(\omega_n \sqrt{1-\zeta^2}) t + \cos^{-1}(\zeta) \right] \right) u(t)$$

2% settling time:

$$t_s = 4\tau = \frac{4}{\zeta\omega_n}$$

Time to peak:

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} = \frac{\pi}{2\pi f_d} = \frac{T_d}{2}$$

% overshoot:

$$PO = \frac{y(t_p) - y_\infty}{y_\infty} \cdot 100\% = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

Rise time:

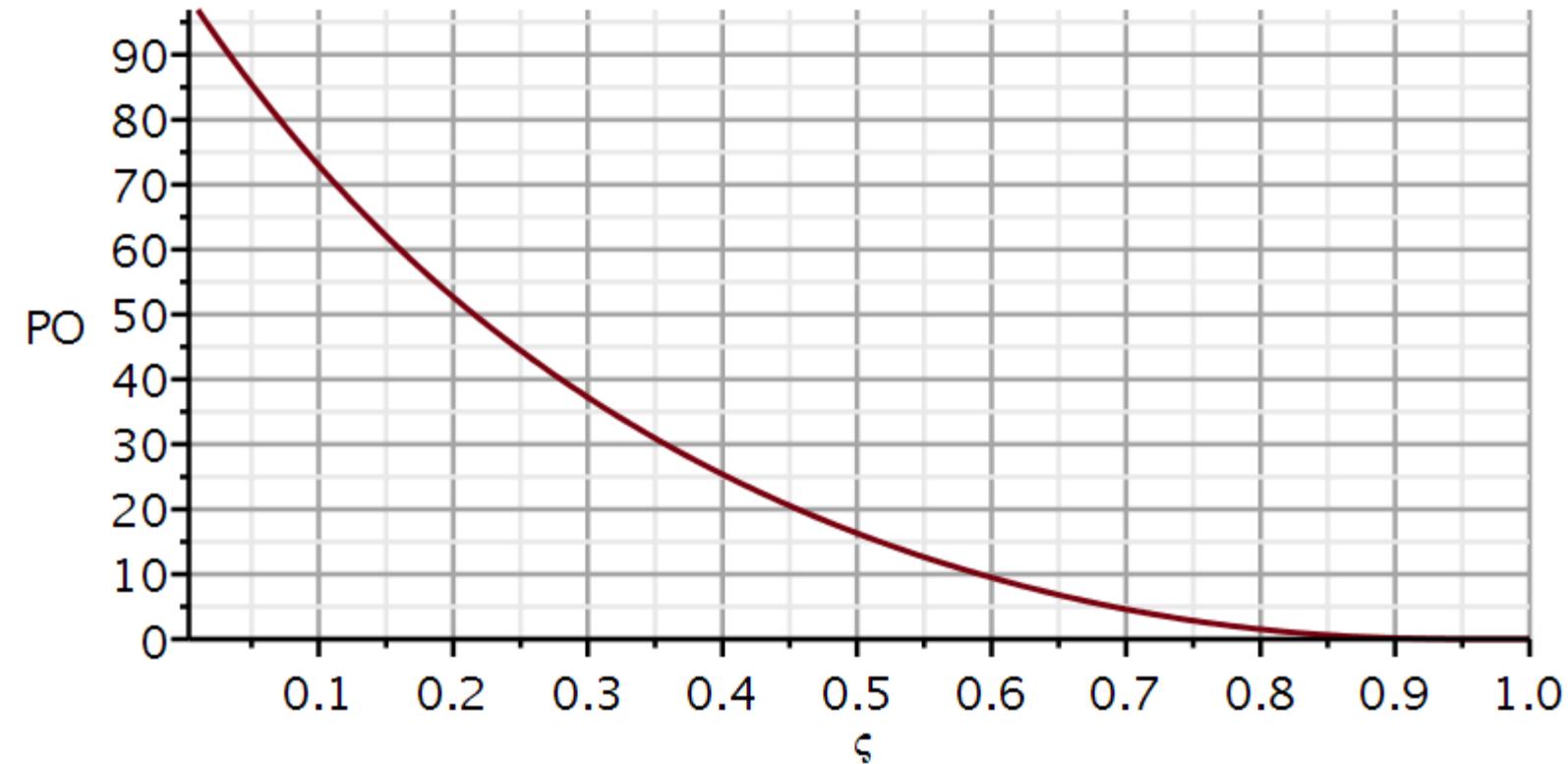
$$t_r \approx \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n}$$

Delay time:

$$t_d \approx \frac{1.1 + 0.125\zeta + 0.469\zeta^2}{\omega_n}$$

Estimating damping ratio from percent overshoot

The damping factor ζ can be determined independently from the **percent overshoot**.



Approach:

$$PO \rightarrow \zeta; t_p \rightarrow \omega_n$$

$$PO = \frac{y(t_p) - y_\infty}{y_\infty} \cdot 100\% = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

Step 1:

Read PO on response curve and convert to percentage.

Step 2:

Calculate damping ratio: $a \stackrel{\text{def}}{=} \ln\left(\frac{PO}{100}\right) \rightarrow \zeta = \frac{|a|}{\sqrt{\pi^2 + a^2}}$

Step 3:

Read time to peak on response curve: t_p

Step 4:

Calculate:

$$\omega_n = \frac{\pi}{t_p \sqrt{1 - \zeta^2}}$$

Remember:

This algorithm is only applicable to a second order **underdamped** system.

Example: System Identification using quantitative features of the step response

Write the differential equation for this system.

$$PO \approx 72\%, NOT 172\%$$

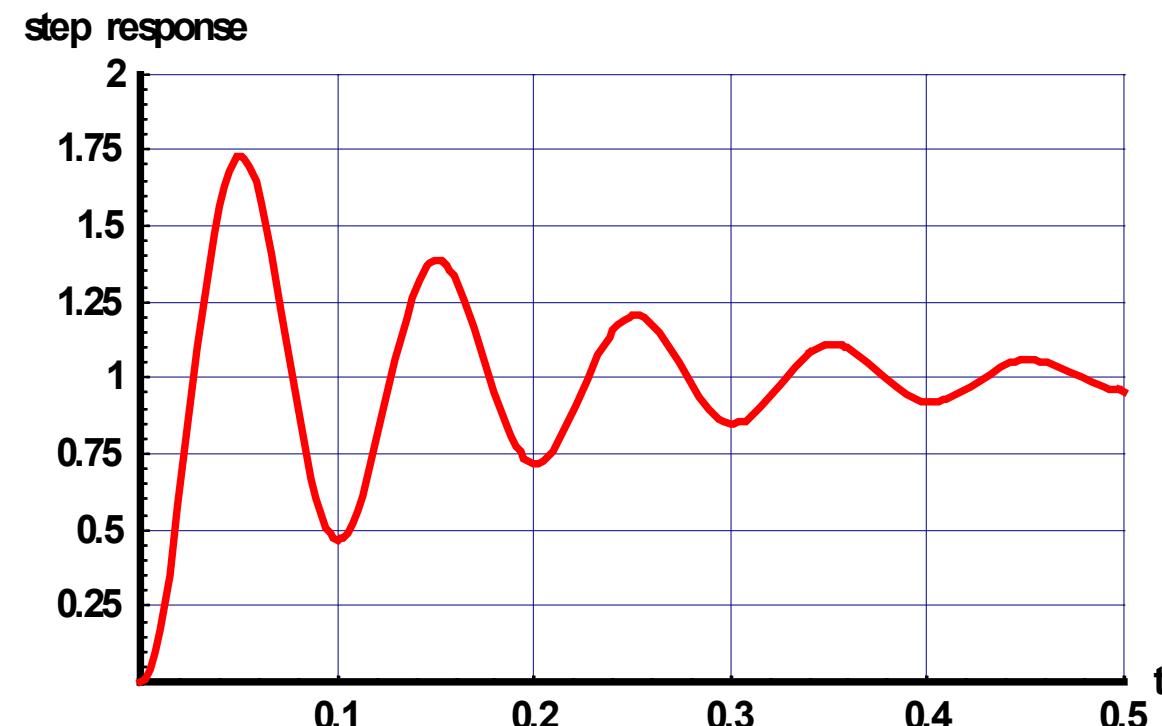
→ $\zeta \approx 0.104$

$$t_p \approx 0.05$$

$$\omega_n \approx 63.2$$

$$a_0 = \omega_n^2 \approx 3990$$

$$a_1 = 2\zeta\omega_n \approx 13.1$$



$$\ddot{y}(t) + 13.1\dot{y}(t) + 3990y(t) = 3990x(t)$$

What about the right-hand side?

Putting it all together

We have seen that, in case of the underdamped system, the system response and its frequency characteristics are all determined by two parameters: ζ and ω_n , which again are determined from the positions of the complex conjugated poles.

It is therefore a very useful skill to be able to anticipate changes in systems response and frequency characteristics caused by simple displacements of the poles.

The many plots on the next slide illustrate what happens to the impulse response, the step response, and the amplitude characteristics, when we move the poles along certain simple trajectories.

Try and see, if you can make heads and tails of the interconnections between plots.

Hint. Try and figure out if moving the poles along a certain trajectory makes the system more or less damped.

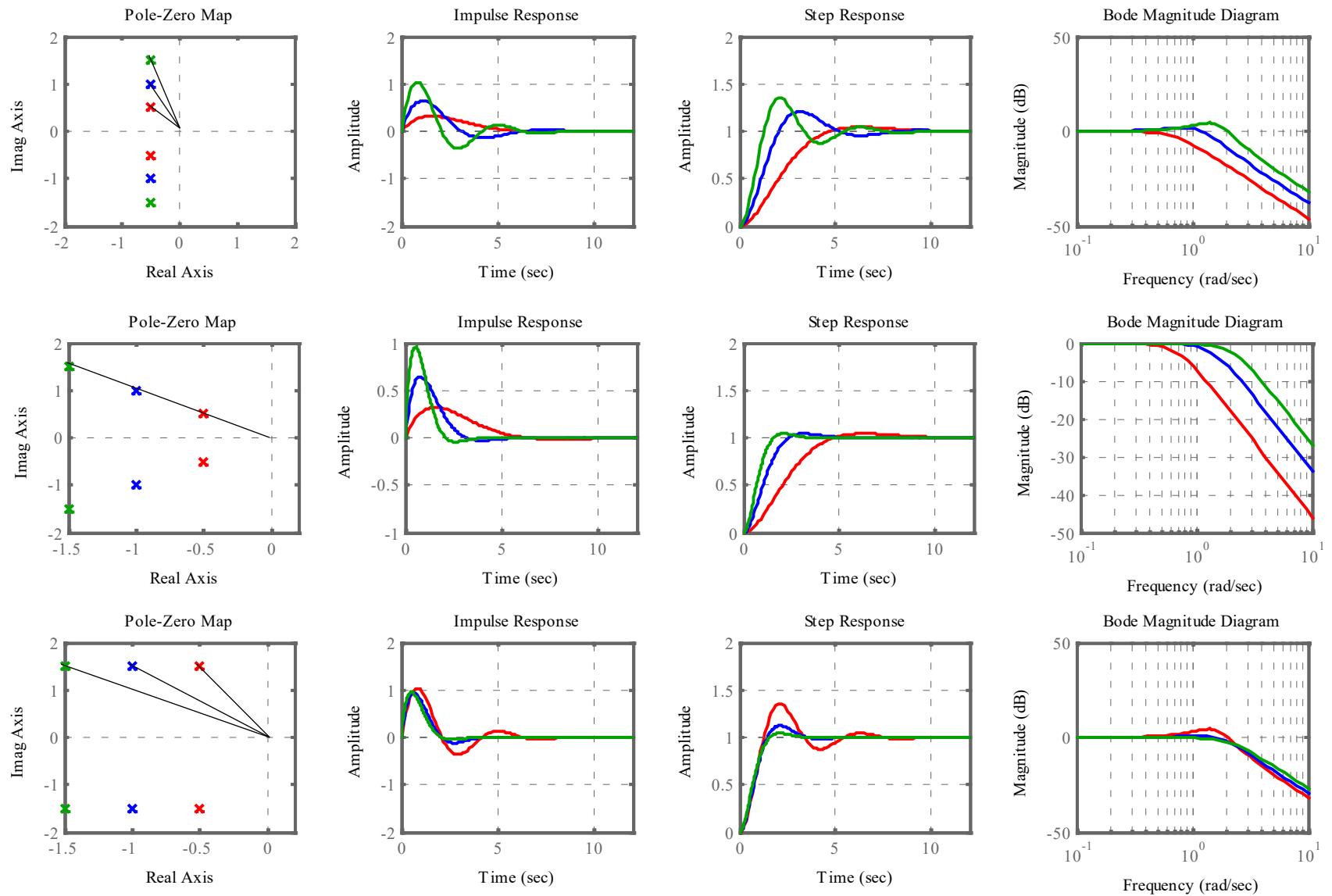
Pole Influence on Dynamic System Characteristics

What is common:

Constant $\zeta \omega_n$

Constant θ
Constant ζ

Constant ω_d



Maple feature in DynamicSystems

Maple's DynamicSystems can calculate features on the step response.

Here we do it ourself first for comparison.

$$\begin{aligned}s_{1,2} &= -p_{1,2} = -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0} = -\zeta\omega_n \pm \sqrt{(\zeta^2 - 1)\omega_n^2} \\ &= -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} = -\zeta\omega_n \pm j\omega_d\end{aligned}$$

Dynamic systems 1

```
restart
with(plots):
with(DynamicSystems):
vin := Step(1, 0, 0):
p1 := -1 + j·1 :: p2 := -1 - j·1 :: expand((s - p1) · (s - p2))
```

$$s^2 + 2s + 2$$

```
a1 := -2·Re(p1)
```

$$a1 := 2$$

```
a0 := (Re(p1))^2 + (Im(p1))^2
```

$$a0 := 2$$

```
 $\zeta := evalf\left(\frac{a1}{2\sqrt{a0}}\right)$ 
```

$$\zeta := 0.7071067810$$

```
 $\omega_n := evalf(\sqrt{a0})$ 
```

$$\omega_n := 1.414213562$$

```
 $t_s := evalf\left(\frac{4}{\zeta \cdot \omega_n}\right)$ 
```

$$t_s := 4.000000004$$

Maple feature in DynamicSystems

$$t_p := \text{evalf}\left(\frac{\pi}{\omega_n \cdot \sqrt{1 - \varsigma^2}}\right)$$

$$t_p := 3.141592654$$

$$PO := 100 \cdot e^{-\varsigma \cdot \frac{\pi}{\sqrt{1 - \varsigma^2}}}$$

$$PO := 4.321391829$$

$$t_r := \text{evalf}\left(\frac{1 - 0.5167 \cdot \varsigma + 2.917 \cdot \varsigma^2}{\omega_n}\right)$$

$$t_r := 1.480072021$$

$$t_d := \text{evalf}\left(\frac{1.1 + 0.125 \cdot \varsigma + 0.469 \cdot \varsigma^2}{\omega_n}\right)$$

$$t_d := 1.006134000$$

$$Ts := 0.05 :$$

$$dur := 10 :$$

$$Ns := \text{round}\left(\frac{dur}{Ts}\right) :$$

$$sys1 := \text{TransferFunction}\left(\frac{(\omega_n)^2}{s^2 + 2 \cdot \varsigma \cdot \omega_n \cdot s + (\omega_n)^2}\right) :$$

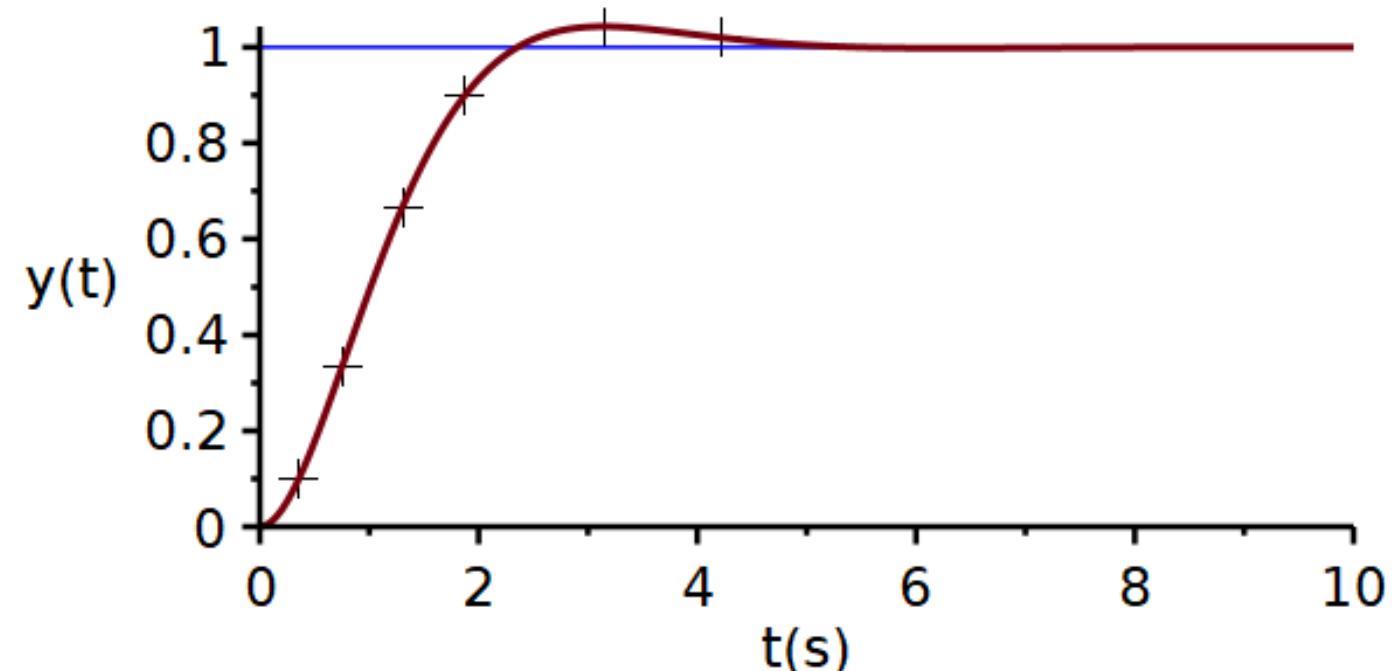
Now define the system:

Using the function StepProperties:

```
prop := StepProperties(sys1) :  
prop  
1., [0.357403497705475, 0.100000000000000], [0.751462782947410, 0.333333333333333],  
[1.30413573538299, 0.666666666666667], [1.87629572625499, 0.900000000000000],  
[3.14159265465485, 1.04321391828416], [4.21618403228512, 1.020000000000000]  
Final value  
prop[1]  
1.  
10% of final value  
prop[2]  
[0.357403497705475, 0.100000000000000]  
33% of final value  
prop[3]  
[0.751462782947410, 0.333333333333333]  
67% of final value  
prop[4]  
[1.30413573538299, 0.666666666666667]  
90% of final value  
prop[5]  
[1.87629572625499, 0.900000000000000]  $t_r := 1.480072021$   
peak value  
prop[6]  
[3.14159265465485, 1.04321391828416]  $t_p := 3.141592654$   
Settling value  
prop[7]  
[4.21618403228512, 1.020000000000000]  $t_s := 4.000000004$ 
```

Maple feature in DynamicSystems

```
plots[display](plot([prop[2..7]], style=point, symbol=cross, color=black, symbolsize=30),  
plot([[0,prop[1]], [dur,prop[1]]], color=blue), ResponsePlot(sys1, Step(), duration=dur,  
numpoints=200, thickness=3.0), axis[2]=[thickness=2.5], axis[1]=[mode=linear, thickness  
=2.5], font=[Helvetica, roman, 18], labels=[ "t(s)", "y(t)" ], labelfont=[ "HELVETICA ", 18 ],  
size=[ 600, 300])
```



Here the step response is plotted together with the measured features marked as a "+".

Bode Plot 1: Conversion of $H(s)$ to Bode plot form

Video 2

Transfer function in factored form

The transfer function can be written as **rational functions**, i.e., as a ratio of two polynomials.

The numerator and denominator can be factored:

There may be zero and poles at the origin:

We can combine all gain factors in one:

Complex conjugated factors are kept as quadratic factors:

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$H(s) = K \frac{(s + b_1)(s + b_2) \cdots (s + b_m)}{(s + a_1)(s + a_2) \cdots (s + a_n)}$$

$$H(s) = K \frac{s^p (s + b_1)(s + b_2) \cdots (s + b_{m-p})}{s^q (s + a_1)(s + a_2) \cdots (s + a_{n-q})}$$

$$H(s) = K \frac{b_1 b_2 \cdots b_{m-p}}{a_1 a_2 \cdots a_{n-q}} \frac{s^p \left(\frac{s}{b_1} + 1\right) \left(\frac{s}{b_2} + 1\right) \cdots \left(\frac{s}{b_{m-p}} + 1\right)}{s^q \left(\frac{s}{a_1} + 1\right) \left(\frac{s}{a_2} + 1\right) \cdots \left(\frac{s}{a_{n-q}} + 1\right)}$$

$$H(s) = K \frac{b_1 b_2^{m_2} b_4^{m_3}}{a_1 a_2^{n_2} a_4^{n_3}} \frac{s^p \left(\frac{s}{b_1} + 1\right) \left(\frac{s}{b_2} + 1\right)^{m_2} \left(\frac{s^2}{b_4} + \frac{b_3}{b_4} s + 1\right)^{m_3}}{s^q \left(\frac{s}{a_1} + 1\right) \left(\frac{s}{a_2} + 1\right)^{n_2} \left(\frac{s^2}{a_4} + \frac{a_3}{a_4} s + 1\right)^{n_3}}$$

Standard form for Bode plot

The s-notation form is valid all over the complex number plane:

We need the frequency characteristic and insert $j\omega \rightarrow s$

We need to plot the amplitude and phase spectra in separate plots:

$$H(s) = K \frac{b_1 b_2^{m_2} b_4^{m_3}}{a_1 a_2^{n_2} a_4^{n_3}} \frac{s^p \left(\frac{s}{b_1} + 1 \right) \left(\frac{s}{b_2} + 1 \right)^{m_2} \left(\frac{s^2}{b_4} + \frac{b_3}{b_4} s + 1 \right)^{m_3}}{s^q \left(\frac{s}{a_1} + 1 \right) \left(\frac{s}{a_2} + 1 \right)^{n_2} \left(\frac{s^2}{a_4} + \frac{a_3}{a_4} s + 1 \right)^{n_3}}$$

$$H(j\omega) = K \frac{b_1 b_2^{m_2} b_4^{m_3}}{a_1 a_2^{n_2} a_4^{n_3}} \frac{(j\omega)^p \left(\frac{j\omega}{b_1} + 1 \right) \left(\frac{j\omega}{b_2} + 1 \right)^{m_2} \left(\frac{(j\omega)^2}{b_4} + \frac{b_3}{b_4} j\omega + 1 \right)^{m_3}}{(j\omega)^q \left(\frac{j\omega}{a_1} + 1 \right) \left(\frac{j\omega}{a_2} + 1 \right)^{n_2} \left(\frac{(j\omega)^2}{a_4} + \frac{a_3}{a_4} j\omega + 1 \right)^{n_3}}$$

$$|H(j\omega)| = K \frac{b_1 b_2^{m_2} b_4^{m_3}}{a_1 a_2^{n_2} a_4^{n_3}} \frac{|j\omega|^p \left| \frac{j\omega}{b_1} + 1 \right| \left| \frac{j\omega}{b_2} + 1 \right|^{m_2} \left| \frac{(j\omega)^2}{b_4} + \frac{b_3}{b_4} j\omega + 1 \right|^{m_3}}{|j\omega|^q \left| \frac{j\omega}{a_1} + 1 \right| \left| \frac{j\omega}{a_2} + 1 \right|^{n_2} \left| \frac{(j\omega)^2}{a_4} + \frac{a_3}{a_4} j\omega + 1 \right|^{n_3}}$$

$$\begin{aligned} \angle H(\omega) &= p \angle(j\omega) + \angle \left(\frac{j\omega}{b_1} + 1 \right) + m_2 \angle \left(\frac{j\omega}{b_2} + 1 \right) + m_3 \angle \left(\frac{(j\omega)^2}{b_4} + \frac{b_3}{b_4} j\omega + 1 \right) \\ &\quad - q \angle(j\omega) - \angle \left(\frac{j\omega}{a_1} + 1 \right) - n_2 \angle \left(\frac{j\omega}{a_2} + 1 \right) - n_3 \angle \left(\frac{(j\omega)^2}{a_4} + \frac{a_3}{a_4} j\omega + 1 \right) \end{aligned}$$

Standard form for Bode plot

Amplitude characteristic:

$$|H(j\omega)| = K \frac{b_1 b_2^{m_2} b_4^{m_3}}{a_1 a_2^{n_2} a_4^{n_3}} \frac{|j\omega|^p \left| \frac{j\omega}{b_1} + 1 \right| \left| \frac{j\omega}{b_2} + 1 \right|^{m_2} \left| \frac{(j\omega)^2}{b_4} + \frac{b_3}{b_4} j\omega + 1 \right|^{m_3}}{|j\omega|^q \left| \frac{j\omega}{a_1} + 1 \right| \left| \frac{j\omega}{a_2} + 1 \right|^{n_2} \left| \frac{(j\omega)^2}{a_4} + \frac{a_3}{a_4} j\omega + 1 \right|^{n_3}}$$

We convert amplitude to decibel and make use of:

$$20 \log_{10} \left(\frac{x \cdot y^m}{z^n} \right) = 20 \log_{10} x + m \cdot 20 \log_{10} y - n \cdot 20 \log_{10} z$$

Now each factor has become a term:

$$\begin{aligned} |H(j\omega)|_{dB} &= 20 \log_{10} K \frac{b_1 b_2^{m_2} b_4^{m_3}}{a_1 a_2^{n_2} a_4^{n_3}} + p \times 20 \log_{10} |j\omega| + 20 \log_{10} \left| \frac{j\omega}{b_1} + 1 \right| \\ &\quad + m_2 \times 20 \log_{10} \left| \frac{j\omega}{b_2} + 1 \right| + m_3 \times 20 \log_{10} \left| \frac{(j\omega)^2}{b_4} + \frac{b_3}{b_4} j\omega + 1 \right| \\ &\quad - q \times 20 \log_{10} |j\omega| - 20 \log_{10} \left| \frac{j\omega}{a_1} + 1 \right| - n_2 \times 20 \log_{10} \left| \frac{j\omega}{a_2} + 1 \right| \\ &\quad - n_3 \times 20 \log_{10} \left| \frac{(j\omega)^2}{a_4} + \frac{a_3}{a_4} j\omega + 1 \right| \end{aligned}$$

Standard form for Bode plot

The advantage of taking \log_{10} is that each factor now becomes a term. We can now draw an amplitude and phase curve for each factor, and then we can add the curves together graphically.

As each term may originate from a specific sub-system, we can easily identify the behavior of each sub-system and then correct those where needed.

Furthermore, if we decide to add a correcting subsystem into our system, we just add its Bode plot to the Bode plot of the original system.

$$|H(j\omega)|_{dB} = \underbrace{20 \log_{10} K \frac{b_1 b_2^{m_2} b_4^{m_3}}{a_1 a_2^{n_2} a_4^{n_3}}}_{\text{constant}} + \underbrace{p \times 20 \log_{10} |j\omega|}_{\text{zero at origin}} + \underbrace{20 \log_{10} \left| \frac{j\omega}{b_1} + 1 \right|}_{\text{first order zero}} \\ + \underbrace{m_2 \times 20 \log_{10} \left| \frac{j\omega}{b_2} + 1 \right|}_{\text{repeated first order zero}} + \underbrace{m_3 \times 20 \log_{10} \left| \frac{(j\omega)^2}{b_4} + \frac{b_3}{b_4} j\omega + 1 \right|}_{\text{repeated second order zero}} \\ - \underbrace{q \times 20 \log_{10} |j\omega|}_{\text{pole at origin}} - \underbrace{20 \log_{10} \left| \frac{j\omega}{a_1} + 1 \right|}_{\text{first order pole}} - \underbrace{n_2 \times 20 \log_{10} \left| \frac{j\omega}{a_2} + 1 \right|}_{\text{repeated first order pole}} \\ - \underbrace{n_3 \times 20 \log_{10} \left| \frac{(j\omega)^2}{a_4} + \frac{a_3}{a_4} j\omega + 1 \right|}_{\text{repeated second order pole}}$$

$$\angle H(\omega) = \underbrace{p \angle(j\omega)}_{\text{zero at origin}} + \underbrace{\angle \left(\frac{j\omega}{b_1} + 1 \right)}_{\text{first order zero}} + \underbrace{m_2 \angle \left(\frac{j\omega}{b_2} + 1 \right)}_{\text{repeated first order zero}} + \underbrace{m_3 \angle \left(\frac{(j\omega)^2}{b_4} + \frac{b_3}{b_4} j\omega + 1 \right)}_{\text{repeated second order zero}} \\ - \underbrace{q \angle(j\omega)}_{\text{pole at origin}} - \underbrace{\angle \left(\frac{j\omega}{a_1} + 1 \right)}_{\text{first order pole}} - \underbrace{n_2 \angle \left(\frac{j\omega}{a_2} + 1 \right)}_{\text{repeated first order pole}} - \underbrace{n_3 \angle \left(\frac{(j\omega)^2}{a_4} + \frac{a_3}{a_4} j\omega + 1 \right)}_{\text{repeated second order pole}}$$

Example 1

Rewrite transfer function into Bode plot form

$$H(s) = 10 \frac{s(s+1)}{(s+2)(s^2 + 2s + 100)^2} = 10 \cdot \frac{1}{2 \cdot 100^2} \cdot \frac{\frac{s}{10} \left(\frac{s}{10} + 1 \right)}{\left(\frac{s}{2} + 1 \right) \left(\left(\frac{s}{10} \right)^2 + \frac{2}{10} \frac{s}{10} + 1 \right)^2}$$

$$H(j\omega) = 5 \cdot 10^{-3} \cdot \frac{j\omega}{1} \cdot \left(\frac{j\omega}{1} + 1 \right) \cdot \left(\frac{j\omega}{2} + 1 \right)^{-1} \cdot \left(\left(\frac{j\omega}{10} \right)^2 + \frac{2}{10} \frac{j\omega}{10} + 1 \right)^{-2}$$

$$\begin{aligned} |H(j\omega)|_{dB} &= 20 \log_{10} 5 \cdot 10^{-3} + 20 \log_{10} \frac{j\omega}{1} + 20 \log_{10} \left(\frac{j\omega}{1} + 1 \right) \\ &\quad - 1 \cdot 20 \log_{10} \left(\frac{j\omega}{2} + 1 \right) - 2 \cdot 20 \log_{10} \left(\left(\frac{j\omega}{10} \right)^2 + \frac{2}{10} \frac{j\omega}{10} + 1 \right) \end{aligned}$$

$$\angle H(\omega) = \angle(j\omega) + \angle \left(\frac{j\omega}{1} + 1 \right) - \angle \left(\frac{j\omega}{2} + 1 \right) - 2 \angle \left(\left(\frac{j\omega}{10} \right)^2 + \frac{2}{10} \frac{j\omega}{10} + 1 \right)$$

Example 2

Rewrite into
standard form
for Bode Plot

$$H(s) = 50 \frac{(s+5)(s^2+s+25)}{s(s+4)^2(s^2+10s+100)} = 50 \cdot \frac{5 \cdot 25}{4^2 \cdot 100} \cdot \frac{\left(\frac{s}{5}+1\right)\left(\left(\frac{s}{5}\right)^2+\frac{1}{5}\frac{s}{5}+1\right)}{\frac{s}{1}\left(\frac{s}{4}+1\right)^2\left(\left(\frac{s}{10}\right)^2+\frac{10}{10}\frac{s}{10}+1\right)}$$

$$H(j\omega) = 50 \cdot \frac{5 \cdot 25}{4^2 \cdot 100} \cdot \left(\frac{j\omega}{5}+1\right) \left(\left(\frac{j\omega}{5}\right)^2+\frac{1}{5}\frac{j\omega}{5}+1\right) \left(\frac{j\omega}{1}\right)^{-1} \left(\frac{j\omega}{4}+1\right)^{-2} \left(\left(\frac{j\omega}{10}\right)^2+\frac{10}{10}\frac{j\omega}{10}+1\right)^{-1}$$

$$|H(j\omega)|_{dB} = 20 \log_{10} \left(50 \cdot \frac{5 \cdot 25}{4^2 \cdot 100} \right) + 20 \log_{10} \left(\frac{j\omega}{5}+1 \right) + 20 \log_{10} \left(\left(\frac{j\omega}{5}\right)^2+\frac{1}{5}\frac{j\omega}{5}+1 \right)$$

$$-20 \log_{10} \left(\frac{j\omega}{1} \right) - 2 \cdot 20 \log_{10} \left(\frac{j\omega}{4}+1 \right) - 20 \log_{10} \left(\left(\frac{j\omega}{10}\right)^2+\frac{10}{10}\frac{j\omega}{10}+1 \right)$$

$$\angle H(\omega) = \angle \left(\frac{j\omega}{5}+1 \right) + \angle \left(\left(\frac{j\omega}{5}\right)^2+\frac{1}{5}\frac{j\omega}{5}+1 \right) - \angle \left(\frac{j\omega}{1} \right) - 2 \angle \left(\frac{j\omega}{4}+1 \right) - \angle \left(\left(\frac{j\omega}{10}\right)^2+\frac{10}{10}\frac{j\omega}{10}+1 \right)$$

Drawing Bode factors Constants and linear factors

Video 2

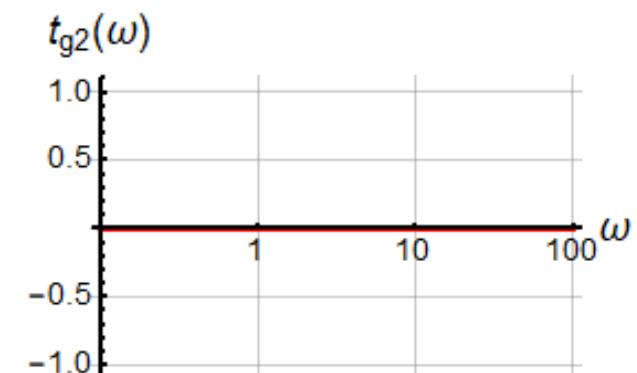
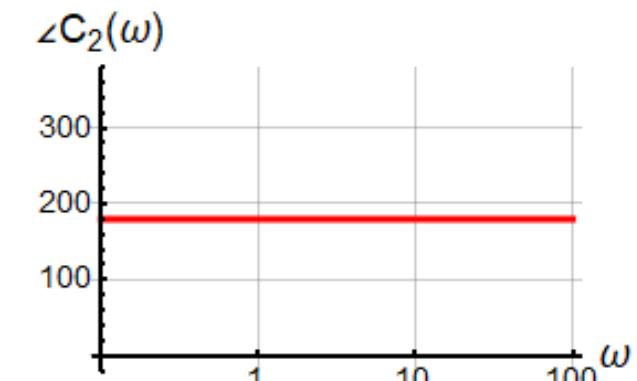
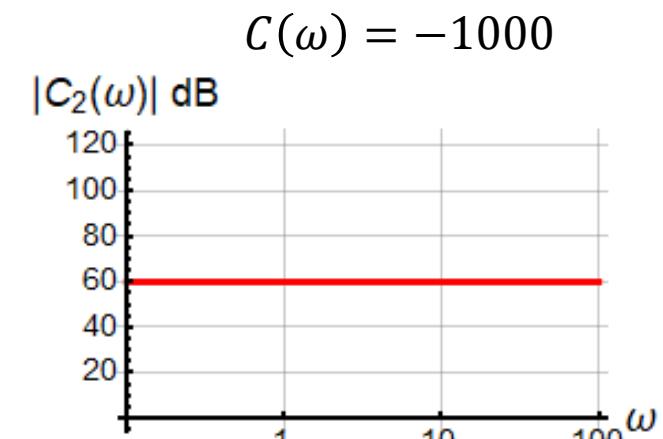
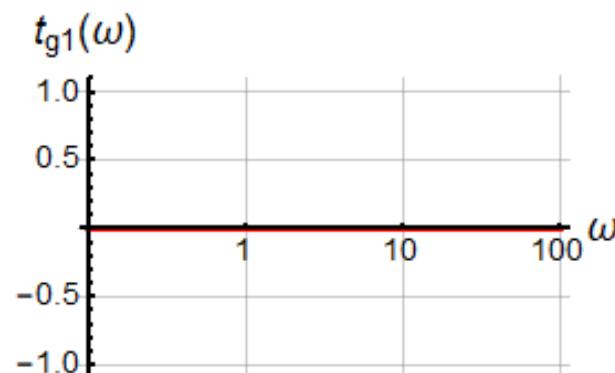
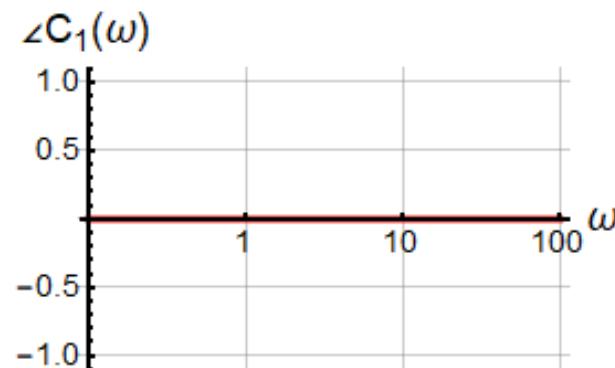
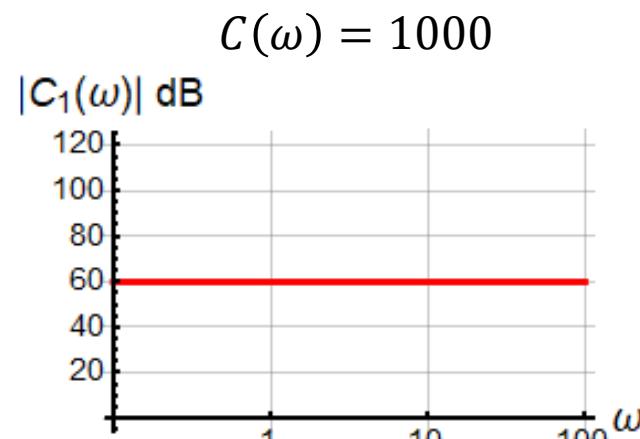
Constant Factor

$$|C(\omega)|_{dB} = 20 \cdot \log \left(K \frac{b_1 b_2^{m_2} b_4^{m_3}}{a_1 a_2^{n_2} a_4^{n_3}} \right)$$

$$\angle C(\omega) = \angle \left(K \frac{b_1 b_2^{m_2} b_4^{m_3}}{a_1 a_2^{n_2} a_4^{n_3}} \right)$$

$$= \begin{cases} 0 & K \frac{b_1 b_2^{m_2} b_4^{m_3}}{a_1 a_2^{n_2} a_4^{n_3}} \geq 0 \\ \pm 180^\circ & K \frac{b_1 b_2^{m_2} b_4^{m_3}}{a_1 a_2^{n_2} a_4^{n_3}} < 0 \end{cases}$$

A negative constant is represented as $|K|\angle 180^\circ$



Single $j\omega$ Factor

$$|\Omega(\omega)|_{dB} = m_1 \cdot 20 \cdot \log|\omega|$$

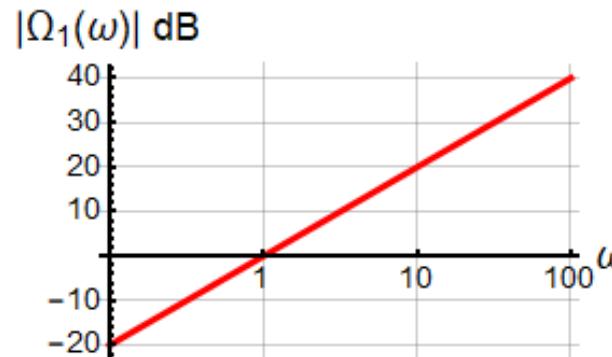
$$\angle\Omega(\omega) = m_1 \cdot \angle j\omega = m_1 \cdot 90^\circ$$

We are to plot the function $m_1 \times 20 \log_{10}|\omega|$ versus $\log_{10}|\omega|$.

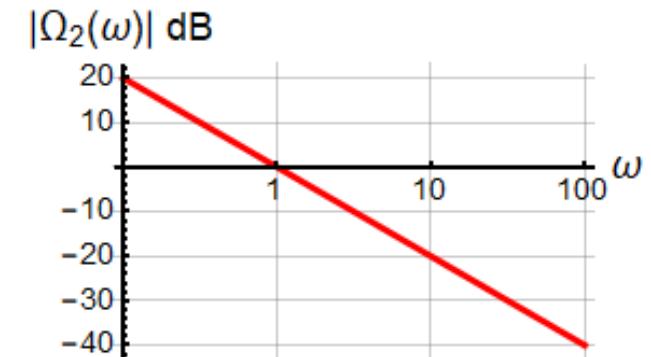
This gives a straight line with slope $\pm m_1 \times \frac{20 \text{ dB}}{\text{decade}}$.

The phase is simply $m_1 \times \angle \pm j = \pm m_1 \times 90^\circ$

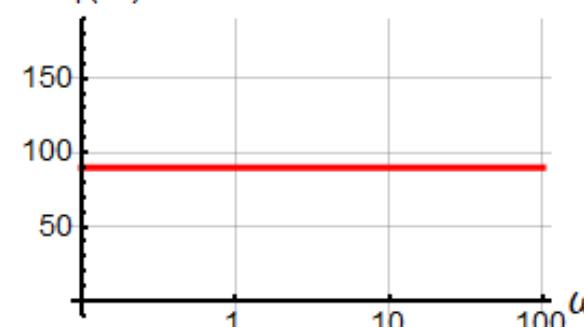
$$m_1 = 1$$



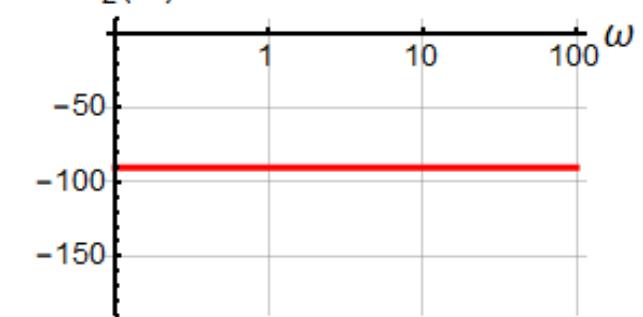
$$m_1 = -1$$



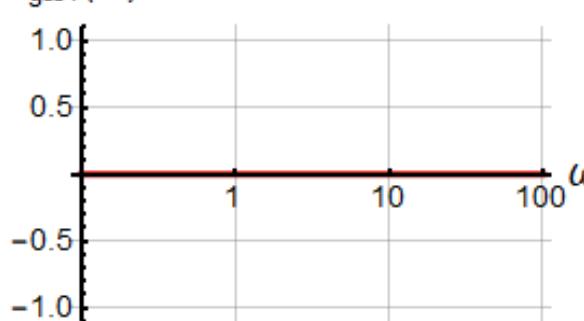
$$\angle\Omega_1(\omega)$$



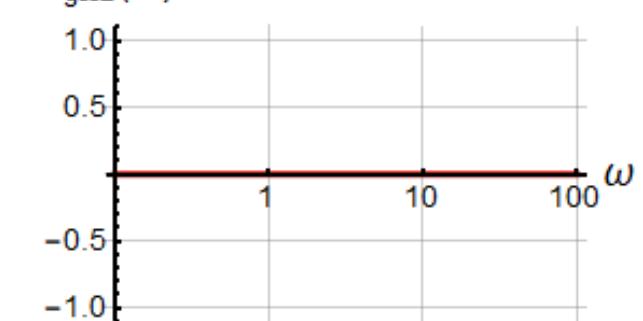
$$\angle\Omega_2(\omega)$$



$$t_{g\Omega 1}(\omega)$$



$$t_{g\Omega 2}(\omega)$$



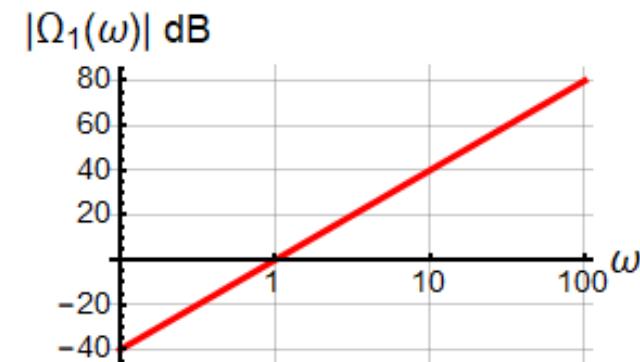
Double $j\omega$ Factor

$$|\Omega(\omega)|_{dB} = m_1 \cdot 20 \cdot \log|\omega|$$

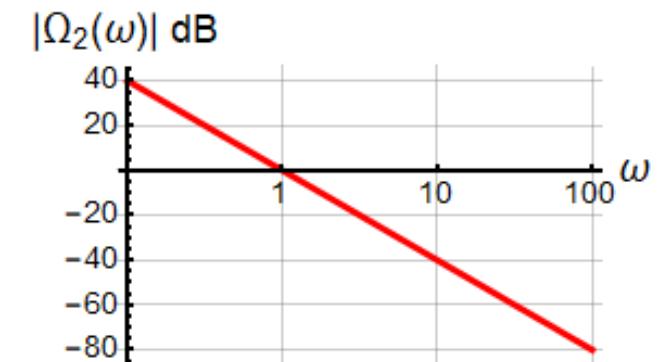
$$\angle\Omega(\omega) = m_1 \cdot \angle j\omega = m_1 \cdot 90^\circ$$

$$(j\omega)^{\pm 2}$$

$$m_1 = 2$$



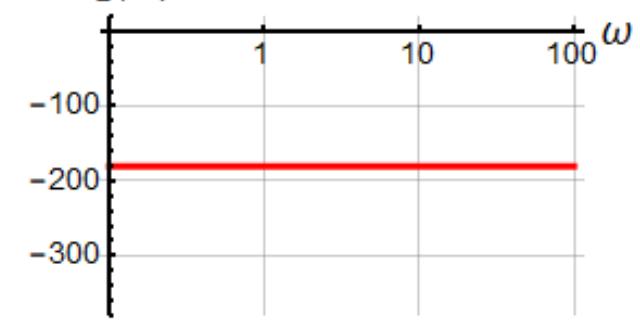
$$m_1 = -2$$



$$\angle\Omega_1(\omega)$$



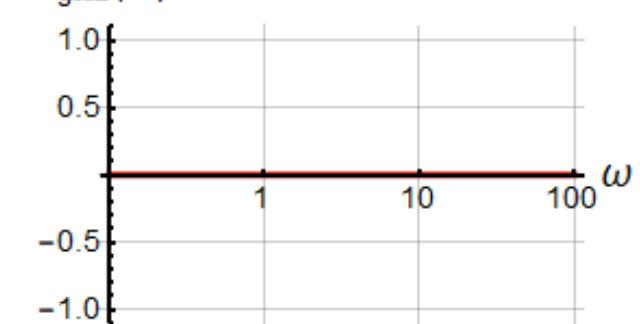
$$\angle\Omega_2(\omega)$$



$$t_{g\Omega 1}(\omega)$$



$$t_{g\Omega 2}(\omega)$$



Linear Factor Amplitude

We can write the amplitude of linear factors as:

$$\left(\frac{j\omega}{a_1} + 1\right)^{m_2} \Rightarrow |L(\omega)|_{dB} = m_2 \cdot 20 \cdot \log \left| \frac{j\omega}{a_1} + 1 \right|$$

To simplify the drawing of the linear function amplitude we consider the two cases:

Low frequency asymptote:

$$\omega \ll a_1 \quad |L(\omega)|_{dB} = m_2 \cdot 20 \cdot \log \left| \frac{j\omega}{a_1} + 1 \right| \approx m_2 \cdot 20 \cdot \log|1| = 0dB$$

High frequency asymptote:

$$\omega \gg a_1 \quad |L(\omega)|_{dB} = m_2 \cdot 20 \cdot \log \left| \frac{j\omega}{a_1} + 1 \right| \approx m_2 \cdot 20 \cdot \log \left| \frac{\omega}{a_1} \right|$$

Break frequency:

$$\omega_c = a_1 \quad |L(\omega_c)|_{dB} = m_2 \cdot 20 \log|j1 + 1| = m_2 \cdot 20 \log|\sqrt{2}| = m_2 \cdot 3dB$$

Low and high frequency asymptotes intersect at break frequency.

Again, we consider the two cases:

$$\left(\frac{j\omega}{a_1} + 1\right)^{m_2} \Rightarrow \angle L(\omega) = m_2 \cdot \angle\left(\frac{j\omega}{a_1} + 1\right) = m_2 \cdot \tan^{-1}\left(\frac{\omega}{a_1}\right)$$

Low frequency asymptote:

$$\omega \ll a_1 \quad \angle L(\omega) = m_2 \cdot \angle\left(\frac{j\omega}{a_1} + 1\right) = m_2 \cdot \angle 1 = 0^\circ$$

High frequency asymptote:

$$\omega \gg a_1 \quad \angle L(\omega) = m_2 \cdot \angle\left(\frac{j\omega}{a_1} + 1\right) = m_2 \cdot \angle\left(\frac{j\omega}{a_1}\right) = m_2 \cdot 90^\circ$$

Break frequency:

$$\omega_c = a_1 \quad \angle L(\omega_c) = m_2 \cdot \angle\left(\frac{j\omega_c}{a_1} + 1\right) = m_2 \cdot \angle(j1 + 1) = m_2 \cdot 45^\circ$$

The transition range is plotted as a straight line between:

$$\frac{\omega_c}{10} \leq \omega \leq 10 \cdot \omega_c$$

Group delay for linear factor

We can calculate the time delay of a frequency from the phase spectrum. It is the slope of the phase curve. The derivative is always a relative measure, as the constant reference disappears in the differentiation. It therefore gives the time delay of frequencies in one range relative to frequencies in another range. The frequencies in one range are thought of as a group of frequencies. Hence the delay is referred to as **group delay**.

A linear factor in the denominator causes a positive time delay. A linear factor in the numerator causes a negative time delay.

The largest time delay occurs at DC, where it is a multiple of the time constant.

No time delay occurs at infinite frequency, where all capacitors are short circuit, with only resistors left in the circuit.

$$H(\omega) = \frac{1}{(j \omega RC + 1)^{n_1}}$$

$$\angle H(\omega) = -n_1 \tan^{-1} \omega RC$$

$$t_g(\omega) = -n_1 \frac{d}{d\omega} \angle H(\omega) = -n_1 \frac{d}{d\omega} (-\tan^{-1} \omega RC)$$

$$t_g(\omega) = n_1 \frac{1}{(\omega RC)^2 + 1} \cdot \frac{d}{d\omega} (\omega RC)$$

$$t_g(\omega) = n_1 \frac{RC}{(\omega RC)^2 + 1}$$

$$t_g(0) = n_1 RC$$

$$t_g(\infty) = 0$$

Single Linear Factor

$$|L(\omega)|_{dB} = m_2 \cdot 20 \cdot \log \left| \frac{j\omega}{a_1} + 1 \right|$$

$$\begin{aligned}\angle L(\omega) &= m_2 \cdot \angle \left(\frac{j\omega}{a_1} + 1 \right) \\ &= m_2 \cdot \tan^{-1} \left(\frac{\omega}{a_1} \right)\end{aligned}$$

$$a_1 = 5 = \frac{1}{\tau}$$

Amplitude slope:

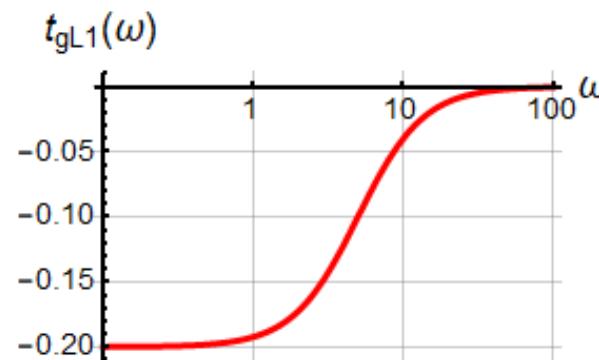
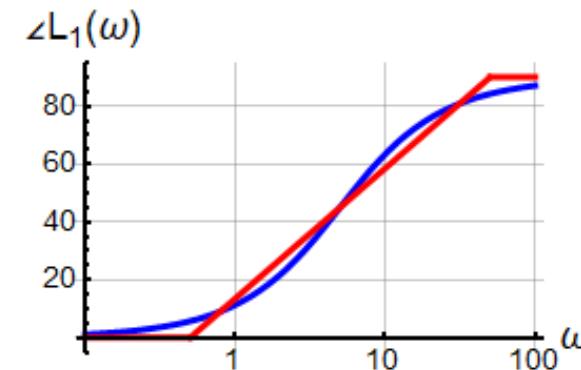
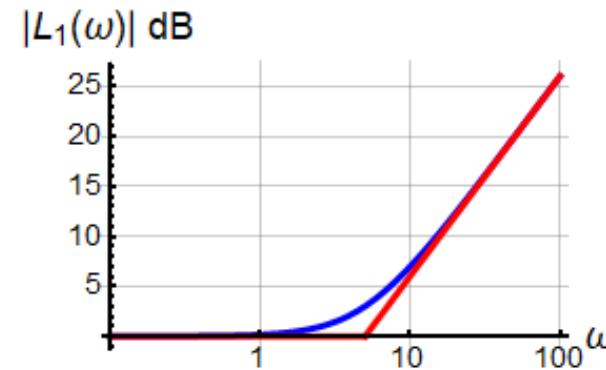
$$m_2 \cdot \frac{20 \text{ dB}}{\text{decade}}$$

$$m_2 \cdot \frac{6 \text{ dB}}{\text{octave}}$$

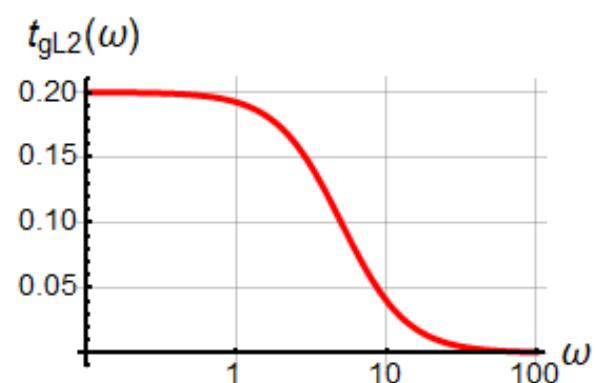
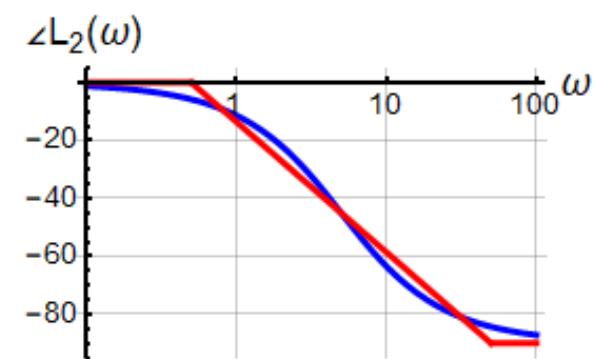
Time delay:

$$t_g = -m_2 \tau = -m_2 \times 0.2$$

$$m_2 = 1$$



$$m_2 = -1$$



Double Linear Factor

$$|L(\omega)|_{dB} = m_2 \cdot 20 \log \left| \frac{j\omega}{a_1} + 1 \right|$$

$$\begin{aligned}\angle L(\omega) &= m_2 \cdot \angle \left(\frac{j\omega}{a_1} + 1 \right) \\ &= m_2 \cdot \tan^{-1} \left(\frac{\omega}{a_1} \right)\end{aligned}$$

$$a_1 = 5 = \frac{1}{\tau}$$

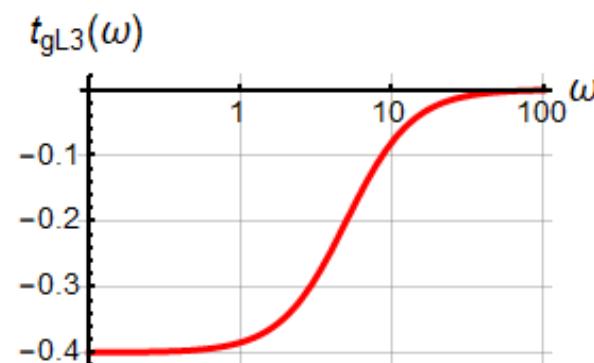
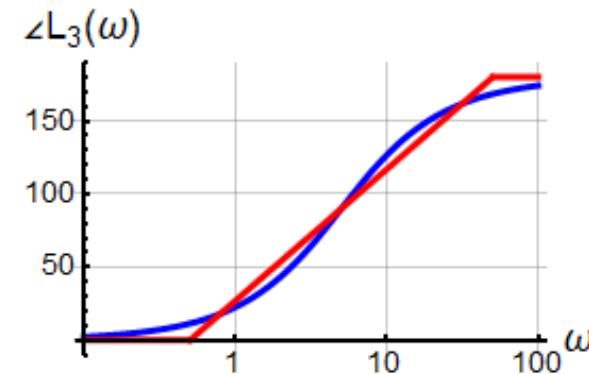
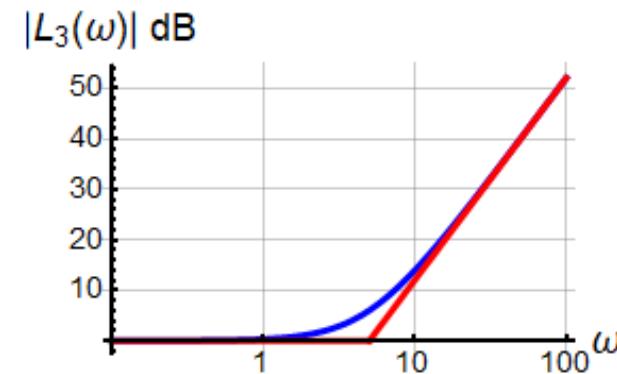
Amplitude slope:

$$m_2 \cdot \frac{20 \text{ dB}}{\text{decade}}$$

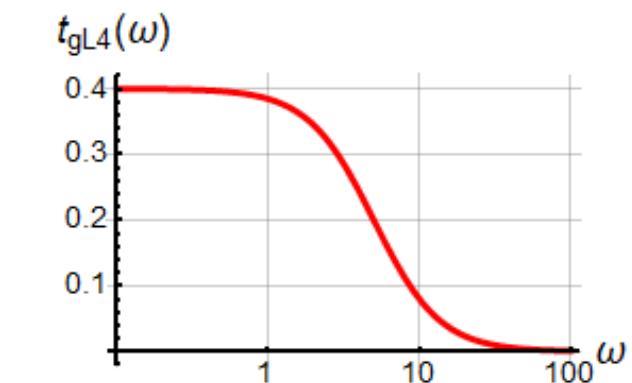
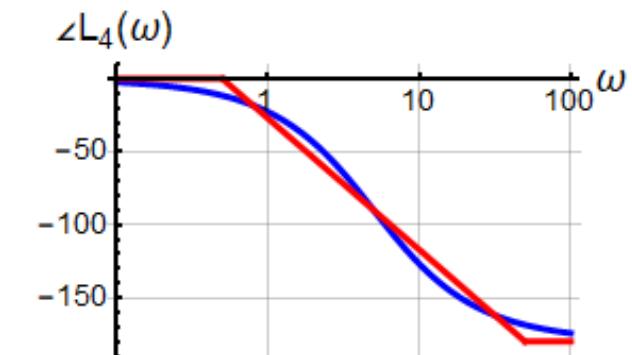
$$m_2 \cdot \frac{6 \text{ dB}}{\text{octave}}$$

Time delay: $t_g = -m_2 \tau = -m_2 \times 0.2$

$$m_2 = 2$$



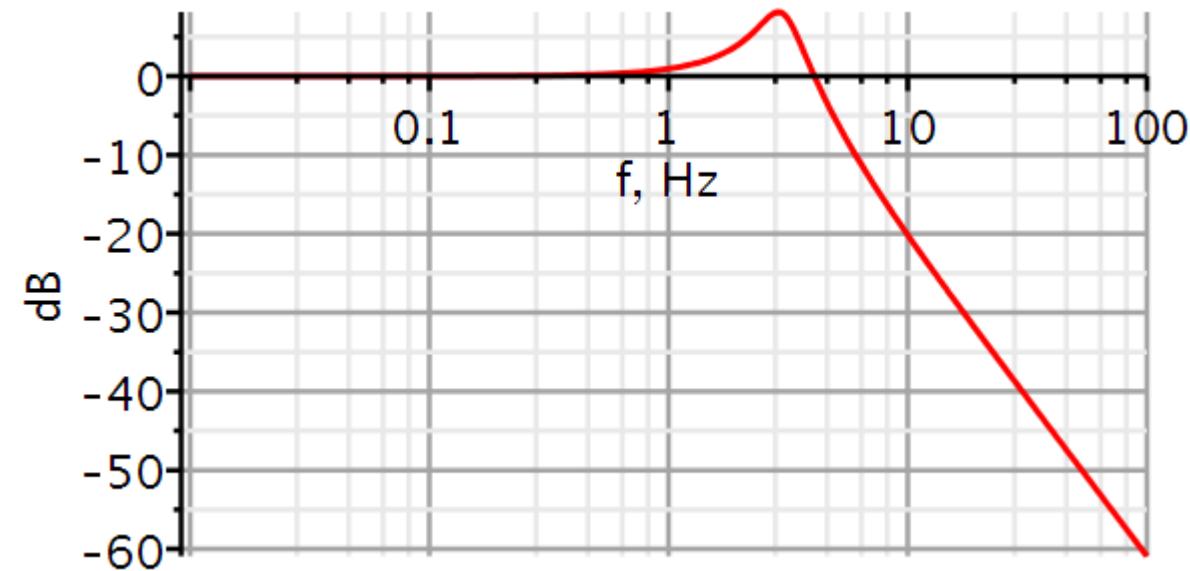
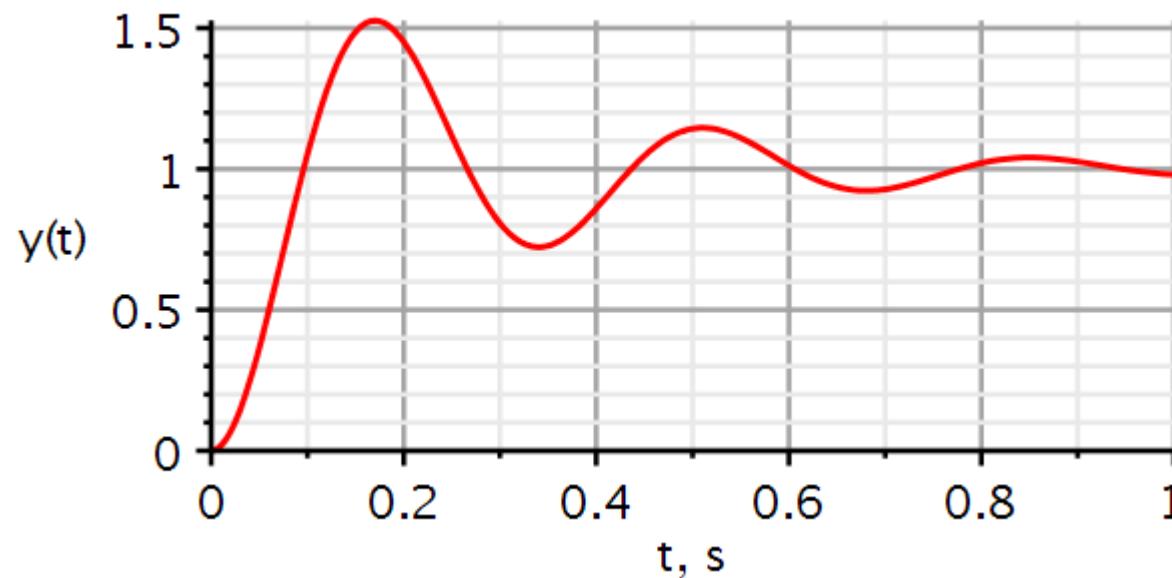
$$m_2 = -2$$



Problems

Problem 1: Systems identification from step response

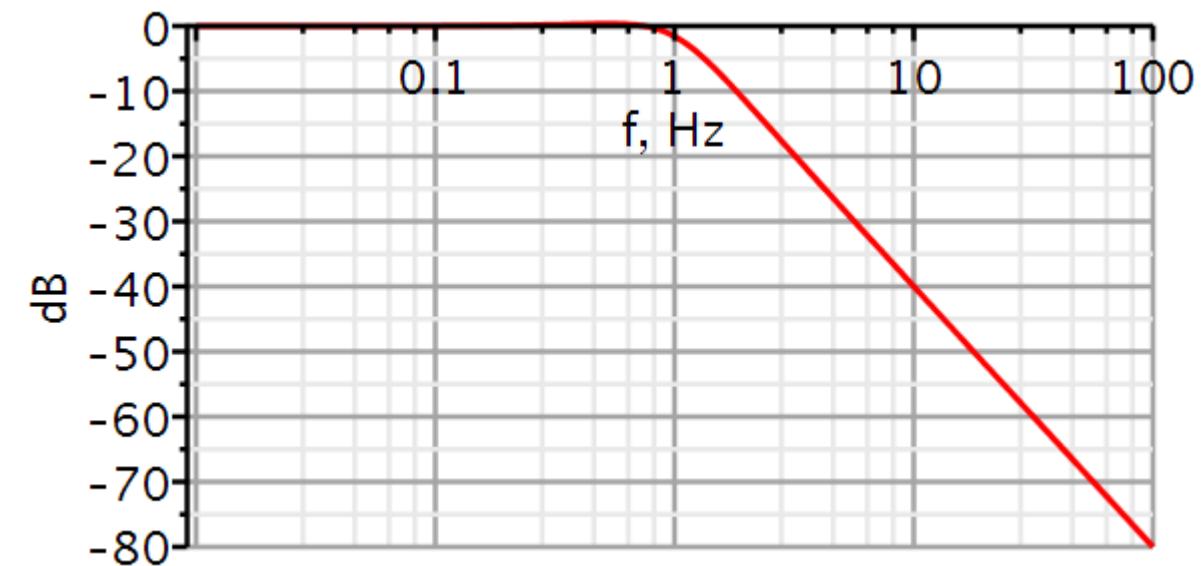
From an unknown system the following step response and amplitude spectrum were measured.



Identify the system parameters, write down the differential equation and plot the amplitude response using the identified systems parameters.

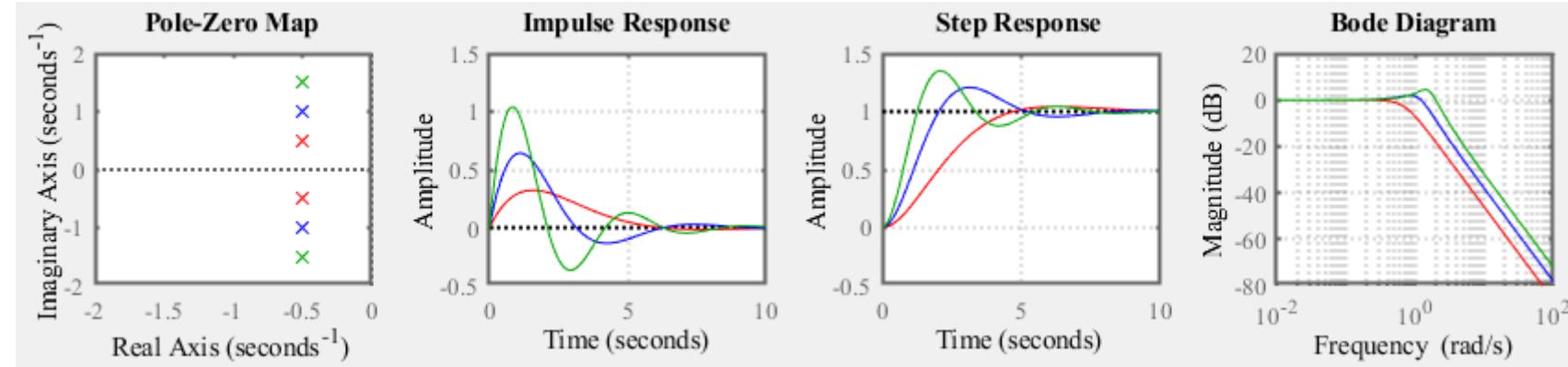
Problem 2: Systems identification from step response

From an unknown system the following step response and amplitude spectrum were measured.

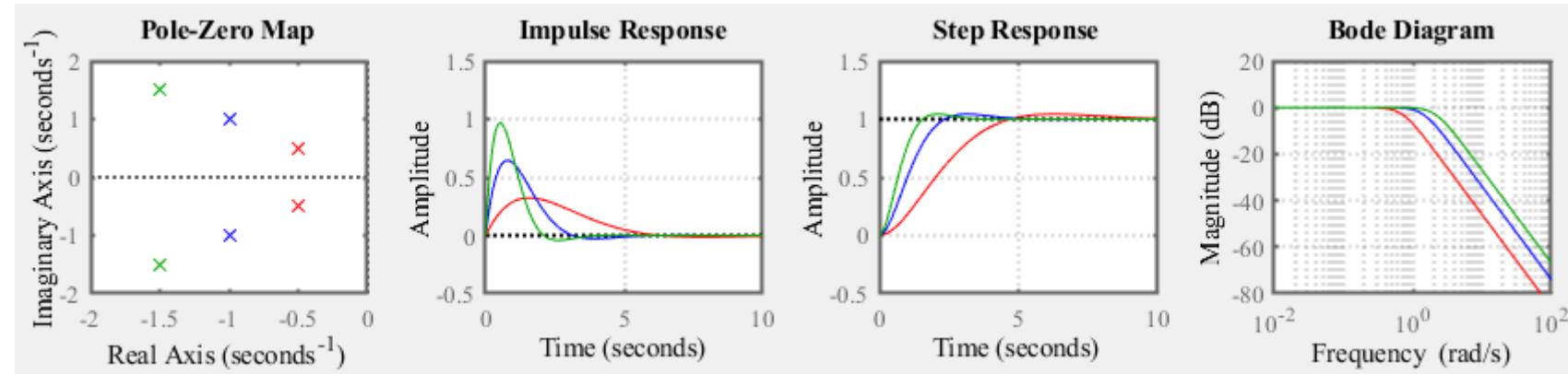


Identify the system parameters, write down the differential equation and plot the amplitude response using the identified systems parameters.

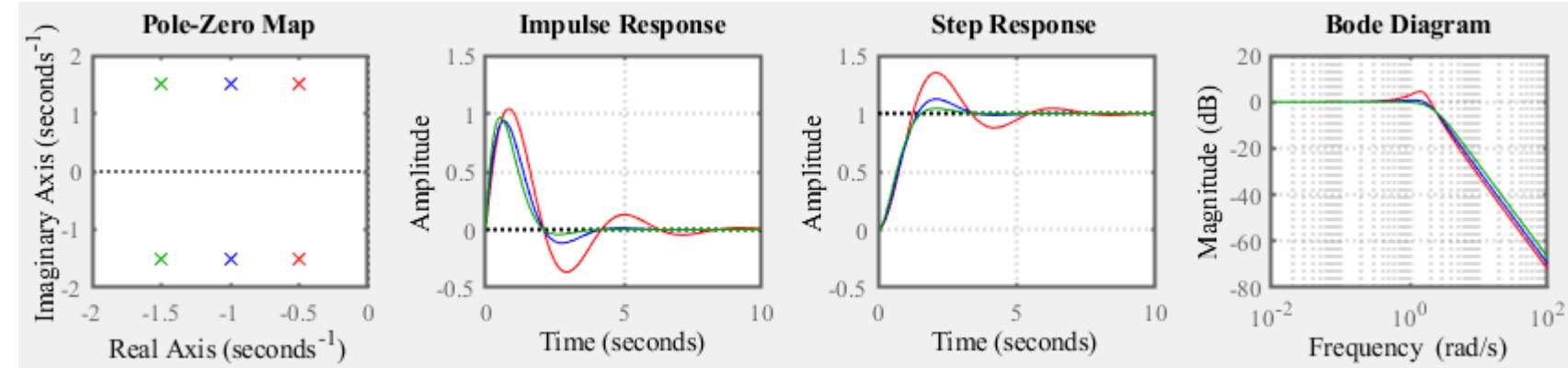
Problem 3: Explain the effect of poles moving



Problem 3: Explain the effect of poles moving



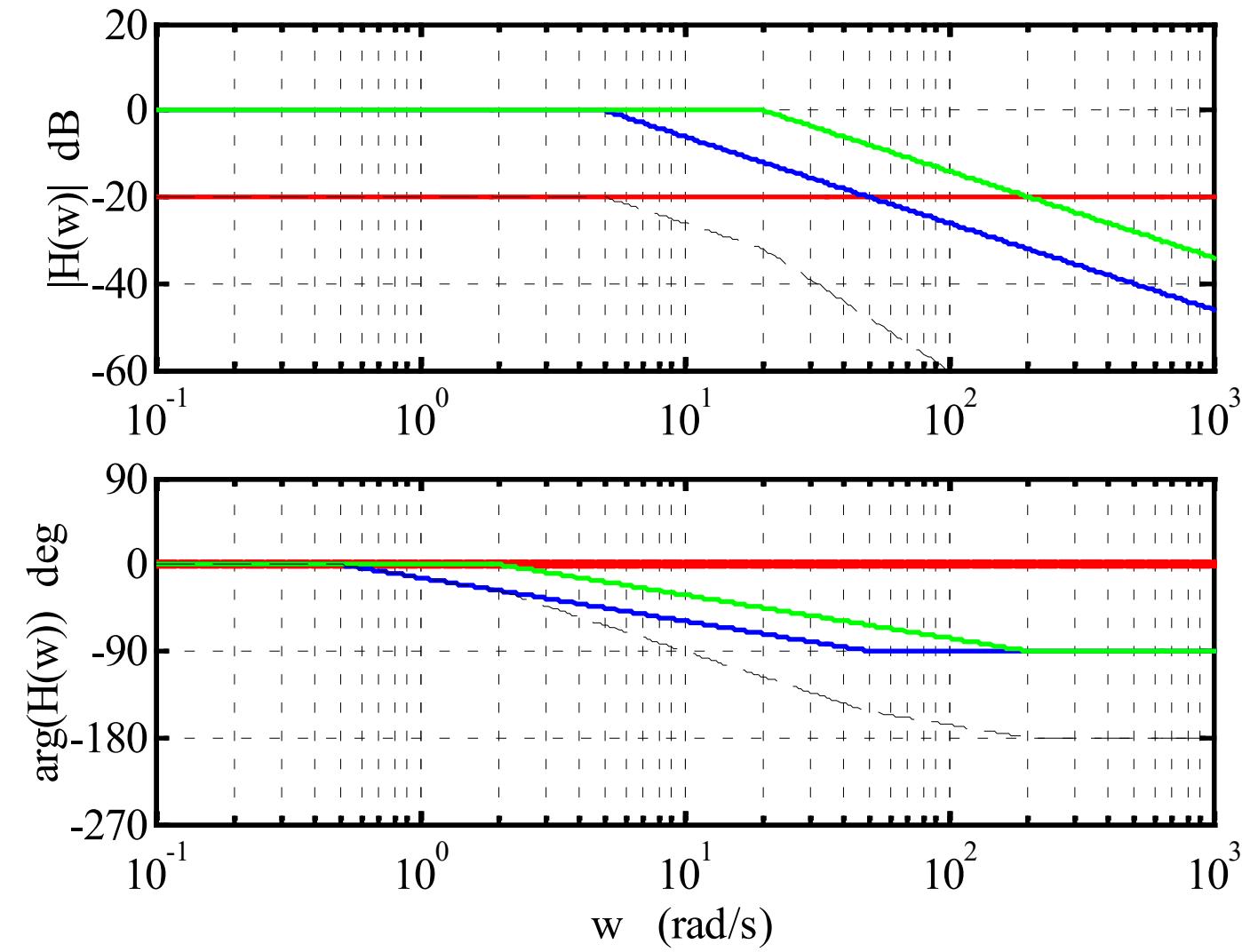
Problem 3: Explain the effect of poles moving



Problem 4: Bode plot

Analyze the Bode plot
and identify the
transfer function.

Also predict the group
delay t_g at $\omega = 0$ for
each linear factor.



Problem 4: Bode plot – redraw using Maple

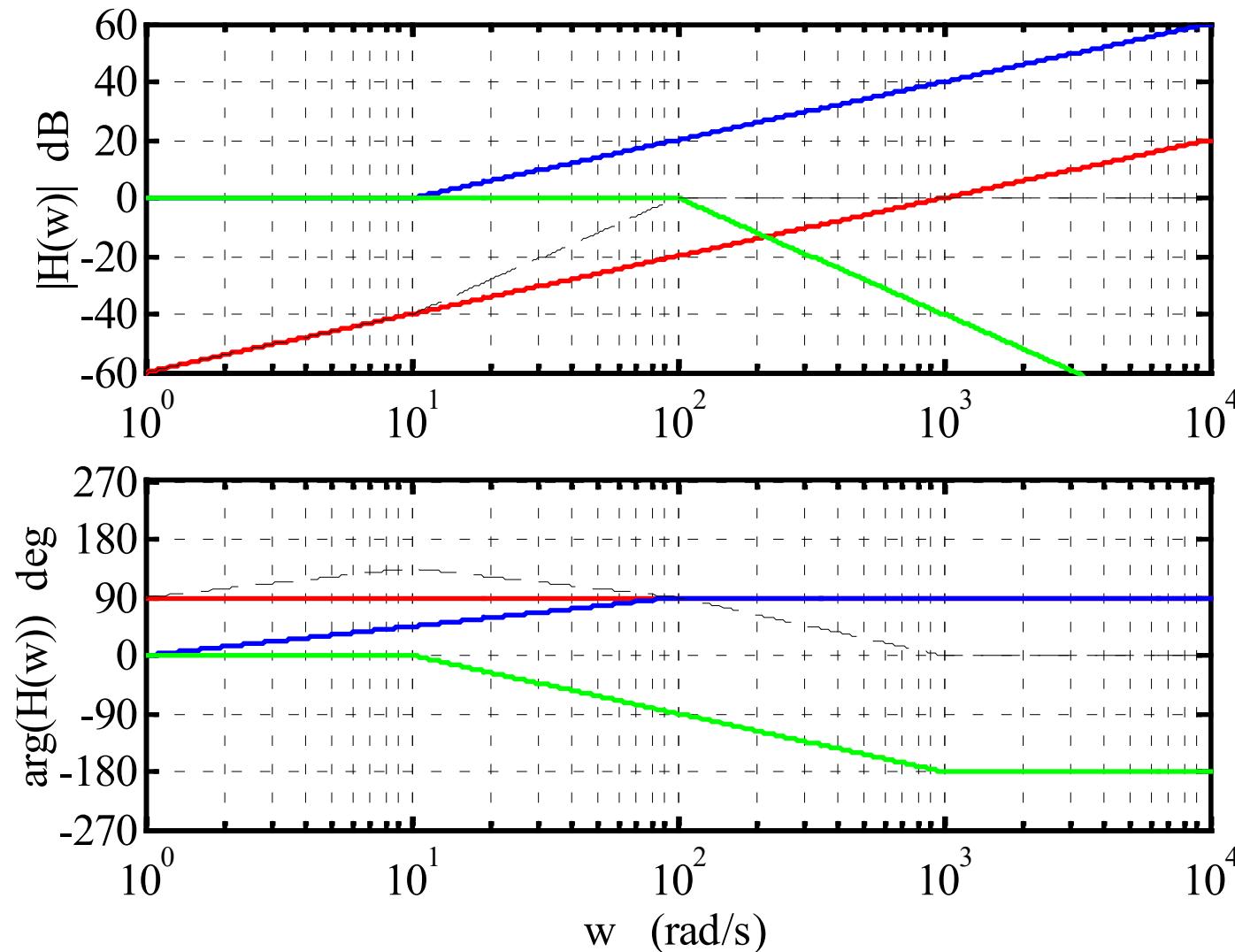
Download Maple file from DTU Learn.

Modify the script to plot the Bode plot curves of
the present system.

Problem 5: Bode plot

Analyze the Bode plot
and identify the transfer
function.

Also predict the group
delay t_g at $\omega = 0$ for
each linear factor.



Problem 5: Bode plot – redraw using Maple

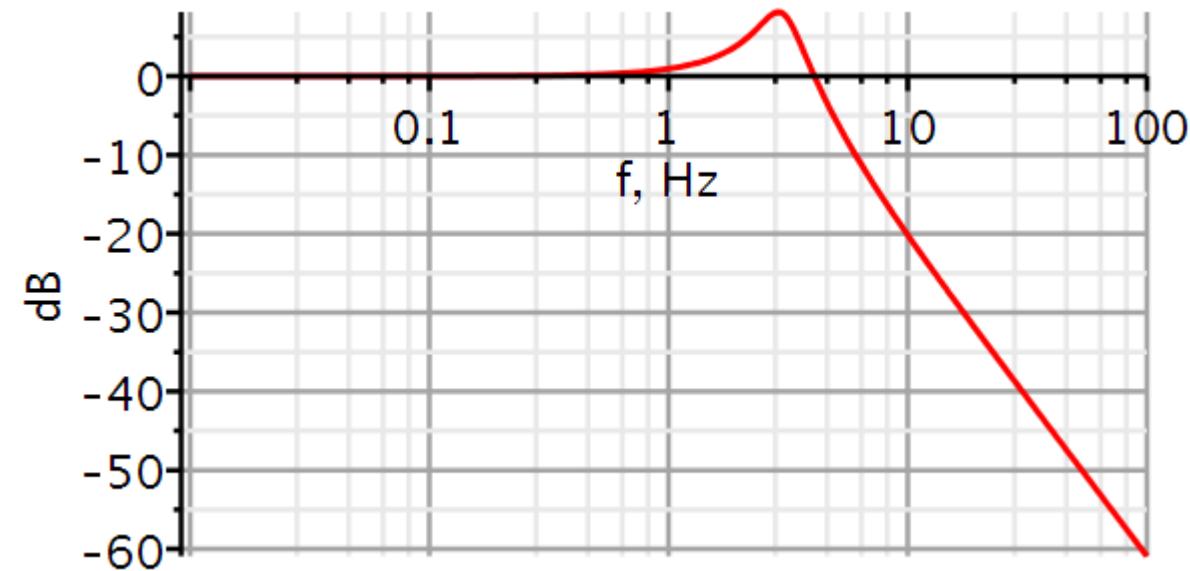
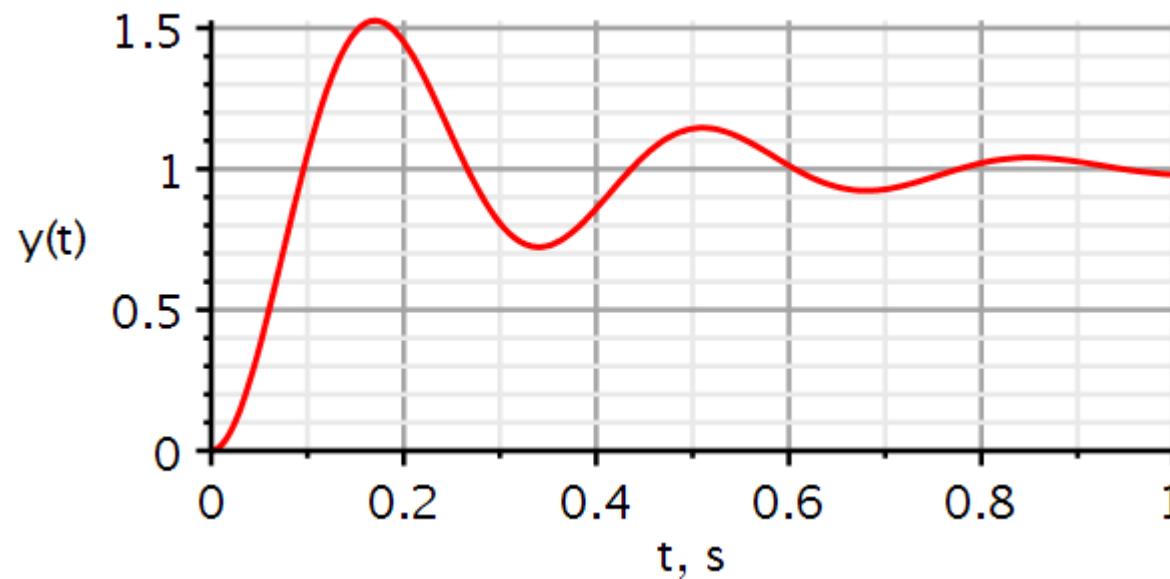
Download Maple file from DTU Learn.

Modify the script to plot the Bode plot curves of
the present system.

Solutions

Problem 1: Systems identification from step response

From an unknown system the following step response and amplitude spectrum were measured.



Identify the system parameters, write down the differential equation and plot the amplitude response using the identified systems parameters.

Problem 1: Systems identification from step response (sol)

$$PO = 52$$

$$a \stackrel{\text{def}}{=} \ln\left(\frac{PO}{100}\right) = \ln 0.52 = -0.654 \quad \rightarrow$$

$$\zeta = \frac{|a|}{\sqrt{\pi^2 + a^2}} = \frac{0.654}{\sqrt{\pi^2 + 0.654^2}} = 0.204$$

$$t_p = 0.168$$

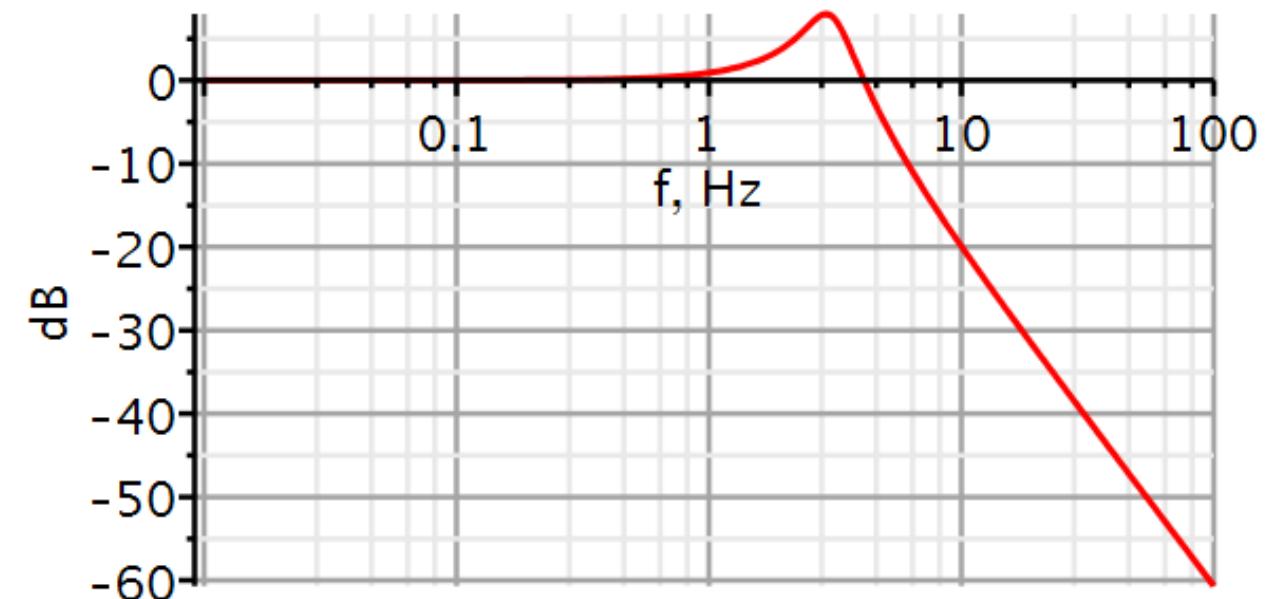
$$\omega_n = \frac{\pi}{t_p \sqrt{1 - \zeta^2}} = \frac{\pi}{0.168 \sqrt{1 - 0.204^2}} = 19.1$$

$$a_1 = 2\zeta\omega_n = 7.79 \quad a_0 = \omega_n^2 = 364.7$$

$$\ddot{y}(t) + 7.79\dot{y}(t) + 364.7y(t) = 364.7x(t)$$

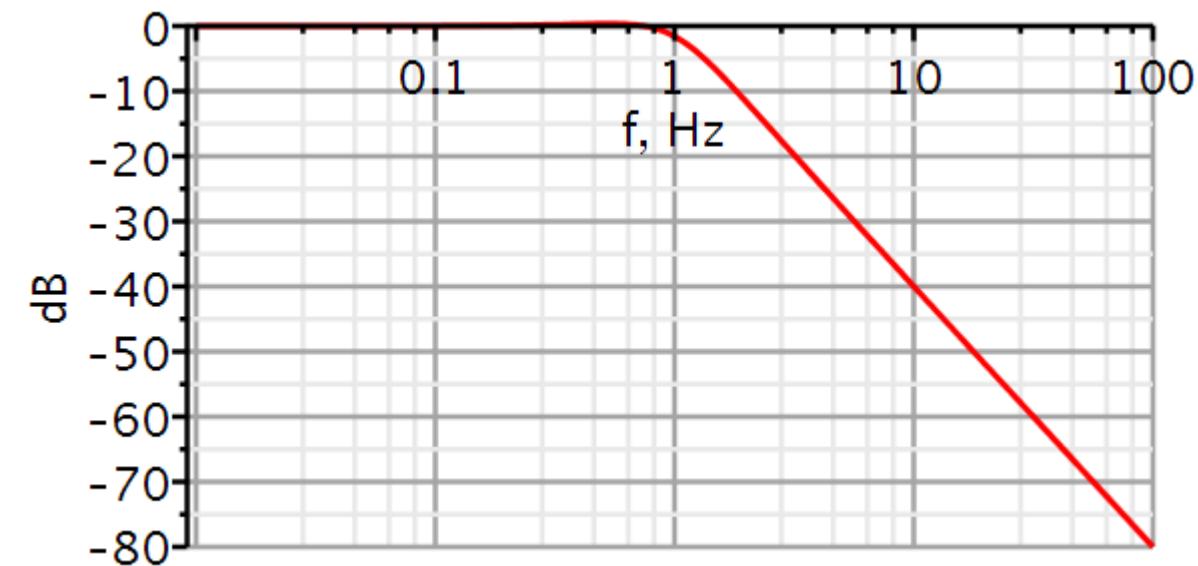
Identified systems

```
sys11est := Coefficients\left(\frac{(19.1)^2}{s^2 + 2 \cdot 0.204 \cdot 19.1 \cdot s + (19.1)^2}\right);  
MagnitudePlot(sys11est, color = [red, blue], thickness = 3, axesfont = ["Helvetica", "ROMAN", 18], axis[2]  
= [thickness = 2.5], axis[1] = [mode = log, thickness = 2.5], labels = ["f, Hz", "dB"], labelfont  
= ["HELVETICA", 18], gridlines, decibels = true, hertz = true, size = [600, 300])
```



Problem 2: Systems identification from step response

From an unknown system the following step response and amplitude spectrum were measured.



Identify the system parameters, write down the differential equation and plot the amplitude response using the identified systems parameters.

Problem 2: Systems identification from step response (sol)

$$PO = 9.5$$

$$a \stackrel{\text{def}}{=} \ln\left(\frac{PO}{100}\right) = \ln 0.095 = -2.354 \quad \rightarrow$$

$$\zeta = \frac{|a|}{\sqrt{\pi^2 + a^2}} = \frac{2.354}{\sqrt{\pi^2 + 2.354^2}} = 0.599$$

$$t_p = 0.615$$

$$\omega_n = \frac{\pi}{t_p\sqrt{1-\zeta^2}} = \frac{\pi}{0.615\sqrt{1-0.599^2}} = 6.379$$

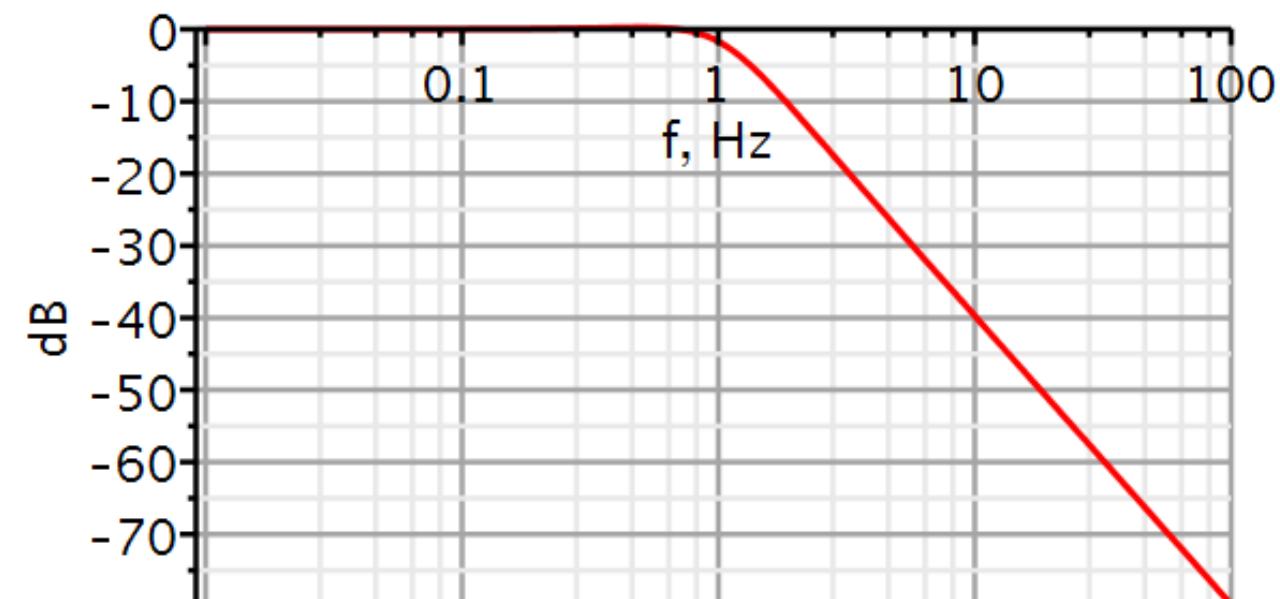
$$a_1 = 2\zeta\omega_n = 7.64 \quad a_0 = \omega_n^2 = 40.69$$

$$\ddot{y}(t) + 7.64\dot{y}(t) + 40.69y(t) = 40.69x(t)$$

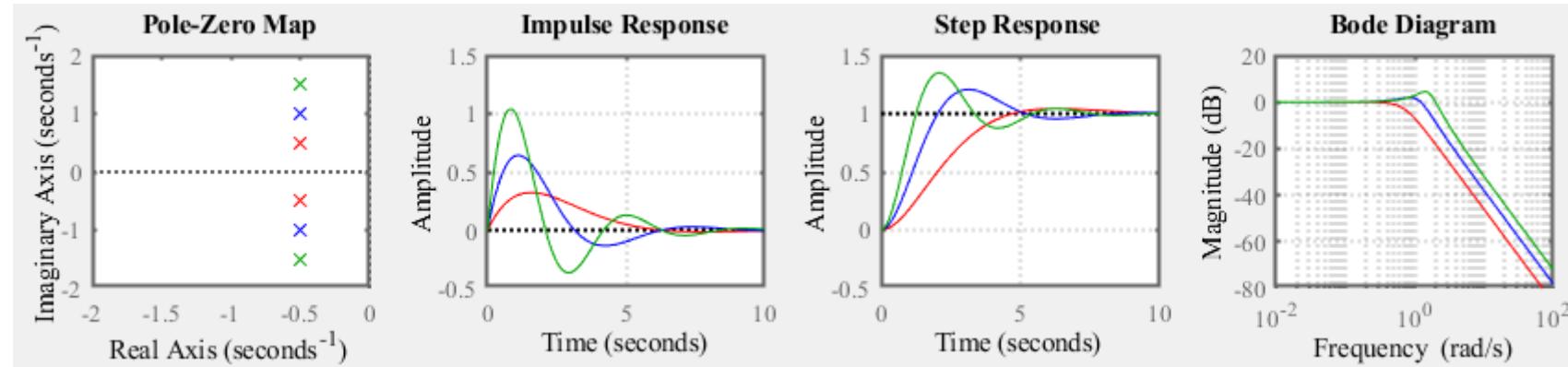
Identified systems

$$sys11est := Coefficients\left(\frac{(6.379)^2}{s^2 + 2 \cdot 0.599 \cdot 6.579 \cdot s + (6.379)^2}\right);$$

`MagnitudePlot(sys11est, color = [red, blue], thickness = 3, axesfont = ["Helvetica", "ROMAN", 18], axis[2] = [thickness = 2.5], axis[1] = [mode = log, thickness = 2.5], labels = ["f, Hz", "dB"], labelfont = ["HELVETICA", 18], gridlines, decibels = true, hertz = true, size = [600, 300])`



Problem 3: Explain the effect of poles moving (sol)



As the poles moves downward from green to red, the angle with the negative real axis becomes smaller and the damping ratio ζ increases. At the same time, the undamped natural frequency ω_n is reduced and the time to peak t_p is increased.

This agrees with the decreasing amplitudes of the oscillations in the impulse and step responses. Also, the frequency of the magnitude peak is decreased.

$$PO = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

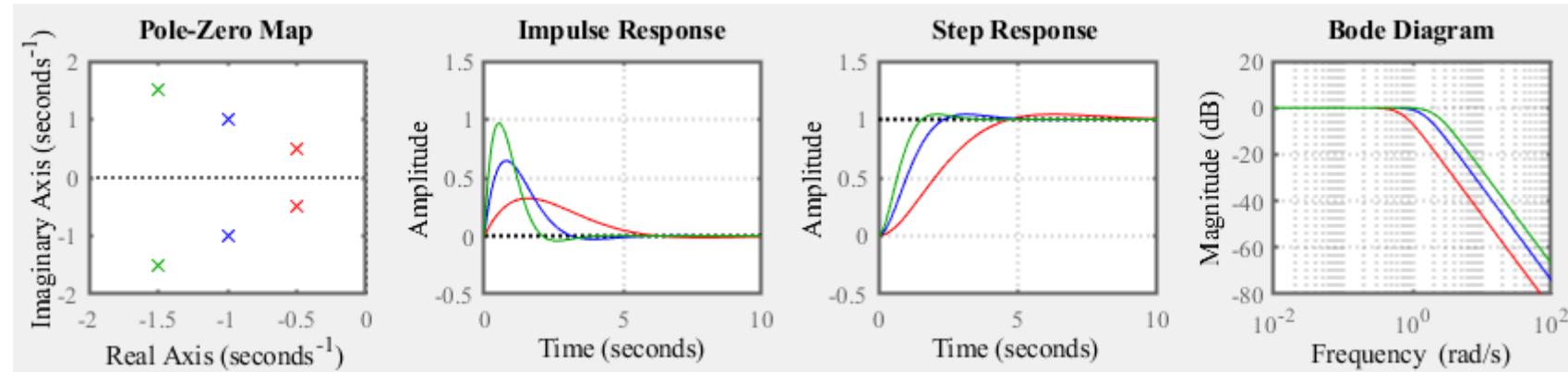
$$\text{If } 0 < \zeta < 1/\sqrt{2}: \quad \omega_{peak} = \omega_n\sqrt{1 - 2\zeta^2}$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1 - \zeta^2}}$$

$$M_{pk} = m_3 \cdot 20 \cdot \log [2\zeta\sqrt{1 - \zeta^2}] dB$$

Problem 3: Explain the effect of poles moving (sol)



Here the natural frequency is reduced when the poles move from green to red. On the other hand, the angle remains unchanged, hence the damping factor is unchanged.

We see that the overshoot in the step response is unchanged, but the time to peak is increased.

There are no peaks in the amplitude spectrum, but the cut-off frequency is reduced because the natural frequency is reduced.

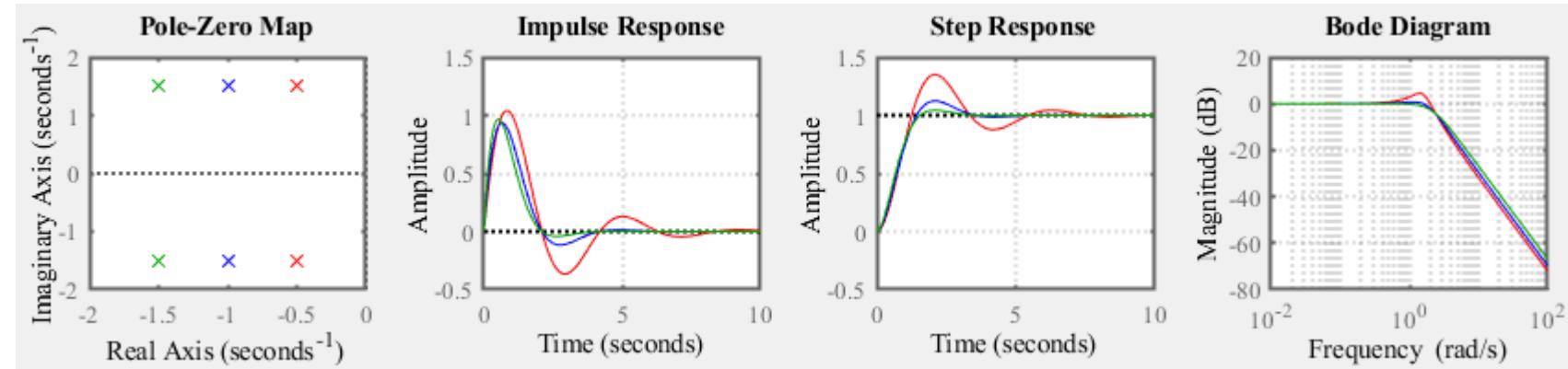
$$PO = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$\text{If } 0 < \zeta < 1/\sqrt{2}: \quad \omega_{peak} = \omega_n \sqrt{1 - 2\zeta^2}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

$$M_{pk} = m_3 \cdot 20 \cdot \log [2\zeta\sqrt{1 - \zeta^2}] dB$$

Problem 3: Explain the effect of poles moving (sol)



Here the damped frequency is constant when the poles move from green to red. On the other hand, the angle increases, hence the damping factor is reduced.

We see that the overshoot in the step response increases, but the time to peak is unchanged. There appears a peak in the amplitude spectrum when the system becomes more underdamped, but the cut-off frequency is *almost* unchanged.

$$PO = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$\text{If } 0 < \zeta < 1/\sqrt{2}: \quad \omega_{peak} = \omega_n\sqrt{1 - 2\zeta^2}$$

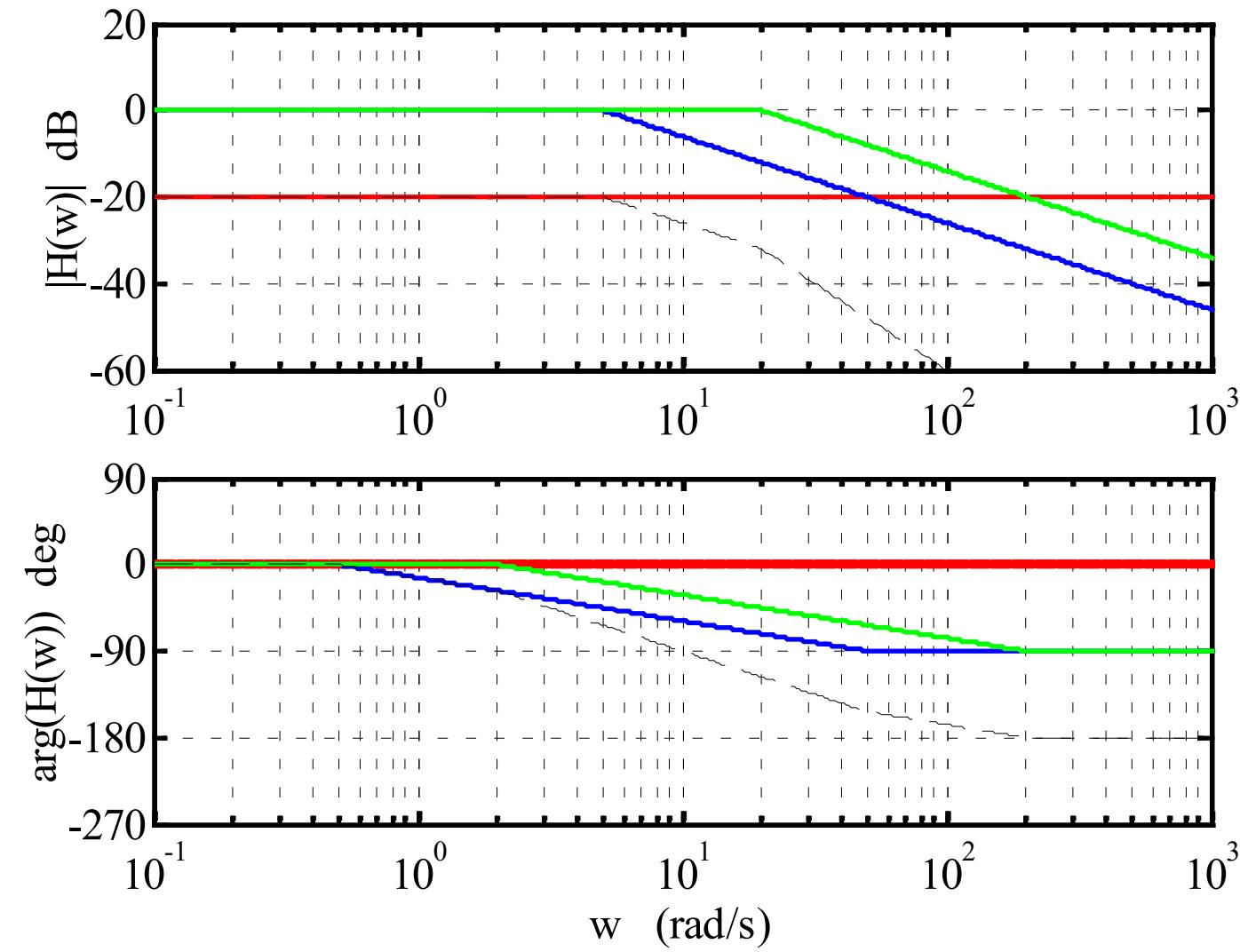
$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1 - \zeta^2}}$$

$$M_{pk} = m_3 \cdot 20 \cdot \log [2\zeta\sqrt{1 - \zeta^2}] dB$$

Problem 4: Bode plot

Analyze the Bode plot
and identify the transfer
function.

Also predict the group
delay t_g at $\omega = 0$ for
each linear factor.



Problem 4: Bode plot

Red curve:

$$C(\omega) = -20 \text{ dB} = 0.1$$

$$t_g = -n_1 \tau$$

Blue curve:

$$L_1(\omega) = \left(\frac{j\omega}{5} + 1 \right)^{-1}$$

$$\tau_1 = 0.2s$$

$$t_{g_1} = -(-1) \times 0.2s = 0.2s$$

Green curve:

$$L_2(\omega) = \left(\frac{j\omega}{20} + 1 \right)^{-1}$$

$$\tau_2 = 0.05s$$

$$t_{g_2} = -(-1) \times 0.05s = 0.05s$$

$$H(\omega) = \frac{0.1}{\left(\frac{j\omega}{5} + 1 \right) \left(\frac{j\omega}{20} + 1 \right)} = \frac{10}{(j\omega + 5)(j\omega + 20)}$$

$$H(s) = \frac{10}{(s + 5)(s + 20)} = \frac{10}{s^2 + 25s + 100}$$

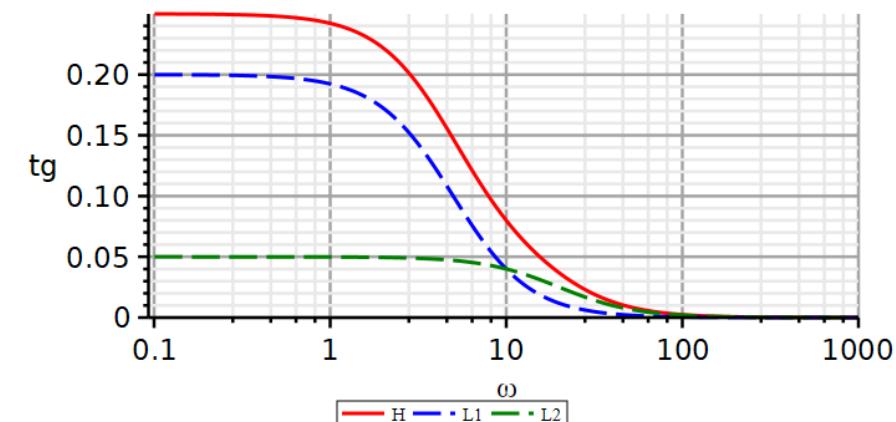
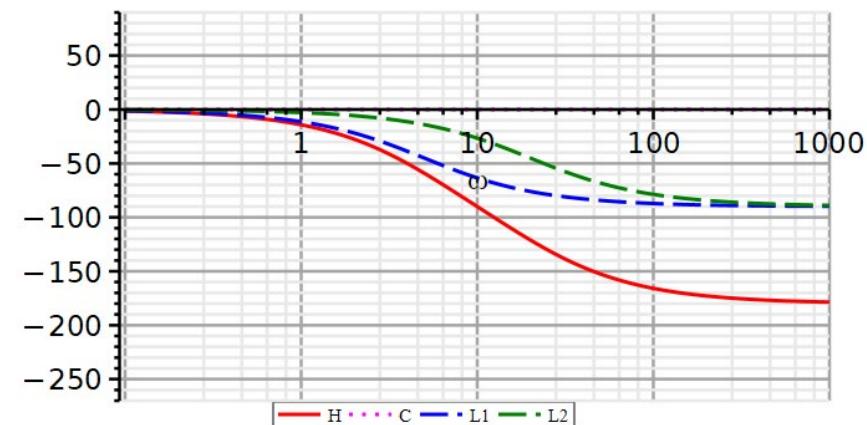
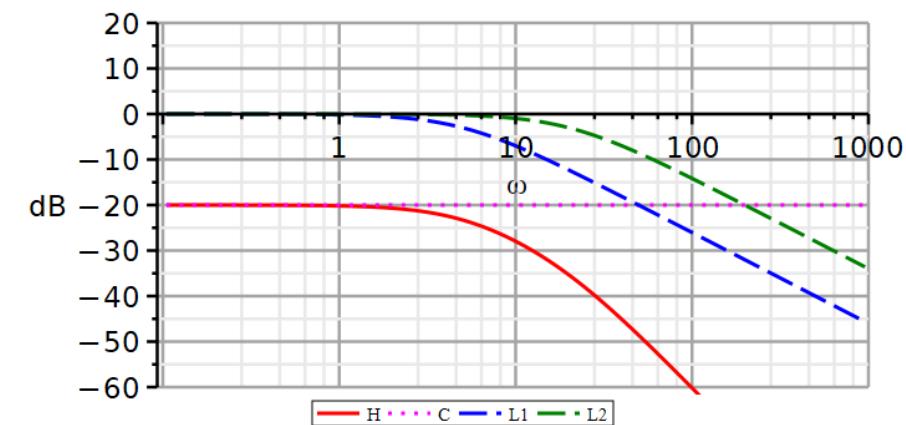
Problem 4: Bode plot – redraw using Maple

Download Maple file from DTU Learn.

Modify the script to plot the Bode plot curves of the present system.

The system phase curve is the sum of the phase curves of each individual factor.

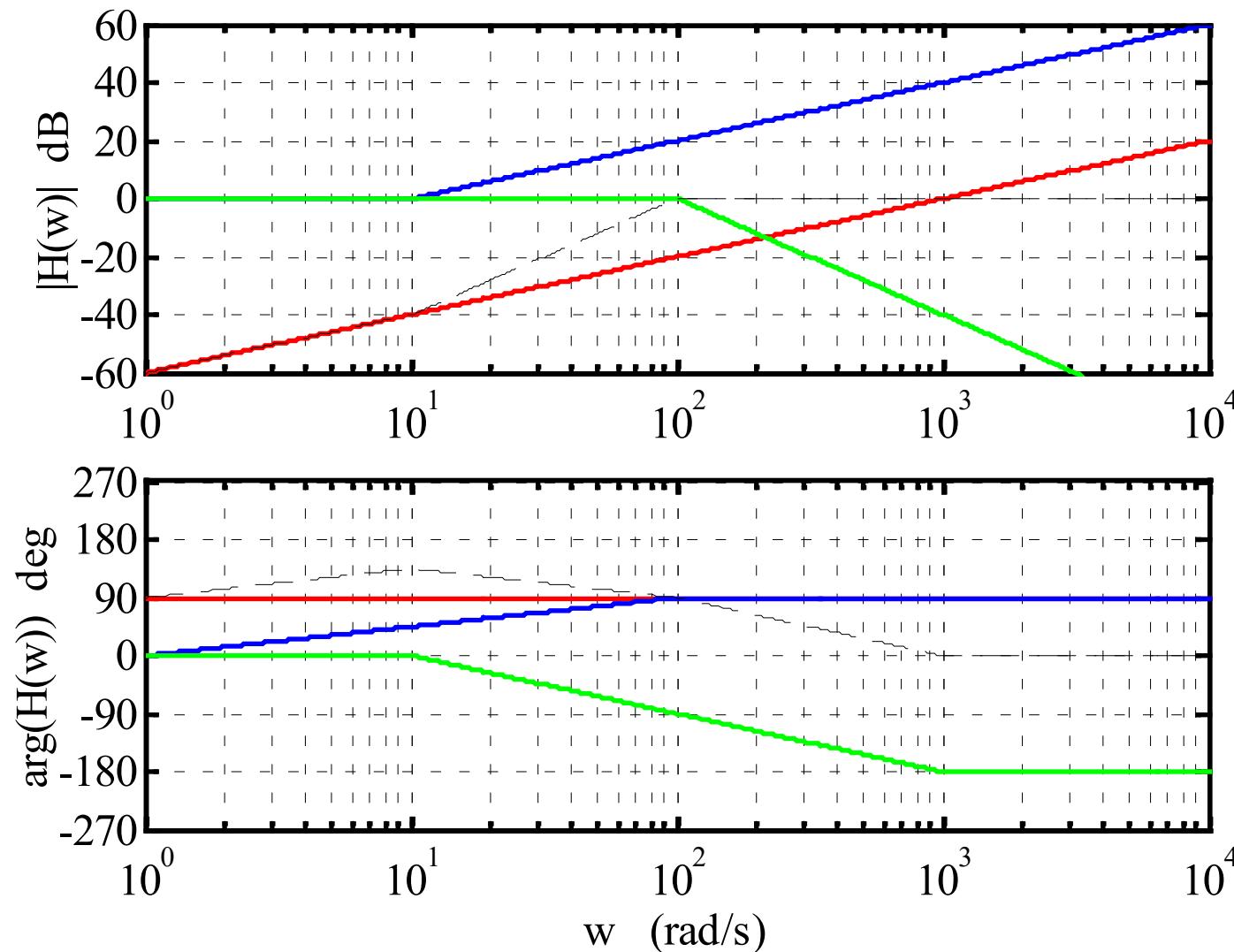
Because the phase curves add together, so does the group delays caused by the individual factors.



Problem 5: Bode plot

Analyze the Bode plot
and identify the transfer
function.

Also predict the group
delay t_g at $\omega = 0$ for
each linear factor.



Problem 5: Bode plot

Red curve:

$$\Omega(\omega) = \frac{j\omega}{10^3}$$

$$t_g = -n_1 \tau$$

Blue curve:

$$L_1(\omega) = \left(\frac{j\omega}{10} + 1 \right) \quad \tau_1 = 0.1s$$

$$t_{g_1} = -1 \times 0.1s = -0.1s$$

Green curve:

$$L_2(\omega) = \left(\frac{j\omega}{100} + 1 \right)^{-2} \quad \tau_2 = 0.01s$$

$$t_{g_2} = -(-2) \times 0.01s = 0.02s$$

$$H(\omega) = \frac{\frac{j\omega}{10^3} \left(\frac{j\omega}{10} + 1 \right)}{\left(\frac{j\omega}{100} + 1 \right)^2} = \frac{j\omega(j\omega + 10)}{(j\omega + 100)^2}$$

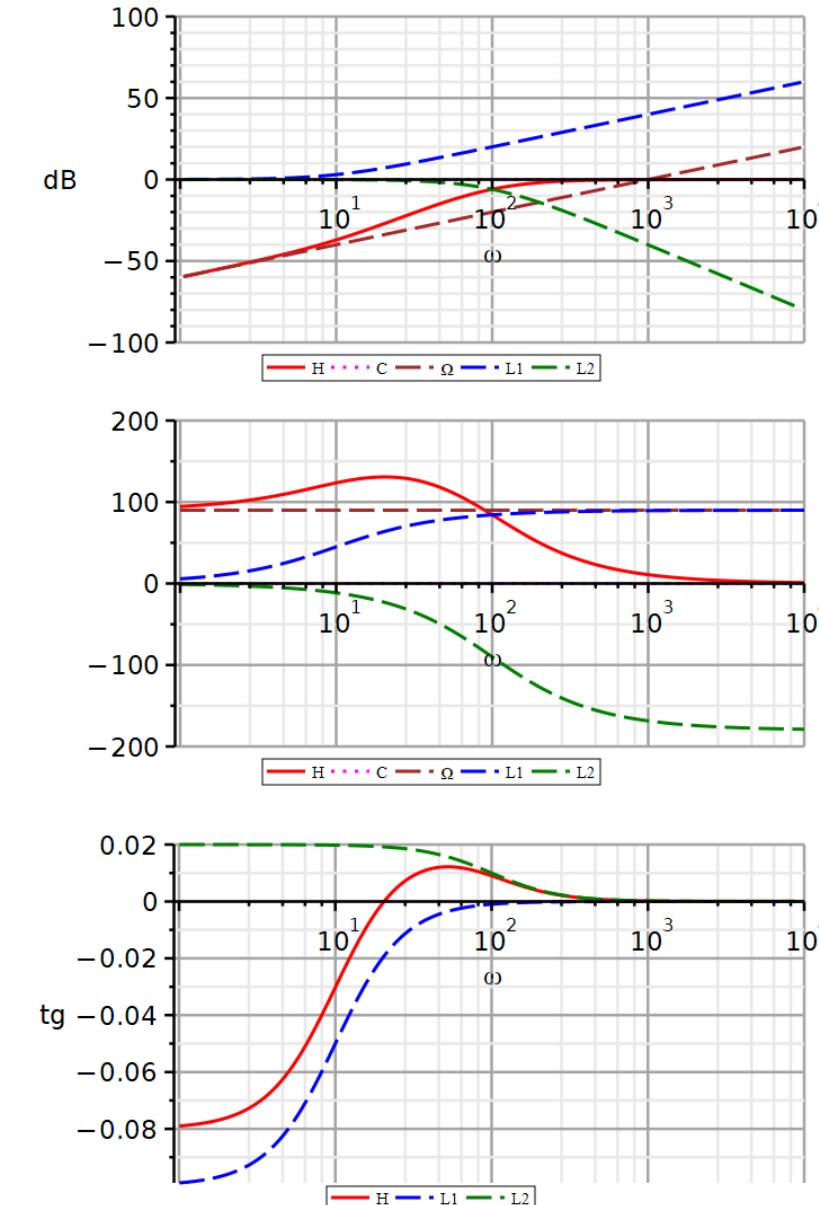
$$H(s) = \frac{s(s + 10)}{(s + 100)^2}$$

Problem 5: Bode plot – redraw using Maple

Download Maple file from DTU Learn.
Modify the script to plot the Bode plot curves of
the present system.

The system phase curve is the sum of the
phase curves of each individual factor.

Because the phase curves add together, so
does the group delays caused by the
individual factors.



22050 Continuous-Time Signals and Linear Systems

Kaj-Åge Henneberg

L10

Bode plot 2: Quadratic factors

Filter design by placement of poles and zeros

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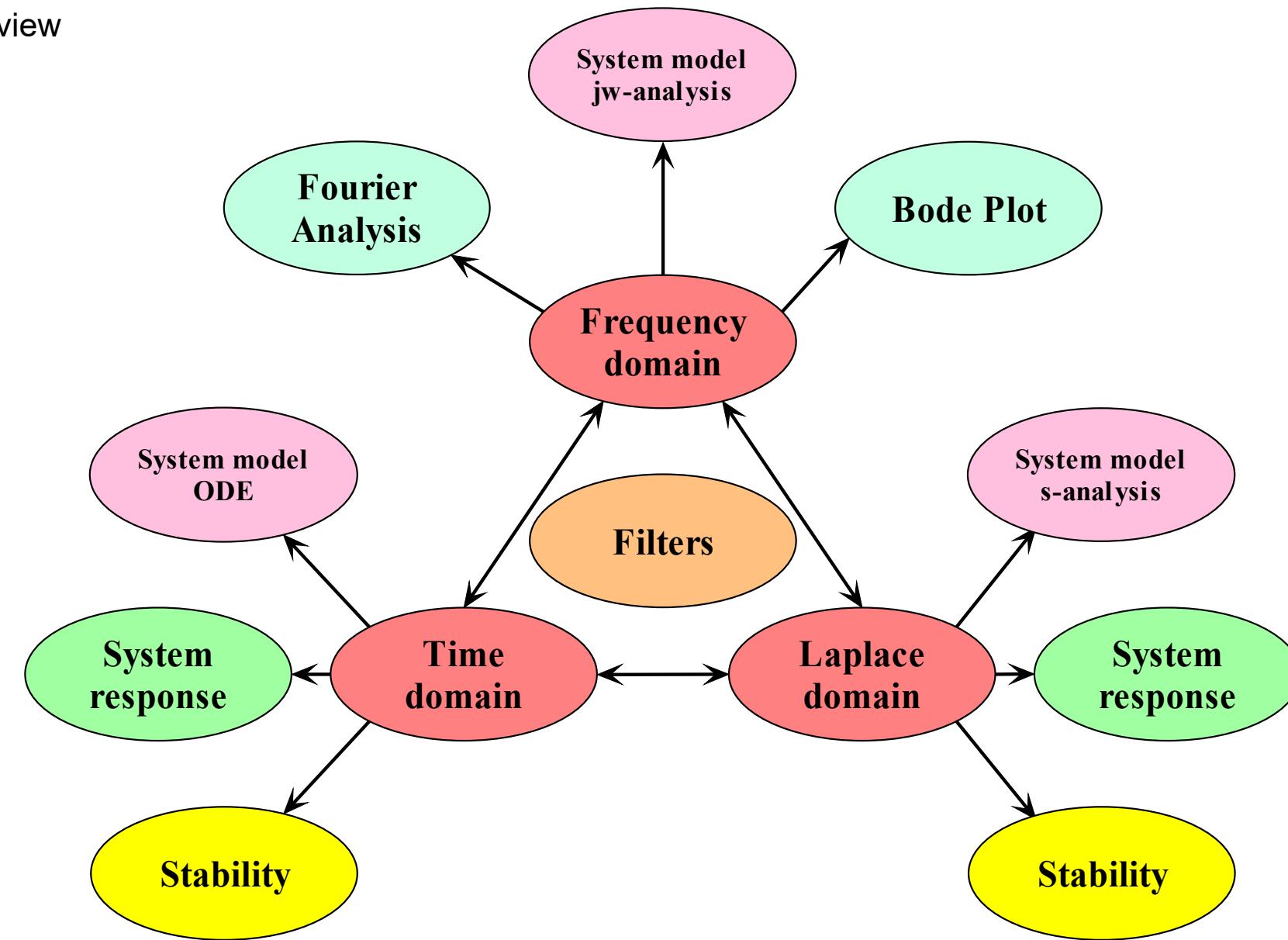
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7.2 Bode plot 2: Quadratic factors

7.4 Filter design by placement of poles and zeros

Bode plot 2 Constants and linear factors

Video

Examples of drawing Bode plots

Draw the amplitude and phase responses for:

$$H(j\omega) = \frac{1}{j\omega + 1}$$

Gain:

DC gain: 1

Cut-off: 100 rad/s

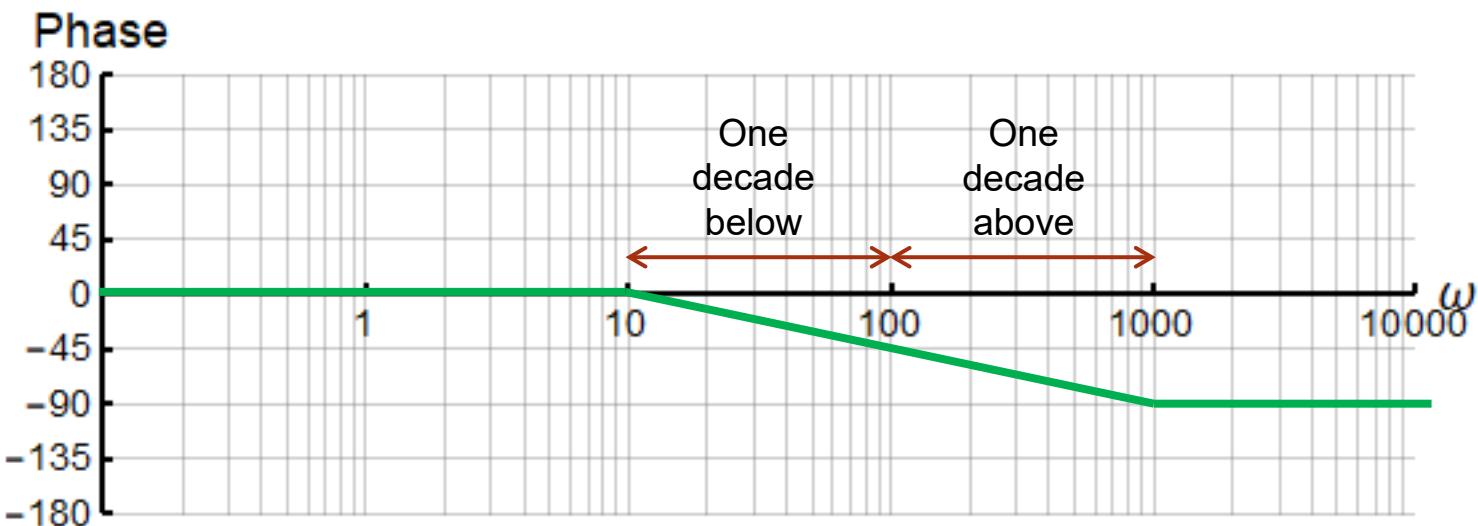
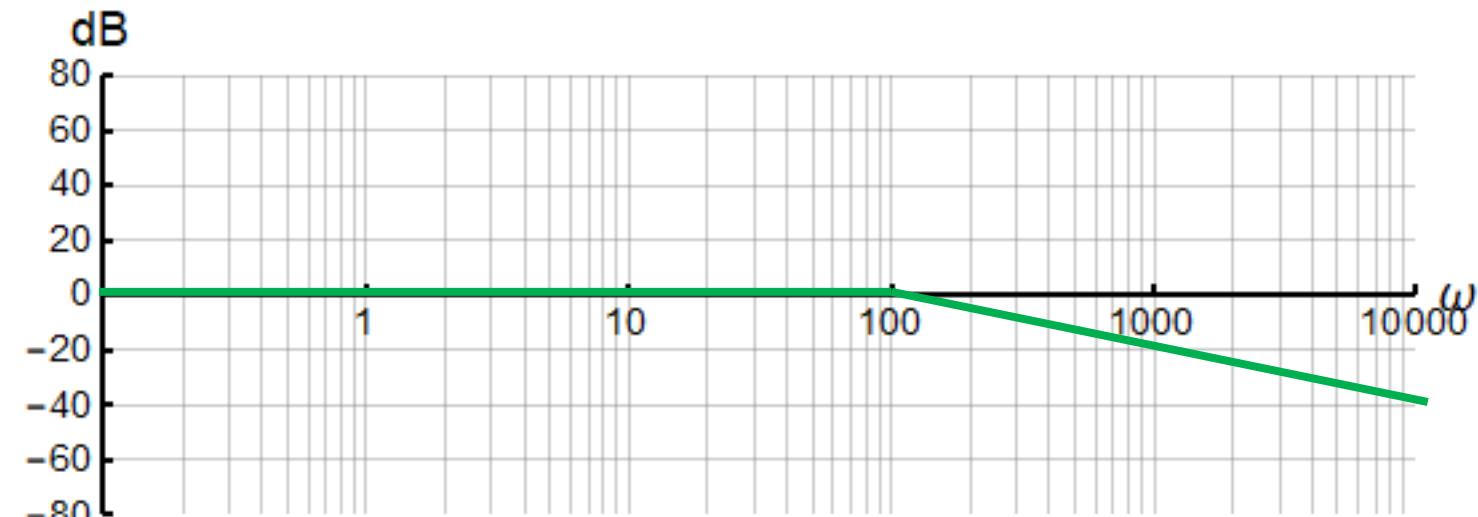
Slope: -20 dB/decade

Phase:

DC phase: 0

HF phase: -90

Transition: 2 decades



Examples of drawing Bode plots

Draw the amplitude and phase responses for:

$$H(j\omega) = \frac{j\omega}{\left(\frac{j\omega}{10} + 1\right)^2}$$

$$\frac{Me^{j\phi}}{Ne^{j\theta}} = \frac{M}{N} e^{j\phi} e^{-j\theta} = \frac{M}{N} e^{j(\phi-\theta)}$$

Gain:

DC gain: 0

Cut-off: 10 rad/s

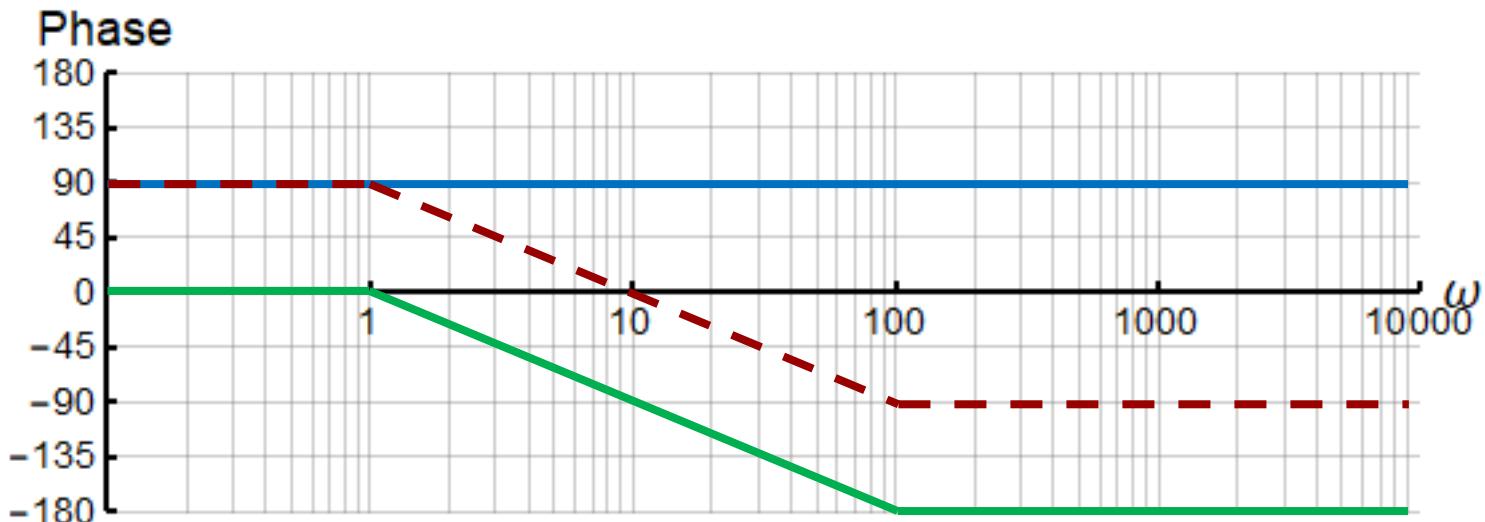
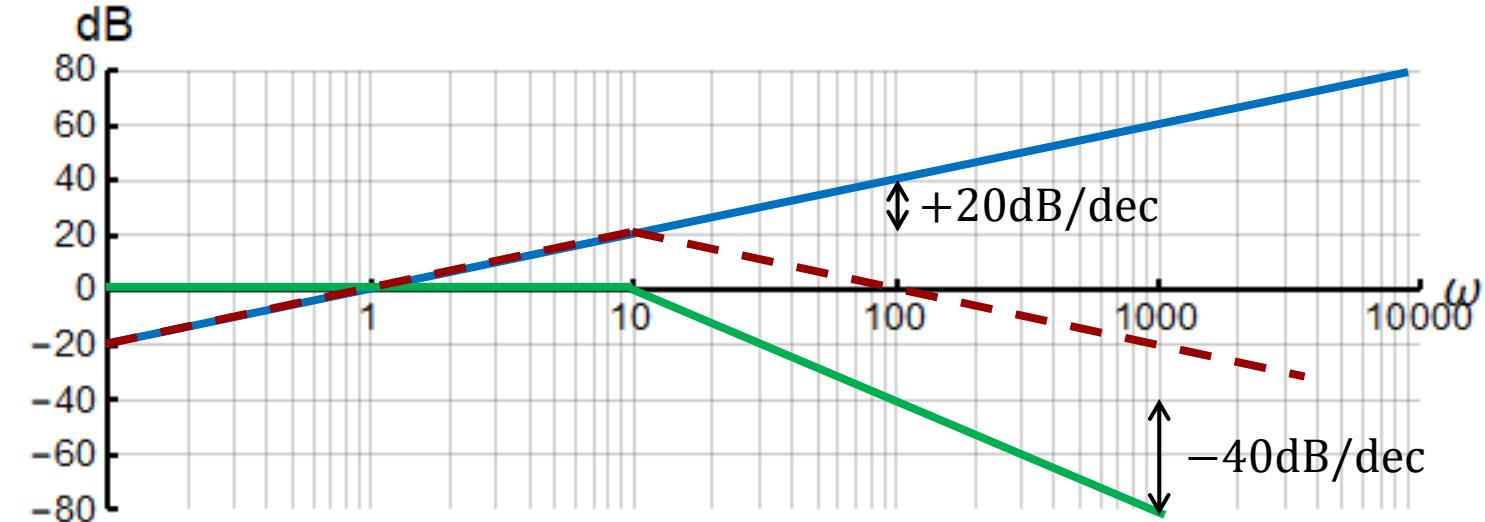
HF Slope: -40 dB/decade

Phase:

DC phase: 90

HF phase: -90

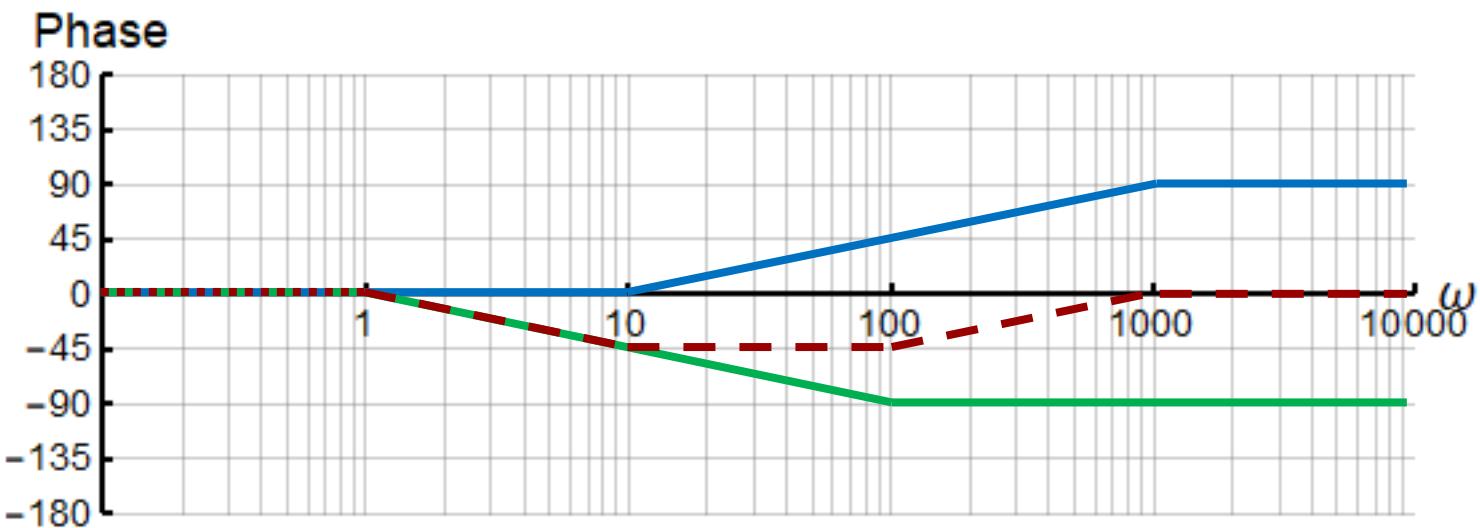
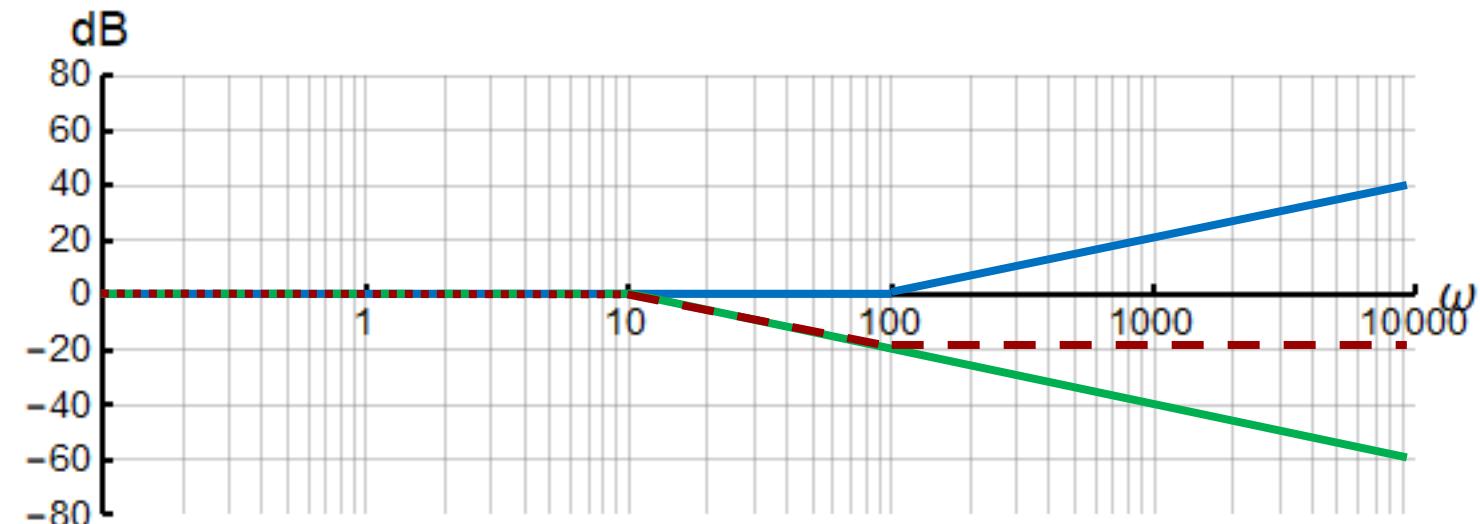
Transition: 2 decades



Examples of drawing Bode plots

Draw the amplitude and phase responses for:

$$H(j\omega) = \frac{\frac{j\omega}{100} + 1}{\frac{j\omega}{10} + 1}$$



Bode plot 2 Quadratic factors

Video

Standard form for Bode plot

The advantage of taking \log_{10} is that each factor now becomes a term. We can now draw an amplitude and phase curve for each factor, and then we can add the curves together graphically.

As each term may originate from a specific sub-system, we can easily identify the behavior of each sub-system and then correct those where needed.

Furthermore, if we decide to add a correcting subsystem into our system, we just add its Bode plot to the Bode plot of the original system.

$$|H(j\omega)|_{dB} = \underbrace{20 \log_{10} K \frac{b_1 b_2^{m_2} b_4}{a_1 a_2^{n_2} a_4^{n_3}}}_{\text{constant}} + \underbrace{p \times 20 \log_{10} |j\omega|}_{\text{zero at origin}} + \underbrace{20 \log_{10} \left| \frac{j\omega}{b_1} + 1 \right|}_{\text{first order zero}} \\ + \underbrace{m_2 \times 20 \log_{10} \left| \frac{j\omega}{b_2} + 1 \right|}_{\text{repeated first order zero}} + \underbrace{m_3 \times 20 \log_{10} \left| \frac{(j\omega)^2}{b_4} + \frac{b_3}{b_4} j\omega + 1 \right|}_{\text{repeated second order zero}} \\ - \underbrace{q \times 20 \log_{10} |j\omega|}_{\text{pole at origin}} - \underbrace{20 \log_{10} \left| \frac{j\omega}{a_1} + 1 \right|}_{\text{first order pole}} - \underbrace{n_2 \times 20 \log_{10} \left| \frac{j\omega}{a_2} + 1 \right|}_{\text{repeated first order pole}} \\ - \underbrace{n_3 \times 20 \log_{10} \left| \frac{(j\omega)^2}{a_4} + \frac{a_3}{a_4} j\omega + 1 \right|}_{\text{repeated second order pole}}$$

$$\angle H(\omega) = \underbrace{p \angle(j\omega)}_{\text{zero at origin}} + \underbrace{\angle \left(\frac{j\omega}{b_1} + 1 \right)}_{\text{first order zero}} + \underbrace{m_2 \angle \left(\frac{j\omega}{b_2} + 1 \right)}_{\text{repeated first order zero}} + \underbrace{m_3 \angle \left(\frac{(j\omega)^2}{b_4} + \frac{b_3}{b_4} j\omega + 1 \right)}_{\text{repeated second order zero}} \\ - \underbrace{q \angle(j\omega)}_{\text{pole at origin}} - \underbrace{\angle \left(\frac{j\omega}{a_1} + 1 \right)}_{\text{first order pole}} - \underbrace{n_2 \angle \left(\frac{j\omega}{a_2} + 1 \right)}_{\text{repeated first order pole}} - \underbrace{n_3 \angle \left(\frac{(j\omega)^2}{a_4} + \frac{a_3}{a_4} j\omega + 1 \right)}_{\text{repeated second order pole}}$$

When the transfer function contains a quadratic factor, we must first examine if it is a true quadratic factor, or if it can be factorized into one double linear factor (critically damped system) or two single linear factors (overdamped system).

We can do this easily by rewriting the quadratic factor from its mathematical form to its physical form:

Undamped
oscillation
frequency:

$$\omega_n = \sqrt{a_0}$$

Damping
factor:

$$\zeta = \frac{a_1}{2\sqrt{a_0}}$$

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_0} = \frac{b_0}{a_0} \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The quadratic factor should be kept as a quadratic factor if:

$$0 < \zeta < 1$$

Otherwise, the quadratic factor should be factorized.

We can write the amplitude of quadratic factors as:

To simplify the drawing of the quadratic function amplitude we consider two cases:

Low frequency asymptote:

$$\omega \ll \omega_n$$

High frequency asymptote:

$$\omega \gg \omega_n$$

Break frequency:

$$\omega_{3dB} = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{(4\zeta^4 - 4\zeta^2 + 2)}}$$

$$|Q(j\omega)|_{dB} = m_3 \cdot 20 \log \left| \left(\frac{j\omega}{\omega_n} \right)^2 + 2j\zeta \left(\frac{\omega}{\omega_n} \right) + 1 \right|$$

$$|Q(j\omega)|_{dB} = m_3 \cdot 20 \log |1| = 0 \text{dB}$$

$$|Q(j\omega)|_{dB} = m_3 \cdot 20 \log \left| \left(-\frac{\omega}{\omega_n} \right)^2 \right| = m_3 \cdot 40 \cdot \log \left| \left(\frac{\omega}{\omega_n} \right) \right|$$

The deviation of the actual amplitude curve from the straight line asymptotes depends on the damping factor, thus we need more information to sketch a good approximation to the amplitude response.

Quadratic Factor Amplitude Peak

The quadratic factor amplitude features a local maximum located at the frequency:

Typo in book
Page 441!

$$\omega_{pk} = \omega_n \sqrt{1 - 2\zeta^2}$$

When: $0 < \zeta < 1/\sqrt{2}$

the amplitude of the peak is:

$$M_{pk,dB} = m_3 \cdot 20 \cdot \log [2\zeta\sqrt{1 - \zeta^2}] \text{ dB}$$

$$M_{peak} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

When: $1/\sqrt{2} \leq \zeta < 1$

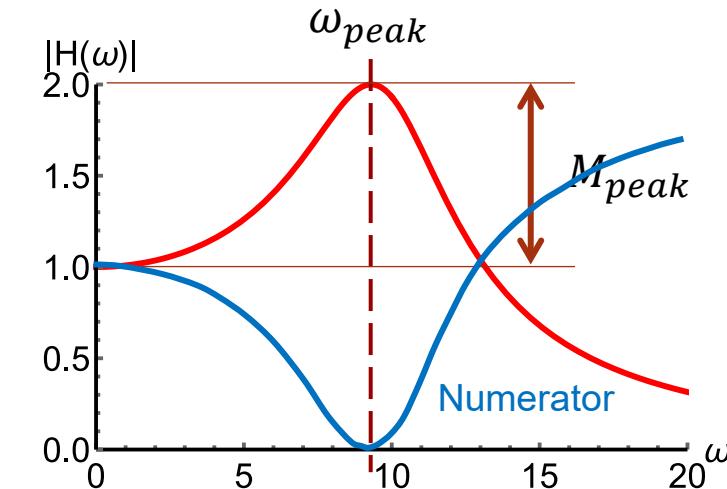
the local peak is located at: $\omega_{pk} = 0$

and the peak amplitude is: $M_{pk} = 0 \text{ dB}$

Resonance frequency: $\omega = \omega_n$

At the resonance frequency, the amplitude is:

$$|Q(j\omega_n)|_{dB} = m_3 \cdot 20 \log |j2\zeta|$$



Quadratic Factor Phase

We can write the phase of quadratic factors as:

$$\angle Q(j\omega) = m_3 \cdot \angle \left| \left(\frac{j\omega}{\omega_n} \right)^2 + 2j\zeta \left(\frac{\omega}{\omega_n} \right) + 1 \right|$$

Again, we consider two cases:

Low frequency asymptote:

$$\omega \ll \omega_n$$

$$\angle Q(j\omega) = m_3 \cdot \angle |j0 + 1| = 0^\circ$$

High frequency asymptote:

$$\omega \gg \omega_n$$

$$\angle Q(j\omega) = m_3 \cdot \angle \left| \left(\frac{j\omega}{\omega_n} \right)^2 \right| = m_3 \cdot \angle \left| - \left(\frac{\omega}{\omega_n} \right)^2 \right| = m_3 \cdot 180^\circ$$

Resonance frequency:

$$\omega = \omega_n$$

$$\angle Q(j\omega_n) = m_3 \cdot \angle |-1 + 2\zeta j + 1| = m_3 \cdot 90^\circ$$

We derive the **group delay** from the phase spectrum:

$$t_g = -\frac{d\theta(\omega)}{d\omega}$$

For a quadratic factor, the group delay is zero at infinite frequency (all capacitors are short circuited).

At the resonance frequency the group delay will feature a peak equal to the effective time constant of the step response envelope.

The group delay at DC is reduced by 2ζ relative to the peak delay.

$$\angle Q(j\omega) = m_3 \cdot \angle \left| \left(\frac{j\omega}{\omega_n} \right)^2 + 2j\zeta \left(\frac{\omega}{\omega_n} \right) + 1 \right|$$

$$t_g(\omega) = -m_3 \frac{d}{d\omega} \tan^{-1} \frac{2\zeta \left(\frac{\omega}{\omega_n} \right)}{1 - \frac{\omega^2}{\omega_n^2}}$$

$$t_g(\omega) = -m_3 \frac{2\zeta \omega_n (\omega^2 + \omega_n^2)}{\omega^4 + (4\zeta^2 - 2)\omega^2 \omega_n^2 + \omega_n^4}$$

$$t_g(\infty) = 0$$

$$t_g(\omega_n) = -m_3 \frac{2\zeta \omega_n (2\omega_n^2)}{(4\zeta^2 - 2)\omega_n^4 + 2\omega_n^4} = -m_3 \frac{4\zeta \omega_n^3}{4\zeta^2 \omega_n^4} = -m_3 \frac{1}{\zeta \omega_n}$$

$$t_g(0) = -m_3 \frac{2\zeta}{\omega_n} \quad \frac{t_g(\omega_n)}{t_g(0)} = \frac{1}{2\zeta}$$

Single second order pole with different damping ratio

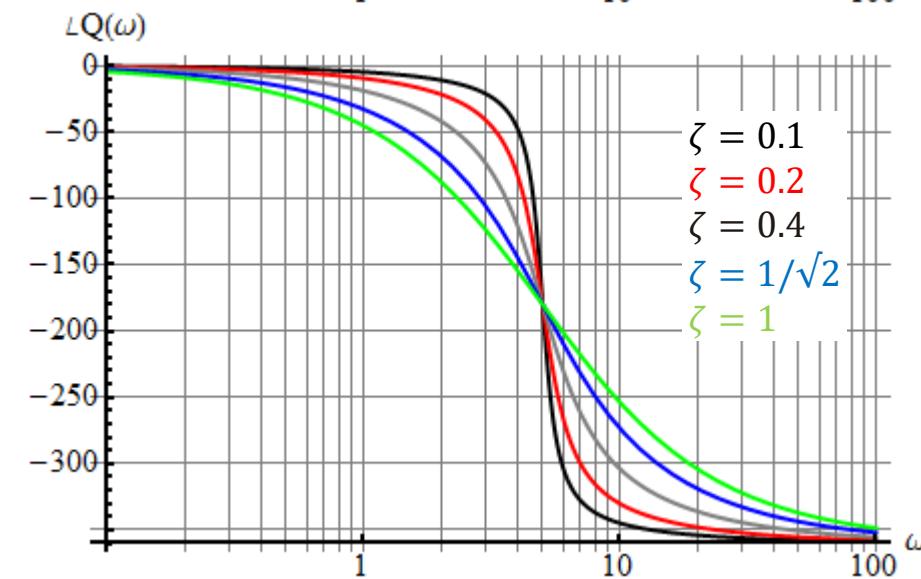
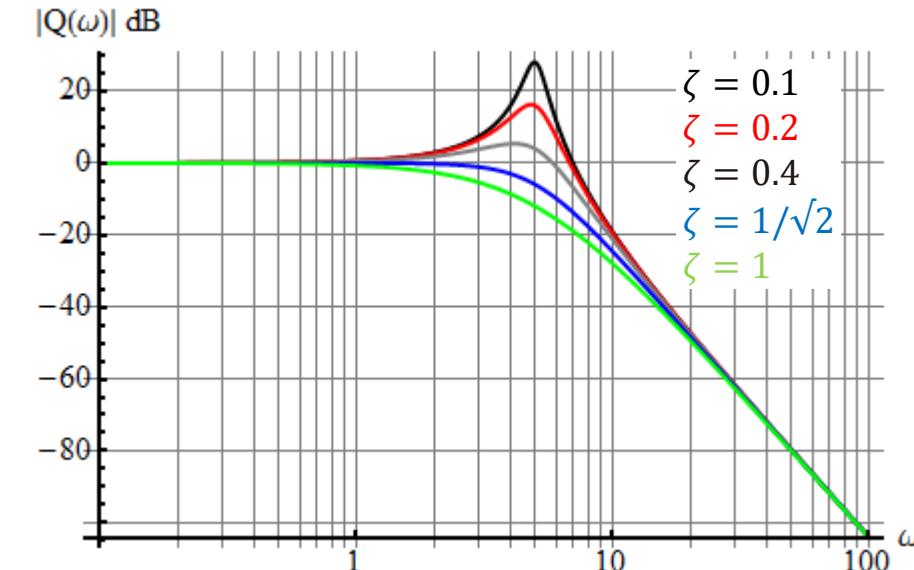
The effect of the damping ratio on the frequency characteristic.

If a system has very little damping it will tend to oscillate at a resonance frequency.

The phase transition band becomes very narrow for small damping ratios.

What is the filter order?

$$\omega_n = 5$$
$$m_3 = 1$$



Quadratic Factor Phase Transition

**Not in
Lathi's book!**

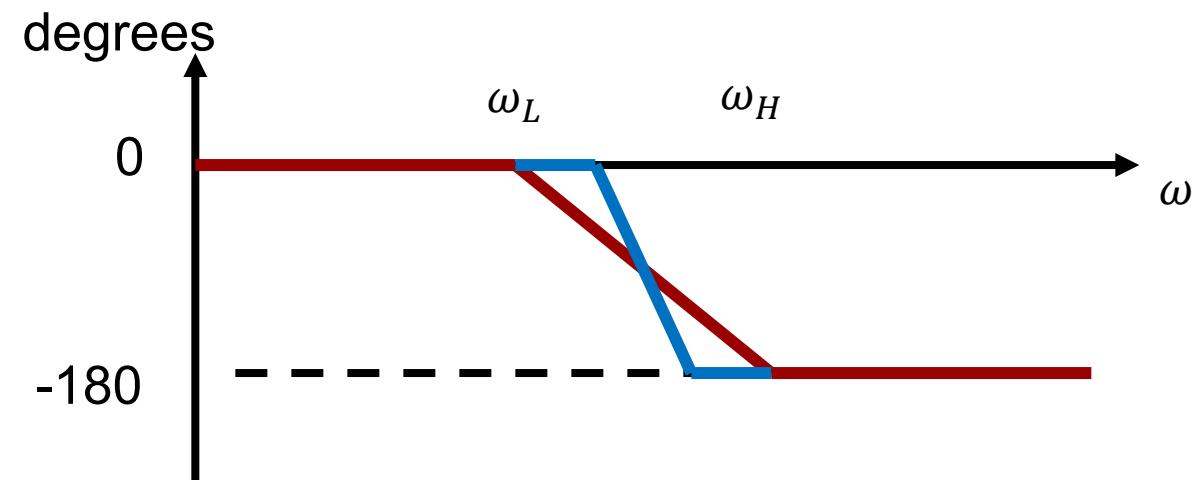
Limits:

$$\begin{aligned}\zeta \rightarrow 1 &\Rightarrow \\ a \rightarrow 1 &\Rightarrow \\ \omega_L &\rightarrow \omega_n/10 \\ \omega_H &\rightarrow 10\omega_n\end{aligned}$$

$$\begin{aligned}\zeta \rightarrow 0 &\Rightarrow \\ \omega_L &\rightarrow \omega_n \\ \omega_H &\rightarrow \omega_n\end{aligned}$$

The low and high frequency boundaries for the phase transition are:

$$\begin{aligned}\omega_L &= 10^{-a}\omega_n \wedge \omega_H = 10^a\omega_n \\ a &= \begin{cases} 1.410\zeta - 0.150\zeta^2 & \zeta \leq 0.2 \\ 1.475\zeta - 0.475\zeta^2 & \zeta > 0.2 \end{cases}\end{aligned}$$



The blue system is more underdamped than the red.

Single Quadratic Factor

$$\left(\left(\frac{j\omega}{\omega_n} \right)^2 + 2j\zeta \frac{\omega}{\omega_n} + 1 \right)^{\pm 1}$$

Amplitude slope:

$$m_3 \cdot \frac{40 \text{dB}}{\text{decade}}$$

$$m_3 \cdot \frac{12 \text{dB}}{\text{octave}}$$

$$t_g(\omega_n) = -m_3 \frac{1}{\zeta \omega_n} = -2$$

$$t_g(0) = -m_3 \frac{2\zeta}{\omega_n} = -0.04$$

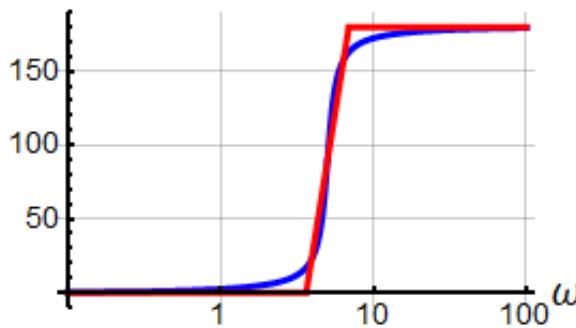
$$\frac{t_g(\omega_n)}{t_g(0)} = \frac{1}{2\zeta^2} = 50$$

$$m_3 = 1$$

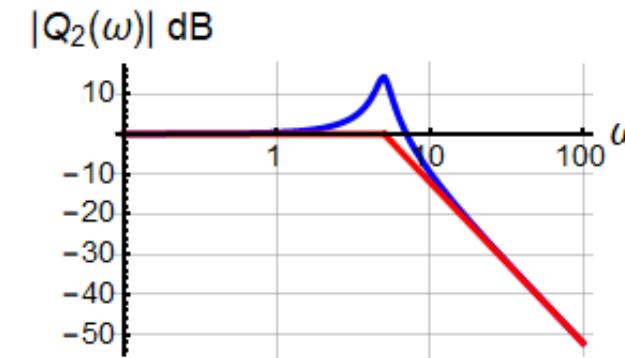


$$\angle Q_1(\omega)$$

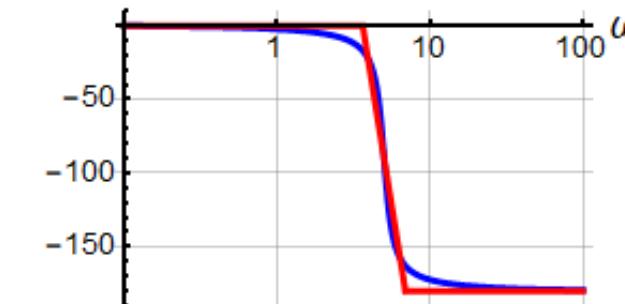
Out[$_$] =



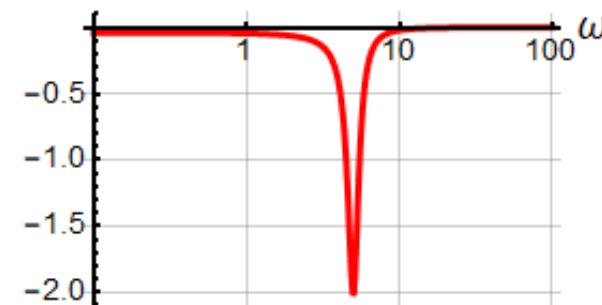
$$m_3 = -1$$



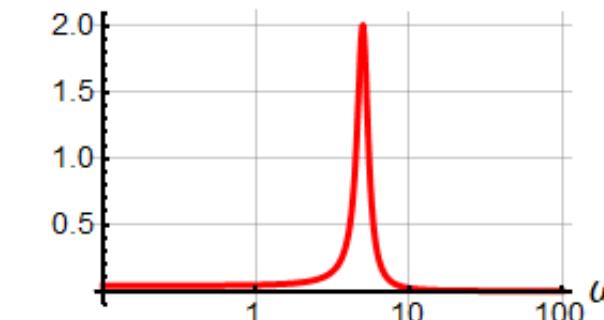
$$\angle Q_2(\omega)$$



$$t_{gQ1}(\omega)$$



$$t_{gQ2}(\omega)$$



$$\begin{aligned} \omega_n &= 5 \\ \zeta &= 0.1 \\ m_3 &= 1 \end{aligned}$$

Double Quadratic Factor

$$\left(\left(\frac{j\omega}{\omega_n} \right)^2 + 2j\zeta \frac{\omega}{\omega_n} + 1 \right)^{\pm 2}$$

Amplitude slope:

$$m_3 \cdot \frac{40 \text{dB}}{\text{decade}}$$

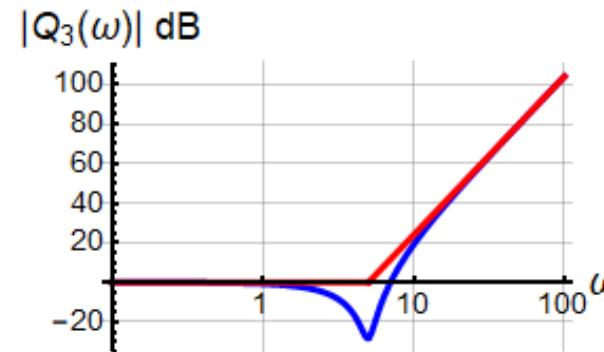
$$m_3 \cdot \frac{12 \text{dB}}{\text{octave}}$$

$$t_g(\omega_n) = -m_3 \frac{1}{\zeta \omega_n} = -4$$

$$t_g(0) = -m_3 \frac{2\zeta}{\omega_n} = -0.08$$

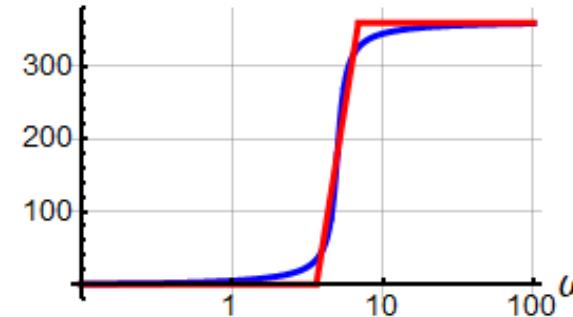
$$\frac{t_g(\omega_n)}{t_g(0)} = \frac{1}{2\zeta^2} = 50$$

$$m_3 = 2$$

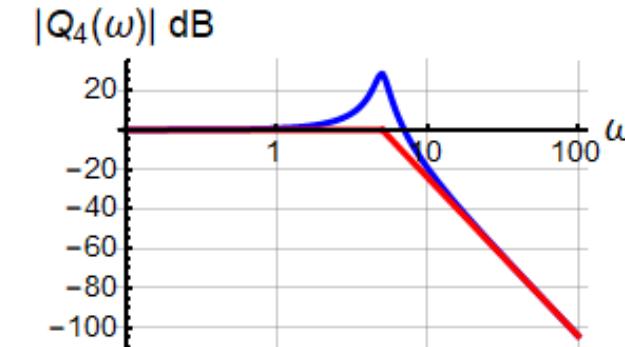


$$\angle Q_3(\omega)$$

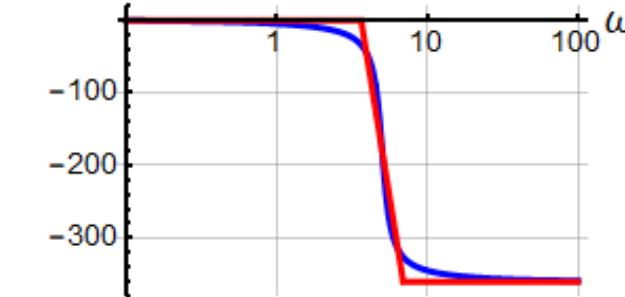
Out[...]=



$$m_3 = -2$$



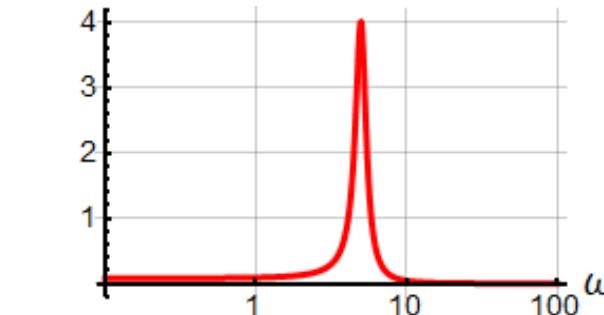
$$\angle Q_4(\omega)$$



$$t_{gQ3}(\omega)$$



$$t_{gQ4}(\omega)$$



$$\omega_n = 5$$

$$\zeta = 0.1$$

$$m_3 = 2$$

Double Quadratic Factor

In the previous lecture we found the 3dB cut-off frequency for a single quadratic factor:

$$\omega_{3dB} = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{(4\zeta^4 - 4\zeta^2 + 2)}}$$

Here is used as an example:

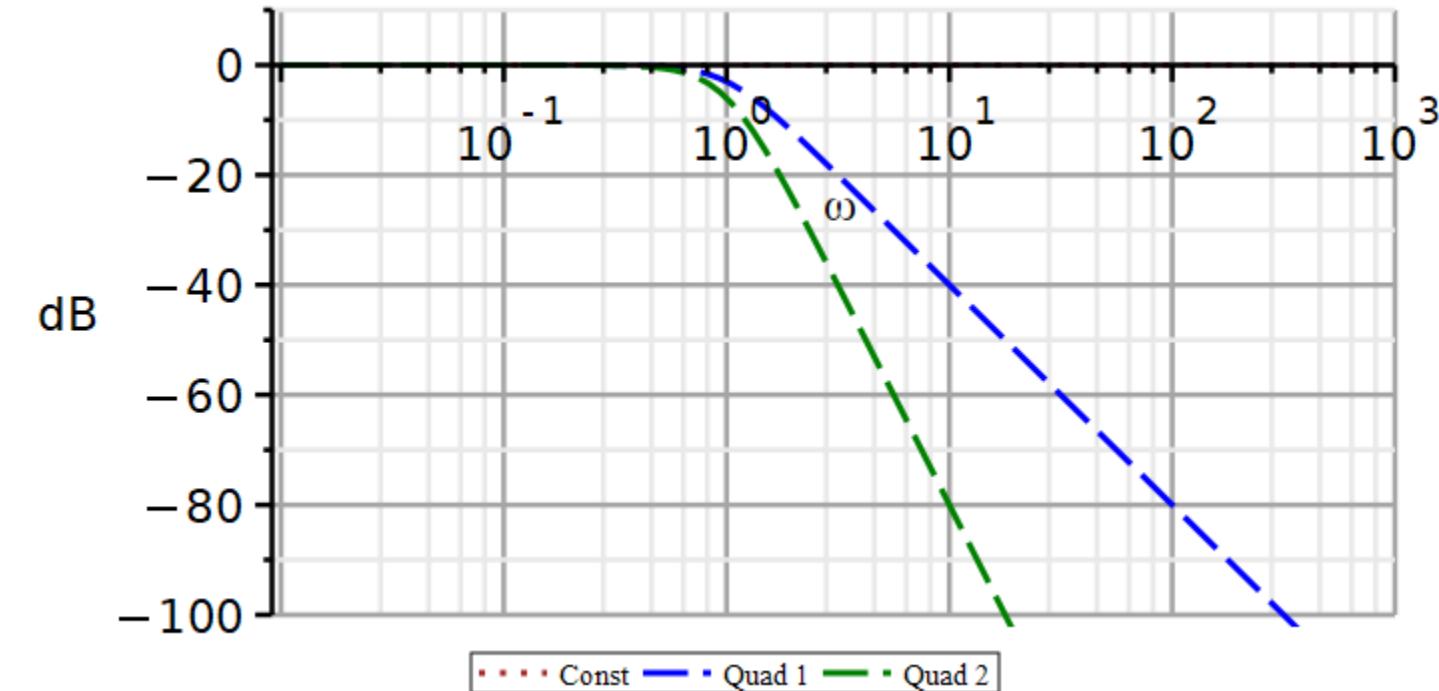
$$\zeta = \frac{1}{\sqrt{2}} \Rightarrow \omega_{3dB} = \omega_n = 1$$

The blue curve is a *single* quadratic factor ($m_3 = -1$). Hence, the amplitude at ω_n is **-3dB**.

The green curve is a *double* quadratic factor ($m_3 = -2$). Hence, the amplitude at ω_n is **-6dB**.

$$|Q(j\omega)|_{dB} = m_3 \cdot 20 \log \left| \left(\frac{j\omega}{1} \right)^2 + 2 \frac{1}{\sqrt{2}} \left(\frac{j\omega}{1} \right) + 1 \right|$$

$$|Q(j\omega_{3dB})|_{dB} = m_3 \cdot (3dB)$$



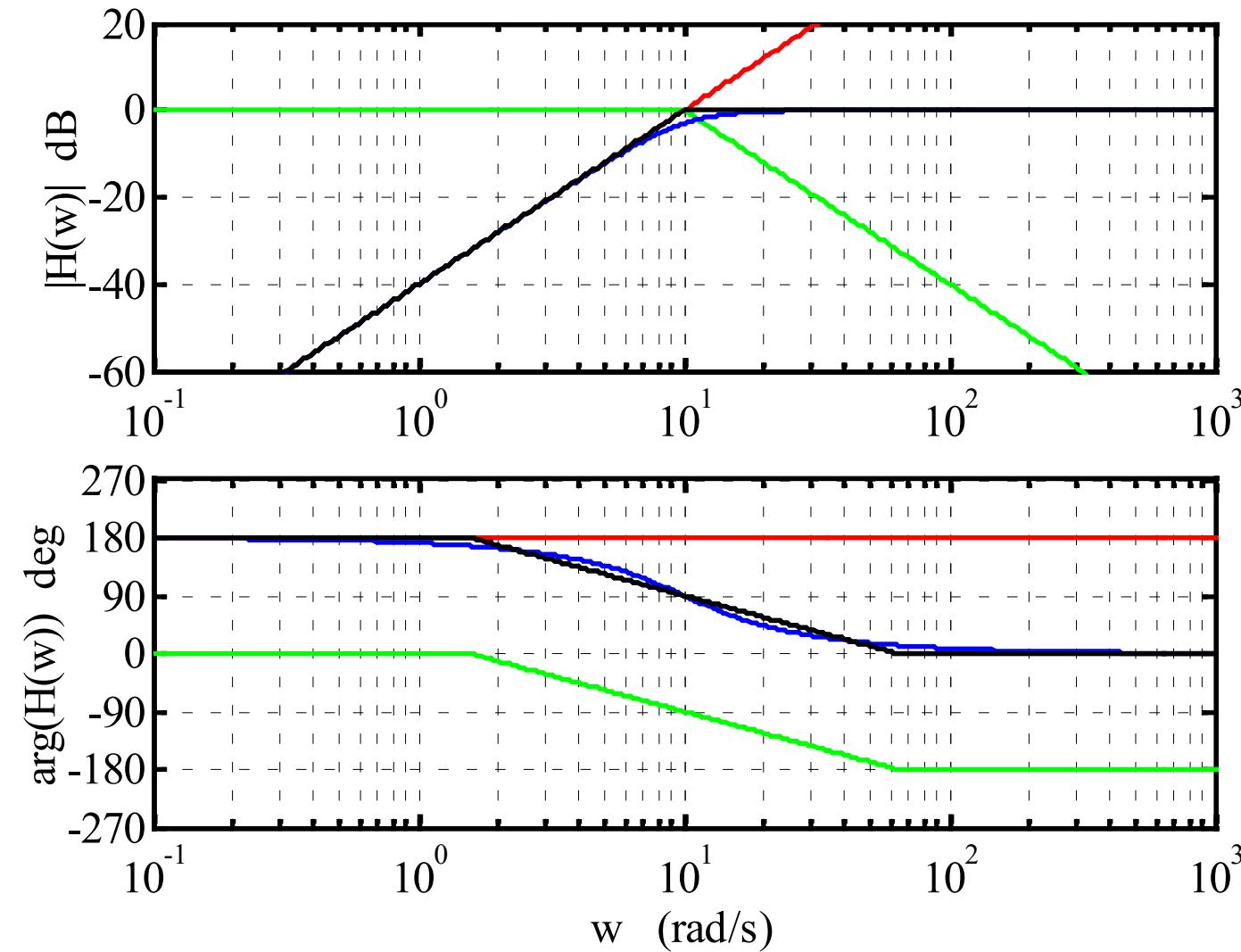
Bode Plot example

Analyze the Bode plot and identify the transfer function.

$$\Omega(\omega) = \left(\frac{j\omega}{10} \right)^2$$

$$Q(\omega) = \left(\left(\frac{j\omega}{10} \right)^2 + 2\zeta \frac{j\omega}{10} + 1 \right)^{-1}$$

Why not: $\left(\frac{j\omega}{10} + 1 \right)^{-2}$



Solution: Bode Plot example

Red curve:

$$\Omega(\omega) = \left(\frac{j\omega}{10} \right)^2$$

Green curve:

$$Q(\omega) = \left(\left(\frac{j\omega}{10} \right)^2 + 2\zeta \frac{j\omega}{10} + 1 \right)^{-1}$$

At what frequency does the phase transition stop:

$$\omega_H = 10 \cdot 10^a = 60 \Rightarrow a = \log_{10}(6) = 0.778$$

Solve for ζ

$$a = 1.475\zeta - 0.475\zeta^2 = 0.778 \Rightarrow \zeta = 0.67 \vee 2.43$$

$$\begin{aligned} H(\omega) &= \frac{(j\omega/10)^2}{(j\omega/10)^2 + 2\zeta(j\omega/10) + 1} \\ &= \frac{(j\omega)^2}{(j\omega)^2 + 2 \cdot 0.67 \cdot 10j\omega + 100} \end{aligned}$$

The curves were
constructed
using:

$$\zeta = \frac{1}{\sqrt{2}} \approx 0.707$$

$$H(s) = \frac{s^2}{(s^2 + 2 \cdot 0.67 \cdot 10 \cdot s + 100)}$$

For underdamped 2nd order systems with a local peak in the amplitude spectrum, we can obtain estimates of $\zeta \wedge \omega_n$ from the amplitude peak and peak frequency:

$$M_{pk} = m_3 \cdot 20 \cdot \log [2\zeta\sqrt{1 - \zeta^2}] \text{ dB}$$

$$\omega_{pk} = \omega_n \sqrt{1 - 2\zeta^2}$$

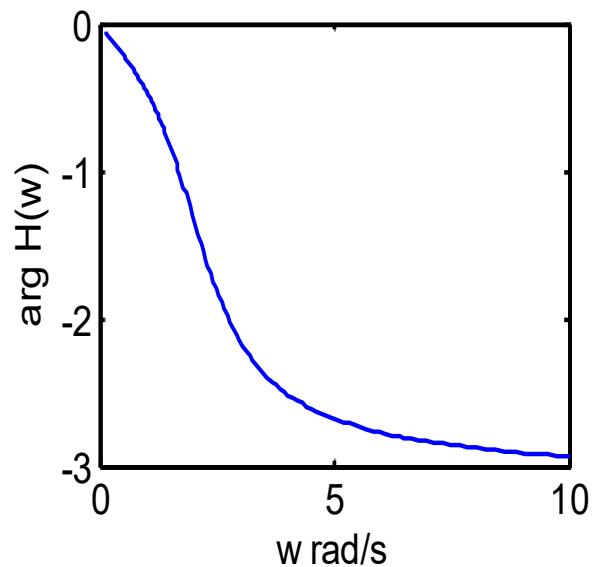
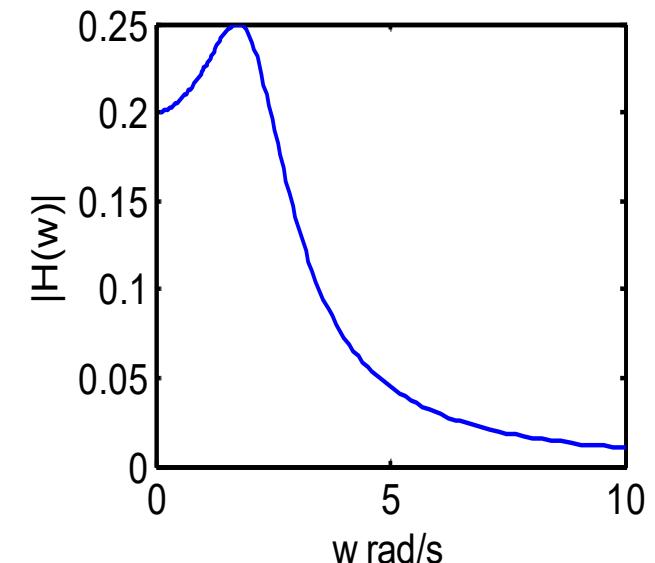
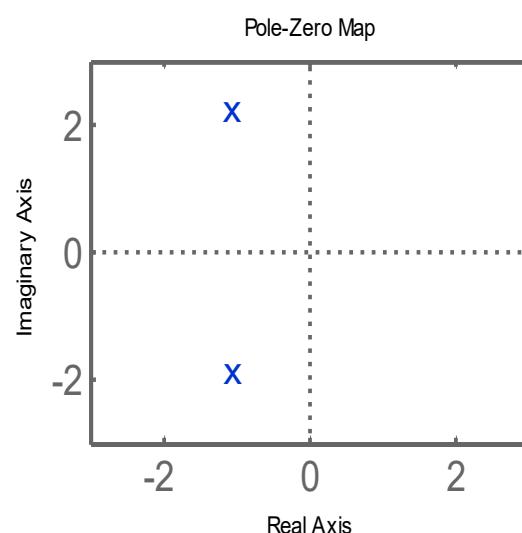
Knowing $\zeta \wedge \omega_n$ we have defined the differential equation, and we can estimate settling time, percent overshoot, delay time, time to peak, and rise time of the step response.

Filter design by placement of poles and zeros

Video

Poles, zeros and frequency characteristic

How can we explain the frequency characteristic from the locations of the poles and zeros?



Graphical Interpretation

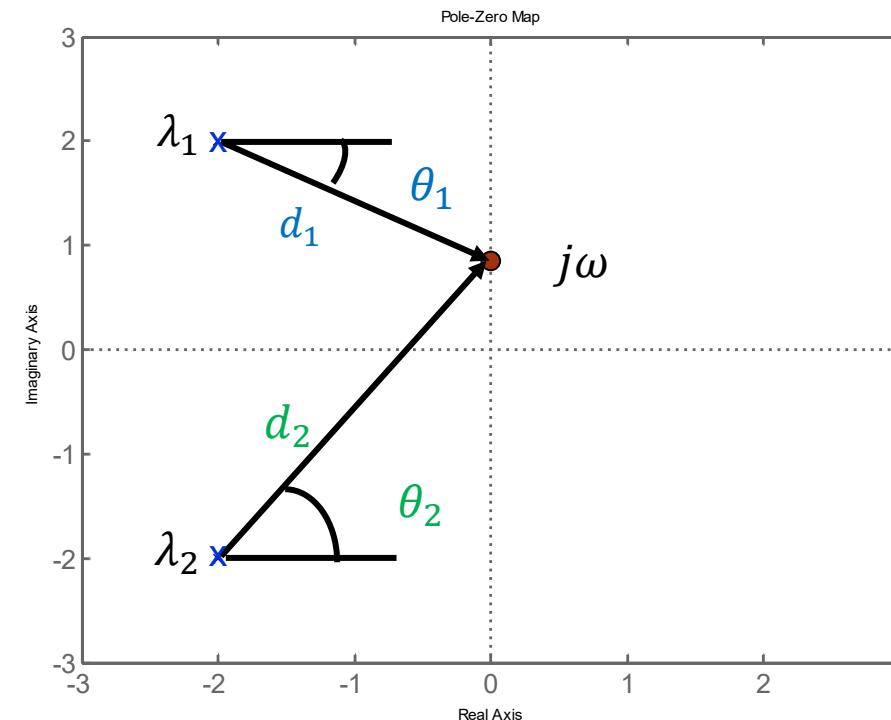
Here we define the transfer function in terms of its poles ($\lambda_1, \lambda_2\dots$).

$$H(s) = \frac{1}{(s - \lambda_1)(s - \lambda_2)}$$

$$H(s)|_{s=j\omega} = \frac{1}{(j\omega - \lambda_1)(j\omega - \lambda_2)}$$

$$= \underbrace{\frac{1}{d_1 d_2}}_{polar\ form} e^{-j(\theta_1 + \theta_2)}$$

d_1 and d_2 are the distance from the poles to a location ($j\omega$) on the imaginary axis.
The angles are as drawn in the diagram.



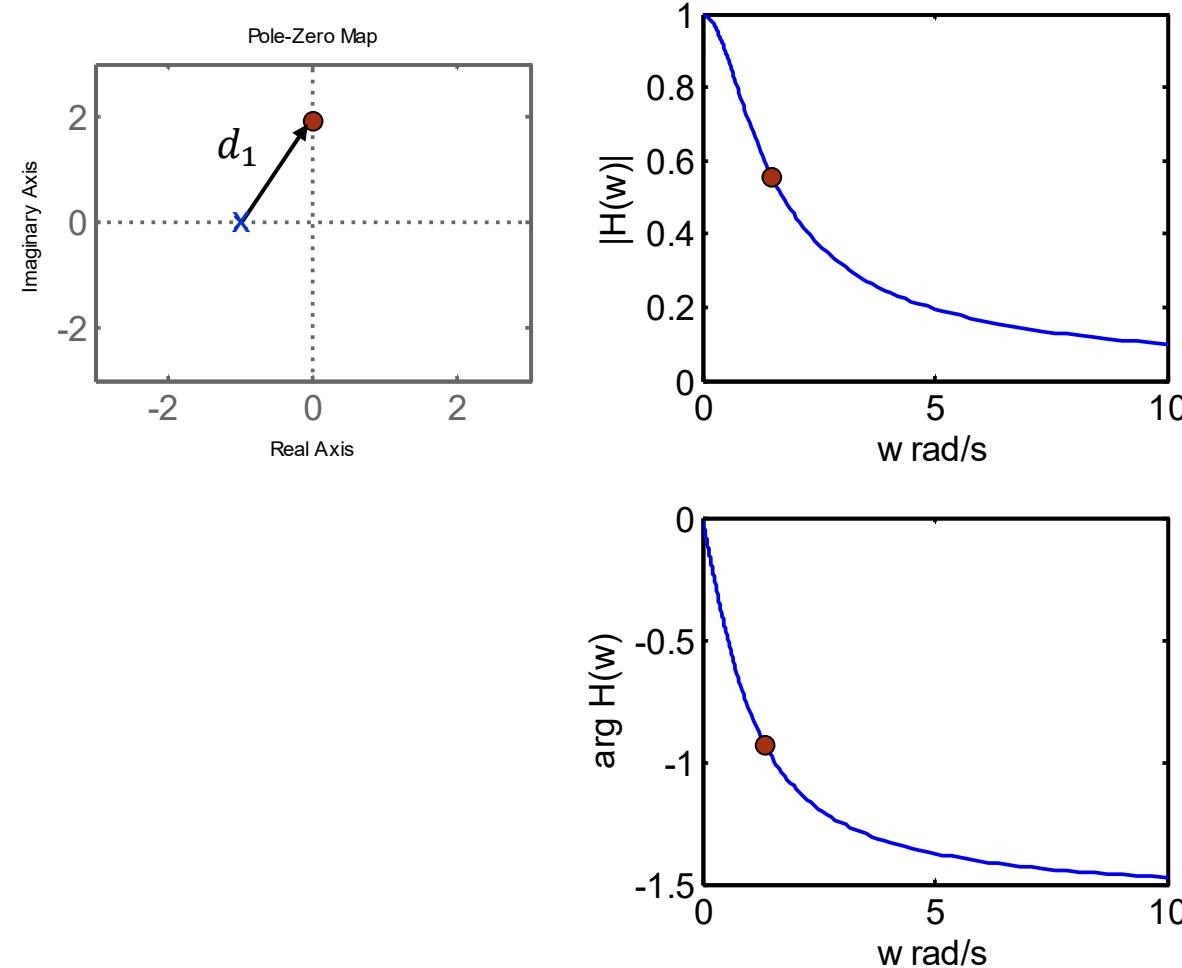
Single real pole

For positive frequencies the angle is positive but the factor is in the denominator, thus the phase is negative.

$$H(s) \Big|_{s=j\omega} = \frac{1}{(j\omega - \lambda_1)}$$

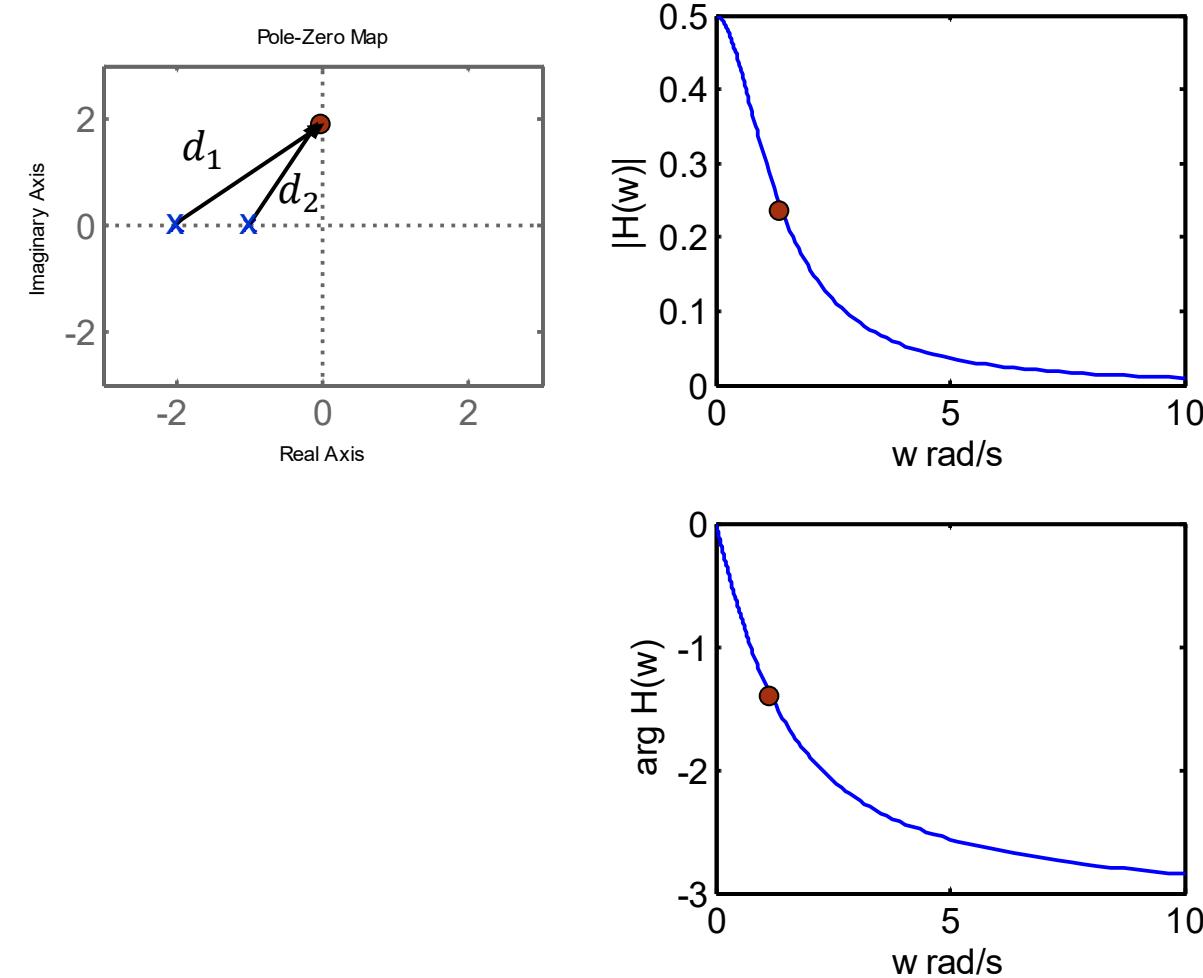
$$= \frac{1}{d_1} e^{-j\theta_1}$$

polar form



Two real poles

We now have two angles which approach 90. Thus, the phase settles at -180 degrees.



$$H(s)\Big|_{s=j\omega} = \frac{1}{(j\omega - \lambda_1)(j\omega - \lambda_2)}$$

$$= \underbrace{\frac{1}{d_1 d_2} e^{-j(\theta_1 + \theta_2)}}_{\text{polar form}}$$

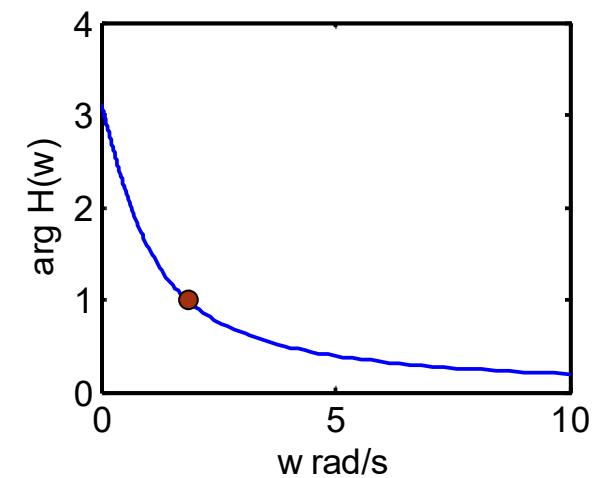
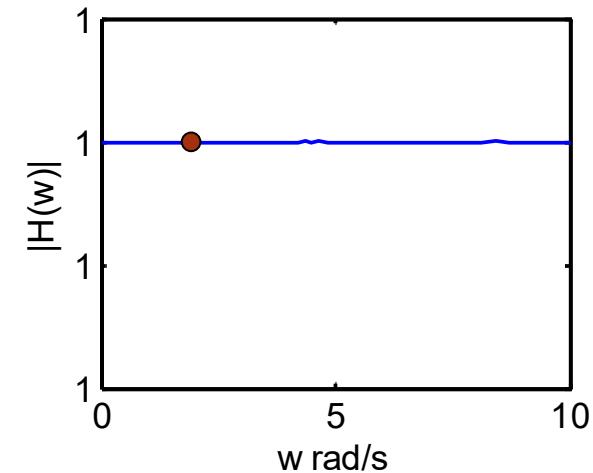
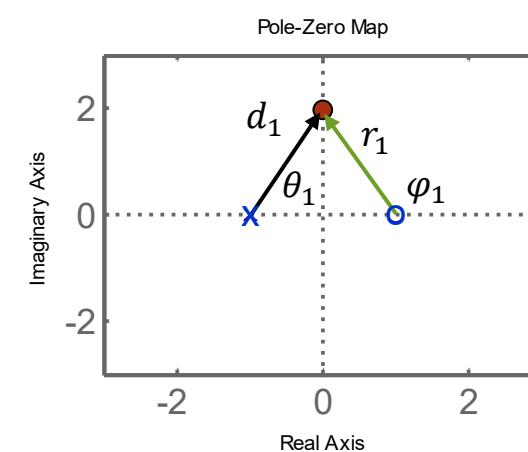
One pole and one positive zero

Amplitude response:

$$\frac{\prod_m r_m}{\prod_n d_n}$$

Phase response:

$$\sum_m \varphi_m - \sum_n \theta_n$$



Why is the amplitude spectrum constant?

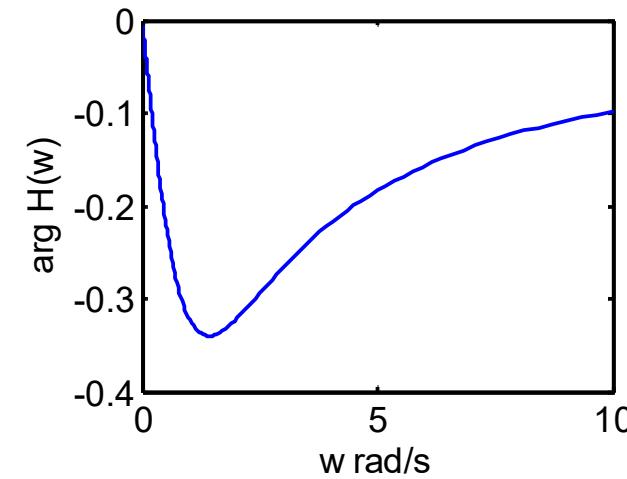
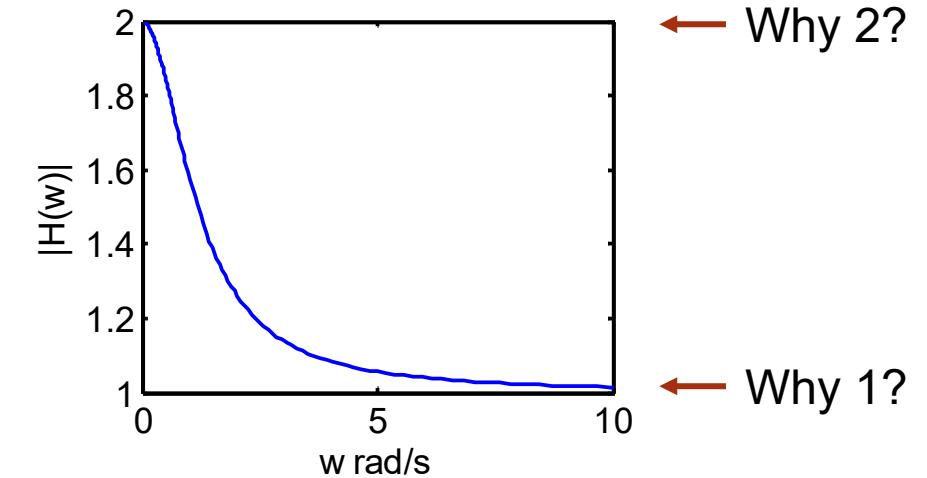
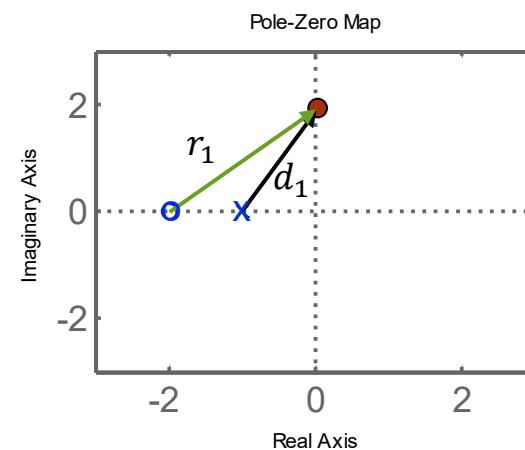
One pole and one negative zero

Amplitude response:

$$\frac{\prod_m r_m}{\prod_n d_n}$$

Phase response:

$$\sum_m \varphi_m - \sum_n \theta_n$$



Why does the phase angle vary much less in this case than in the previous case?

Amplitude response and zero sign

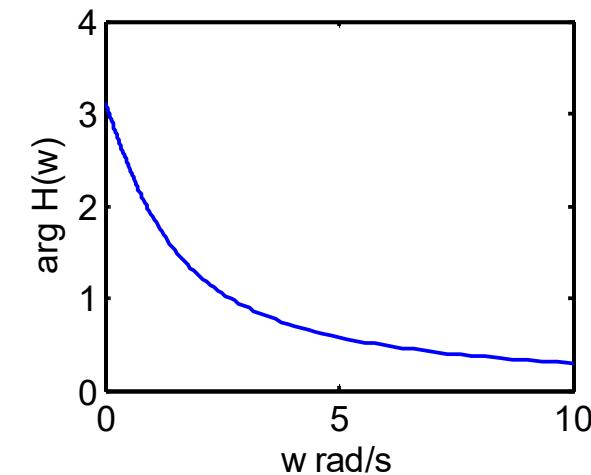
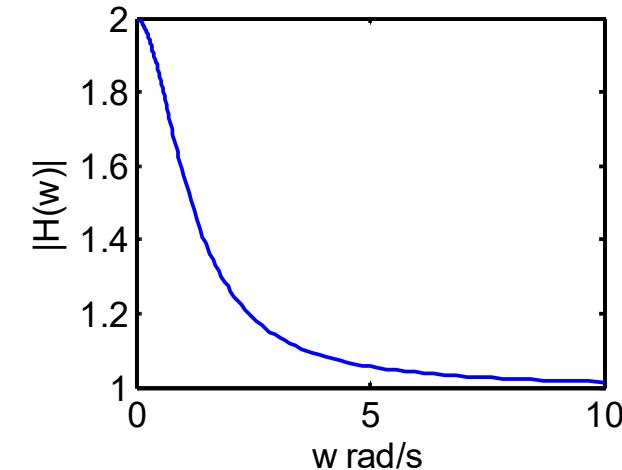
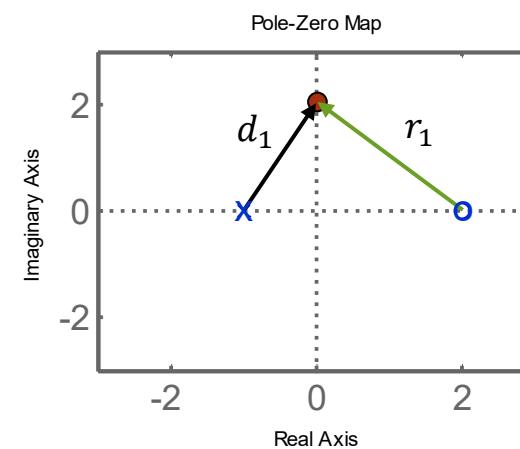
Amplitude
response:

$$\frac{\prod_m r_m}{\prod_n d_n}$$

Phase
response:

$$\sum_m \varphi_m - \sum_n \theta_n$$

What effect does the sign of the zero have on the amplitude response?
Compare with previous slide.



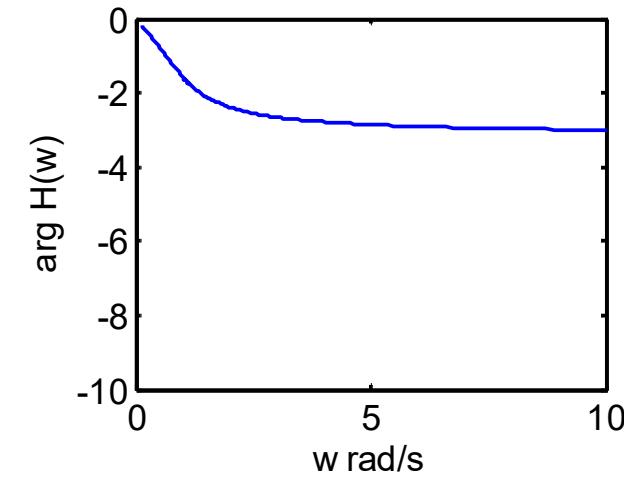
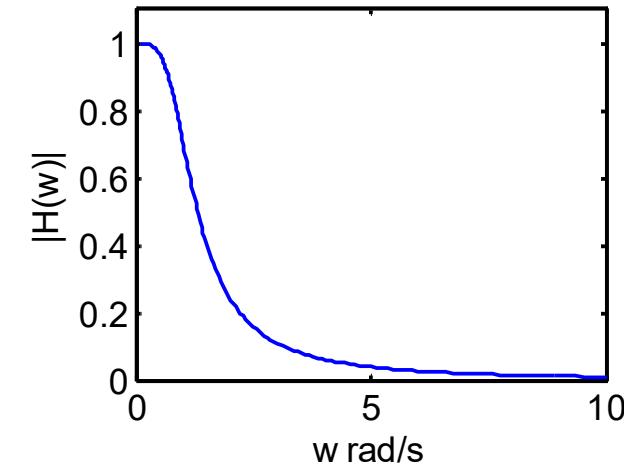
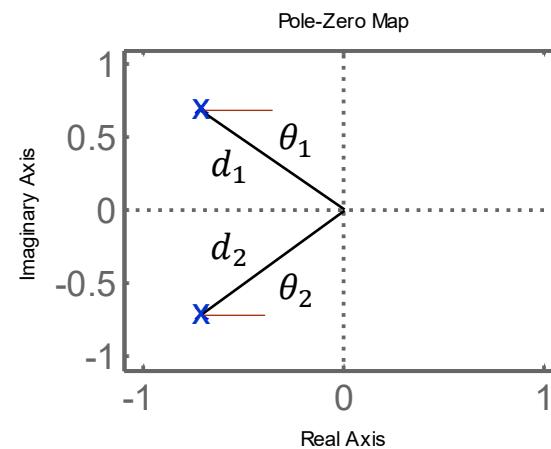
2nd order lowpass filter

Amplitude
response:

$$\frac{1}{\prod_n d_n}$$

Phase
response:

$$0 - \sum_n \theta_n$$



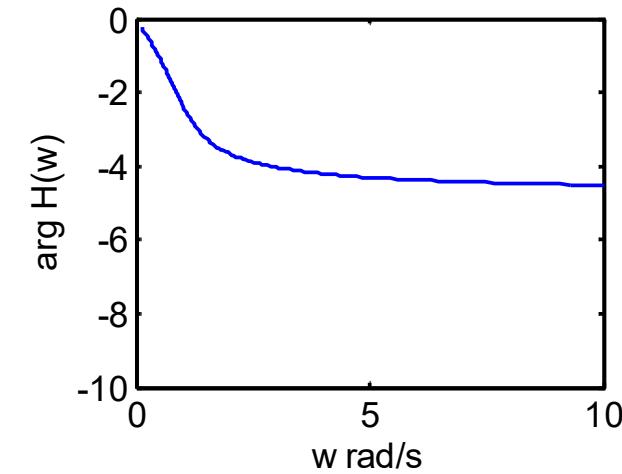
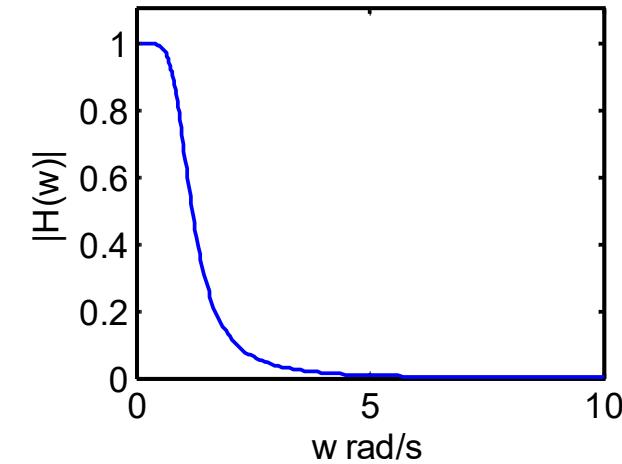
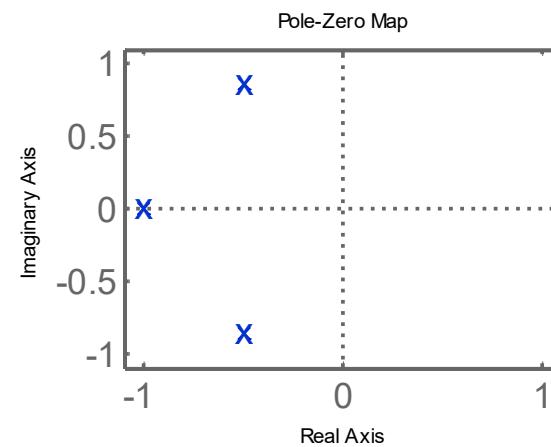
3rd order lowpass filter

Amplitude
response:

$$\frac{1}{\prod_n d_n}$$

Phase
response:

$$0 - \sum_n \theta_n$$



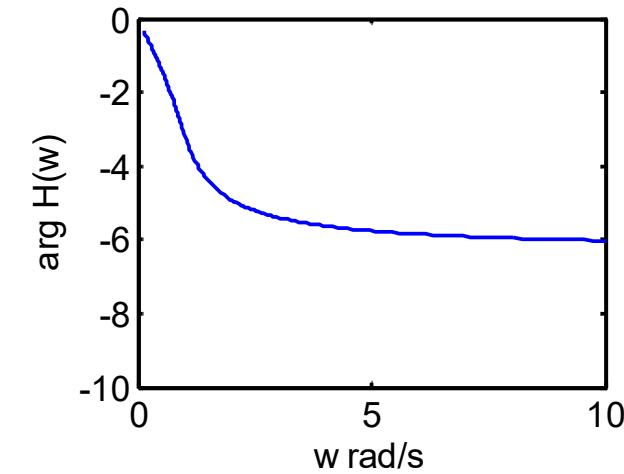
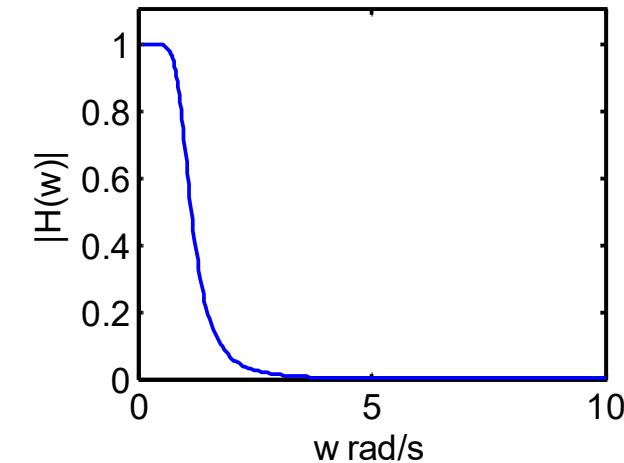
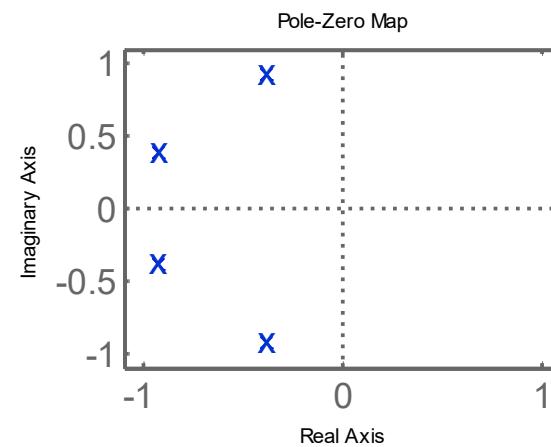
4th order lowpass filter

Amplitude
response:

$$\frac{1}{\prod_n d_n}$$

Phase
response:

$$0 - \sum_n \theta_n$$



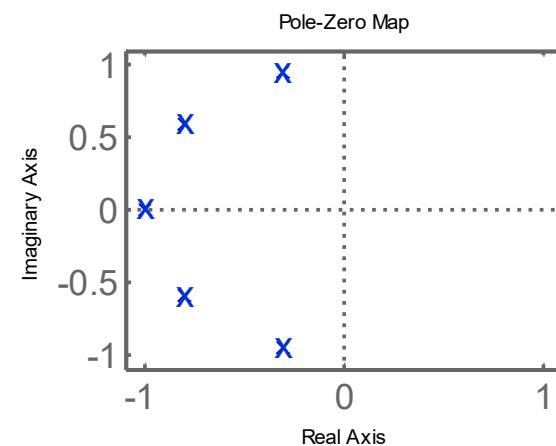
5th order lowpass filter

Amplitude
response:

$$\frac{1}{\prod_n d_n}$$

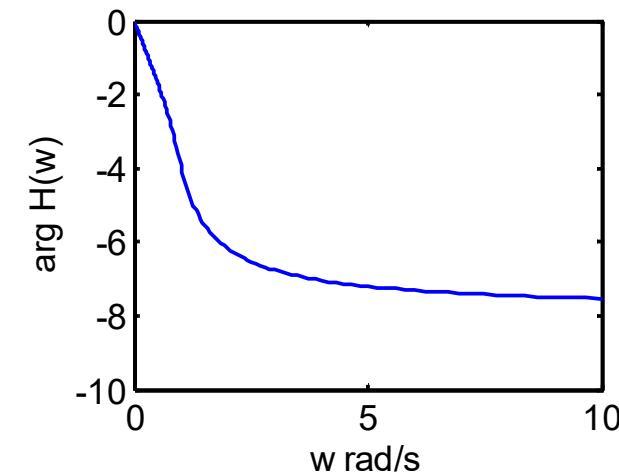
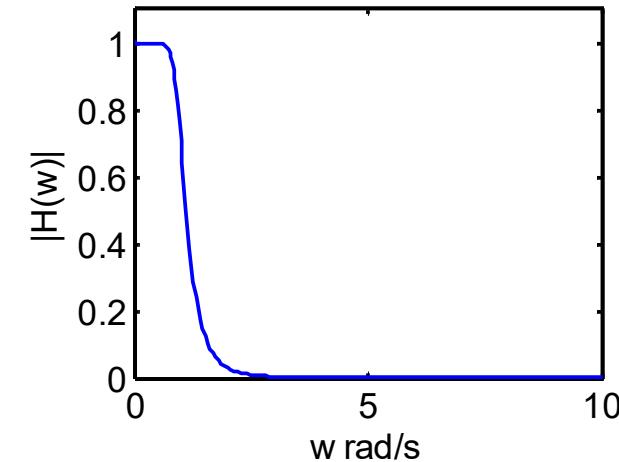
Phase
response:

$$0 - \sum_n \theta_n$$



Notice that filters of odd orders have poles on the negative real axis.

All these filters have poles on the unit circle and are of the **Butterworth type**.



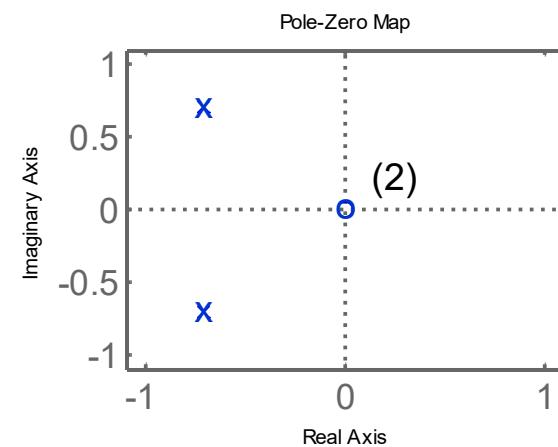
2nd order highpass filter

Amplitude
response:

$$\frac{\prod_m r_m}{\prod_n d_n}$$

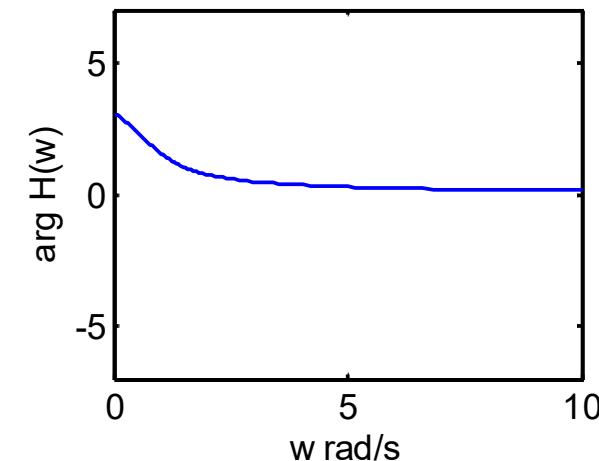
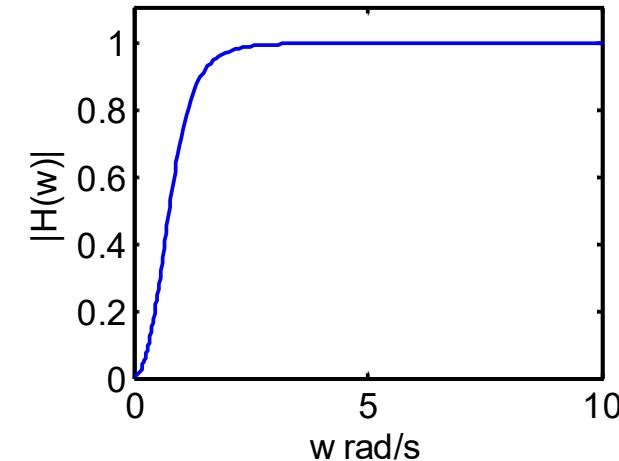
Phase
response:

$$\sum_m \varphi_m - \sum_n \theta_n$$



Why does the amplitude
go to zero in the origin?

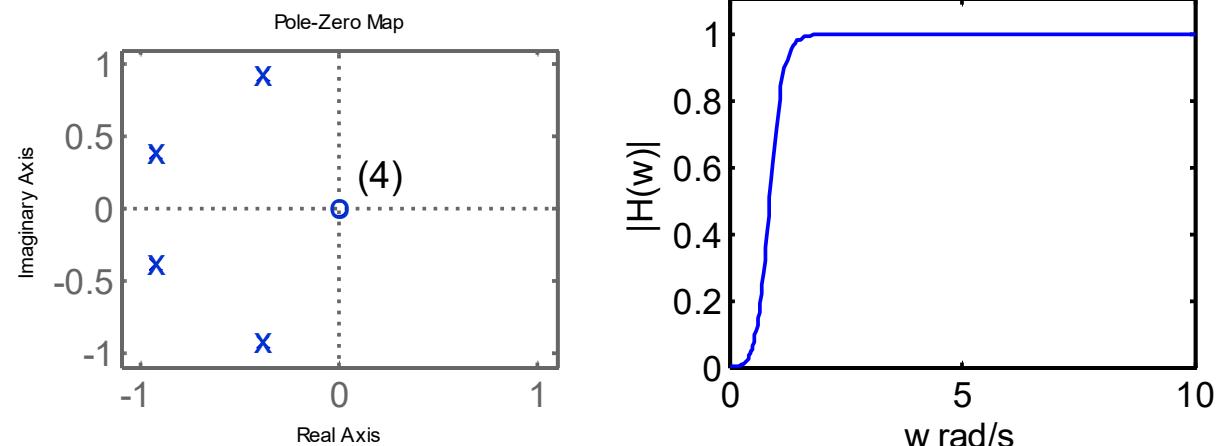
Why does the phase
response jump from -180
to 180 degrees?



4th order highpass filter

Amplitude
response:

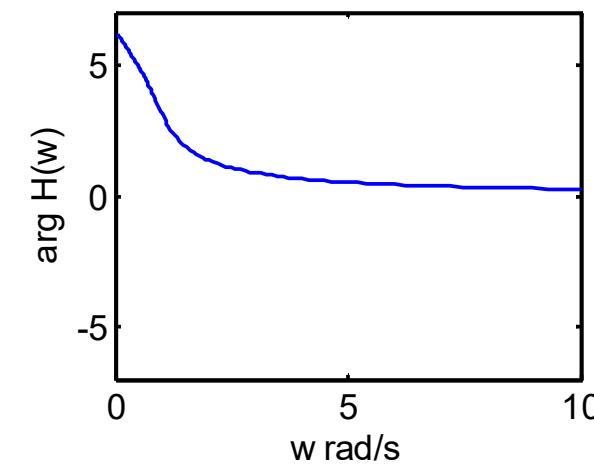
$$\frac{\prod_m r_m}{\prod_n d_n}$$



Phase
response:

$$\sum_m \varphi_m - \sum_n \theta_n$$

What is the phase jump
in 0 rad/s?

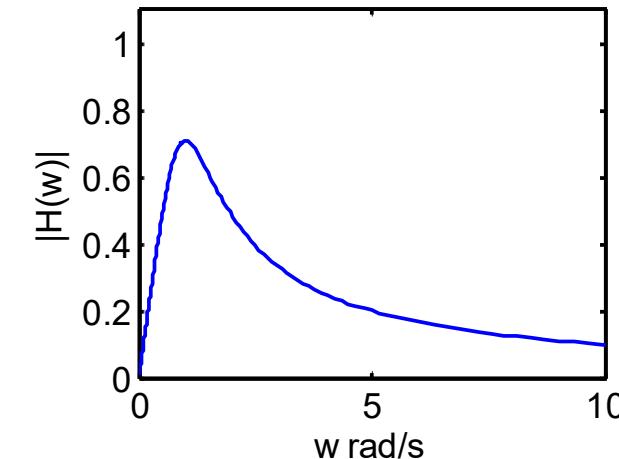
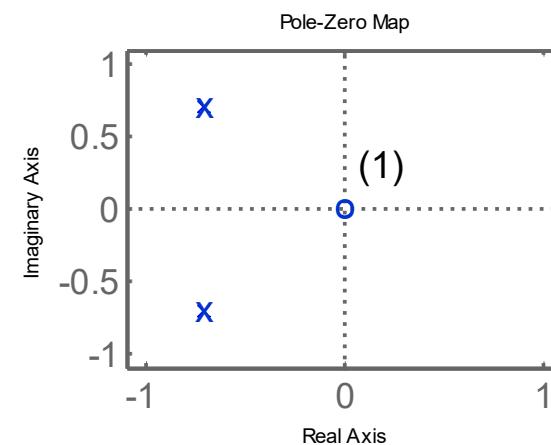


$$4 \cdot 180^\circ = 720^\circ$$

2nd order bandpass filter

Amplitude
response:

$$\frac{\prod_m r_m}{\prod_n d_n}$$

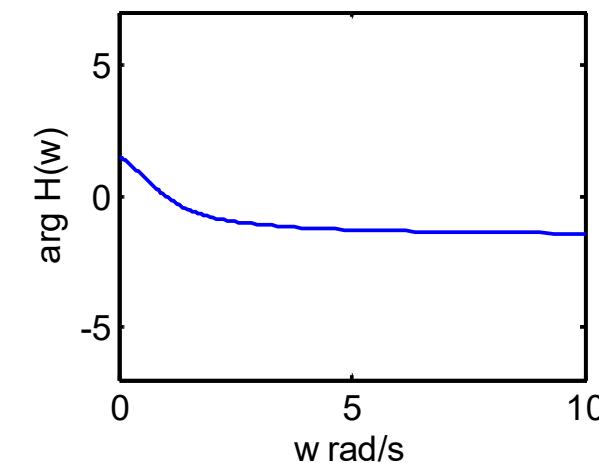


As $\omega \rightarrow \infty$ how fast does the
magnitude decrease, 1st or 2nd order?

Phase
response:

$$\sum_m \varphi_m - \sum_n \theta_n$$

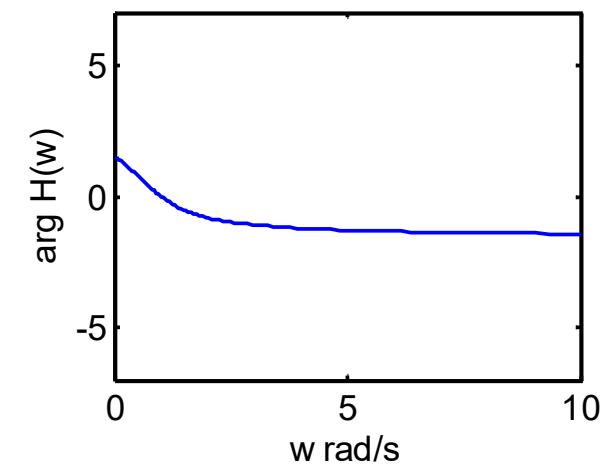
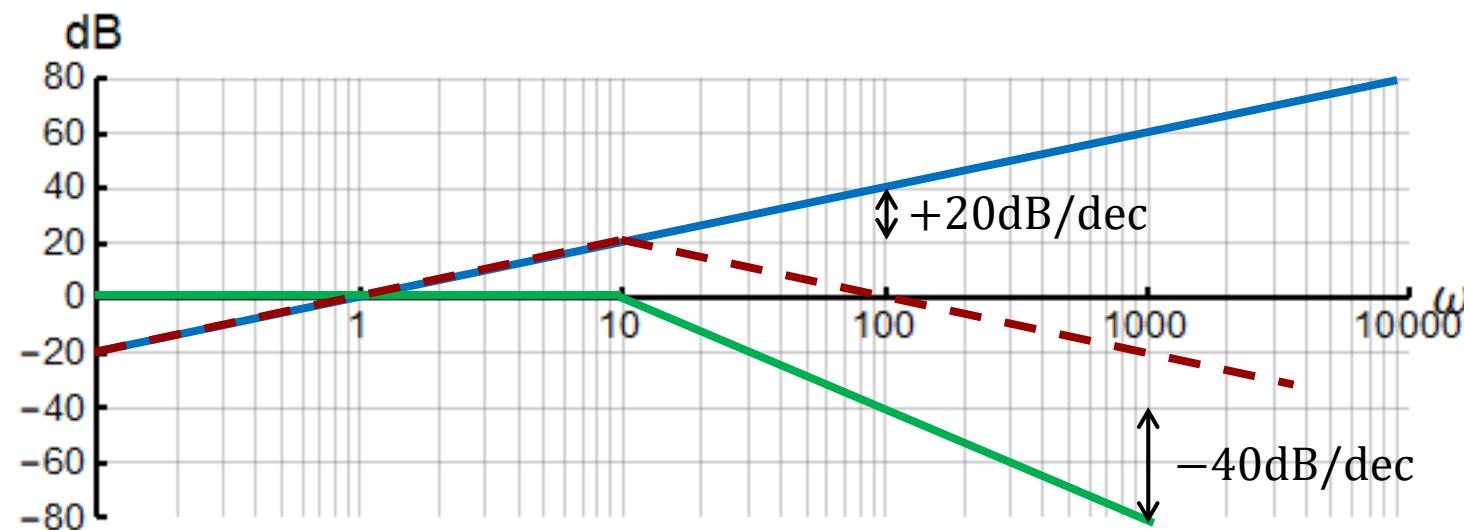
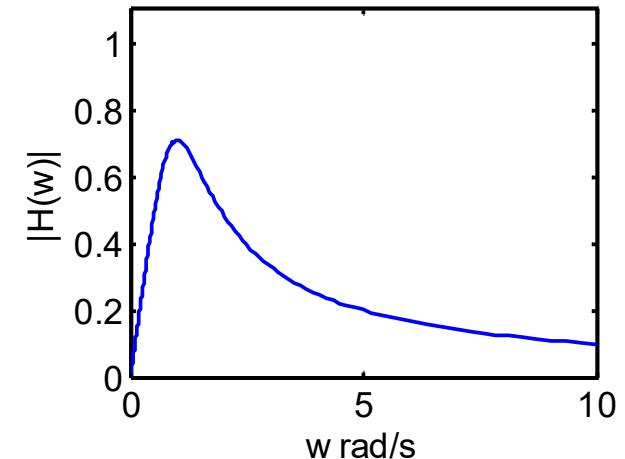
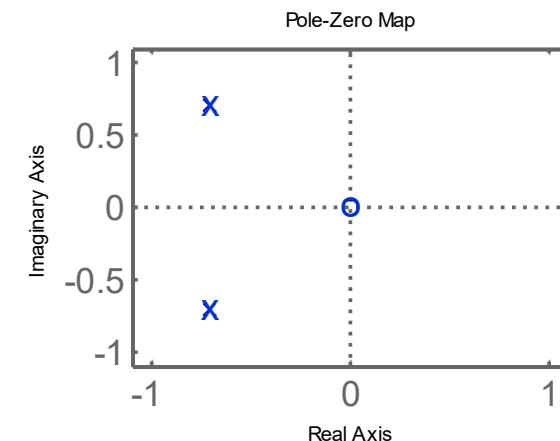
What is the phase jump
in 0 rad/s?



$$1 \cdot 180^\circ = 180^\circ$$

2nd order bandpass filter

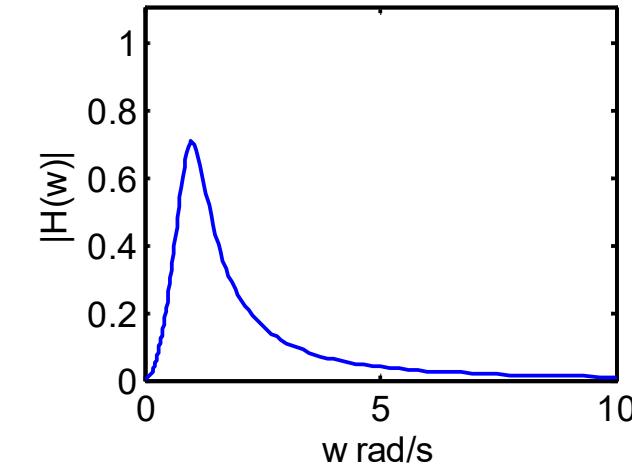
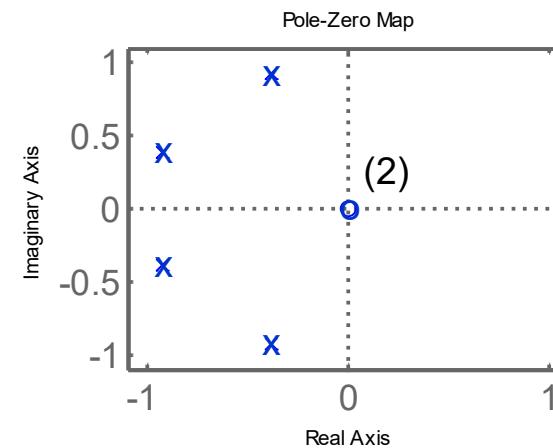
Does this amplitude decay agree with Bode plot?



4th order bandpass filter

Amplitude
response:

$$\frac{\prod_m r_m}{\prod_n d_n}$$



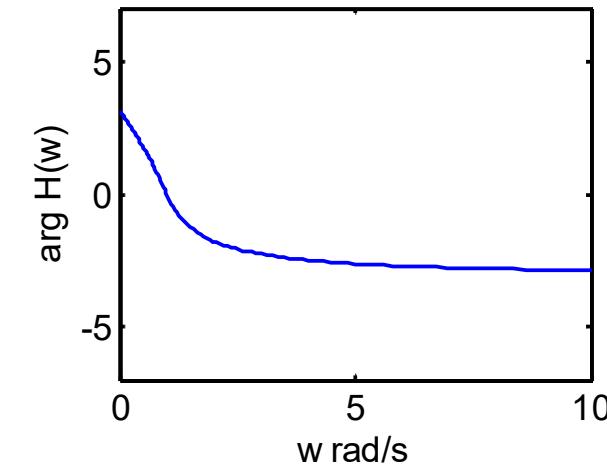
As $\omega \rightarrow \infty$ how fast does the magnitude decrease, 1st or 2nd order?

Phase
response:

$$\sum_m \varphi_m - \sum_n \theta_n$$

What is the phase jump in 0 rad/s?

$$2 \cdot 180^\circ = 360^\circ$$



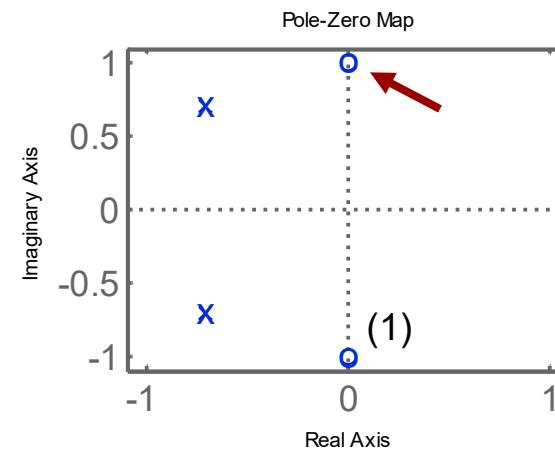
2nd order bandstop filter

Amplitude
response:

$$\frac{\prod_m r_m}{\prod_n d_n}$$

Phase
response:

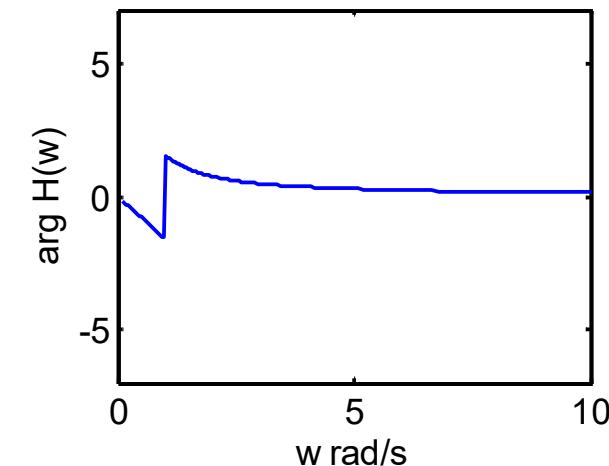
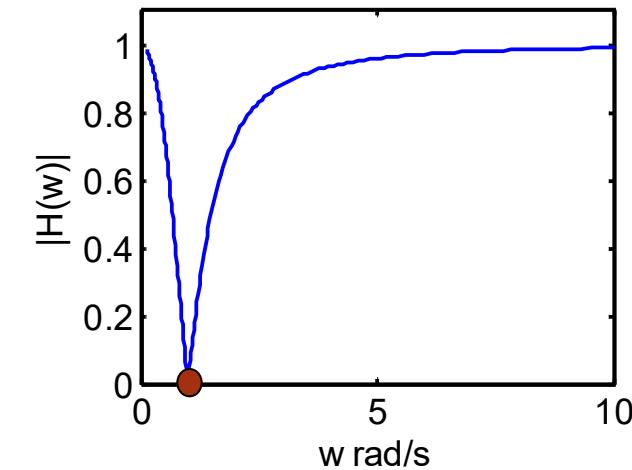
$$\sum_m \varphi_m - \sum_n \theta_n$$



What creates the dip in
amplitude?

What is the phase jump
in 1 rad/s?

$$1 \cdot 180^\circ = 180^\circ$$



Twin-T notch filter

$$H(s) = \frac{(RCs + 1)((RC)^2 s^2 + 1)}{(RCs + 1)((RC)^2 s^2 + 4(1 - \alpha)(RC)s + 1)}$$

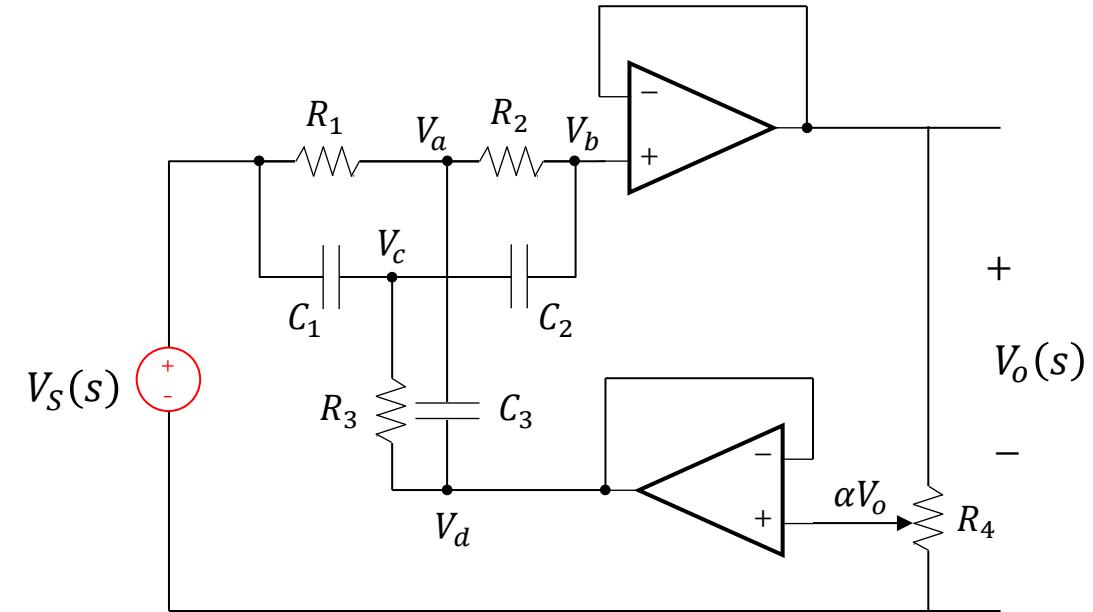
We like to study the poles of the quadratic factor.

$$((RC)^2 s^2 + 4(1 - \alpha)(RC)s + 1) = 0$$

We get: $s_{1,2} = \frac{-2 + 2\alpha \pm \sqrt{4\alpha^2 - 8\alpha + 3}}{RC}$

$$R = 68\text{k}\Omega \quad C = 47\text{nF}$$

Inserting numbers in the expression for the poles:



$$\alpha = 0.2: \quad s_{1,2} = -109.8, \quad -891.4$$

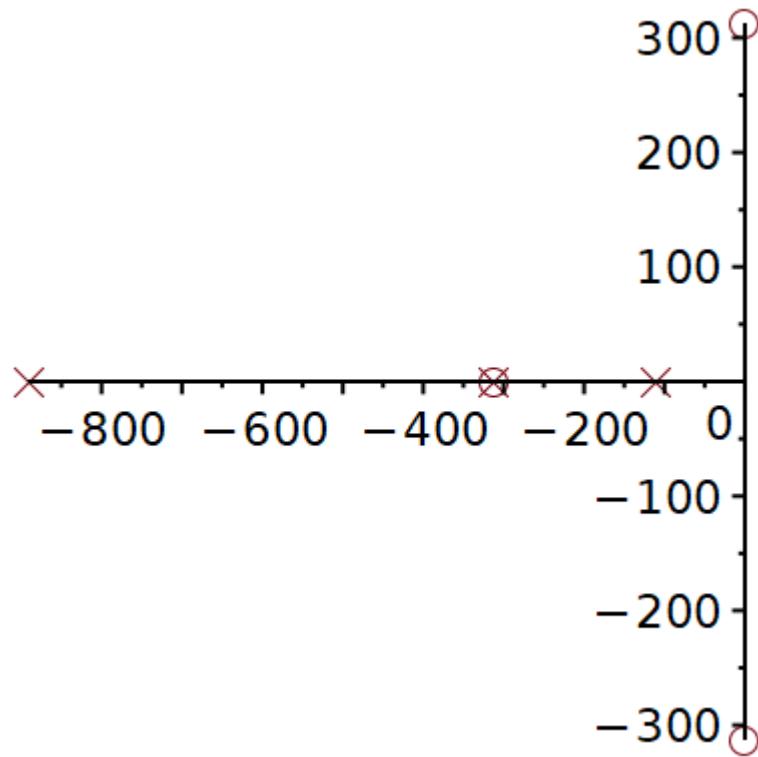
$$\alpha = 0.5: \quad s_{1,2} = -312.9, \quad -312.9$$

$$\alpha = 0.8: \quad s_{1,2} = -125.2 \pm j286.8$$

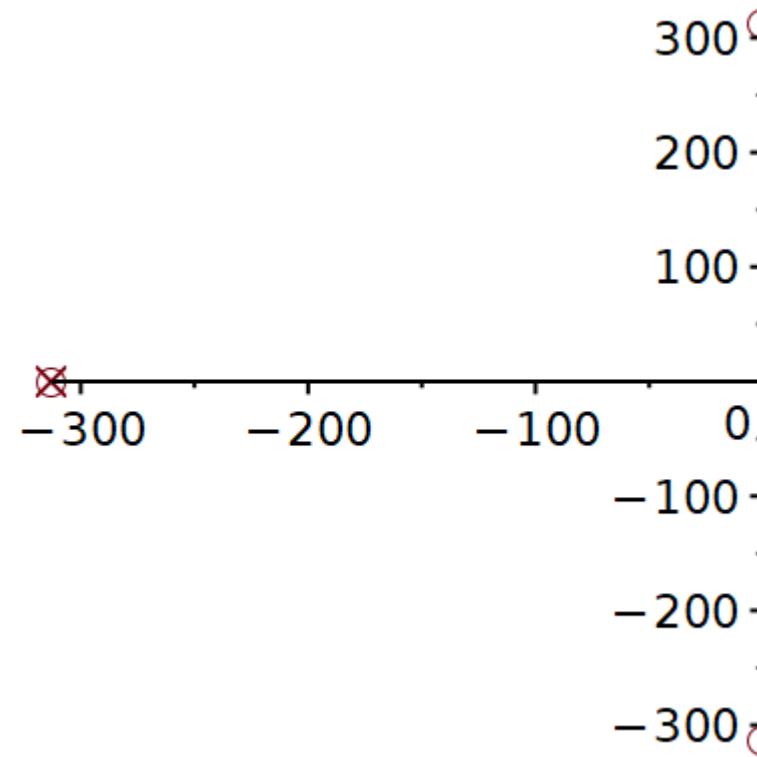
Twin-T notch filter

$$H(s) = \frac{(RCs + 1)((RC)^2 s^2 + 1)}{(RCs + 1)((RC)^2 s^2 + 4(1 - \alpha)(RC)s + 1)}$$

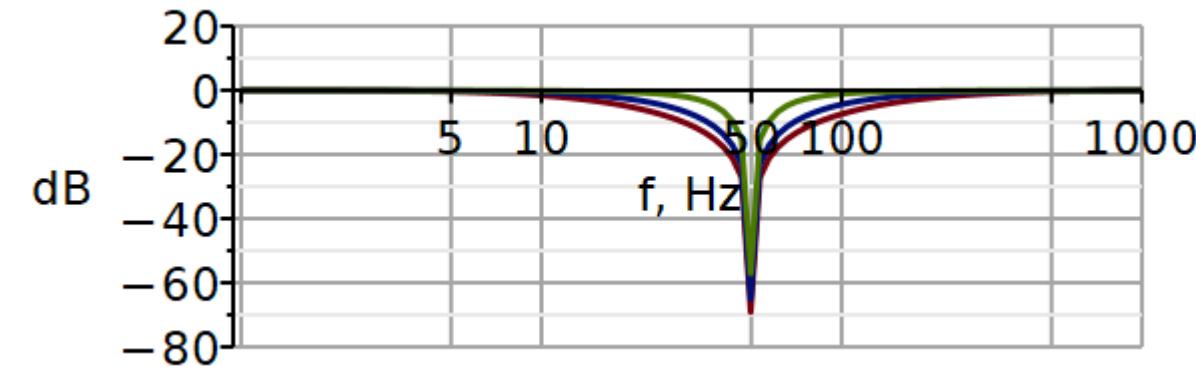
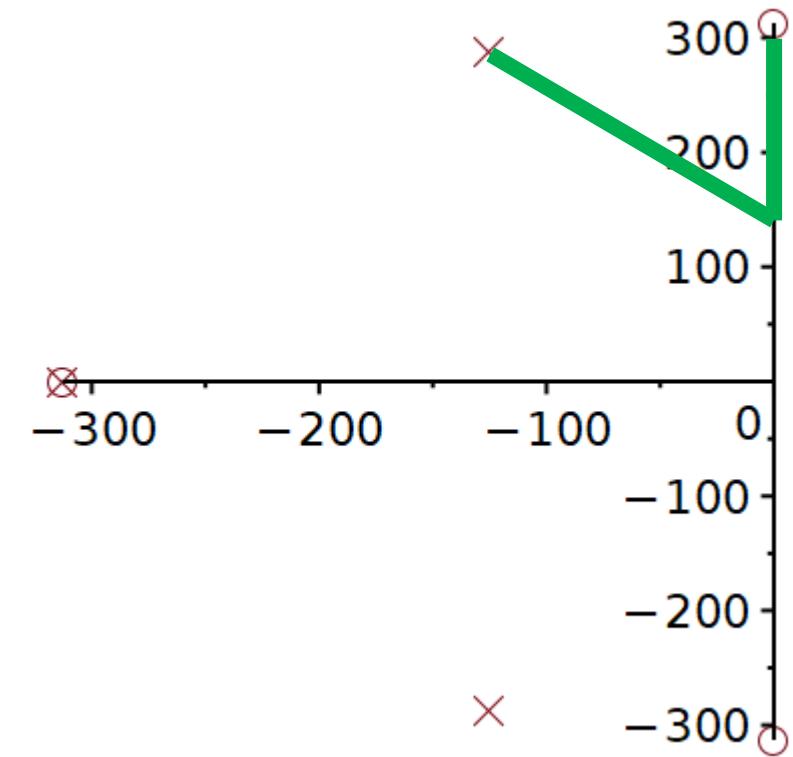
$$\alpha = 0.2, \zeta = 1.6$$



$$\alpha = 0.5, \zeta = 1$$



$$\alpha = 0.8, \zeta = 0.4$$



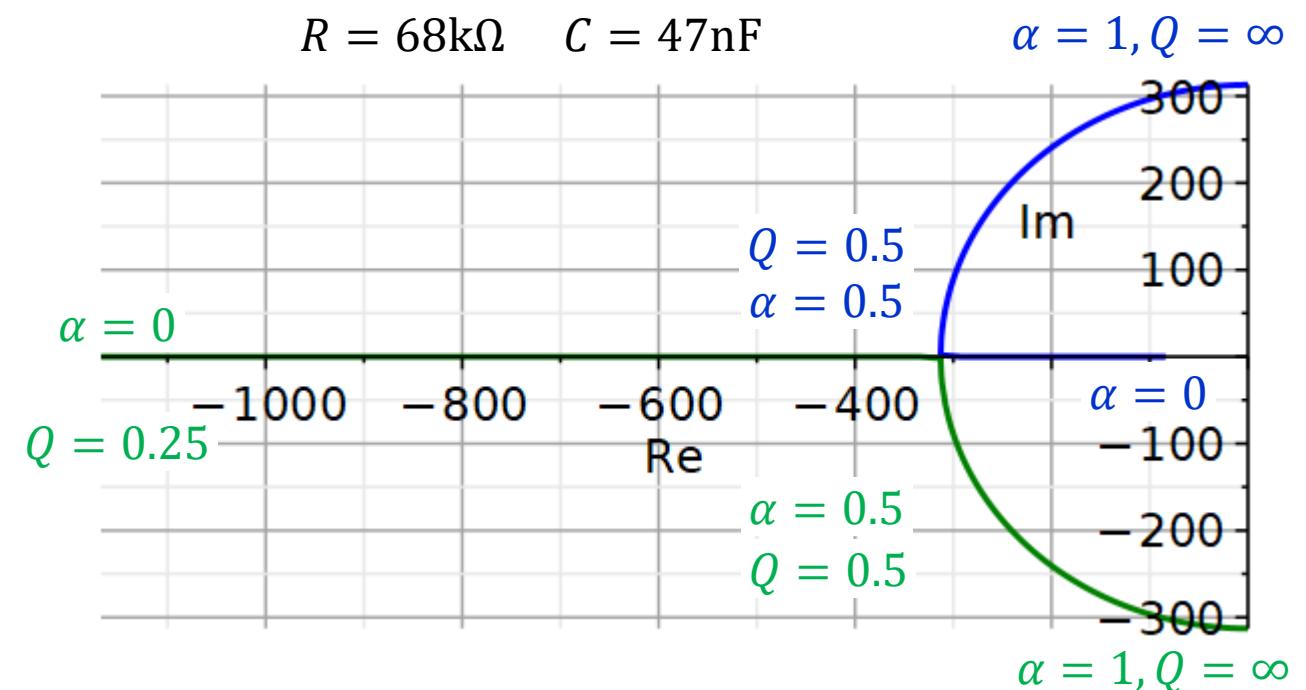
Twin-T notch filter

It is interesting to plot the **pole trajectories** when α is adjusted.

$$f_1(\alpha, R, C) = \frac{-2 + 2\alpha + \sqrt{4\alpha^2 - 8\alpha + 3}}{RC}$$

$$f_2(\alpha, R, C) = \frac{-2 + 2\alpha - \sqrt{4\alpha^2 - 8\alpha + 3}}{RC}$$

The adjustment of α only changes the damping factor ζ in the quadratic factor in the denominator. In this case, the complex conjugated poles move along a semicircle.



$$\zeta = 2(1 - \alpha) \quad Q = \frac{1}{4(1 - \alpha)}$$

```

plot1 := plot([Re(f1(alpha, 68E3, 47E-9)), Im(f1(alpha, 68E3, 47E-9)), alpha=0..1], color="Blue") :
plot2 := plot([Re(f2(alpha, 68E3, 47E-9)), Im(f2(alpha, 68E3, 47E-9)), alpha=0..1], color="Green") :
plots[display](plot1, plot2, thickness=3, font=[Helvetica, roman, 18], axis[2]=[thickness=1.5],
axis[1]=[thickness=1.5], labels=["Re", "Im"], labelfont=[["HELVETICA", 18]], numpoints
= 100, gridlines, size=[600, 300])

```

Problems

Download Maple file from DTU Learn, week 10/exercises.

Adapt the Maple file to plot Bode plot curves for the total transfer function and for the individual factors.

Problem 1

Use the downloaded Maple file to plot the amplitude and phase characteristics of the following transfer function as well as its individual factors.

Start by converting the transfer function to standard Bode format.

In the Maple file:

- Edit the functions to be plotted
- Edit the legends to fit the number of curves
- Edit the line styles to fit the number of curves
- Edit the plotting statements to fit the number of curves
- Edit the frequency range if needed

$$G(j\omega) = \frac{10^4(j\omega + 2)}{(j\omega + 10)(j\omega + 100)}$$

Problem 2

Use the downloaded Maple file to plot the amplitude and phase characteristics of the following transfer function as well as its individual factors.

$$H(s) = \frac{100s}{s^2 + 20s + 10000}$$

Start by converting the transfer function to standard Bode format.

In the Maple file:

- Edit the functions to be plotted
- Edit the legends to fit the number of curves
- Edit the line styles to fit the number of curves
- Edit the plotting statements to fit the number of curves
- Edit the frequency range if needed

Problem 3 – Sallen-Key Lowpass Filter

Use the downloaded Maple file to plot the amplitude and phase characteristics of the following transfer function as well as its individual factors.

Start by converting the transfer function to standard Bode format.

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_0}$$

$$a_1 = 2827.53$$

$$a_0 = 394815$$

$$b_0 = 789629$$

In the Maple file:

- Edit the functions to be plotted
- Edit the legends to fit the number of curves
- Edit the line styles to fit the number of curves
- Edit the plotting statements to fit the number of curves
- Edit the frequency range if needed

Problem 4 – Sallen-Key Highpass filter

Use the downloaded Maple file to plot the amplitude and phase characteristics of the following transfer function as well as its individual factors.

$$H(s) = \frac{2s^2}{\left(s + \frac{\pi}{10}\right)^2}$$

Start by converting the transfer function to standard Bode format.

In the Maple file:

- Edit the functions to be plotted
- Edit the legends to fit the number of curves
- Edit the line styles to fit the number of curves
- Edit the plotting statements to fit the number of curves
- Edit the frequency range if needed

Problem 5 – Sallen-Key Bandpass filter

Use the downloaded Maple file to plot the amplitude and phase characteristics of the following transfer function as well as its individual factors.

Start by converting the transfer function to standard Bode format.

$$H(s) = \frac{b_1 s}{s^2 + a_1 s + a_0}$$

$$a_1 = 31.41$$

$$a_0 = 98695$$

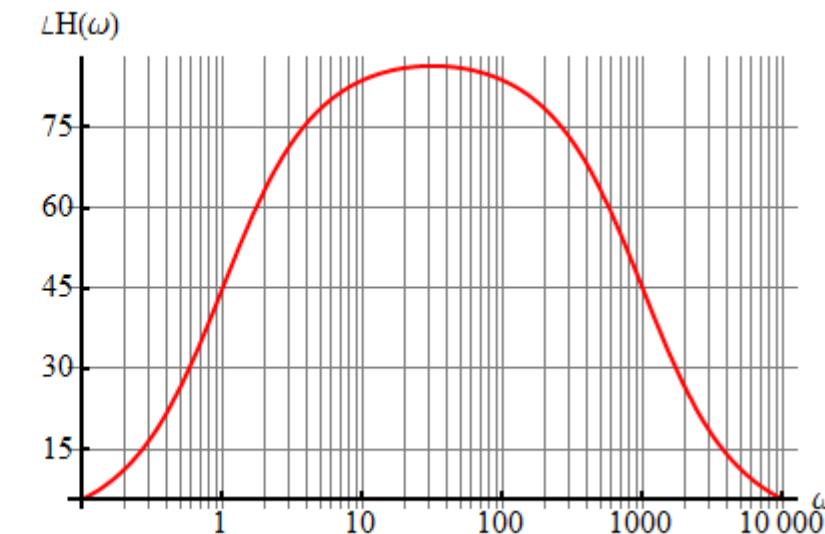
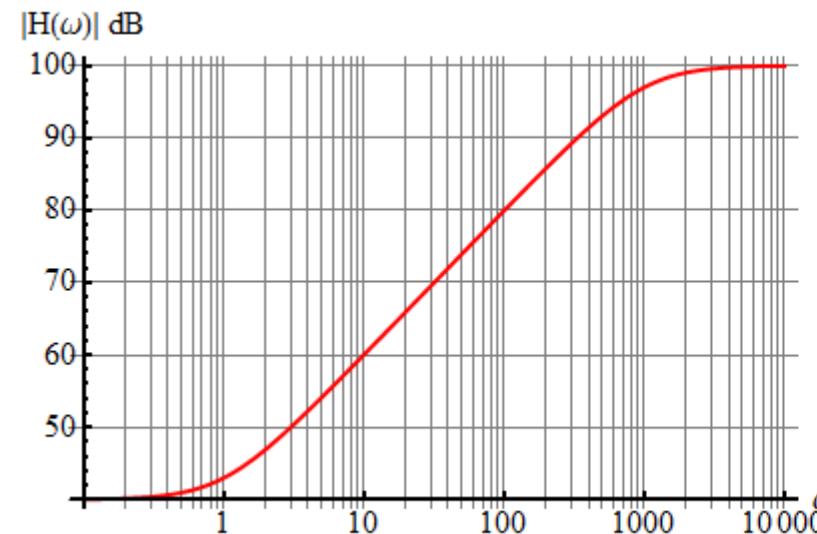
In the Maple file:

$$b_1 = 628.3$$

- Edit the functions to be plotted
- Edit the legends to fit the number of curves
- Edit the line styles to fit the number of curves
- Edit the plotting statements to fit the number of curves
- Edit the frequency range if needed

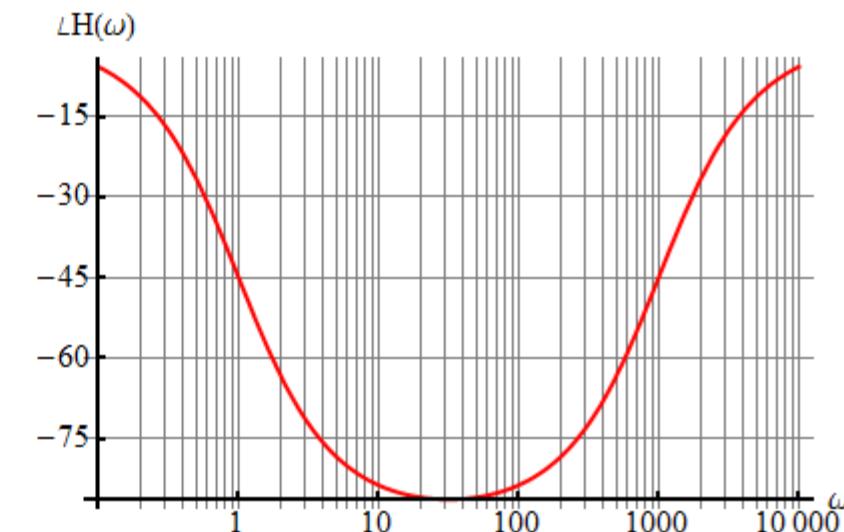
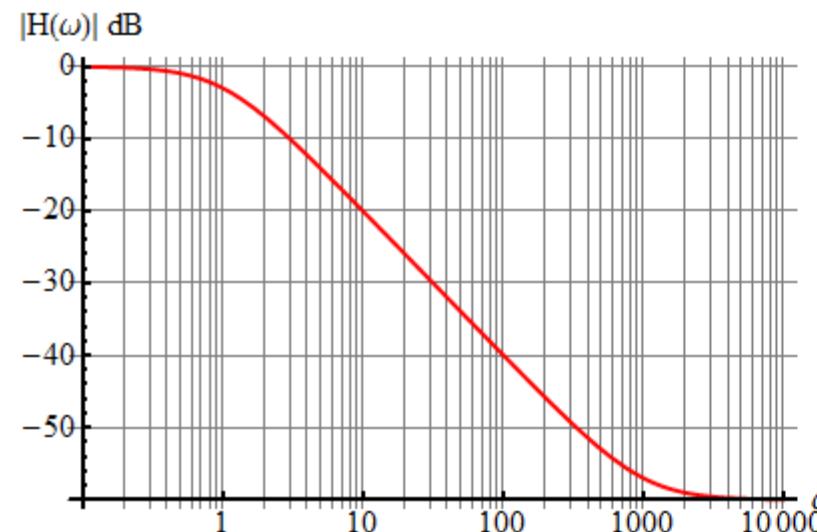
Problem 6 – Systems identification

Use the frequency characteristics to write the transfer function.



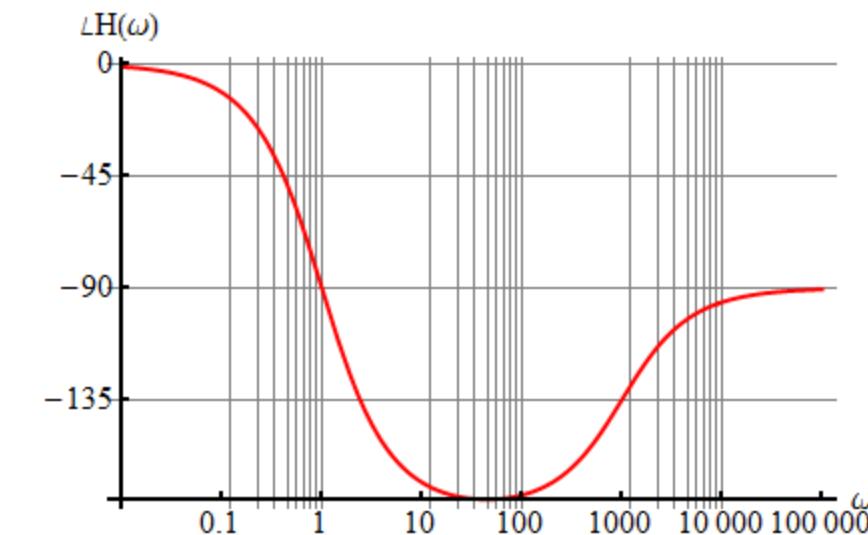
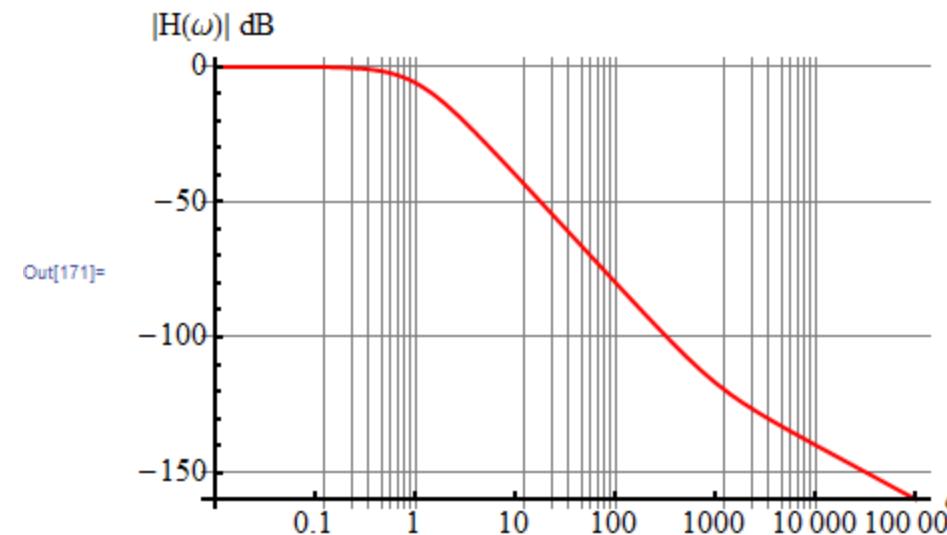
Problem 7 – Systems identification

Use the frequency characteristics to write the transfer function.



Problem 8 – Systems identification

Use the frequency characteristics to write the transfer function.



Solutions

Problem 1

Use the downloaded Maple file to plot the amplitude and phase characteristics of the following transfer function as well as its individual factors.

Start by converting the transfer function to standard Bode format.

In the Maple file:

- Edit the functions to be plotted
- Edit the legends to fit the number of curves
- Edit the line styles to fit the number of curves
- Edit the plotting statements to fit the number of curves
- Edit the frequency range if needed

$$G(j\omega) = \frac{10^4(j\omega + 2)}{(j\omega + 10)(j\omega + 100)}$$

Problem 1 (sol)

Use the downloaded Maple file to plot the amplitude and phase characteristics of the following transfer function as well as its individual factors.

Start by converting the transfer function to standard Bode format.

In the Maple file:

- Edit the functions to be plotted
- Edit the legends to fit the number of curves
- Edit the line styles to fit the number of curves
- Edit the plotting statements to fit the number of curves
- Edit the frequency range if needed

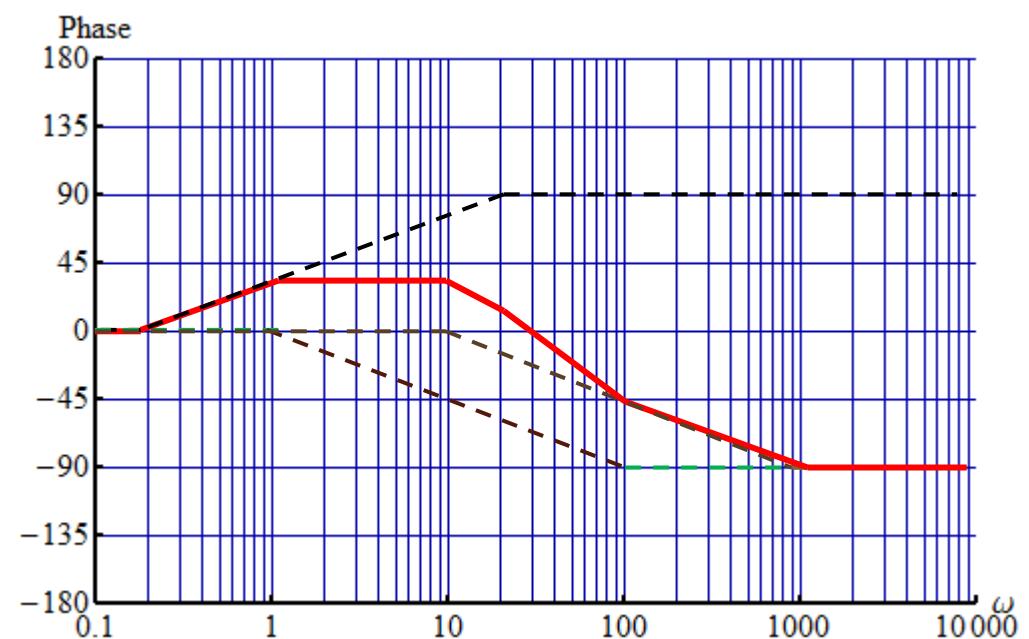
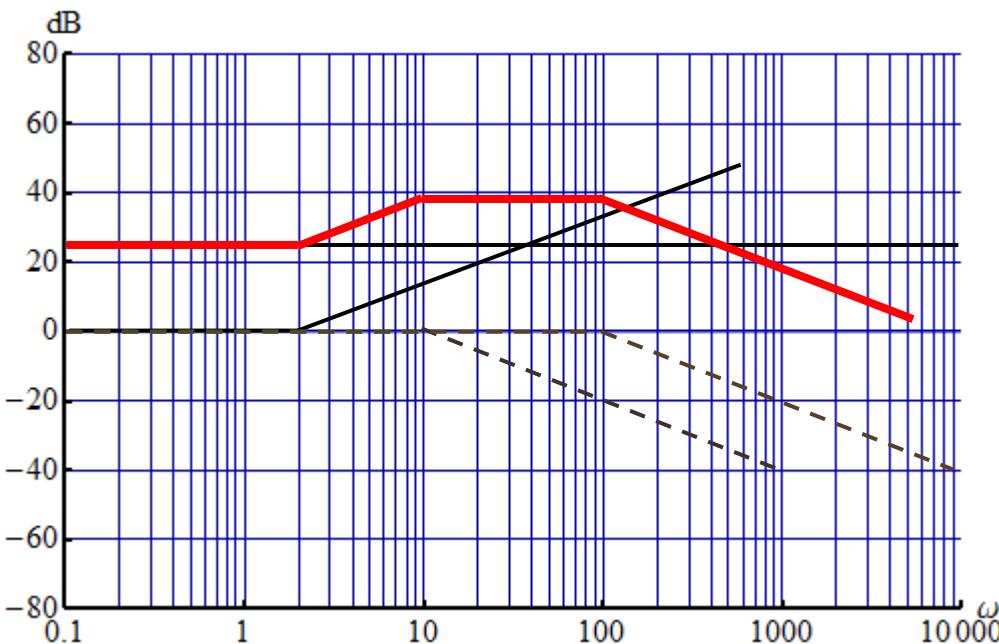
$$G(j\omega) = \frac{10^4(j\omega + 2)}{(j\omega + 10)(j\omega + 100)}$$

$$G(j\omega) = \frac{2 \times 10^4}{10 \times 100} \frac{\left(\frac{j\omega}{2} + 1\right)}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{100} + 1\right)}$$

$$G(j\omega) = 20 \frac{\left(\frac{j\omega}{2} + 1\right)}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{100} + 1\right)}$$

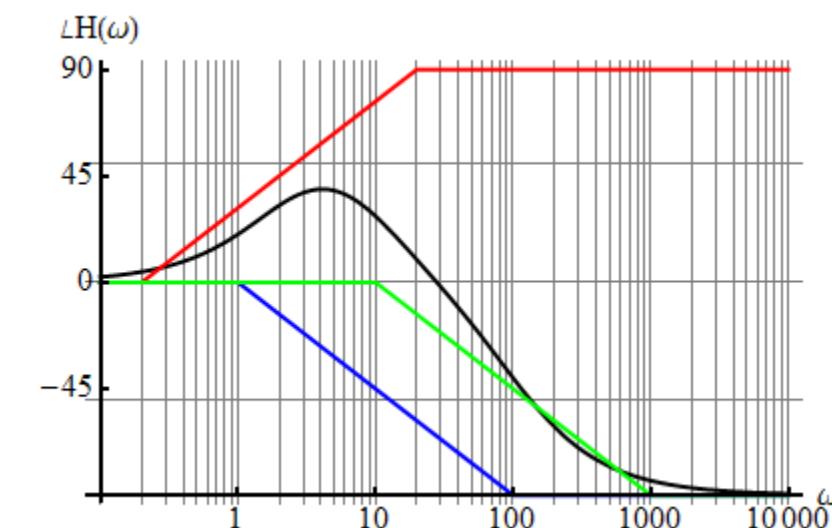
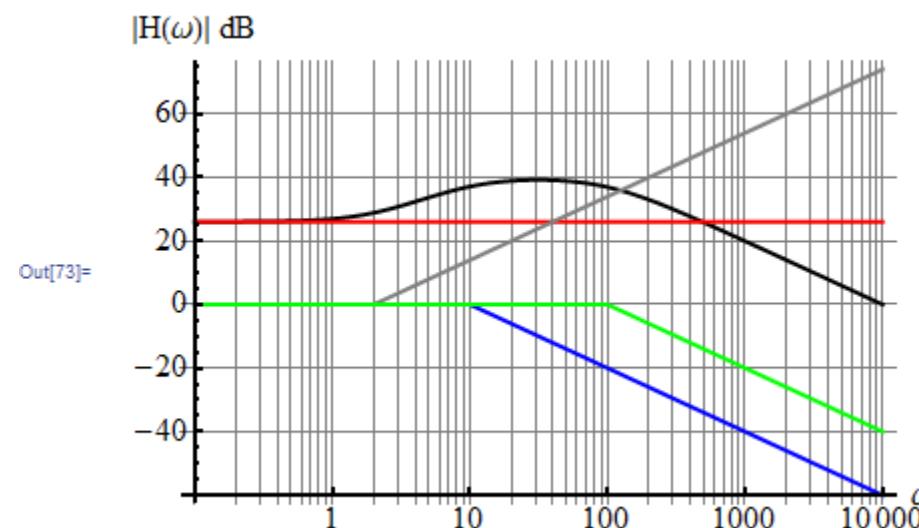
Problem 1 (sol – by hand)

$$G(j\omega) = 20 \frac{\left(\frac{j\omega}{2} + 1\right)}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{100} + 1\right)}$$



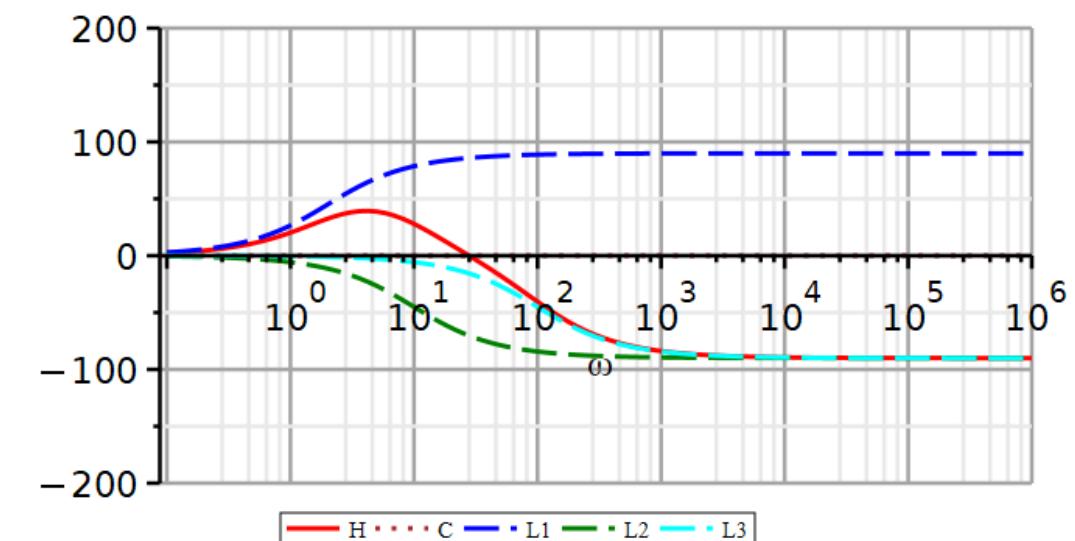
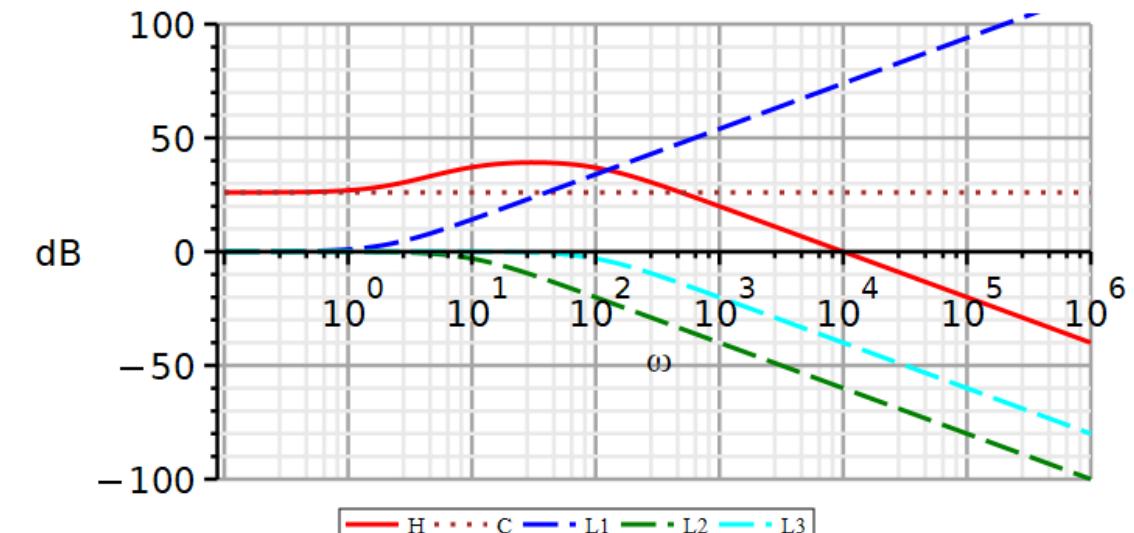
Problem 1 (sol – by Mathematica)

$$G(j\omega) = 20 \frac{\left(\frac{j\omega}{2} + 1\right)}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{100} + 1\right)}$$



Problem 1 (sol by Maple)

$$G(j\omega) = 20 \frac{\left(\frac{j\omega}{2} + 1\right)}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{100} + 1\right)}$$



Problem 2

Use the downloaded Maple file to plot the amplitude and phase characteristics of the following transfer function as well as its individual factors.

$$H(s) = \frac{100s}{s^2 + 20s + 10000}$$

Start by converting the transfer function to standard Bode format.

In the Maple file:

- Edit the functions to be plotted
- Edit the legends to fit the number of curves
- Edit the line styles to fit the number of curves
- Edit the plotting statements to fit the number of curves
- Edit the frequency range if needed

Problem 2 (sol)

Use the downloaded Maple file to plot the amplitude and phase characteristics of the following transfer function as well as its individual factors.

Start by converting the transfer function to standard Bode format.

In the Maple file:

- Edit the functions to be plotted
- Edit the legends to fit the number of curves
- Edit the line styles to fit the number of curves
- Edit the plotting statements to fit the number of curves
- Edit the frequency range if needed

$$H(s) = \frac{100s}{s^2 + 20s + 10000}$$

$$H(j\omega) = \frac{100j\omega}{(j\omega)^2 + 20j\omega + 10000}$$

$$H(j\omega) = \frac{\frac{j\omega}{100}}{\left(\frac{j\omega}{100}\right)^2 + \frac{20}{100} \frac{j\omega}{100} + 1}$$

$$H(j\omega) = \frac{\frac{j\omega}{100}}{\left(\frac{j\omega}{100}\right)^2 + 2 \times 0.1 \frac{j\omega}{100} + 1}$$

Damping factor: $\zeta = 0.1$

Undamped resonance frequency: $\omega_n = 100 \text{ rad/s}$

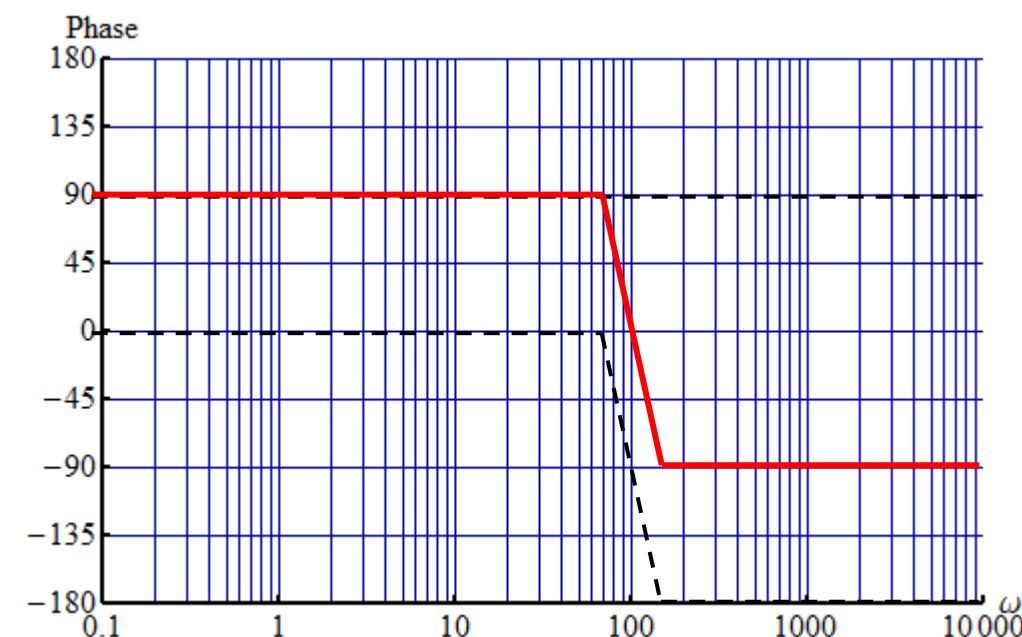
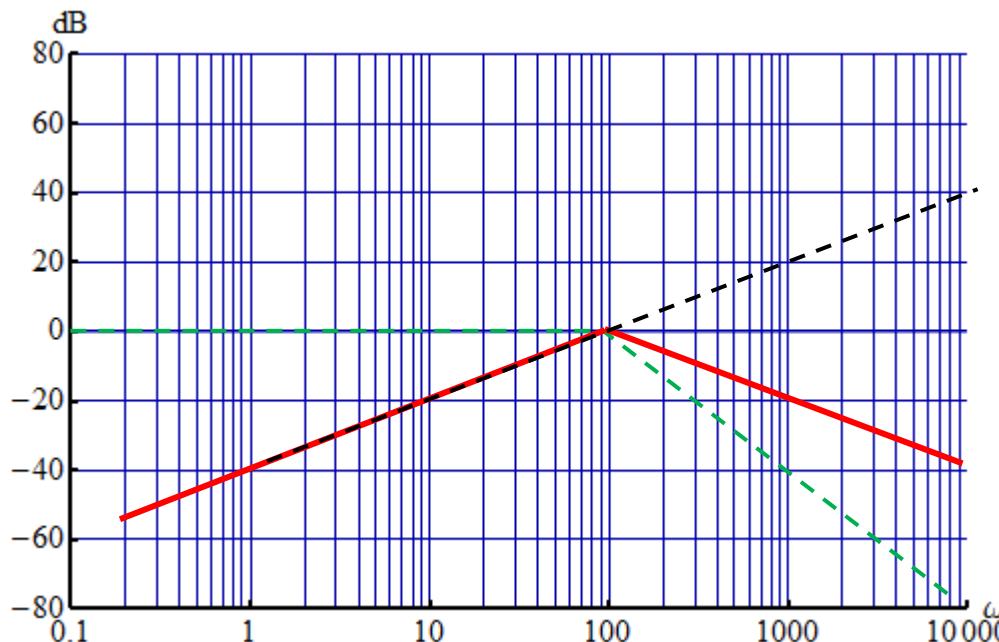
Problem 2 (sol – by hand)

$$\zeta \leq 0.2: \alpha = 1.410 \zeta - 0.150 \zeta^2 \quad \alpha = 0.1395$$

$$H(j\omega) = \frac{j\omega}{\left(\frac{j\omega}{100}\right)^2 + 2 \times 0.1 \frac{j\omega}{100} + 1}$$

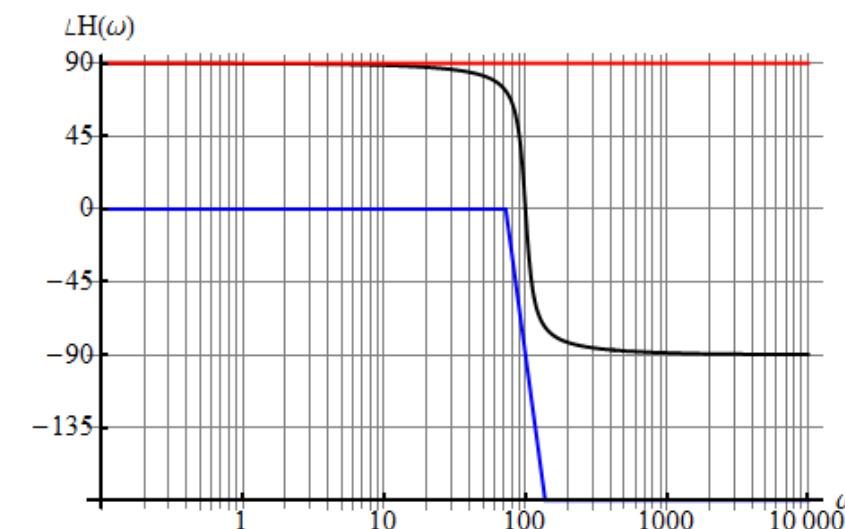
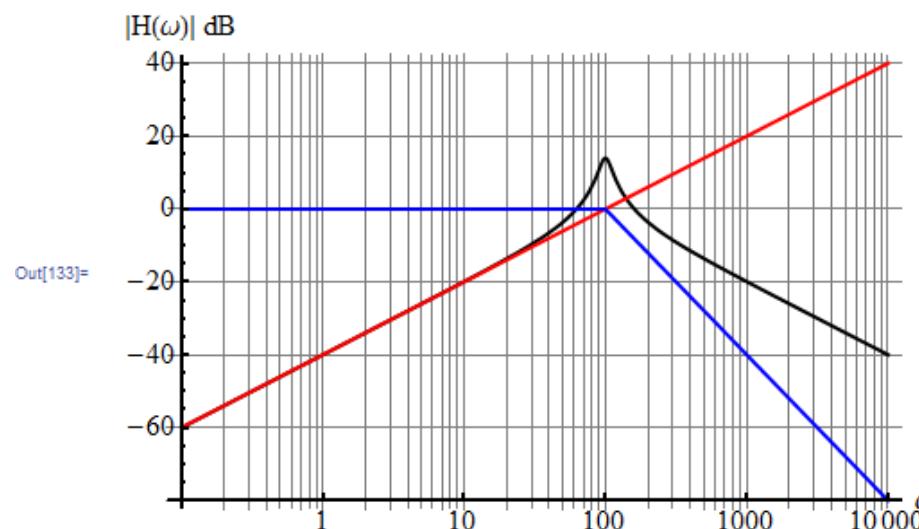
$$\frac{100 \text{ rad/s}}{10^{0.1395}} = 72.5 \text{ rad/s}$$

$$10^{0.1395} 100 \text{ rad/s} = 137.88 \text{ rad/s}$$



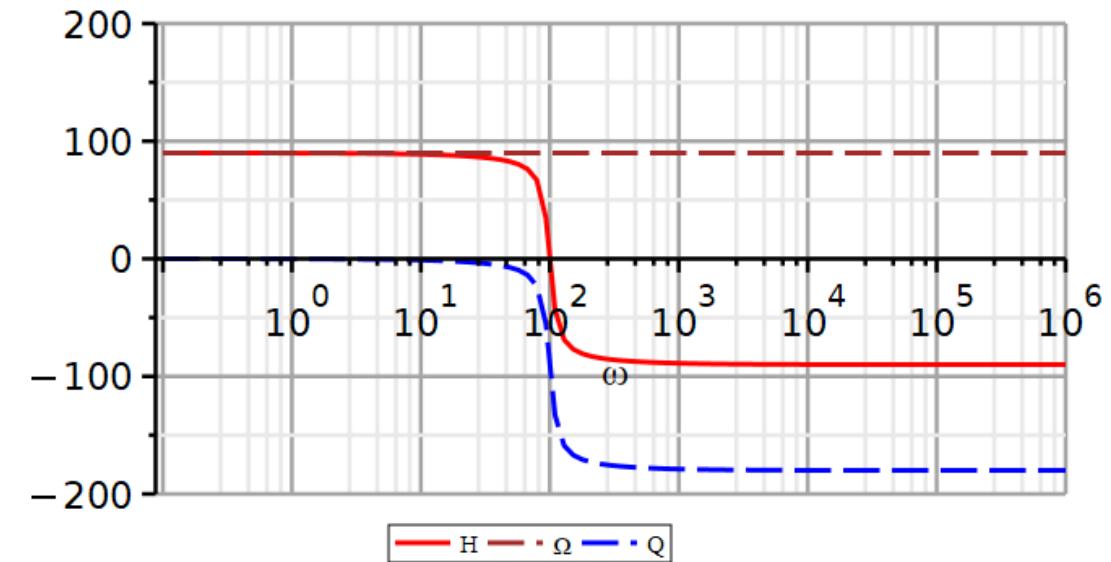
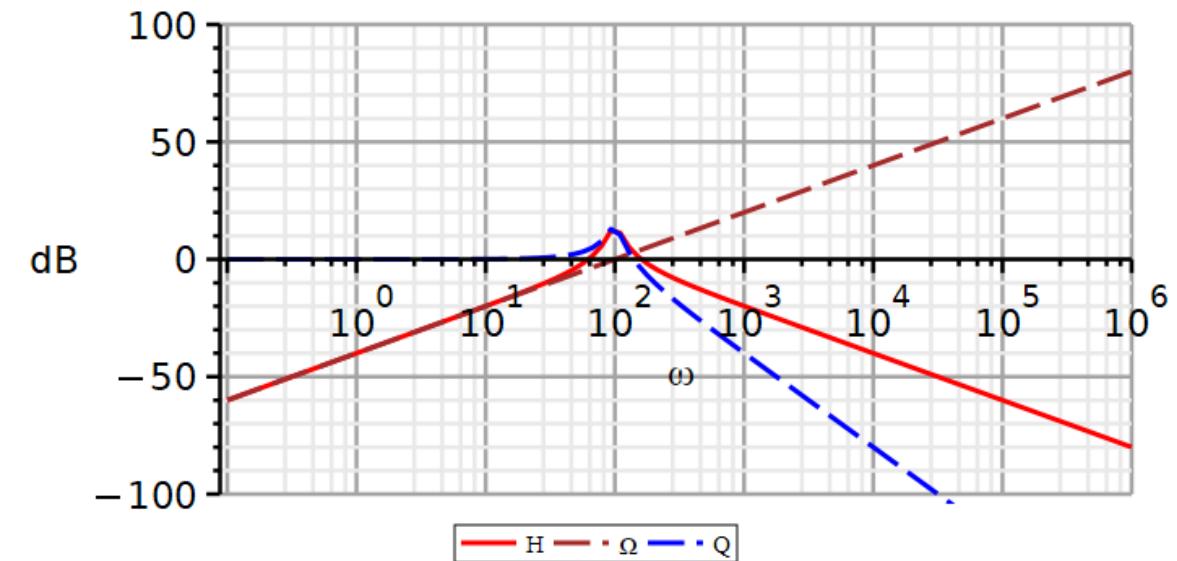
Problem 2 (sol by Mathematica)

$$H(j\omega) = \frac{\frac{j\omega}{100}}{\left(\frac{j\omega}{100}\right)^2 + 2 \times 0.1 \frac{j\omega}{100} + 1}$$



Problem 2 (sol by Maple)

$$H(j\omega) = \frac{\frac{j\omega}{100}}{\left(\frac{j\omega}{100}\right)^2 + 2 \times 0.1 \frac{j\omega}{100} + 1}$$



Problem 3 – Sallen-Key Lowpass Filter

Use the downloaded Maple file to plot the amplitude and phase characteristics of the following transfer function as well as its individual factors.

Start by converting the transfer function to standard Bode format.

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_0}$$

$$a_1 = 2827.53$$

$$a_0 = 394815$$

$$b_0 = 789629$$

In the Maple file:

- Edit the functions to be plotted
- Edit the legends to fit the number of curves
- Edit the line styles to fit the number of curves
- Edit the plotting statements to fit the number of curves
- Edit the frequency range if needed

Problem 3 – Sallen-Key Lowpass Filter (sol)

Use the downloaded Maple file to plot the amplitude and phase characteristics of the following transfer function as well as its individual factors.

Start by converting the transfer function to standard Bode format.

In the Maple file:

- Edit the functions to be plotted
- Edit the legends to fit the number of curves
- Edit the line styles to fit the number of curves
- Edit the plotting statements to fit the number of curves
- Edit the frequency range if needed

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_0}$$

$$H(s) = \frac{789629}{(s + 147.3)(s + 2680)}$$

$$H(s) = \frac{2}{\left(\frac{s}{147.3} + 1\right)\left(\frac{s}{2680} + 1\right)}$$

$$H(j\omega) = \frac{2}{\left(\frac{j\omega}{147.3} + 1\right)\left(\frac{j\omega}{2680} + 1\right)}$$

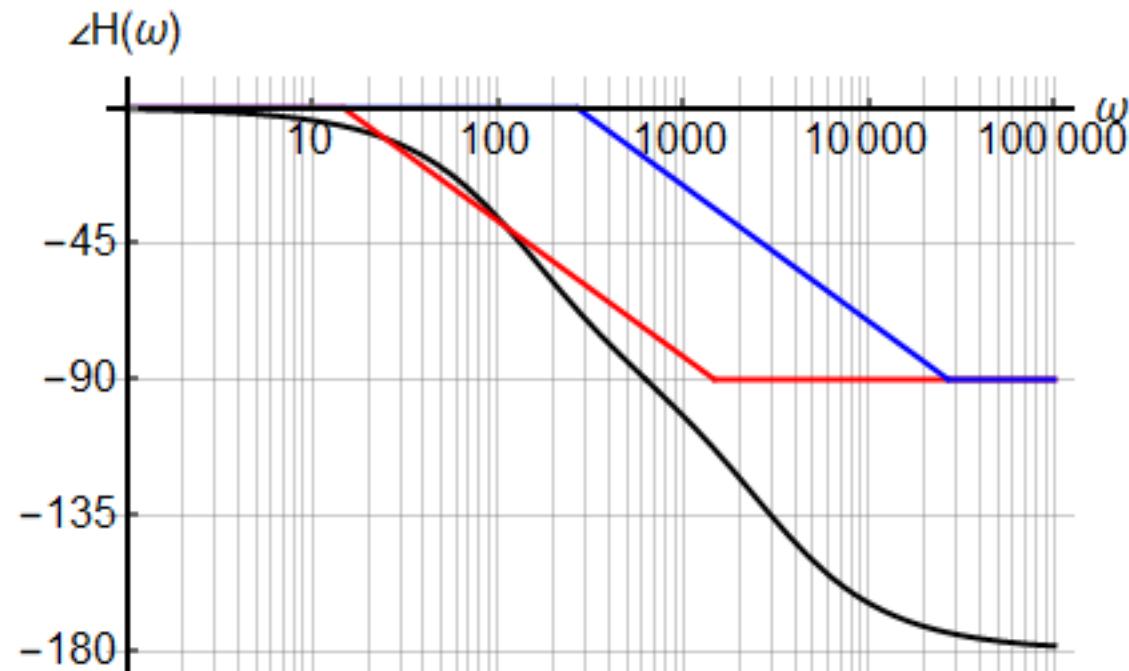
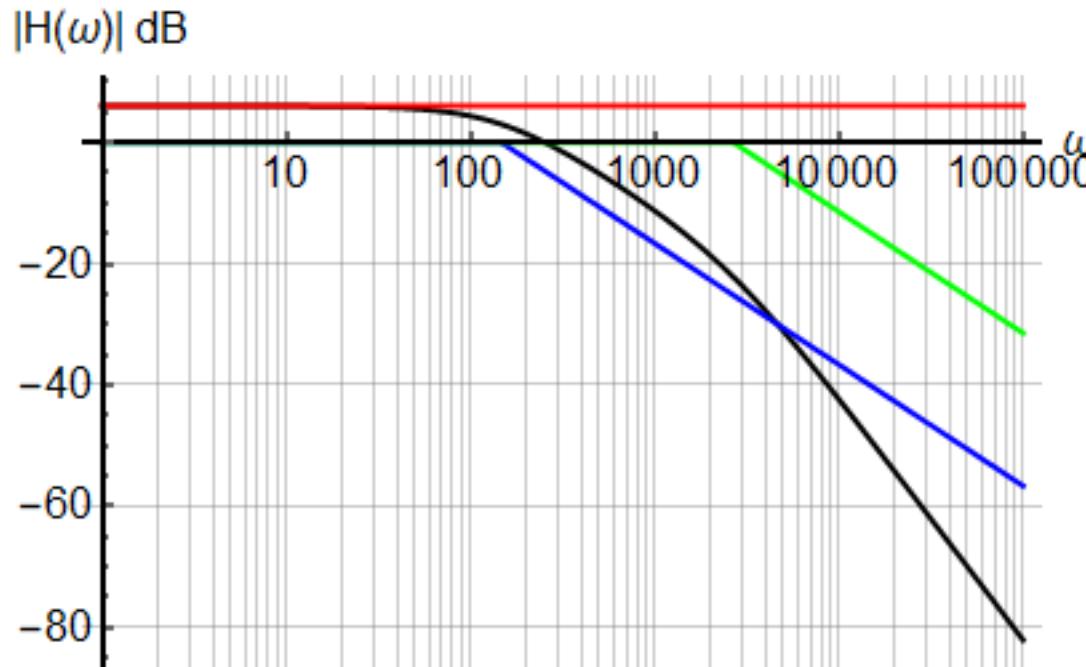
$$a_1 = 2827.53$$

$$a_0 = 394815$$

$$b_0 = 789629$$

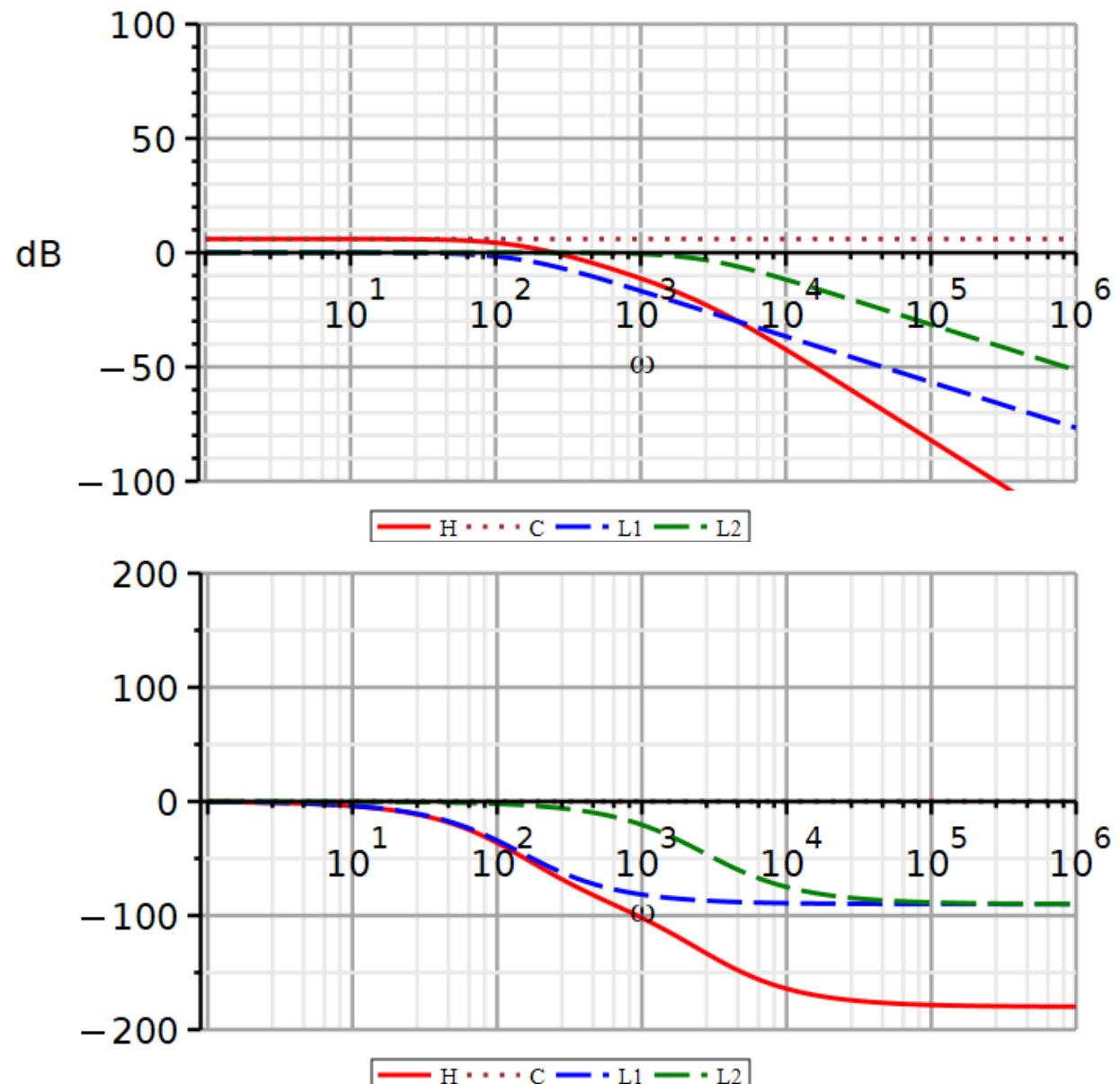
Problem 3 – Sallen-Key Lowpass Filter (sol – by Mathematica)

$$H(j\omega) = \frac{2}{\left(\frac{j\omega}{147.3} + 1\right)\left(\frac{j\omega}{2680} + 1\right)}$$



Problem 3 – Sallen-Key Lowpass Filter (sol – by Maple)

$$H(j\omega) = \frac{2}{\left(\frac{j\omega}{147.3} + 1\right)\left(\frac{j\omega}{2680} + 1\right)}$$



Problem 4 – Sallen-Key Highpass filter

Use the downloaded Maple file to plot the amplitude and phase characteristics of the following transfer function as well as its individual factors.

$$H(s) = \frac{2s^2}{\left(s + \frac{\pi}{10}\right)^2}$$

Start by converting the transfer function to standard Bode format.

In the Maple file:

- Edit the functions to be plotted
- Edit the legends to fit the number of curves
- Edit the line styles to fit the number of curves
- Edit the plotting statements to fit the number of curves
- Edit the frequency range if needed

Problem 4 – Sallen-Key Highpass filter (sol)

Use the downloaded Maple file to plot the amplitude and phase characteristics of the following transfer function as well as its individual factors.

Start by converting the transfer function to standard Bode format.

In the Maple file:

- Edit the functions to be plotted
- Edit the legends to fit the number of curves
- Edit the line styles to fit the number of curves
- Edit the plotting statements to fit the number of curves
- Edit the frequency range if needed

$$H(s) = \frac{2s^2}{\left(s + \frac{\pi}{10}\right)^2}$$

$$H(s) = \frac{2}{\left(\frac{\pi}{10}\right)^2} \frac{s^2}{\left(\frac{s}{\pi} + 1\right)^2}$$

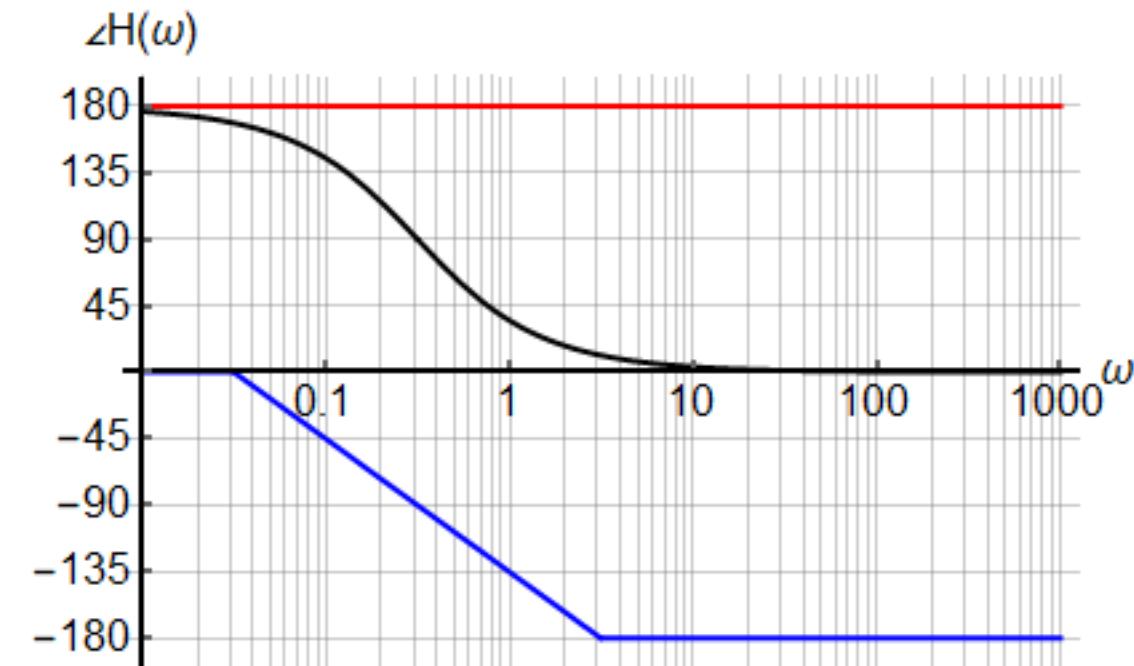
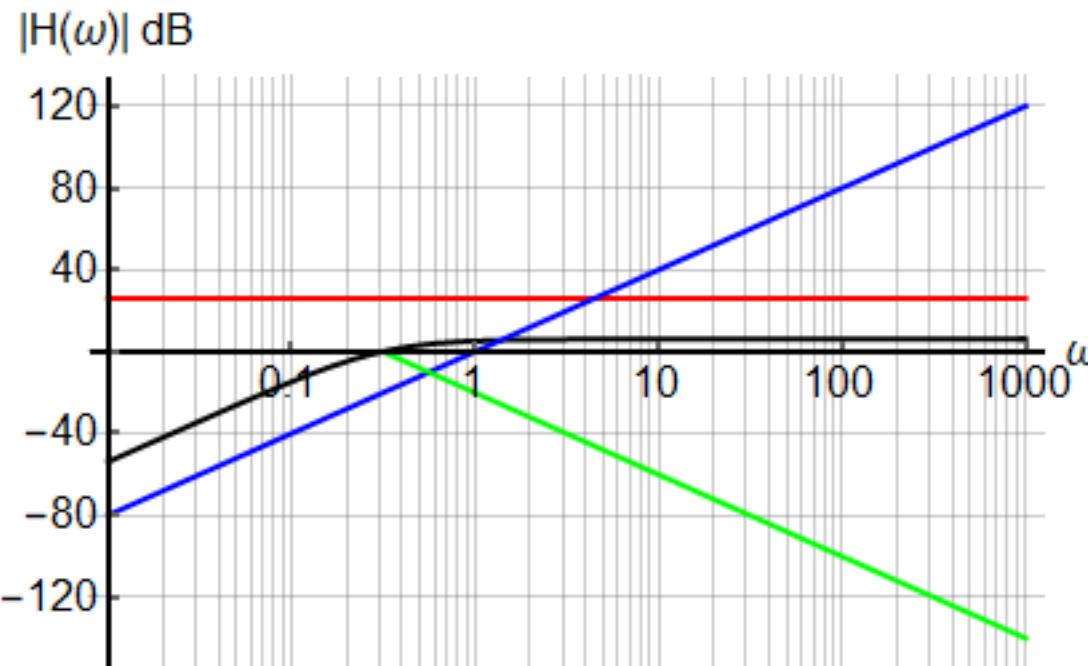
$$H(j\omega) = \frac{2}{\left(\frac{\pi}{10}\right)^2} \frac{(j\omega)^2}{\left(\frac{j\omega}{\pi} + 1\right)^2}$$

Alternatively:

$$H(j\omega) = 2 \frac{\left(\frac{j\omega}{\pi}\right)^2}{\left(\frac{j\omega}{\pi} + 1\right)^2}$$

Problem 4 – Sallen-Key Highpass filter (sol by Mathematica)

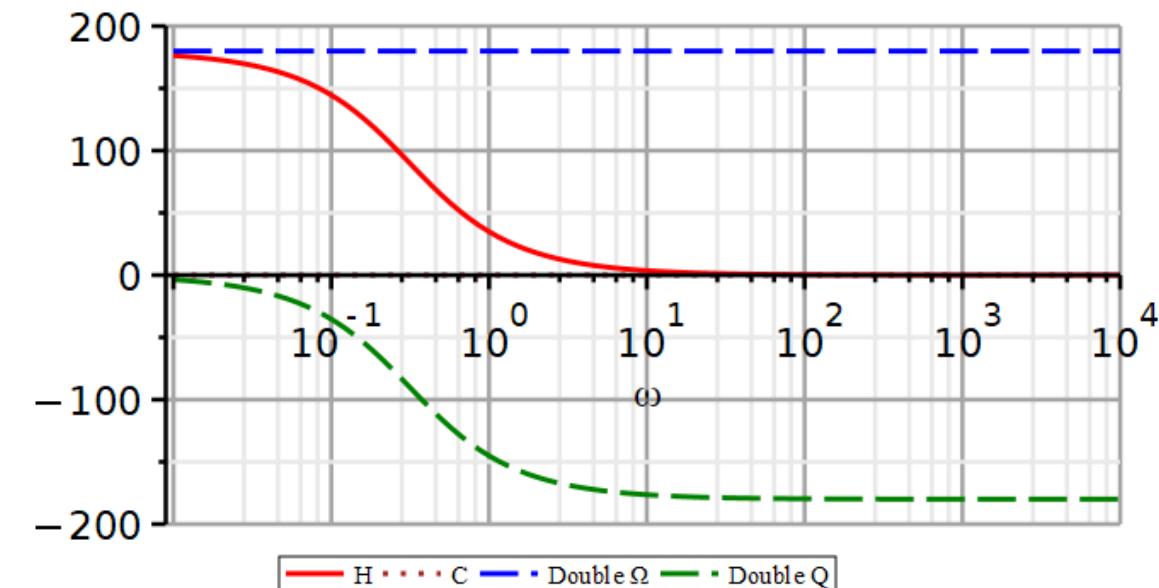
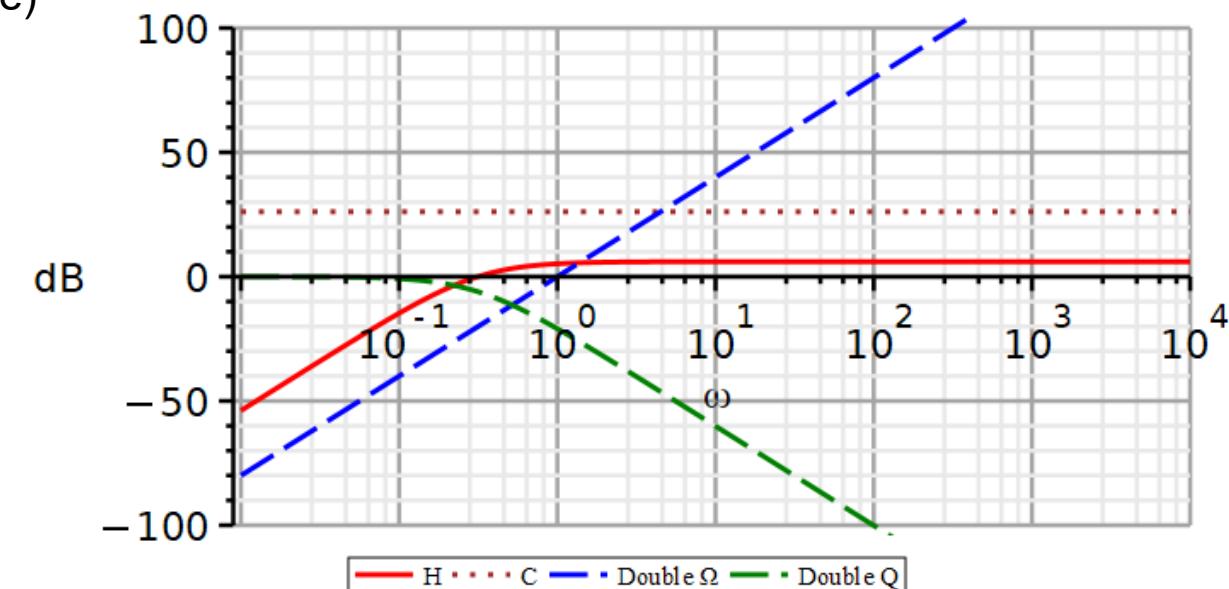
$$H(j\omega) = \frac{2}{\left(\frac{\pi}{10}\right)^2} \frac{(j\omega)^2}{\left(\frac{j\omega}{\pi} + 1\right)^2}$$
$$\frac{2}{\left(\frac{\pi}{10}\right)^2} = 20.26$$



Problem 4 – Sallen-Key Highpass filter (sol by Maple)

$$H(j\omega) = \frac{2}{\left(\frac{\pi}{10}\right)^2} \frac{(j\omega)^2}{\left(\frac{j\omega}{\pi} + 1\right)^2}$$

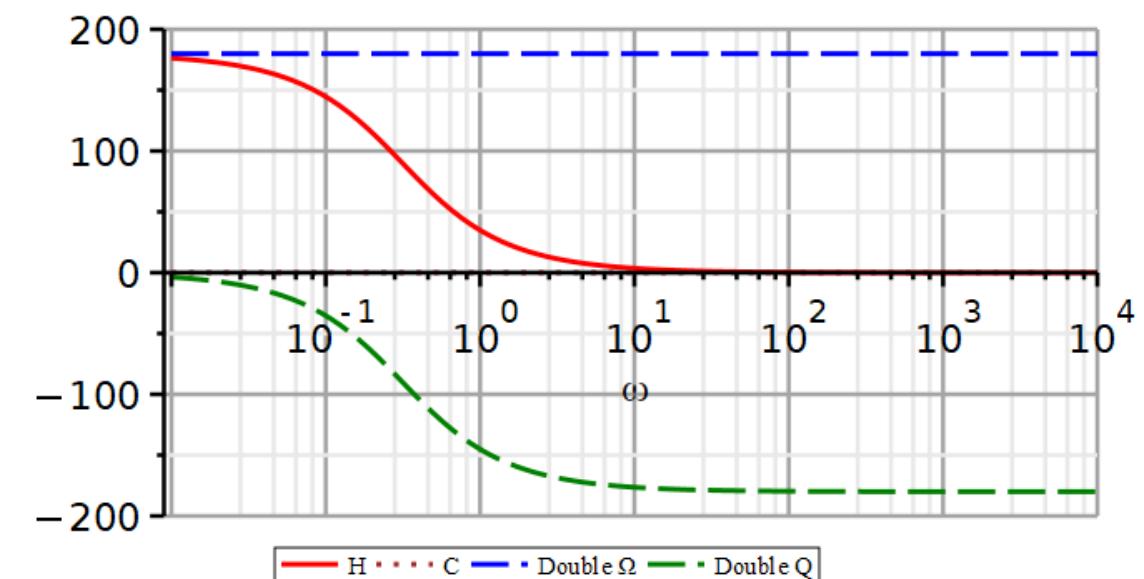
$$\frac{2}{\left(\frac{\pi}{10}\right)^2} = 20.26 \approx 26dB$$



Problem 4 – Sallen-Key Highpass filter (sol by Maple)

An alternative form:

$$H(j\omega) = \frac{\left(\frac{j\omega}{\pi}\right)^2}{\left(\frac{j\omega}{\pi} + 1\right)^2}$$



Problem 5 – Sallen-Key Bandpass filter

Use the downloaded Maple file to plot the amplitude and phase characteristics of the following transfer function as well as its individual factors.

Start by converting the transfer function to standard Bode format.

$$H(s) = \frac{b_1 s}{s^2 + a_1 s + a_0}$$

$$a_1 = 31.41$$

$$a_0 = 98695$$

In the Maple file:

$$b_1 = 628.3$$

- Edit the functions to be plotted
- Edit the legends to fit the number of curves
- Edit the line styles to fit the number of curves
- Edit the plotting statements to fit the number of curves
- Edit the frequency range if needed

Problem 5 – Sallen-Key Bandpass filter (sol)

$$a_1 = 31.41$$

$$a_0 = 98695$$

$$b_1 = 628.3$$

$$\omega_n = 314.157$$

$$\zeta = 0.05$$

$$\omega_L = 10^{-a} \omega_n \wedge \omega_H = 10^a \omega_n$$

$$a = \begin{cases} 1.410\zeta - 0.150\zeta^2 & \zeta \leq 0.2 \\ 1.475\zeta - 0.475\zeta^2 & \zeta > 0.2 \end{cases}$$

$$a = 0.0701 \Rightarrow \omega_L = 267.3, \omega_H = 369.2$$

$$H(s) = \frac{b_1 s}{s^2 + a_1 s + a_0}$$

$$H(s) = \frac{b_1 s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

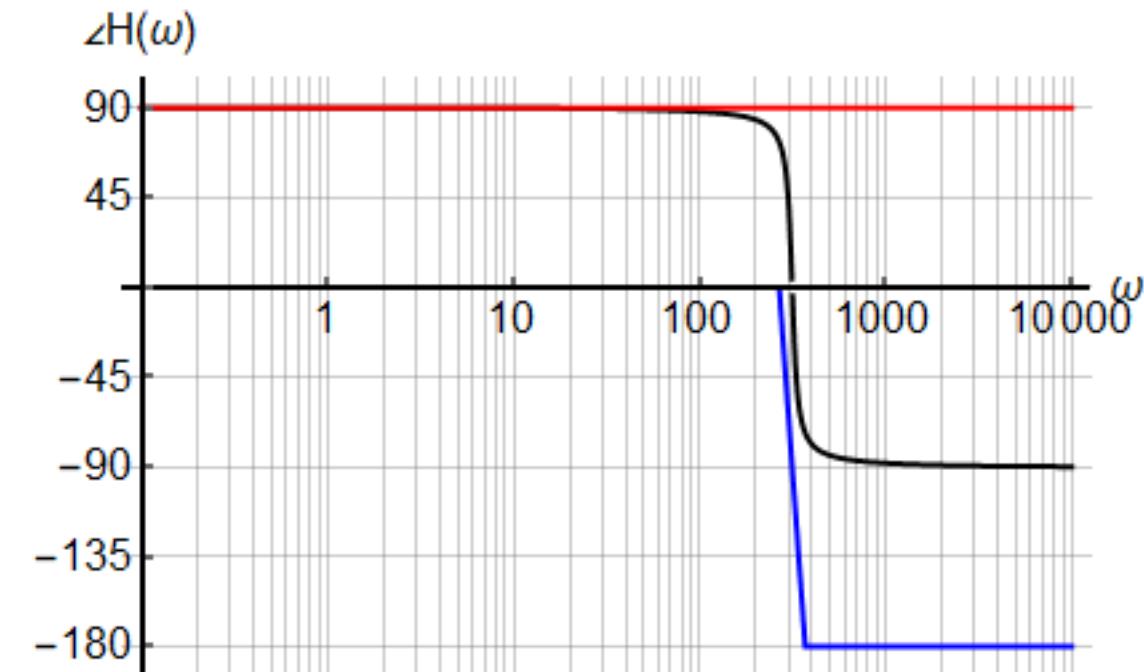
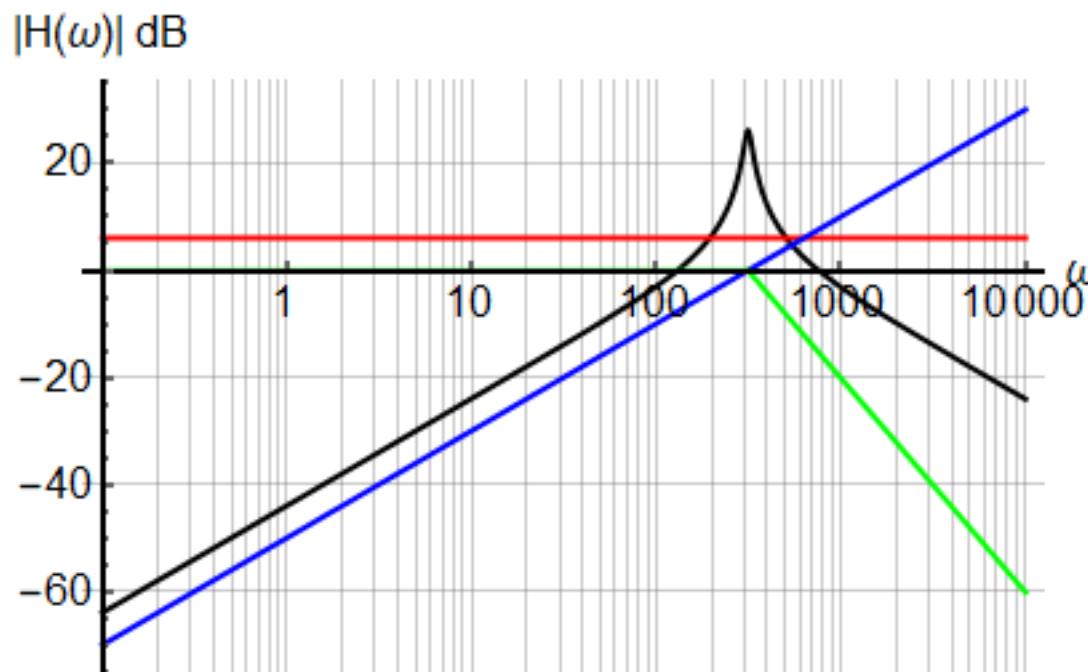
$$H(s) = \frac{b_1}{\omega_n^2} \frac{s}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta \frac{s}{\omega_n} + 1}$$

$$H(s) = \frac{b_1}{\omega_n} \frac{\frac{s}{\omega_n}}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta \frac{s}{\omega_n} + 1}$$

$$H(j\omega) = \frac{b_1}{\omega_n} \frac{\frac{j\omega}{\omega_n}}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1}$$

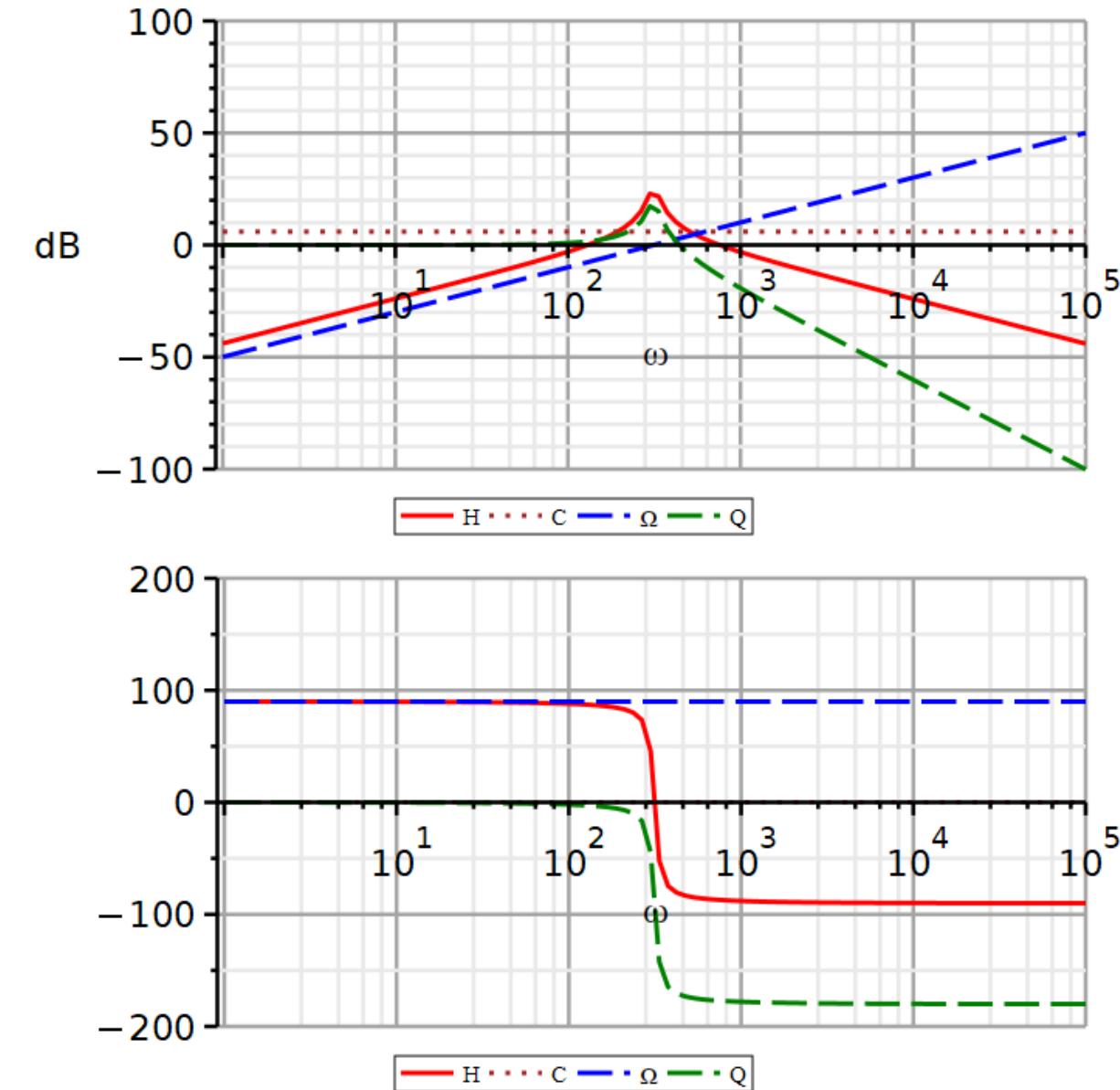
Problem 5 – Sallen-Key Bandpass filter (sol by Mathematica)

$$H(j\omega) = \frac{628}{314} \cdot \frac{\frac{j\omega}{314}}{\left(\frac{j\omega}{314}\right)^2 + 2\zeta \frac{j\omega}{314} + 1} = 2 \frac{\frac{j\omega}{314}}{\left(\frac{j\omega}{314}\right)^2 + 2\zeta \frac{j\omega}{314} + 1}$$



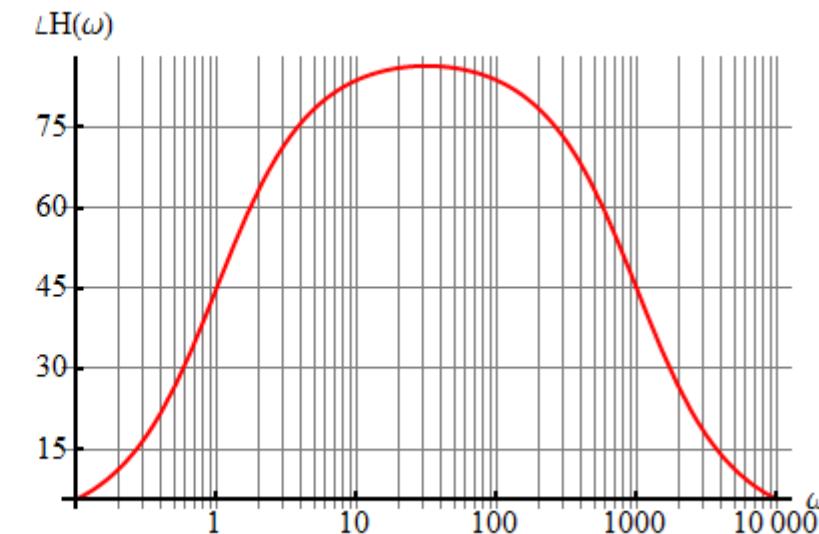
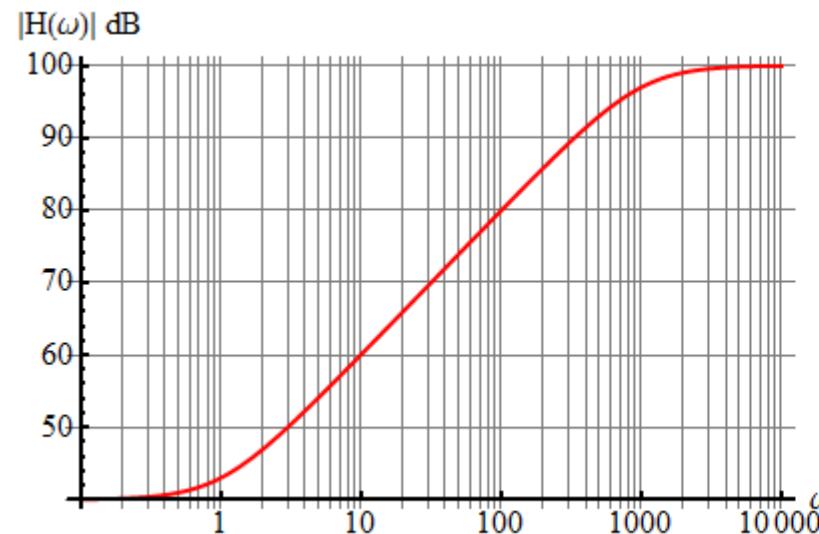
Problem 5 – Sallen-Key Bandpass filter (sol by Maple)

$$H(j\omega) = \frac{2 \frac{j\omega}{314}}{\left(\frac{j\omega}{314}\right)^2 + 2\zeta \frac{j\omega}{314} + 1}$$



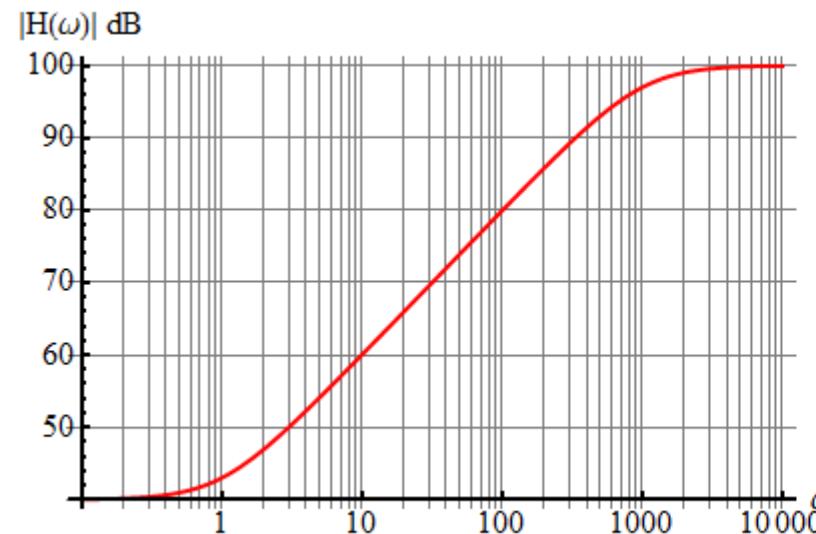
Problem 6

Use the frequency characteristics to write the transfer function.

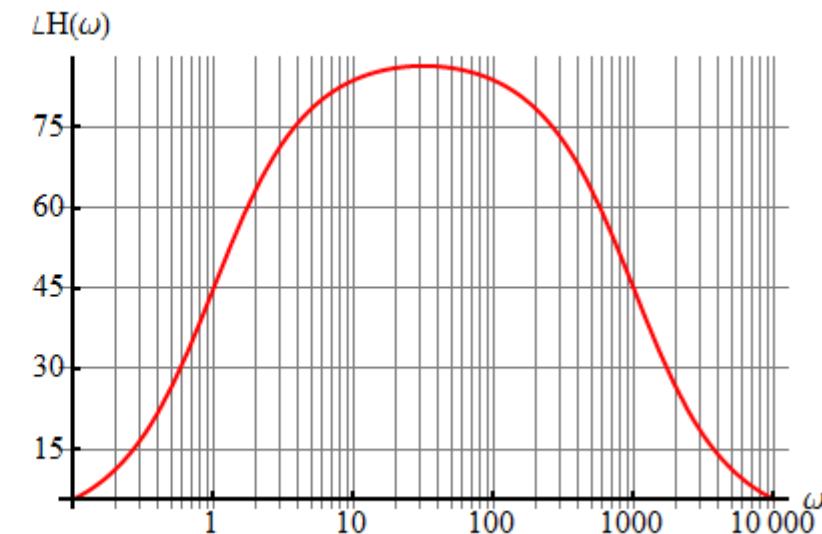


Problem 6 (sol)

Use the frequency characteristics to write the transfer function.



$$H(\omega) = 100 \frac{\frac{j\omega}{1000} + 1}{\frac{j\omega}{1000} + 1}$$

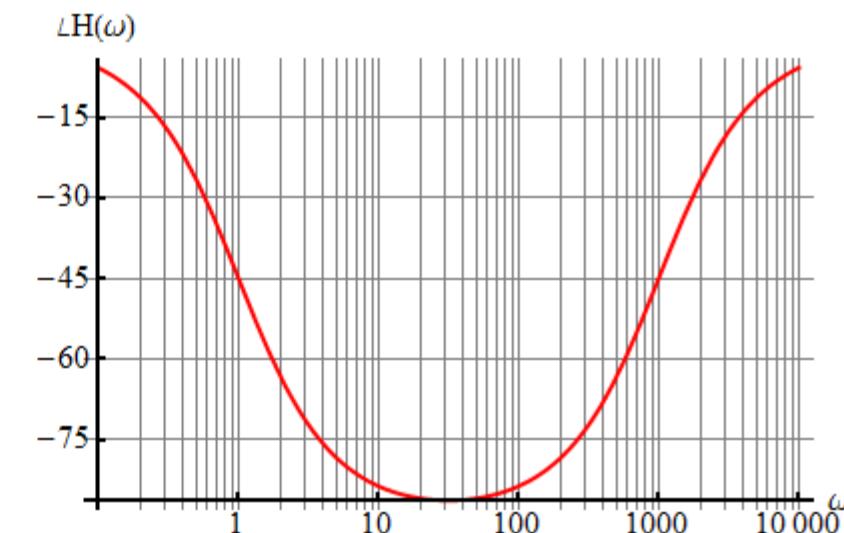
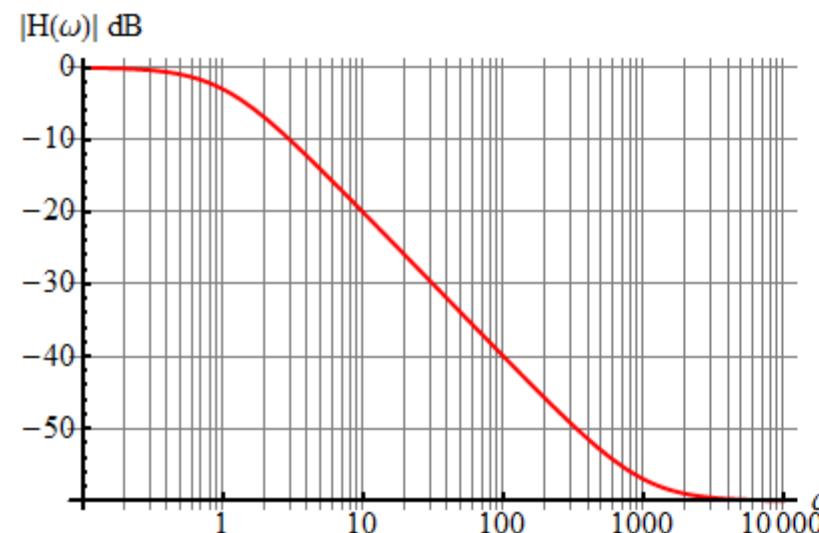


$$H(s) = 10^5 \frac{s + 1}{s + 1000}$$

You should understand on what grounds this solution is constructed.

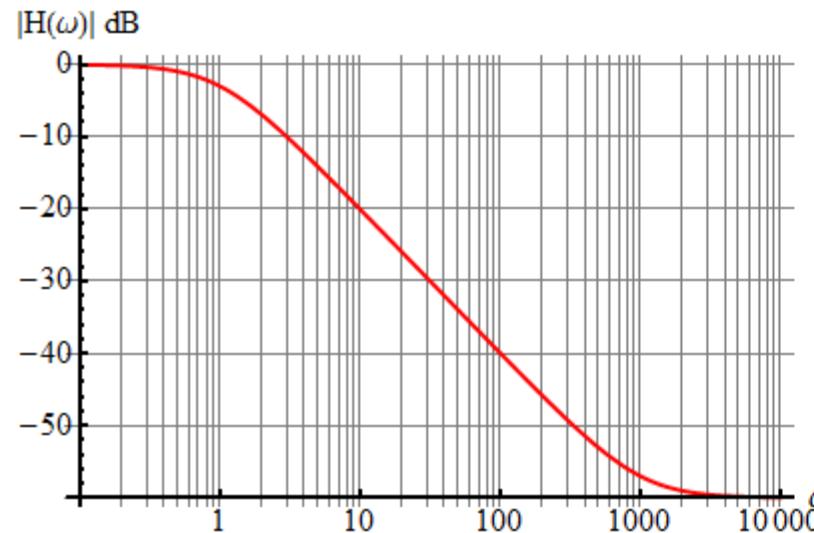
Problem 7

Use the frequency characteristics to write the transfer function.

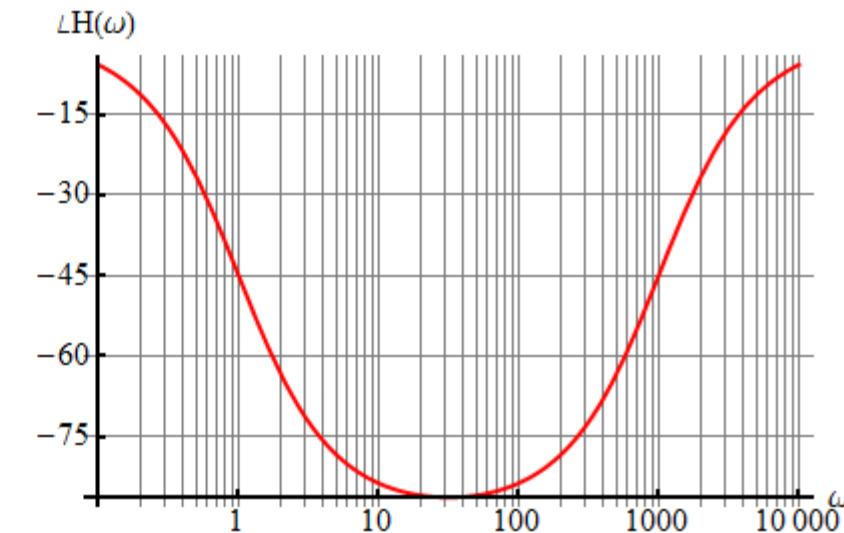


Problem 7 (sol)

Use the frequency characteristics to write the transfer function.



$$H(\omega) = \frac{j\omega}{1000} + 1$$

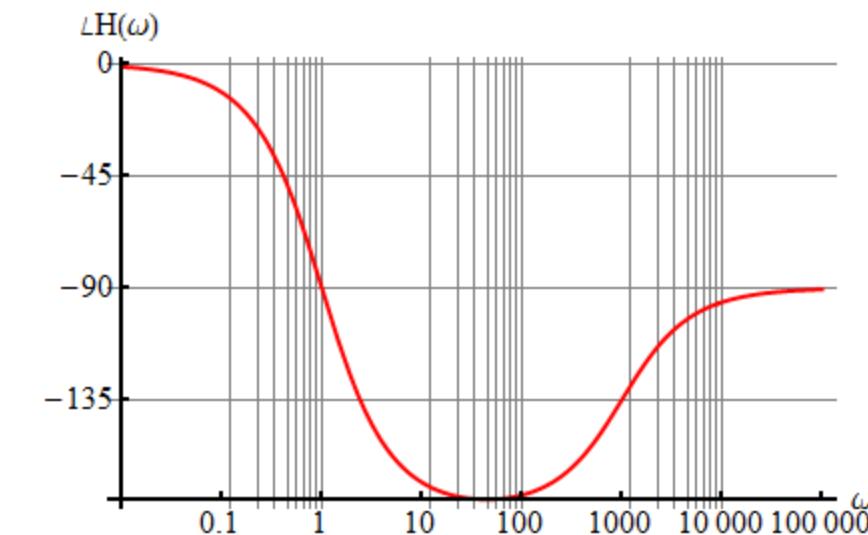
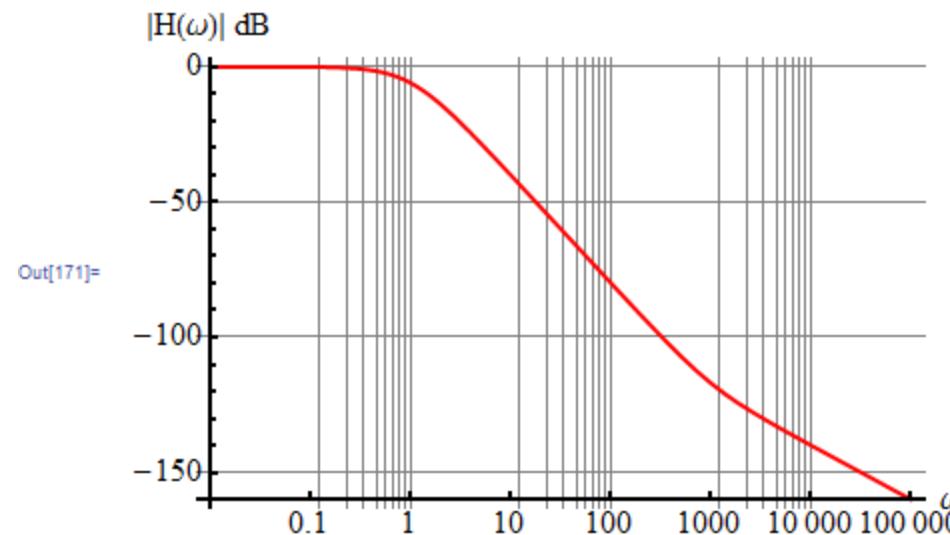


$$H(s) = 10^{-3} \frac{s + 1000}{s + 1}$$

You should understand on what grounds this solution is constructed.

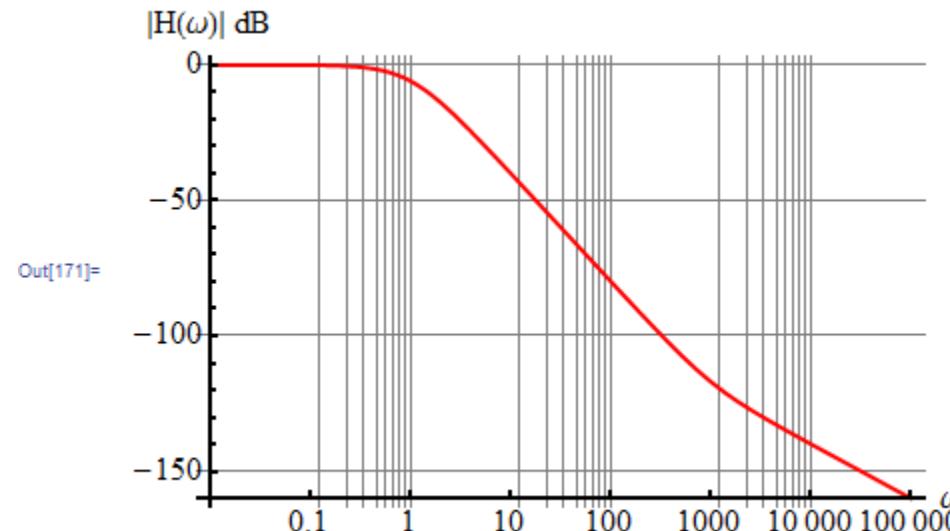
Problem 8

Use the frequency characteristics to write the transfer function.

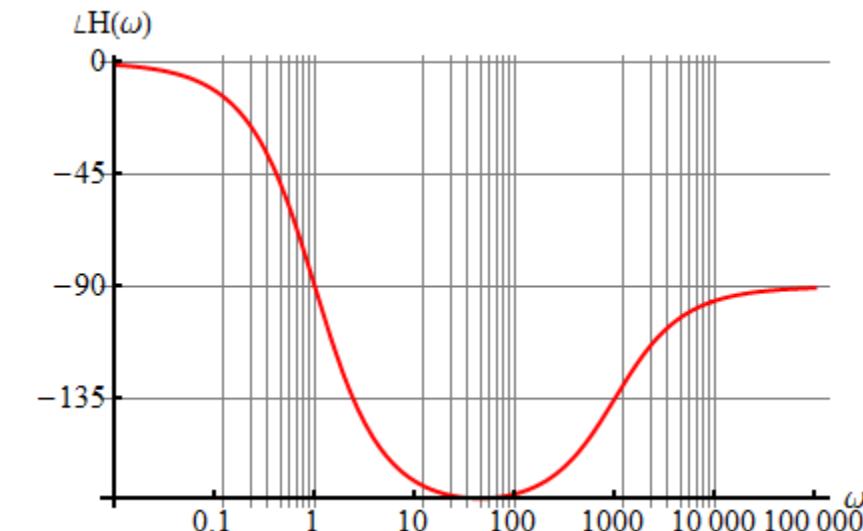


Problem 8 (sol)

Use the frequency characteristics to write the transfer function.



$$H(\omega) = \frac{\frac{j\omega}{1000} + 1}{\left(\frac{j\omega}{1} + 1\right)^2}$$



$$H(s) = 10^{-3} \frac{s + 1000}{(s + 1)^2}$$

The phase changes over two decades, suggesting it could be double pole. But it is only an estimate. It could also be a quadratic factor with a damping factor ζ very close to but slightly less than 1.

22050 Continuous-Time Signals and Linear Systems

Kaj-Åge Henneberg

L11

Butterworth filter design – part 1

Defining properties

Pole symmetries in Butterworth filters

Sallen-Key circuit as a Butterworth filter

Sensitivity analysis

Sensitivity-optimized filter design

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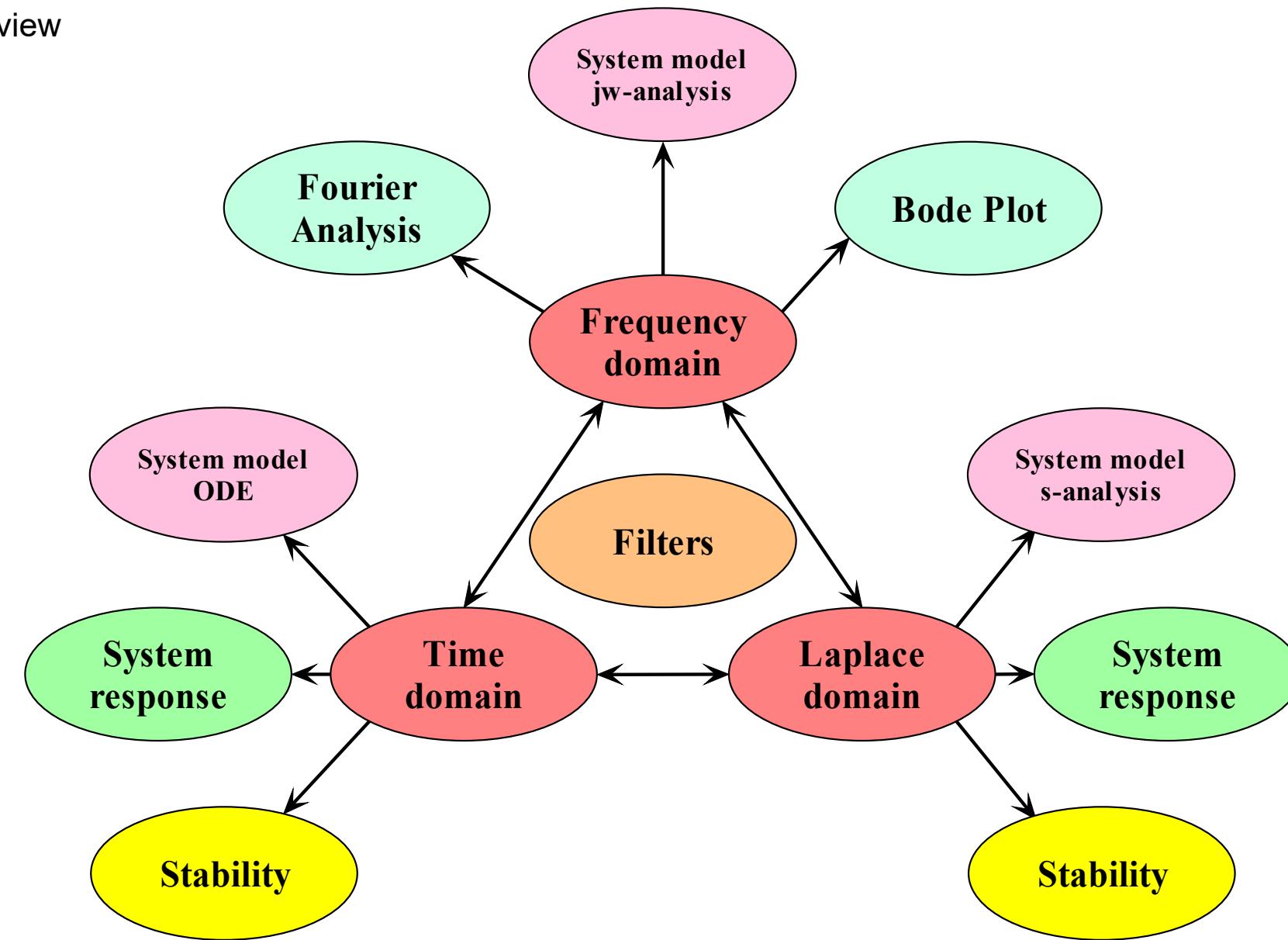
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- In course assignments, the reader is permitted to write the same equations and draw the same diagrams, but should do this using their own tools.



- Defining properties of Butterworth filters
- Butterworth Transfer function (Lathi 7.5)
 - Defining equation
 - Pole positions and symmetries
 - Calculating coefficients
- Design of Butterworth lowpass filter using a Sallen-Key circuit
- Sensitivity to change in component values
- Strategy for sensitivity-optimized filter design
- Designing a sensitivity-optimized frequency-normalized lowpass filter
- Frequency scaling
- Impedance scaling

Butterworth filters Defining properties

Video

Filter design specifications

The objective of a filter is to let frequencies in the passband through the filter with a specified maximum variation in gain (from G_0 to G_p), while attenuating signals in the stopband by a specified attenuation G_s .

G_0 : Gain at $\omega = 0$

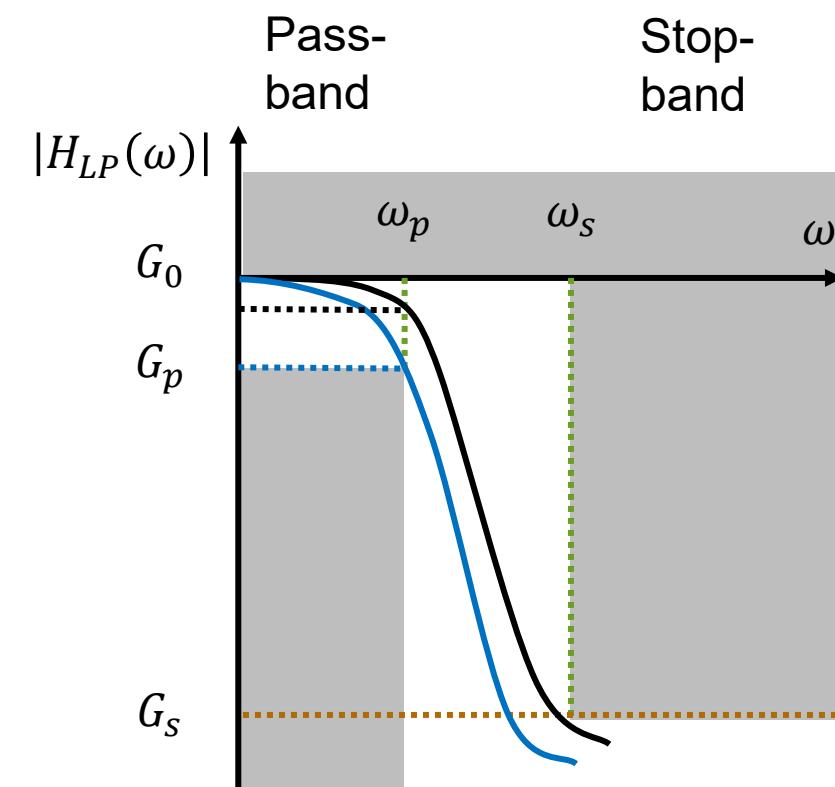
G_p : Gain at $\omega = \omega_p$

G_s : Gain at $\omega = \omega_s$

The gain at DC can be set as wanted, but most often it is set to 1 (0dB).

The gain at the end of the passband can be set as wanted. If it is set to -3dB, then $\omega_p = \omega_c = \omega_{3dB}$.

ω_p does not have to be the 3dB cut-off frequency. It could be the frequency, where the gain has decreased by 1dB. In this case $\omega_c > \omega_p$.



The blue curve satisfies specifications for the attenuation G_p at ω_p .

The black curve satisfies specifications for the attenuation G_s at ω_s .

The amplitude response of an n'th order **Butterworth filter** is by definition:

The gain at the cut-off frequency ω_c is –3dB regardless of the filter order n.

We notice that the defining equation produces a **lowpass** frequency response. Other types of filters (highpass, bandpass, stopband) can be obtained by frequency transformation methods. Even in these cases we start by designing a lowpass filter.

It appears at first glance that the above definition ignores completely the phase characteristic. We will discover, that this is not the case.

$$|H(j\omega)| \stackrel{\text{def}}{=} \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

$$|H(j\omega_c)| \stackrel{\text{def}}{=} \frac{1}{\sqrt{1 + (1)^{2n}}} = \frac{1}{\sqrt{2}}$$

Defining properties of the Butterworth filter

To be general, we add a pass-band gain parameter ϵ . We also substitute $\omega_p \rightarrow \omega_c$.

We define the magnitude squared:

This function can be expanded into a Maclaurin series, which is a Taylor series expanded about $\omega = 0$.

For a first order filter:

$$|H(j\omega)| \stackrel{\text{def}}{=} \frac{1}{\sqrt{1 + \epsilon^2 \left(\frac{\omega}{\omega_p}\right)^2}}$$

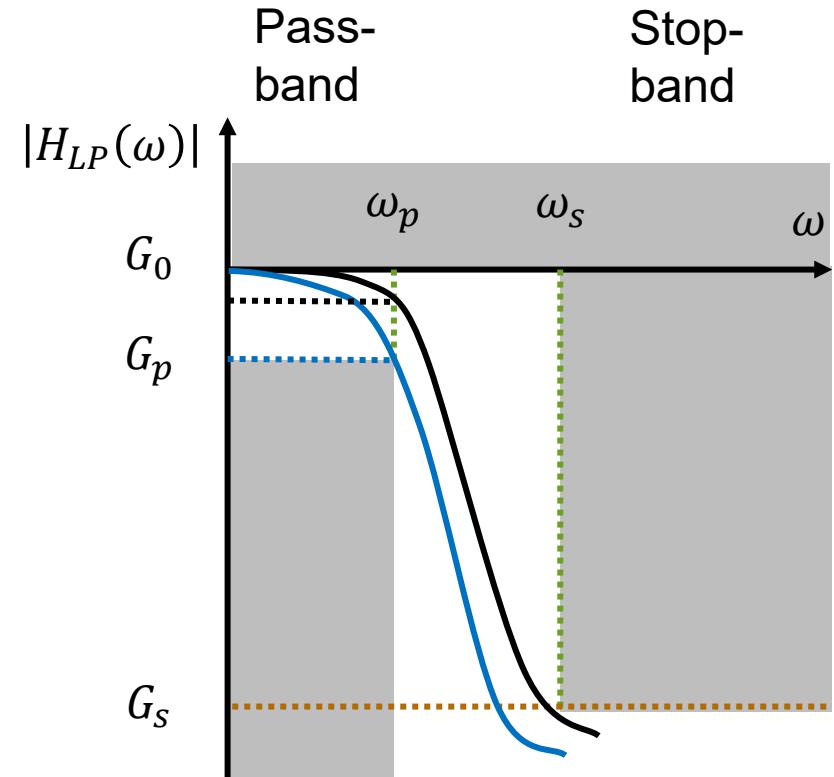
$$G(j\omega) \stackrel{\text{def}}{=} \frac{1}{1 + \epsilon^2 \left(\frac{\omega}{\omega_p}\right)^2}$$

$$G(j\omega) = \sum_{k=0}^{\infty} (-1)^k \epsilon^{2k} \left(\frac{\omega}{\omega_p}\right)^{2nk}$$

$$G(j\omega) = \frac{1}{1 + \epsilon^2 \left(\frac{\omega}{\omega_p}\right)^2} = 1 - \epsilon^2 \left(\frac{\omega}{\omega_p}\right)^2 + \epsilon^4 \left(\frac{\omega}{\omega_p}\right)^4 - \epsilon^6 \left(\frac{\omega}{\omega_p}\right)^6 + \dots$$

For a second order filter:

$$G(j\omega) = \frac{1}{1 + \epsilon^2 \left(\frac{\omega}{\omega_p}\right)^4} = 1 - \epsilon^2 \left(\frac{\omega}{\omega_p}\right)^4 + \epsilon^4 \left(\frac{\omega}{\omega_p}\right)^8 - \epsilon^6 \left(\frac{\omega}{\omega_p}\right)^{12} + \dots$$



Defining properties of the Butterworth filter

We want to investigate the **slope** of the squared magnitude function at DC.

For convenience, all coefficients are collected in A_k :

Taking the m 'th derivative:

If $m < 2n$ then all terms will contain ω to a non-zero power.

If $m = 2n$:

We observe that the Butterworth squared magnitude function has zero slope at DC to the $2n-1$ derivative.

$$G(j\omega) = \sum_{k=0}^{\infty} (-1)^k \epsilon^{2k} \left(\frac{\omega}{\omega_p}\right)^{2nk} = \sum_{k=0}^{\infty} A_k \omega^{2nk}$$

$$G(j\omega) = A_0 + A_1 \omega^{2n} + A_2 \omega^{4n} + \dots$$

$$G^{(m)}(j\omega) = B_1 \omega^{2n-m} + B_2 \omega^{4n-m} + \dots$$

$$G^{(m)}(j0) = 0, \quad m < 2n$$

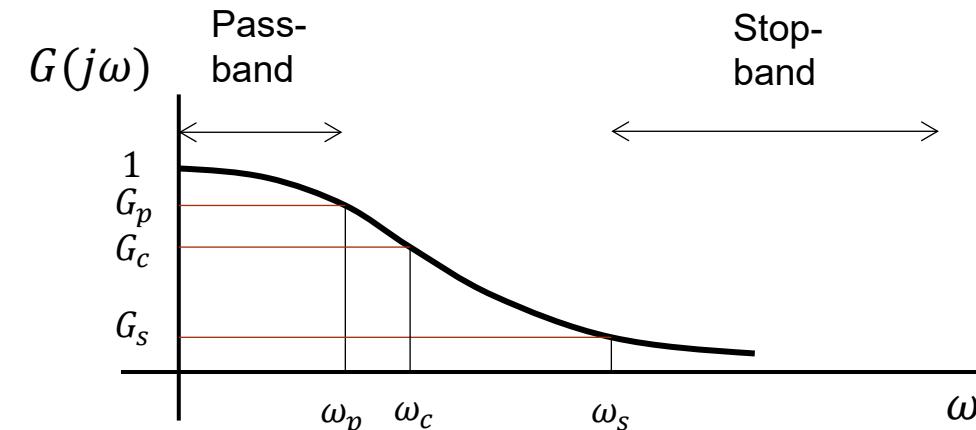
$$G^{(m=2n)}(j\omega) = B_1 \omega^0 + B_2 \omega^{4n-2n} + \dots$$

$$G^{(m)}(j0) = B_1, \quad m = 2n$$

The Butterworth filter has a maximally flat amplitude response at $\omega = 0$. This is the Hallmark of the Butterworth filter.

Amplitude specifications

We now apply the passband and stopband specifications to the squared magnitude.



$$G(j\omega) \stackrel{\text{def}}{=} \frac{1}{1 + \epsilon^2 \left(\frac{\omega}{\omega_p}\right)^{2n}}$$

$$\frac{1}{1 + \epsilon^2} < G(j\omega) < 1, \quad 0 < \omega < \omega_p$$

The specifications for the **passband** states how much gain is allowed to vary within a specified passband.

The specifications for the **stopband** states the minimum attenuation for signal frequencies within the stopband.

$$G(j\omega) < \frac{1}{1 + \epsilon^2 \left(\frac{\omega_s}{\omega_p}\right)^{2n}}, \quad \omega > \omega_s$$

Amplitude specifications

Using the gain parameter ϵ we can specify, how much the gain is allowed to vary in the frequency range $0 < \omega < \omega_p$, independent of filter order n .

The frequency ω_p is that frequency, where the squared magnitude is:

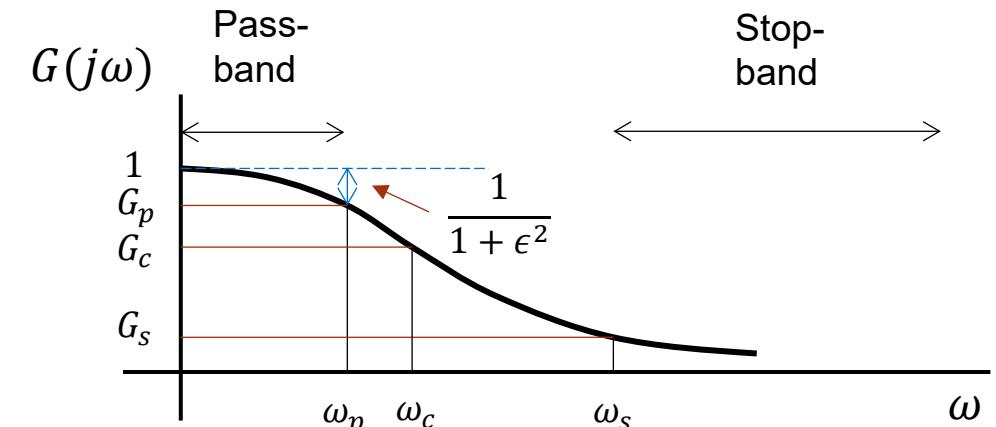
$\epsilon^2 = 0.1$: Squared magnitude drops from 1 to 0.9

$\epsilon^2 = 1$: Squared magnitude drops from 1 to 0.5

Magnitude drops from 1 to $\frac{1}{\sqrt{2}}$, from 0dB to -3dB

If we want ω_p to be the 3dB cut-off frequency (ω_c), then we must set $\epsilon = 1$ and the squared magnitude $G_p = 1/2$. This is the most used choice for the Butterworth filter.

This is equivalent to defining:



$$G(j\omega_p) \stackrel{\text{def}}{=} \frac{1}{1 + \epsilon^2 \left(\frac{\omega_p}{\omega_c}\right)^{2n}} = \frac{1}{1 + \epsilon^2}$$

$$G(j\omega) \stackrel{\text{def}}{=} \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}$$

$$\epsilon \stackrel{\text{def}}{=} \left(\frac{\omega_p}{\omega_c}\right)^n$$

Butterworth filters Transfer function

Video

Frequency-normalized Butterworth filter

The design procedure uses tables of filter transfer function coefficients. To minimize the need for such tables, all filters, regardless of cut-off frequency, is initially designed as a **frequency-normalized lowpass filter with a cut-off frequency of 1 rad/s**.

For this reason, we only need one coefficient table for all types of Butterworth filters.

We cannot design a system from the amplitude response function ($|H(j\omega)|$) alone because that would make the phase arbitrary.

We need to consider the **squared magnitude** $G(j\omega)$ of the defining function.

We have set $\epsilon = 1$ so that we can define $\omega_p = \omega_c$ as the 3dB cut-off frequency.

For further insight we need to study the poles of $G(s)$.

$$G(j\omega) \stackrel{\text{def}}{=} \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}$$

$$G(j\omega) \stackrel{\text{def}}{=} \frac{1}{1 + \left(\frac{j\omega}{j\omega_c}\right)^{2n}}$$

$$G(s) = \frac{1}{1 + \left(\frac{s}{j\omega_c}\right)^{2n}}$$

Butterworth poles

We will seek the transfer function for the frequency normalized lowpass filter, i.e., $\omega_c \stackrel{\text{def}}{=} 1\text{rad/s}$

$$G(s) = \frac{1}{1 + \left(\frac{s}{j\omega_c}\right)^{2n}}$$

$$G(s) = \frac{1}{1 + \left(\frac{s}{j}\right)^{2n}}$$

The denominator is zero (poles) when :

$$\left(\frac{s}{j}\right)^{2n} = -1 \Rightarrow s_k^{2n} = -(j)^{2n} \\ = e^{j\pi(2k-1)} \times \left(e^{\frac{j\pi}{2}}\right)^{2n} \\ = e^{j\pi(2k+n-1)}$$

Taking the $2n$ root on both sides, we get the **poles**:

$$s_k = e^{\frac{j\pi}{2n}(2k+n-1)}, \quad k = 1, 2, 3, \dots, 2n$$

$G(s)$ has $2n$ poles lying evenly distributed on the **unit circle**.

Pole symmetries

$$G(s) = \frac{1}{1 + \left(\frac{s}{j}\right)^{2n}}$$

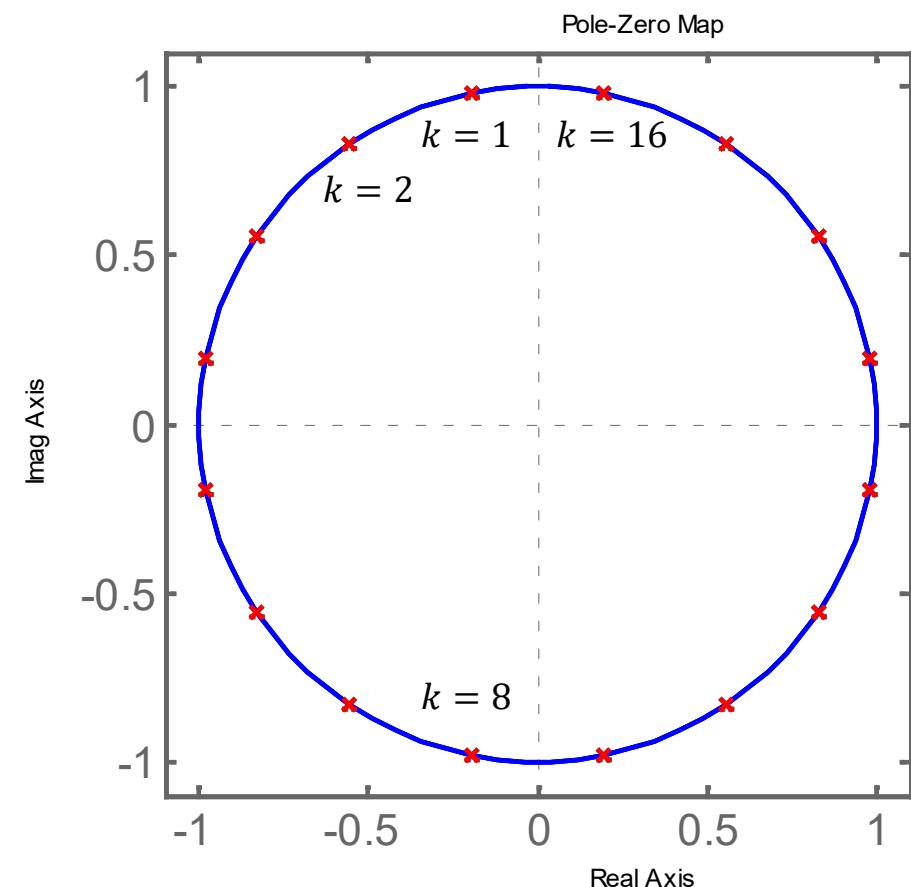
$$s_k = e^{\frac{j\pi}{2n}(2k+n-1)}, \quad k = 1, 2, 3, \dots, 2n$$

Observation:

The poles of $G(s)$ are evenly distributed on the unit circle.

In the interest of stability, we cannot implement a filter transfer function $H(s)$ that contains **poles in the right half-plane**.

We need to **split up the poles in two groups**: those that lie in the left half-plane and those that lie in the right half-plane.



Pole symmetries

Let us study pole symmetries a bit closer.
 We will investigate a 2nd order filter ($n = 2$).
 However, the symmetry will exist for all
 values of n .

We will think of $G(s)$ in its factorized form:

$$s_k = e^{\frac{j\pi}{2n}(2k+n-1)}, \quad k = 1, 2, \dots, n$$

$$s_k = e^{\frac{j\pi}{4}(2k+1)}, \quad n = 2, \quad k = 1, 2, 3, 4$$

For a 2nd order filter, $G(s)$ will have 4 poles.

135°

$$s_1 = e^{\frac{j\pi 3}{4}} = \cos \frac{3}{4}\pi + j \sin \frac{3}{4}\pi$$

Subtracting 2π

$225^\circ = -135^\circ$

$$s_2 = e^{\frac{j\pi 5}{4}} = \cos \frac{5}{4}\pi + j \sin \frac{5}{4}\pi = \cos \frac{-3}{4}\pi + j \sin \frac{-3}{4}\pi = e^{-\frac{j\pi 3}{4}}$$

Subtracting π + negating

$315^\circ = -45^\circ$

$$s_3 = e^{\frac{j\pi 7}{4}} = \cos \frac{7}{4}\pi + j \sin \frac{7}{4}\pi = -\cos \frac{3}{4}\pi - j \sin \frac{3}{4}\pi = -e^{\frac{j\pi 3}{4}}$$

Subtracting π + negating

$405^\circ = 45^\circ$

$$s_4 = e^{\frac{j\pi 9}{4}} = \cos \frac{9}{4}\pi + j \sin \frac{9}{4}\pi = -\cos \frac{5}{4}\pi - j \sin \frac{5}{4}\pi = -e^{-\frac{j\pi 3}{4}}$$

Pole symmetries

We have discovered the following pole symmetries.

We observe that s_1 and s_2 are complex conjugated poles in the left half-plane.

We observe that s_3 and s_4 are complex conjugated poles in the right half-plane.

We also see that each pole in the right half-plane is the negative of a diagonal twin in the left half-plane.

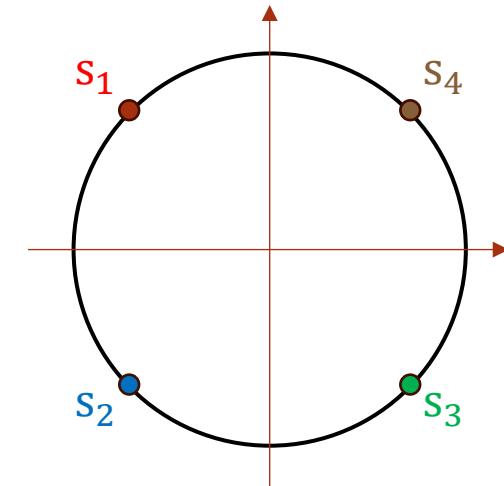
We also see that s_3 is the negative complex conjugated of s_2 and s_4 is the negative complex conjugated of s_1 .

$$s_1 = e^{\frac{j\pi 3}{4}}$$

$$s_2 = e^{\frac{j\pi 5}{4}} = e^{-\frac{j\pi 3}{4}} = (e^{\frac{j\pi 3}{4}})^* = s_1^*$$

$$s_3 = e^{\frac{j\pi 7}{4}} = -e^{\frac{j\pi 3}{4}} = -s_1 = -s_2^*$$

$$s_4 = e^{\frac{j\pi 9}{4}} = -e^{-\frac{j\pi 3}{4}} = -s_2 = -s_1^*$$



$$G(s) = \underbrace{\frac{1}{(s - s_1)(s - s_2)}}_{left\ half-plane} \underbrace{\frac{1}{(s - s_3)(s - s_4)}}_{right\ half-plane}$$

$$G(s) = \underbrace{\frac{1}{(s - s_1)(s - s_2)}}_{left\ half-plane} \underbrace{\frac{1}{(s + s_1)(s + s_2)}}_{right\ half-plane}$$

$$G(s) = \underbrace{\frac{1}{(s - s_1)(s - s_2)}}_{left\ half-plane} \underbrace{\frac{1}{(-s - s_1)(-s - s_2)}}_{right\ half-plane} = \underbrace{H(s)}_{left} \underbrace{H(-s)}_{right}$$

Pole symmetries

The poles of $H(s)$ lies in the left half plane. It is stable and has the impulse response:

Using the **bilateral Laplace transform**, we can find the Laplace transform of the time-reflected signal $h(-t)$:

We make the substitution $\tau = -t$ and $d\tau = -dt$:

Since reversing the integration limits introduces a negative sign, we get:

We finally get:

This result is a proof of the time reversal theorem for the bilateral Laplace transform:

$$h(t) \leftrightarrow H(s)$$

$$\mathcal{L}\{h(-t)\} = \int_{-\infty}^{\infty} h(-t)e^{-st} dt$$

$$\mathcal{L}\{h(-t)\} = \int_{\infty}^{-\infty} h(\tau)e^{s\tau}(-d\tau)$$

$$\mathcal{L}\{h(-t)\} = \int_{-\infty}^{\infty} h(\tau)e^{s\tau} d\tau$$

$$\mathcal{L}\{h(-t)\} = \int_{-\infty}^{\infty} h(\tau)e^{-(s)\tau} d\tau = H(-s)$$

$$h(-t) \leftrightarrow H(-s)$$

Pole symmetries

The defining function $G(j\omega)$ had no phase.

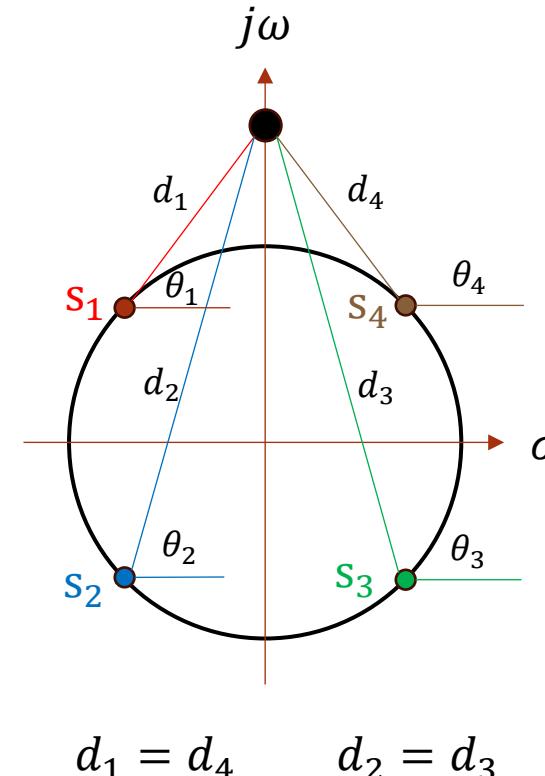
We are now in a position to explain why.

From the sketch we observe that poles mirrored about the imaginary axis have phase angles that add up to π (180°).

It follows from the math that $H(j\omega)$ and $H(-j\omega)$ have phase angles that are different by π (180°). In other words, their angles add to zero.

Observation:

Although the defining function $G(j\omega)$ had no phase, the frequency characteristic $H(j\omega)$ of the Butterworth transfer function has a phase angle.



$$H(j\omega) = \frac{1}{d_1 e^{j\theta_1}} \cdot \frac{1}{d_2 e^{j\theta_2}} = \frac{1}{d_1 d_2} e^{-j(\theta_1 + \theta_2)}$$

$$H(-j\omega) = \frac{1}{d_3 e^{j\theta_3}} \cdot \frac{1}{d_4 e^{j\theta_4}}$$

$$\theta_3 = 180^\circ - \theta_2 \quad \theta_4 = 180^\circ - \theta_1$$

$$\begin{aligned} H(-j\omega) &= \frac{1}{d_3 e^{j\theta_3}} \cdot \frac{1}{d_4 e^{j\theta_4}} \\ &= \frac{1}{d_2 e^{j(180^\circ - \theta_2)} d_1 e^{j(180^\circ - \theta_1)}} \\ &= \frac{1}{d_1 d_2} e^{j(\theta_1 + \theta_2)} \end{aligned}$$

$$G(j\omega) = H(j\omega)H(-j\omega) = \frac{1}{d_1 d_2} e^{-j(\theta_1 + \theta_2)} \frac{1}{d_1 d_2} e^{j(\theta_1 + \theta_2)} = \frac{1}{(d_1 d_2)^2}$$

Poles of stable filter

Using only the poles in the left half-plane, k must run from 1 to n , not from 1 to $2n$:

The transfer function is:

$$s_k = e^{\frac{j\pi}{2n}(2k+n-1)}, \quad k = 1, 2, 3, \dots, n$$

$$H(s) = \frac{1}{(s - s_1)(s - s_2) \cdots (s - s_n)}$$

	Pole indices					
	n	k=1	k=2	k=3	k=4	k=5
O r d e r	1	$e^{j\pi}$				
	2	$e^{j\pi 3/4}$	$e^{j\pi 5/4}$			
	3	$e^{j\pi 4/6}$	$e^{j\pi}$	$e^{j\pi 8/6}$		
	4	$e^{j\pi 5/8}$	$e^{j\pi 7/8}$	$e^{j\pi 9/8}$	$e^{j\pi 11/8}$	
	5	$e^{j\pi 6/10}$	$e^{j\pi 8/10}$	$e^{j\pi}$	$e^{j\pi 12/10}$	$e^{j\pi 14/10}$

Poles of Butterworth filter

n	k=1	k=2	k=3	k=4	k=5
1	$e^{j\pi}$				
2	$e^{j\pi 3/4}$	$e^{j\pi 5/4}$			
3	$e^{j\pi 4/6}$	$e^{j\pi}$	$e^{j\pi 8/6}$		
4	$e^{j\pi 5/8}$	$e^{j\pi 7/8}$	$e^{j\pi 9/8}$	$e^{j\pi 11/8}$	
5	$e^{j\pi 6/10}$	$e^{j\pi 8/10}$	$e^{j\pi}$	$e^{j\pi 12/10}$	$e^{j\pi 14/10}$

Making use of pole symmetries:

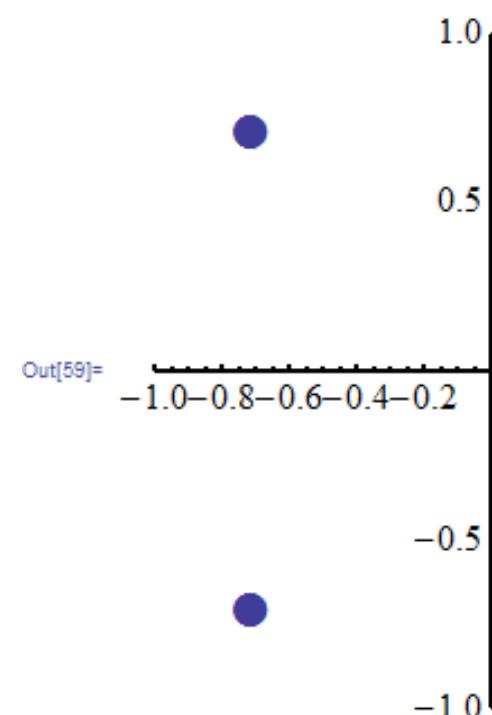
n	k=1	k=2	k=3	k=4	k=5
1	$e^{j\pi}$				
2	$e^{j\pi 3/4}$	$e^{-j\pi 3/4}$			
3	$e^{j\pi 4/6}$	$e^{j\pi}$	$e^{-j\pi 4/6}$		
4	$e^{j\pi 5/8}$	$e^{j\pi 7/8}$	$e^{-j\pi 7/8}$	$e^{-j\pi 5/8}$	
5	$e^{j\pi 6/10}$	$e^{j\pi 8/10}$	$e^{j\pi}$	$e^{-j\pi 8/10}$	$e^{-j\pi 6/10}$

Poles in s-plane for Butterworth filters

The angle between poles depends on n:

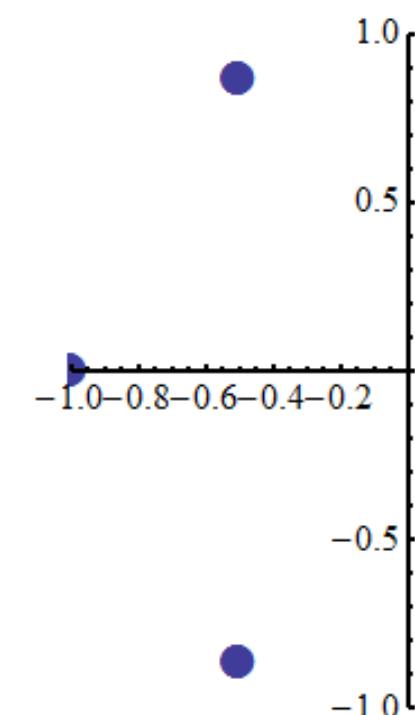
$$\Delta\phi = \frac{180}{n}$$

$n = 2$



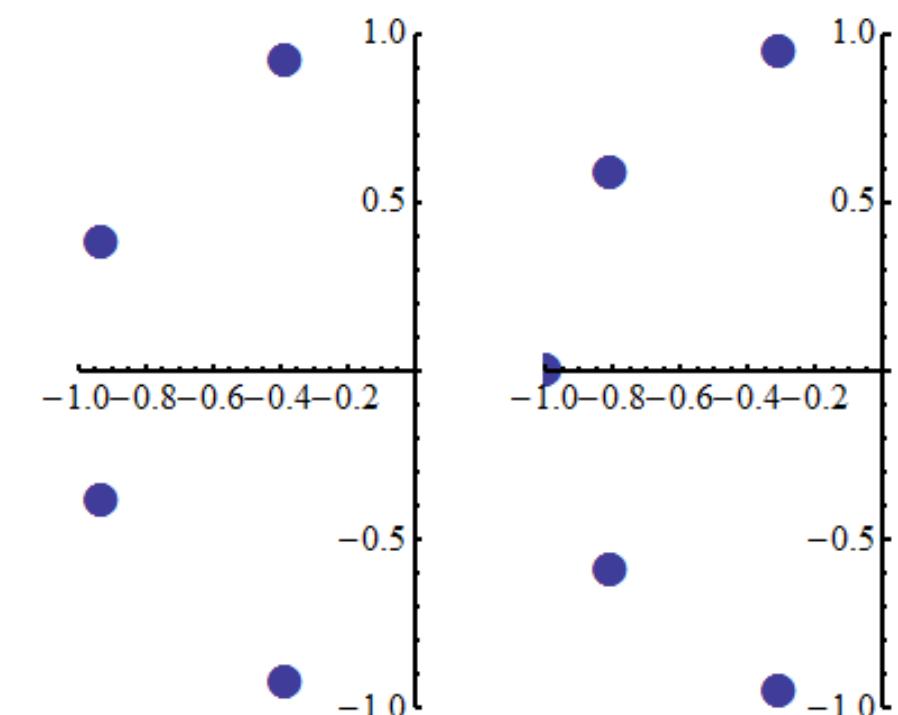
$$\Delta\phi = 90^\circ$$

$n = 3$



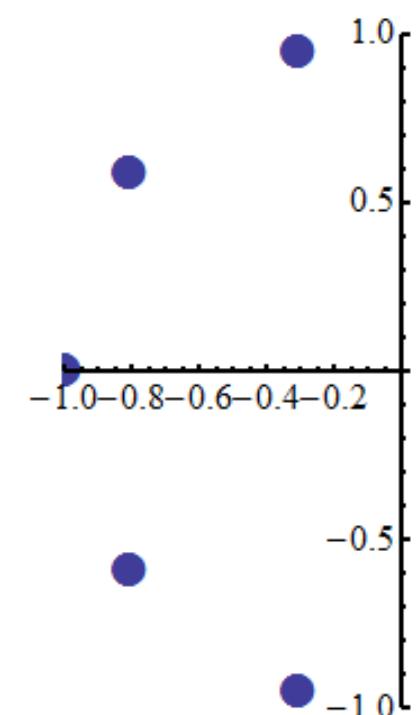
$$\Delta\phi = 60^\circ$$

$n = 4$



$$\Delta\phi = 45^\circ$$

$n = 5$



$$\Delta\phi = 36^\circ$$

As many poles are complex conjugated, we can pair these factors in **quadratic factors**:

$$H(s) = \frac{1}{(s - s_1)(s - s_2) \cdots (s - s_n)}$$

$$H(s) = \frac{1}{(s - s_1) \underbrace{(s^2 + a_{11}s + a_{10})}_{stage\ 1} \underbrace{(s^2 + a_{21}s + a_{20})}_{stage\ 2} \cdots}$$

Linear factor: 1st order filter circuit,

Quadratic factor: 2nd order filter circuit, such as the **Multiple feedback circuit** or the **Sallen-Key circuit**.

For the k 'th filter stage, we must derive the polynomial coefficients a_{k1} and a_{k0} .

Polynomial coefficients for Butterworth filters

Odd order:

$$s_{k'} = e^{\frac{j\pi}{2n}(2k'+n-1)}, \quad k' = 1, 2, 3, \dots, n$$

Linear indexing

Mapping from linear pole indexing to complex conjugated pairs:

linear	paired
1,5	1,-1
2,4	2,-2
3	3

$$H(s) = \frac{1}{\left(s - e^{\frac{j\pi}{2n}\left(\frac{2(n+1)}{2}+n-1\right)}\right)} \cdot \prod_{k=1}^{\frac{n-1}{2}} \frac{1}{\left(s - e^{\frac{j\pi}{2n}(2k+n-1)}\right)\left(s - e^{\frac{j\pi}{2n}(2(n+1-k)+n-1)}\right)}$$

Paired indexing

$$H(s) = \frac{1}{(s - e^{j\pi})} \cdot \prod_{k=1}^{\frac{n-1}{2}} \frac{1}{\left(s - e^{\frac{j\pi}{2n}(2k+n-1)}\right)\left(s - e^{-\frac{j\pi}{2n}(2k+n-1)}\right)}$$

Paired indexing

$$H(s) = \frac{1}{s + 1} \cdot \prod_{k=1}^{\frac{n-1}{2}} \frac{1}{s^2 + a_{k1}s + a_{k0}}$$

Paired indexing

$$a_{k0} = e^{\frac{j\pi}{2n}(2k+n-1)} \cdot e^{-\frac{j\pi}{2n}(2k+n-1)} = 1$$

$$a_{k1} = -e^{\frac{j\pi}{2n}(2k+n-1)} - e^{-\frac{j\pi}{2n}(2k+n-1)} = -2 \cos\left(\frac{\pi}{2n}(2k + n - 1)\right)$$

Polynomial coefficients for Butterworth filters

Even order:

$$s_{k'} = e^{\frac{j\pi}{2n}(2k'+n-1)}, \quad k' = 1, 2, 3, \dots, n \quad \text{Linear indexing}$$

$$H(s) = \prod_{k=1}^{\frac{n}{2}} \frac{1}{(s - e^{\frac{j\pi}{2n}(2k+n-1)})(s - e^{\frac{j\pi}{2n}(2(n+1-k)+n-1)})} \quad \text{Paired indexing}$$

$$H(s) = \prod_{k=1}^{\frac{n}{2}} \frac{1}{(s - e^{\frac{j\pi}{2n}(2k+n-1)})(s - e^{-\frac{j\pi}{2n}(2k+n-1)})} \quad \text{Paired indexing}$$

$$H(s) = \prod_{k=1}^{\frac{n}{2}} \frac{1}{s^2 + a_{k1}s + a_{k0}} \quad \text{Paired indexing}$$

$$a_{k0} = e^{\frac{j\pi}{2n}(2k+n-1)} \cdot e^{-\frac{j\pi}{2n}(2k+n-1)} = 1$$

$$a_{k1} = -e^{\frac{j\pi}{2n}(2k+n-1)} - e^{-\frac{j\pi}{2n}(2k+n-1)} = -2 \cos\left(\frac{\pi}{2n}(2k+n-1)\right)$$

We are now able to tabulate the coefficients for each stage in a higher order Butterworth filter.

$$a_{k1} = -2 \cos\left(\frac{\pi}{2n}(2k + n - 1)\right)$$

$$a_{k0} = 1$$

For odd order filters, the first stage is assumed to be the first order filter. For this stage $a_{k2} = 0$.

$$\omega_{nk} = \sqrt{a_{k0}} = 1$$

$$\zeta_k = \frac{a_{k1}}{2\sqrt{a_{k0}}} = \frac{a_{k1}}{2}$$

$$H(s) = \prod_{k=1}^L H_k(s) \quad H_k(s) = \frac{b_{k0}}{a_{k2}s^2 + a_{k1}s + a_{k0}}$$

n	k	a_{k2}	a_{k1}	a_{k0}
1	1	0.0	1.0	1.0
2	1	1.0	1.41421	1.0
3	1	0.0	1.0	1.0
	2	1.0	1.0	1.0
4	1	1.0	0.765367	1.0
	2	1.0	1.84776	1.0
5	1	0.0	1.0	1.0
	2	1.0	0.618034	1.0
	3	1.0	1.61803	1.0

Butterworth transfer functions

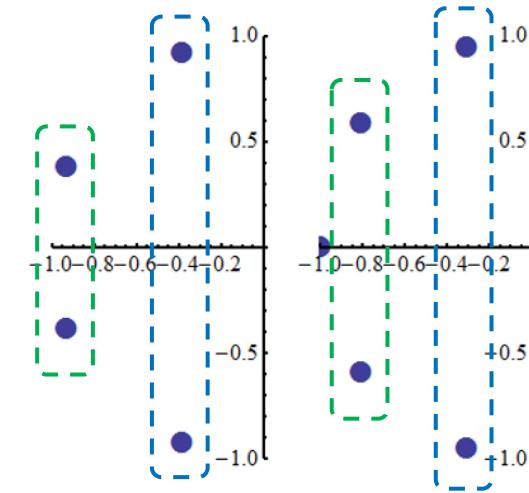
n	k	a_{k2}	a_{k1}	a_{k0}
1	1	0.0	1.0	1.0
2	1	1.0	1.41421	1.0
3	1	0.0	1.0	1.0
	2	1.0	1.0	1.0
4	1	1.0	0.765367	1.0
	2	1.0	1.84776	1.0
5	1	0.0	1.0	1.0
	2	1.0	0.618034	1.0
	3	1.0	1.61803	1.0

$$n = 2: \quad H_2(s) = \frac{1}{s^2 + 1.41421s + 1}$$

$$n = 3: \quad H_3(s) = \frac{1}{s + 1} \cdot \frac{1}{s^2 + 1.0s + 1}$$

$$n = 4: \quad H_4(s) = \underbrace{\frac{1}{s^2 + 0.765367s + 1}}_{\text{stage 1}} \cdot \underbrace{\frac{1}{s^2 + 1.84776s + 1}}_{\text{stage 2}}$$

$$n = 5: \quad H_5(s) = \frac{1}{s + 1} \cdot \underbrace{\frac{1}{s^2 + 0.618034s + 1}}_{\text{stage 1}} \cdot \underbrace{\frac{1}{s^2 + 1.61803s + 1}}_{\text{stage 2}}$$

**Observation:**

For the 4th and 5th order filters, the two quadratic factors have different coefficients. Hence the filter stages must have different component values. One 2nd order stage takes care of one pole pair, the other stage takes care of the other pole pair. **How do we see which pole pair belongs to which stage? Hint: ζ**

Butterworth filters Matching to Sallen-Key circuit

Video

2nd order Sallen-Key low pass filter

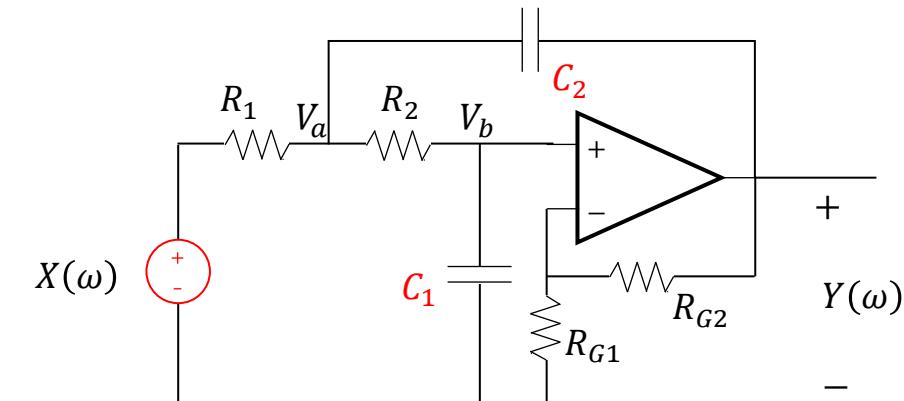
We will implement the Butterworth filter using a Sallen-Key lowpass circuit.

Be aware, that it is not the circuit that makes it a Butterworth filter, it is the values of the coefficients in the frequency characteristic.

By matching coefficients in the two equations, we obtain two equations:

$$R_1 R_2 C_1 C_2 = \frac{1}{a_0}$$

$$(R_2 C_1 + R_1 C_1 + R_1 C_2 (1 - K)) = \frac{a_1}{a_0}$$



$$H_{LP}(j\omega) = \frac{\frac{K}{R_1 R_2 C_1 C_2}}{(j\omega)^2 + j\omega \left(\frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} + \frac{(1 - K)}{R_2 C_1} \right) + \frac{1}{R_1 R_2 C_1 C_2}}$$

$$H_{LP}(j\omega) = \frac{K}{R_1 R_2 C_1 C_2 (j\omega)^2 + j\omega (R_2 C_1 + R_1 C_1 + R_1 C_2 (1 - K)) + 1}$$

$$H_{LP}(j\omega) = \frac{K}{\frac{1}{a_0} (j\omega)^2 + \frac{a_1}{a_0} (j\omega) + 1}$$

2nd order Sallen-Key low pass filter

Isolating R_2 in the first equation and inserting it into the second, we obtain an equation for R_1 .

It is a quadratic equation.

$$R_2 = \frac{1}{a_0 R_1 C_1 C_2} \rightarrow (R_2 C_1 + R_1 C_1 + R_1 C_2 (1 - K)) = \frac{a_1}{a_0}$$

$$\frac{1}{a_0 R_1 C_2} + R_1 C_1 + R_1 C_2 (1 - K) = \frac{a_1}{a_0}$$

We choose the positive solution:

$$R_1 = \frac{a_1 - \sqrt{a_1^2 - 4a_0 \left(\frac{C_1}{C_2} + 1 - K \right)}}{2a_0 (C_1 + C_2 (1 - K))}$$

We can then solve for R_2 :

$$R_2 = \frac{a_1 + \sqrt{a_1^2 - 4a_0 \left(\frac{C_1}{C_2} + 1 - K \right)}}{2a_0 C_1}$$

2nd order Sallen-Key low pass filter

For the resistors to have **real values**, we must ensure that the **discriminants are positive**.

This puts a constraint on the capacitor values:

Another constraint on the capacitor values ensures that the **resistors are positive**.

We can combine the two constraints:

$$R_1 = \frac{a_1 - \sqrt{a_1^2 - 4a_0 \left(\frac{C_1}{C_2} + 1 - K \right)}}{2a_0(C_1 + C_2(1 - K))}$$

$$R_2 = \frac{a_1 + \sqrt{a_1^2 - 4a_0 \left(\frac{C_1}{C_2} + 1 - K \right)}}{2a_0C_1}$$

$$a_1^2 - 4a_0 \left(\frac{C_1}{C_2} + 1 - K \right) > 0 \Rightarrow \frac{a_1^2}{4a_0} > \frac{C_1}{C_2} + 1 - K \Rightarrow \frac{a_1^2}{4a_0} + K - 1 > \frac{C_1}{C_2}$$

$$C_1 + C_2(1 - K) > 0 \Rightarrow C_1 > C_2(1 - K)$$

$$C_2(K - 1) < C_1 < C_2 \left(\frac{a_1^2}{4a_0} + K - 1 \right)$$

So, if we first pick a value for C_2 , we can pick a value for C_1 within a bounded range.

2nd order Sallen-Key low pass filter

Until now we have followed an approach that will provide a filter circuit that is realizable. At least we have avoided complex-valued or negative-valued resistors.

On the other hand, we have **not paid any attention to optimization** of some features of the system.

For electrical circuits, the uncertainty in performance associated with **variation in component values** is an important concern.

This variation can come about if a **malfuctioning component** must be replaced by a new component with a slightly different value, or because the **component value drifts** due to environmental factors (temperature, humidity, etc.).

We can apply **sensitivity analysis** to discover how sensitive a given circuit is to changes in component values.

$$C_2(K - 1) < C_1 < C_2 \left(\frac{a_1^2}{4a_0} + K - 1 \right)$$

$$R_1 = \frac{a_1 - \sqrt{a_1^2 - 4a_0 \left(\frac{C_1}{C_2} + 1 - K \right)}}{2a_0(C_1 + C_2(1 - K))}$$

$$R_2 = \frac{a_1 + \sqrt{a_1^2 - 4a_0 \left(\frac{C_1}{C_2} + 1 - K \right)}}{2a_0C_1}$$

Butterworth filters

Sensitivity of $H(j\omega)$ to drift in component values

Video

Sensitivity analysis of Sallen-Key lowpass filter

A second order transfer function is defined by two parameters, the resonance frequency ω_n and the damping ratio ζ .

In filter circuits, the **quality factor** Q is used instead of the damping ratio:

Inaccurate component values or large temperature drift can cause the real-life filter to have a frequency characteristic that differs from the intended. Different circuit topologies have different sensitivities.

A sensitivity analysis is called for in critical applications and one circuit topology may be a better candidate than others because of lower sensitivity to **component drift**.

$$H_{LP}(j\omega) = \frac{Ka_0}{(j\omega)^2 + j\omega a_1 + a_0}$$

$$H_{LP}(j\omega) = \frac{K\omega_n^2}{(j\omega)^2 + j\omega 2\zeta\omega_n + \omega_n^2}$$

$$H_{LP}(j\omega) = \frac{K\omega_n^2}{(j\omega)^2 + j\omega \frac{\omega_n}{Q} + \omega_n^2}$$

$$\frac{1}{Q} \stackrel{\text{def}}{=} 2\zeta \Rightarrow Q \stackrel{\text{def}}{=} \frac{1}{2\zeta} = \frac{1}{2} \frac{2\sqrt{a_0}}{a_1} = \frac{\sqrt{a_0}}{a_1}$$

Control theory

Filter theory

ζ	Q
0	∞
$1/\sqrt{2}$	$1/\sqrt{2}$
1	$1/2$

Sensitivity analysis of Sallen-Key lowpass filter

Sallen-Key frequency characteristic:

In sensitivity analysis, we can derive the sensitivity for $H(\omega)$ and obtain insight into what frequency range is most sensitive to change in component values.

We can also derive sensitivities for the circuit parameters, such as resonance frequency (ω_n) and the **Quality factor** (Q).

To calculate the change in Q due to change in component values, we need expressions for Q :

$$H_{LP}(j\omega) = \frac{\frac{K}{R_1 R_2 C_1 C_2}}{(j\omega)^2 + j\omega \left(\frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} + \frac{(1-K)}{R_2 C_1} \right) + \frac{1}{R_1 R_2 C_1 C_2}}$$

$$\omega_n = (R_1 R_2 C_1 C_2)^{-1/2}$$

$$\frac{1}{Q} \stackrel{\text{def}}{=} 2\zeta \Rightarrow Q \stackrel{\text{def}}{=} \frac{1}{2\zeta} = \frac{1}{2} \frac{2\sqrt{a_0}}{a_1} = \frac{\sqrt{a_0}}{a_1}$$

$$Q = \frac{\sqrt{R_1 R_2 C_1 C_2}}{R_2 C_1 + R_1 C_1 + (1-K)R_1 C_2}$$

$$\frac{1}{Q} = \sqrt{\frac{R_2 C_1}{R_1 C_2}} + \sqrt{\frac{R_1 C_1}{R_2 C_2}} + (1-K) \sqrt{\frac{R_1 C_2}{R_2 C_1}}$$

Sensitivity analysis of Sallen-Key lowpass filter

Sensitivity of a function y to a parameter x is defined as:

$$S_x^y \stackrel{\text{def}}{=} \lim_{\Delta x \rightarrow 0} \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{x}{y} \frac{\partial y}{\partial x} = \frac{\partial \ln y}{\partial \ln x} \quad \frac{\partial \ln y}{\partial y} = \frac{1}{y} \Rightarrow \partial \ln y = \frac{\partial y}{y}$$

If $S_x^y = 5$, it implies that a 1% increase in x will produce a 5% *increase* in y .

If $S_x^y = -2$, it implies that a 1% increase in x will produce a 2% *decrease* in y .

Using this concept, we can analyze how many percent the amplitude and phase characteristics will change, if a component value is increased by 1%.

Sensitivity analysis of Sallen-Key lowpass filter

Our usual form:

$$H_{LP}(j\omega) = \frac{\frac{K}{R_1 R_2 C_1 C_2}}{(j\omega)^2 + j\omega \left(\frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} + \frac{(1-K)}{R_2 C_1} \right) + \frac{1}{R_1 R_2 C_1 C_2}}$$

An equivalent form:

$$H_{LP}(j\omega) = \frac{\frac{K}{R_1 R_2}}{(j\omega)^2 C_1 C_2 + j\omega \left(\frac{C_1}{R_1} + \frac{C_1}{R_2} + \frac{C_2(1-K)}{R_2} \right) + \frac{1}{R_1 R_2}} \stackrel{\text{def}}{=} \frac{N}{D}$$

Generic expression for the sensitivity function:

$$S_x^H = \frac{x}{H} \cdot \frac{\partial H}{\partial x} = x \left(\frac{N'}{N} - \frac{D'}{D} \right) \quad N' \stackrel{\text{def}}{=} \frac{\partial N}{\partial x}, D' \stackrel{\text{def}}{=} \frac{\partial D}{\partial x}$$

Using polar form in the sensitivity function:

$$H(j\omega) = |H(j\omega)| e^{j\theta(\omega)}$$

Taking the derivative of a nested function:

$$S_x^H = \frac{x}{H} \cdot \frac{\partial H}{\partial x} = \frac{x}{|H(j\omega)| e^{j\theta(\omega)}} \left(\frac{\partial |H(j\omega)|}{\partial x} e^{j\theta(\omega)} + |H(j\omega)| \frac{\partial e^{j\theta(\omega)}}{\partial x} \right)$$

$$S_x^H = \frac{x}{H} \cdot \frac{\partial H}{\partial x} = \frac{x}{|H(j\omega)| e^{j\theta(\omega)}} \left(\frac{\partial |H(j\omega)|}{\partial x} e^{j\theta(\omega)} + |H(j\omega)| e^{j\theta(\omega)} \frac{\partial (\theta(\omega))}{\partial x} \right)$$

Multiplying into the parenthesis and simplifying:

$$S_x^H = \frac{x}{H} \cdot \frac{\partial H}{\partial x} = \frac{x}{|H(j\omega)|e^{j\theta(\omega)}} \left(\frac{\partial |H(j\omega)|}{\partial x} e^{j\theta(\omega)} + |H(j\omega)| e^{j\theta(\omega)} \frac{\partial (j\theta(\omega))}{\partial x} \right)$$

The sensitivity of $H(j\omega)$ to a parameter will be a complex-valued function of ω .

$$S_x^H = \frac{x}{H} \cdot \frac{\partial H}{\partial x} = \underbrace{\frac{x}{|H(j\omega)|} \frac{\partial |H(j\omega)|}{\partial x}}_{S_x^{|H|}} + j|\theta| \underbrace{\frac{x}{|\theta|} \frac{\partial |\theta|(\omega)}{\partial x}}_{S_x^{|\theta|}}$$

$$S_x^H = S_x^{|H|} + j|\theta| S_x^{|\theta|} = x \left(\frac{N'}{N} - \frac{D'}{D} \right)$$

We see that the real part of the derived sensitivity is the sensitivity of the magnitude function.

$$S_x^{|H|} = \operatorname{Re} \left\{ x \left(\frac{N'}{N} - \frac{D'}{D} \right) \right\} \quad \text{Magnitude sensitivity}$$

The imaginary part divided by the phase (in radians) is the sensitivity function for the phase.

$$S_x^{|\theta|} = \frac{1}{|\theta|} \operatorname{Im} \left\{ x \left(\frac{N'}{N} - \frac{D'}{D} \right) \right\} \quad \text{Phase sensitivity}$$

Sensitivity analysis of Sallen-Key lowpass filter

The sensitivity to change in C_1

$$S_{C_1}^H = \frac{\omega^2 C_1 C_2 - j\omega C_1 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)}{D} = \begin{cases} 0 & \omega \rightarrow 0 \\ -1 & \omega \rightarrow \infty \end{cases}$$

The sensitivity to change in C_2

$$S_{C_2}^H = \frac{\omega^2 C_1 C_2 - j\omega \frac{C_2}{R_2} (1 - K)}{D} = \begin{cases} 0 & \omega \rightarrow 0 \\ -1 & \omega \rightarrow \infty \end{cases}$$

The sensitivity to change in R_1

$$S_{R_1}^H = \frac{\omega^2 C_1 C_2 - j\omega \left(\frac{C_1}{R_2} + \frac{C_2}{R_2} (1 - K) \right)}{D} = \begin{cases} 0 & \omega \rightarrow 0 \\ -1 & \omega \rightarrow \infty \end{cases}$$

The sensitivity to change in R_2

$$S_{R_2}^H = \frac{\omega^2 C_1 C_2 - j\omega \frac{C_1}{R_1}}{D} = \begin{cases} 0 & \omega \rightarrow 0 \\ -1 & \omega \rightarrow \infty \end{cases}$$

The sensitivity to change in K

$$S_K^H = \frac{-\omega^2 C_1 C_2 + j\omega \left(\frac{C_1}{R_2} + \frac{C_2}{R_2} + \frac{C_1}{R_1} \right) + \frac{1}{R_1 R_2}}{D} = \begin{cases} 1 & \omega \rightarrow 0 \\ 1 & \omega \rightarrow \infty \end{cases}$$

All expressions have the same denominator:

$$D = (j\omega)^2 C_1 C_2 + j\omega \left(\frac{C_1}{R_1} + \frac{C_1}{R_2} + \frac{C_2(1 - K)}{R_2} \right) + \frac{1}{R_1 R_2}$$

Sensitivity analysis of Sallen-Key lowpass filter

Implementation in Maple.

We define a numerator function and a denominator function:

Then we define a general sensitivity function to which we can pass any component symbol
($x = R_1, R_2, C_1, C_2, K$)

Then we call the function, passing C_1 , and obtains an expression for the sensitivity to C_1 . This is a function of ω and can be plotted for a range of frequencies, if we know the values of the components.

$$H := \omega \rightarrow \frac{\frac{K}{R1 \cdot R2}}{(j\omega)^2 \cdot C1 \cdot C2 + (j\omega) \cdot \left(\frac{C1}{R2} + \frac{C1}{R1} + \frac{C2 \cdot (1 - K)}{R2} \right) + \frac{1}{R1 \cdot R2}} :$$

$$Den := \omega \rightarrow (j\omega)^2 \cdot C1 \cdot C2 + (j\omega) \cdot \left(\frac{C1}{R2} + \frac{C1}{R1} + \frac{C2 \cdot (1 - K)}{R2} \right) + \frac{1}{R1 \cdot R2} :$$

$$Num := \omega \rightarrow \frac{K}{R1 \cdot R2} :$$

$$SH := (x, \omega) \rightarrow x \cdot \left(\frac{\left(\frac{\partial}{\partial x}(Num(\omega)) \right)}{Num(\omega)} - \frac{\left(\frac{\partial}{\partial x}(Den(\omega)) \right)}{Den(\omega)} \right) :$$

$$SH(C1, \omega)$$

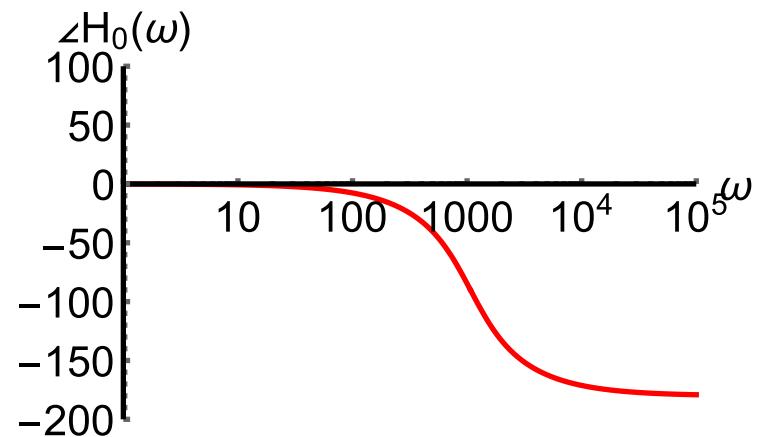
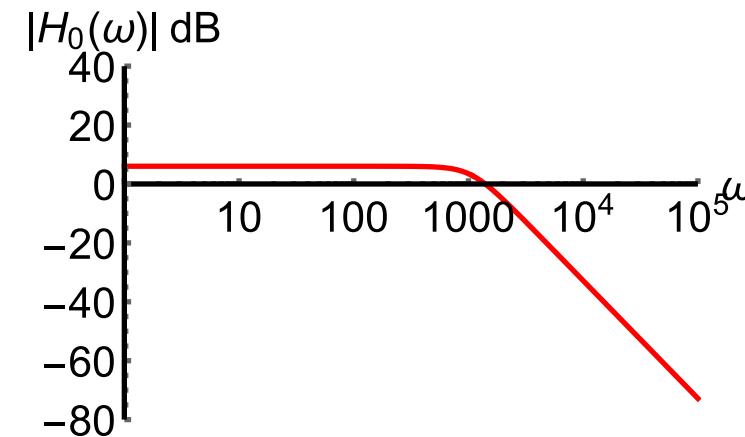
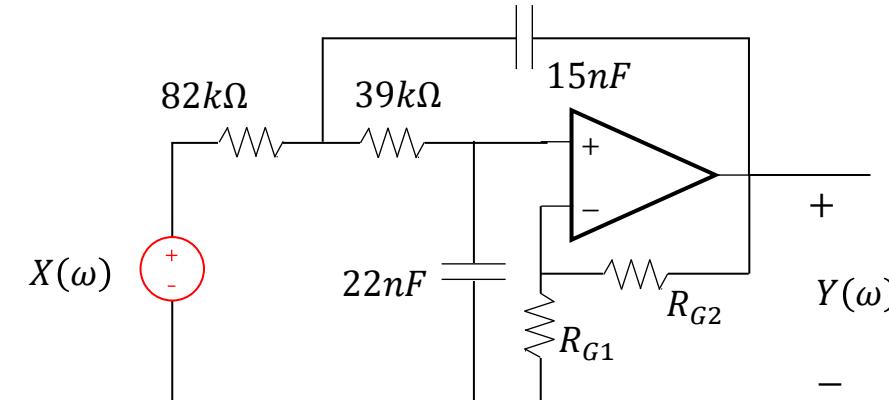
$$- \frac{C1 \left(-\omega^2 C2 + I\omega \left(\frac{1}{R2} + \frac{1}{R1} \right) \right)}{-\omega^2 C1 C2 + I\omega \left(\frac{C1}{R2} + \frac{C1}{R1} + \frac{C2 (1 - K)}{R2} \right) + \frac{1}{R1 R2}}$$

$$SHC1 := \omega \rightarrow - \frac{C1 \left(-\omega^2 C2 + I\omega \left(\frac{1}{R2} + \frac{1}{R1} \right) \right)}{-\omega^2 C1 C2 + I\omega \left(\frac{C1}{R2} + \frac{C1}{R1} + \frac{C2 (1 - K)}{R2} \right) + \frac{1}{R1 R2}} :$$

Example: 2nd order Sallen-Key lowpass filter

Example lowpass filter.

We will use these component values to analyze the sensitivity of the frequency characteristics to the components.



Sensitivity analysis of Sallen-Key lowpass filter

To plot the sensitivity of the **magnitude** we just take the **real part of $S_x^H(\omega)$** .

To plot the sensitivity of the **phase** we take the **imaginary part of $S_x^H(\omega)$** and **divide by the absolute value of the phase in radians**.

```
Sp1 := plot([Re(SHC1(ω)), Re(SHC2(ω)), Re(SHR1(ω)), Re(SHR2(ω)), Re(SHK(ω))], ω = 1 .. 4E3,
           -3 .. 3, color = ["Red", "Brown", "Blue", "Green", "Black"], thickness = 3, font = [Helvetica, roman, 18],
           axis[2] = [thickness = 2.5], axis[1] = [thickness = 2.5], labels = ["ω, rad/s", typeset(Sx^H)], labelfont
           = [Helvetica, 18], numpoints = 100, gridlines, size = [600, 200]):
```

```
Sp2 := plot([Im(SHC1(ω))/radian(H(ω)), Im(SHC2(ω))/radian(H(ω)), Im(SHR1(ω))/radian(H(ω)),
           Im(SHR2(ω))/radian(H(ω)), Im(SHK(ω))/radian(H(ω))], ω = 1 .. 4E3, -3 .. 3, color = ["Red", "Brown", "Blue", "Green", "Black"], thickness = 3,
           font = [Helvetica, roman, 18], axis[2] = [thickness = 2.5], axis[1] = [thickness = 2.5], labels
           = ["ω, rad/s", typeset(Sx^θ)], labelfont = [Helvetica, 18], numpoints = 100, gridlines, size = [600, 200]):
```

```
plotstack := Array(1 .. 2, 1 .. 1):
plotstack[1, 1] := Sp1:
plotstack[2, 1] := Sp2:
display(plotstack, aligncolumns = [1])
```

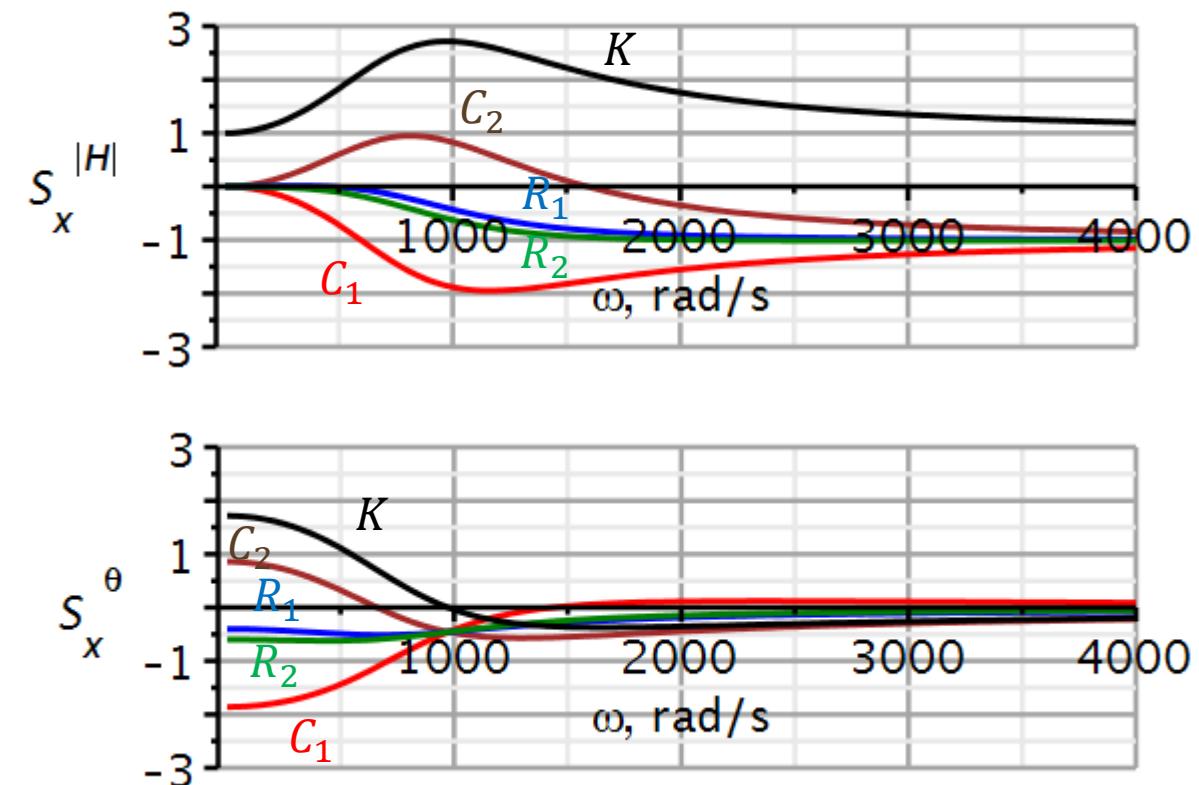
Sensitivity analysis of Sallen-Key lowpass filter

We see, that the largest sensitivity is to the gain K . A 1% increase in K will increase the magnitude by almost 3% at the 3dB cut-off frequency.

This suggest that a unity gain filter would be better since in that case there are no gain resistors that can drift.

We also notice that similar changes in C_1 and C_2 will change the amplitude in opposite directions, thus tending to cancel out. This is not the case for similar drift in the two resistors.

We confirm that the magnitude sensitivity agrees with the low- and high-frequency asymptotes derived previously.



Butterworth filters Sensitivity of Q and ω_n

Video

Sensitivity analysis of Sallen-Key lowpass filter

Now we will investigate the sensitivity of the resonance frequency ω_n and the quality factor Q .

$$H_{LP}(j\omega) = \frac{\frac{K}{R_1 R_2 C_1 C_2}}{(j\omega)^2 + j\omega \left(\frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} + \frac{(1-K)}{R_2 C_1} \right) + \frac{1}{R_1 R_2 C_1 C_2}}$$

$$H_{LP}(j\omega) = \frac{\frac{K}{\omega_n^2}}{(j\omega)^2 + j\omega \frac{\omega_n}{Q} + \omega_n^2}$$

$$\omega_n = (R_1 R_2 C_1 C_2)^{-1/2}$$

$$\frac{1}{Q} \stackrel{\text{def}}{=} 2\zeta \Rightarrow Q \stackrel{\text{def}}{=} \frac{1}{2\zeta} = \frac{1}{2} \frac{2\sqrt{a_0}}{a_1} = \frac{\sqrt{a_0}}{a_1}$$

$$Q = \frac{\sqrt{R_1 R_2 C_1 C_2}}{R_2 C_1 + R_1 C_1 + (1-K)R_1 C_2}$$

Using Maple, we can obtain sensitivity functions:

$$S_x^y \stackrel{\text{def}}{=} \frac{x}{y} \frac{\partial y}{\partial x}$$

Observation:

All components have the same impact on the resonance frequency, and the **component values are irrelevant**.

DC gain K has no impact on the resonance frequency.

The sensitivity of the Q factor is component dependent and needs further investigation.

$$S_{R_1}^{\omega_n} = -\frac{1}{2}$$

$$S_{R_2}^{\omega_n} = -\frac{1}{2}$$

$$S_{C_1}^{\omega_n} = -\frac{1}{2}$$

$$S_{C_2}^{\omega_n} = -\frac{1}{2}$$

$$S_K^{\omega_n} = 0$$

$$S_{R_1}^Q = \frac{1}{2} \cdot \frac{(K-1)R_1C_2 + C_1(R_2 - R_1)}{(K-1)R_1C_2 + C_1(R_1 + R_2)}$$

$$S_{R_2}^Q = -\frac{1}{2} \cdot \frac{(K-1)R_1C_2 + C_1(R_2 - R_1)}{(K-1)R_1C_2 + C_1(R_1 + R_2)}$$

$$S_{C_1}^Q = \frac{1}{2} \cdot \frac{(K-1)R_1C_2 + C_1(R_2 + R_1)}{(K-1)R_1C_2 - C_1(R_1 + R_2)}$$

$$S_{C_2}^Q = -\frac{1}{2} \cdot \frac{(K-1)R_1C_2 + C_1(R_2 + R_1)}{(K-1)R_1C_2 - C_1(R_1 + R_2)}$$

$$S_K^Q = \frac{K R_1 C_2}{(K-1)R_1C_2 + (R_1 + R_2)C_1}$$

Sensitivity analysis using Maple

$$\omega_n := \sqrt{\frac{1}{R1 \cdot R2 \cdot C1 \cdot C2}} :$$

$$Q := \frac{\sqrt{R1 \cdot R2 \cdot C1 \cdot C2}}{C1 \cdot (R1 + R2) + R1 \cdot C2 \cdot (1 - K)} :$$

$$S(y, x) := \frac{x}{y} \frac{\partial}{\partial x}(y) :$$

$$S(\omega_n, R1)$$

$$-\frac{1}{2}$$

$$S(\omega_n, R2)$$

$$-\frac{1}{2}$$

$$S(\omega_n, C1)$$

$$-\frac{1}{2}$$

$$S(\omega_n, C2)$$

$$-\frac{1}{2}$$

$$SQR1 := \text{simplify}(S(Q, R1))$$

$$SQR1 := \frac{(C1 + C2(1 - K))R1 - R2C1}{(-2C1 + (2K - 2)C2)R1 - 2R2C1}$$

$$SQR2 := \text{simplify}(S(Q, R2))$$

$$SQR2 := \frac{(-C1 + (K - 1)C2)R1 + R2C1}{(-2C1 + (2K - 2)C2)R1 - 2R2C1}$$

$$SQCI := \text{simplify}(S(Q, C1))$$

$$SQCI := \frac{(C1 + (K - 1)C2)R1 + R2C1}{(-2C1 + (2K - 2)C2)R1 - 2R2C1}$$

$$SQC2 := \text{simplify}(S(Q, C2))$$

$$SQC2 := \frac{(-C1 + C2(1 - K))R1 - R2C1}{(-2C1 + (2K - 2)C2)R1 - 2R2C1}$$

$$SQK := \text{simplify}(S(Q, K))$$

$$SQK := -\frac{KC2R1}{(-C1 + (K - 1)C2)R1 - R2C1}$$

From our analysis of the sensitivity of the frequency characteristics we observed that the sensitivity to gain is the largest. **Hence, we would like the gain to be unity, so that it is independent of component values.**

$$K \stackrel{\text{def}}{=} 1$$

Observations:

The sensitivities to the two resistors have opposite sign. The same is the case for the capacitors. If they drift equally, their drift cancels out.

If $R_1 \stackrel{\text{def}}{=} R_2$, then the sensitivity of Q to the resistors becomes zero.

$$S_{R_1}^Q = \frac{1}{2} \cdot \frac{(K-1)R_1C_2 + C_1(R_2 - R_1)}{(K-1)R_1C_2 + C_1(R_1 + R_2)} = \frac{1}{2} \cdot \frac{(R_2 - R_1)}{(R_1 + R_2)}$$

$$S_{R_2}^Q = -\frac{1}{2} \cdot \frac{(K-1)R_1C_2 + C_1(R_2 - R_1)}{(K-1)R_1C_2 + C_1(R_1 + R_2)} = -\frac{1}{2} \cdot \frac{(R_2 - R_1)}{(R_1 + R_2)}$$

$$S_{C_1}^Q = \frac{1}{2} \cdot \frac{(K-1)R_1C_2 + C_1(R_2 + R_1)}{(K-1)R_1C_2 - C_1(R_1 + R_2)} = -\frac{1}{2} \cdot \frac{(R_2 + R_1)}{(R_1 + R_2)} = -\frac{1}{2}$$

$$S_{C_2}^Q = -\frac{1}{2} \cdot \frac{(K-1)R_1C_2 + C_1(R_2 + R_1)}{(K-1)R_1C_2 - C_1(R_1 + R_2)} = \frac{1}{2} \cdot \frac{(R_2 + R_1)}{(R_1 + R_2)} = \frac{1}{2}$$

$$S_K^Q = \frac{KR_1C_2}{(K-1)R_1C_2 + (R_1 + R_2)C_1} = \frac{KR_1C_2}{(R_1 + R_2)C_1}$$

So, for the present filter it is a good idea to match resistor values.

$$K = 1, R_1 = R_2$$

For this particular circuit:

- Matching resistor values will nullify the sensitivity of Q to the resistors.
- Matching capacitor values will not have the same effect.

We will match resistor values, when choosing component values.

$$S_{R_1}^Q = \frac{1}{2} \cdot \frac{(R_2 - R_1)}{(R_1 + R_2)} = 0$$

$$S_{R_2}^Q = -\frac{1}{2} \cdot \frac{(R_2 - R_1)}{(R_1 + R_2)} = 0$$

$$S_{C_1}^Q = -\frac{1}{2}$$

Observation:

We are analyzing the Sallen-Key lowpass filter. Our conclusions only apply to this circuit.

$$S_{C_2}^Q = \frac{1}{2}$$

For a Sallen-Key highpass filter, our conclusions will be different.

Butterworth filters Sensitivity-optimized filter design

Video

Design for minimum sensitivity

We will use what we have learned from the sensitivity analysis to design the Sallen-Key lowpass filter.

However, we will start with **the frequency-normalized filter**. We will use a hat to indicate a component value for the frequency-normalized filter.

$$H_{LP}(j\omega) = \frac{\frac{K}{R_1 R_2 C_1 C_2}}{(j\omega)^2 + j\omega \left(\frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} + \frac{(1-K)}{R_2 C_1} \right) + \frac{1}{R_1 R_2 C_1 C_2}}$$

$$H_{LP}(j\hat{\omega}) = \frac{\frac{K}{\hat{R}_1 \hat{R}_2 \hat{C}_1 \hat{C}_2}}{(j\hat{\omega})^2 + j\hat{\omega} \left(\frac{1}{\hat{R}_1 \hat{C}_2} + \frac{1}{\hat{R}_2 \hat{C}_2} + \frac{(1-K)}{\hat{R}_2 \hat{C}_1} \right) + \frac{1}{\hat{R}_1 \hat{R}_2 \hat{C}_1 \hat{C}_2}}$$

Sensitivity analysis of Sallen-Key lowpass filter

We can use the conclusions from the sensitivity analysis as a strategy for calculating component values.

$$Q = \frac{\sqrt{\hat{R}_1 \hat{R}_2 \hat{C}_1 \hat{C}_2}}{\hat{R}_2 \hat{C}_1 + \hat{R}_1 \hat{C}_1 + (1 - K) \hat{R}_1 \hat{C}_2}$$

Setting $K = 1$ in Q :

$$Q = \frac{\sqrt{\hat{R}_1 \hat{R}_1 \hat{C}_1 \hat{C}_2}}{\hat{R}_1 \hat{C}_1 + \hat{R}_1 \hat{C}_1}$$

Setting $\hat{R}_2 \stackrel{\text{def}}{=} \hat{R}_1$ in Q :

$$Q = \frac{1}{2} \frac{\hat{R}_1 \sqrt{\hat{C}_1 \hat{C}_2}}{\hat{R}_1 \hat{C}_1}$$

Simplifying:

$$Q = \frac{1}{2} \sqrt{\frac{\hat{C}_2}{\hat{C}_1}} = \frac{\sqrt{a_0}}{a_1}$$

This gives us a way to choose capacitor values. We choose a ratio \hat{C}_2/\hat{C}_1 to match the known number on the right-hand side.

$$\frac{\hat{C}_2}{\hat{C}_1} = \frac{4a_0}{a_1^2}$$

If we limit the possible values of the capacitors to a certain series, e.g., E-6, then we can look up the best combination in a look-up table.

Sensitivity analysis of Sallen-Key lowpass filter

Inserting the theoretical ratio of the capacitors into the expressions for the resistors:

$$\frac{\hat{C}_1}{\hat{C}_2} = \frac{a_1^2}{4a_0} \quad K = 1$$

The theoretically desired ratio of capacitors agrees with matching resistors.

$$\begin{aligned}\hat{R}_1 &= \frac{a_1 - \sqrt{a_1^2 - 4a_0 \left(\frac{\hat{C}_1}{\hat{C}_2} + 1 - K \right)}}{2a_0 \left(\hat{C}_1 + \hat{C}_2(1 - K) \right)} \\ &= \frac{a_1 - \sqrt{a_1^2 - 4a_0 \frac{a_1^2}{4a_0}}}{2a_0 \hat{C}_1} = \frac{a_1}{2a_0 \hat{C}_1} \\ \hat{R}_2 &= \frac{a_1 + \sqrt{a_1^2 - 4a_0 \left(\frac{\hat{C}_1}{\hat{C}_2} + 1 - K \right)}}{2a_0 \hat{C}_1} \\ &= \frac{a_1 + \sqrt{a_1^2 - 4a_0 \frac{a_1^2}{4a_0}}}{2a_0 \hat{C}_1} = \frac{a_1}{2a_0 \hat{C}_1}\end{aligned}$$

From this result we obtain an expression for the theoretically optimal time constant involving C_1 :

$$\hat{R}_1 \hat{C}_1 = \hat{R}_2 \hat{C}_1 = \frac{a_1}{2a_0}$$

If we must choose the resistors from a given series, e.g., E-12, then we can only achieve an approximate time constant.

Design procedure for low sensitivity

Example:

$$a_0 = 1$$

$$a_1 = \sqrt{2}$$

We need to find capacitor values to obtain a specific ratio. We can search a table (next slide) for possible ratios and choose the best ratio.

Having decided on \hat{C}_1 , the product $\hat{R}_1\hat{C}_1$ must have a certain value.

The best value can be looked up in a table (2nd next slide).

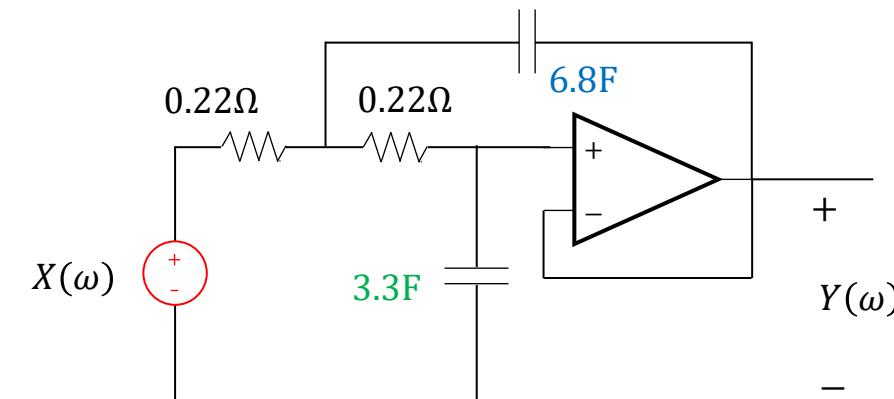
$$\frac{\hat{C}_2}{\hat{C}_1} = \frac{4a_0}{a_1^2} = \frac{4 \cdot 1}{2} = 2 \approx \frac{6.8}{3.3}$$

From a look-up table, next slide

$$\hat{R}_1 \hat{C}_1 = \hat{R}_2 \hat{C}_1 = \frac{a_1}{2a_0} = \frac{\sqrt{2}}{2 \cdot 1} = \frac{1}{\sqrt{2}} = 0.707 \Rightarrow \hat{R}_1 = \frac{0.707}{3.3} = 0.214$$

Our choice of component values for the normalized filter.

$$\hat{C}_1 = 3.3F, \hat{C}_2 = 6.8F, \hat{R}_1 = 0.22\Omega, \hat{R}_2 = 0.22\Omega$$



Design procedure for low sensitivity

$C_2 \rightarrow$

$$\frac{\hat{C}_2}{\hat{C}_1} = \frac{4a_0}{a_1^2} = \frac{4 \cdot 1}{2} = 2 \approx \frac{6.8}{3.3}$$

C2/C1	1	1,2	1,5	1,8	2,2	2,7	3,3	3,9	4,7	5,6	6,8	8,2	10	12	15	18	22	27	33	39	47	56	68	82	100
1	1,0000	1,2000	1,5000	1,8000	2,2000	2,7000	3,3000	3,9000	4,7000	5,6000	6,8000	8,2000	10,0000	12,0000	15,0000	18,0000	22,0000	27,0000	33,0000	39,0000	47,0000	56,0000	68,0000	82,0000	100,0000
1,2	0,8333	1,0000	1,2500	1,5000	1,8333	2,2500	2,7500	3,2500	3,9167	4,6667	5,6667	6,8333	8,3333	10,0000	12,5000	15,0000	18,3333	22,5000	27,5000	32,5000	39,1667	46,6667	56,6667	68,3333	83,3333
1,5	0,6667	0,8000	1,0000	1,2000	1,4667	1,8000	2,2000	2,6000	3,1333	3,7333	4,5333	5,4667	6,6667	8,0000	10,0000	12,0000	14,6667	18,0000	22,0000	26,0000	31,3333	37,3333	45,3333	54,6667	66,6667
1,8	0,5556	0,6667	0,8333	1,0000	1,2222	1,5000	1,8333	2,1667	2,6111	3,1111	3,7778	4,5556	5,5556	6,6667	8,3333	10,0000	12,2222	15,0000	18,3333	21,6667	26,1111	31,1111	37,7778	45,5556	55,5556
2,2	0,4545	0,5455	0,6818	0,8182	1,0000	1,2273	1,5000	1,7727	2,1364	2,5455	3,0909	3,7273	4,5455	5,4545	6,8182	8,1818	10,0000	12,2727	15,0000	17,7273	21,3636	25,4545	30,9091	37,2727	45,4545
2,7	0,3704	0,4444	0,5556	0,6667	0,8148	1,0000	1,2222	1,4444	1,7407	2,0741	2,5185	3,0370	3,7037	4,4444	5,5556	6,6667	8,1481	10,0000	12,2222	14,4444	17,4074	20,7407	25,1852	30,3704	37,0370
3,3	0,3030	0,3636	0,4545	0,5455	0,6667	0,8182	1,0000	1,1818	1,4242	1,6970	2,0606	2,4848	3,0303	3,6364	4,5455	5,4545	6,6667	8,1818	10,0000	11,8182	14,2424	16,9697	20,6061	24,8485	30,3030
3,9	0,2564	0,3077	0,3846	0,4615	0,5641	0,6923	0,8462	1,0000	1,2051	1,4359	1,7436	2,1026	2,5641	3,0769	3,8462	4,6154	5,6410	6,9231	8,4615	10,0000	12,0513	14,3590	17,4359	21,0256	25,6410
4,7	0,2128	0,2553	0,3191	0,3830	0,4681	0,5745	0,7021	0,8298	1,0000	1,1915	1,4468	1,7447	2,1277	2,5532	3,1915	3,8298	4,6809	5,7447	7,0213	8,2979	10,0000	11,9149	14,4681	17,4468	21,2766
5,6	0,1786	0,2143	0,2679	0,3214	0,3929	0,4821	0,5893	0,6964	0,8393	1,0000	1,2143	1,4643	1,7857	2,1429	2,6786	3,2143	3,9286	4,8214	5,8929	6,9643	8,3929	10,0000	12,1429	14,6429	17,8571
6,8	0,1471	0,1765	0,2206	0,2647	0,3235	0,3971	0,4853	0,5735	0,6912	0,8235	1,0000	1,2059	1,4706	1,7647	2,2059	2,6471	3,2353	3,9706	4,8529	5,7353	6,9118	8,2353	10,0000	12,0588	14,7059
8,2	0,1220	0,1463	0,1829	0,2195	0,2683	0,3293	0,4024	0,4756	0,5732	0,6829	0,8293	1,0000	1,2195	1,4634	1,8293	2,1951	2,6829	3,2927	4,0244	4,7561	5,7317	6,8293	8,2927	10,0000	12,1951
10	0,1000	0,1200	0,1500	0,1800	0,2200	0,2700	0,3300	0,3900	0,4700	0,5600	0,6800	0,8200	1,0000	1,2000	1,5000	1,8000	2,2000	2,7000	3,3000	3,9000	4,7000	5,6000	6,8000	8,2000	10,0000
12	0,0833	0,1000	0,1250	0,1500	0,1833	0,2250	0,2750	0,3250	0,3917	0,4667	0,5667	0,6833	0,8333	1,0000	1,2500	1,5000	1,8333	2,2500	2,7500	3,2500	3,9167	4,6667	5,6667	6,8333	8,3333
15	0,0667	0,0800	0,1000	0,1200	0,1467	0,1800	0,2200	0,2600	0,3133	0,3733	0,4533	0,5467	0,6667	0,8000	1,0000	1,2000	1,4667	1,8000	2,2000	2,6000	3,1333	3,7333	4,5333	5,4667	6,6667
18	0,0556	0,0667	0,0833	0,1000	0,1222	0,1500	0,1833	0,2167	0,2611	0,3111	0,3778	0,4556	0,5556	0,6667	0,8333	1,0000	1,2222	1,5000	1,8333	2,1667	2,6111	3,1111	3,7778	4,5556	5,5556
22	0,0455	0,0545	0,0682	0,0818	0,1000	0,1227	0,1500	0,1773	0,2136	0,2545	0,3091	0,3727	0,4545	0,5455	0,6818	0,8182	1,0000	1,2273	1,5000	1,7727	2,1364	2,5455	3,0909	3,7273	4,5455
27	0,0370	0,0444	0,0556	0,0667	0,0815	0,1000	0,1222	0,1444	0,1741	0,2074	0,2519	0,3037	0,3704	0,4444	0,5556	0,6667	0,8148	1,0000	1,2222	1,4444	1,7407	2,0741	2,5185	3,0370	3,7037
33	0,0303	0,0364	0,0455	0,0545	0,0667	0,0818	0,1000	0,1182	0,1424	0,1697	0,2061	0,2485	0,3030	0,3636	0,4545	0,5455	0,6667	0,8182	1,0000	1,1818	1,4242	1,6970	2,0606	2,4848	3,0303
39	0,0256	0,0308	0,0385	0,0462	0,0564	0,0692	0,0846	0,1000	0,1205	0,1436	0,1744	0,2103	0,2564	0,3077	0,3846	0,4615	0,5641	0,6923	0,8462	1,0000	1,2051	1,4359	1,7436	2,1026	2,5641
47	0,0213	0,0255	0,0319	0,0383	0,0468	0,0574	0,0702	0,0830	0,1000	0,1191	0,1447	0,1745	0,2128	0,2553	0,3191	0,3830	0,4681	0,5745	0,7021	0,8298	1,0000	1,1915	1,4468	1,7447	2,1277
56	0,0179	0,0214	0,0268	0,0321	0,0393	0,0482	0,0589	0,0696	0,0839	0,1000	0,1214	0,1464	0,1786	0,2143	0,2679	0,3214	0,3929	0,4821	0,5893	0,6964	0,8393	1,0000	1,2143	1,4643	1,7857
68	0,0147	0,0176	0,0221	0,0265	0,0324	0,0397	0,0485	0,0574	0,0691	0,0824	0,1000	0,1206	0,1471	0,1765	0,2206	0,2647	0,3235	0,3971	0,4853	0,5735	0,6912	0,8235	1,0000	1,2059	1,4706
82	0,0122	0,0146	0,0183	0,0220	0,0268	0,0329	0,0402	0,0476	0,0573	0,0683	0,0829	0,1000	0,1220	0,1463	0,1829	0,2195	0,2683	0,3293	0,4024	0,4756	0,5732	0,6829	0,8293	1,0000	1,2195
100	0,0100	0,0120	0,0150	0,0180	0,0220	0,0270	0,0330	0,0390	0,0470	0,0560	0,0680	0,0820	0,1000	0,1200	0,1500	0,1800	0,2200	0,2700	0,3300	0,3900	0,4700	0,5600	0,6800	0,8200	1,0000

Design procedure for low sensitivity

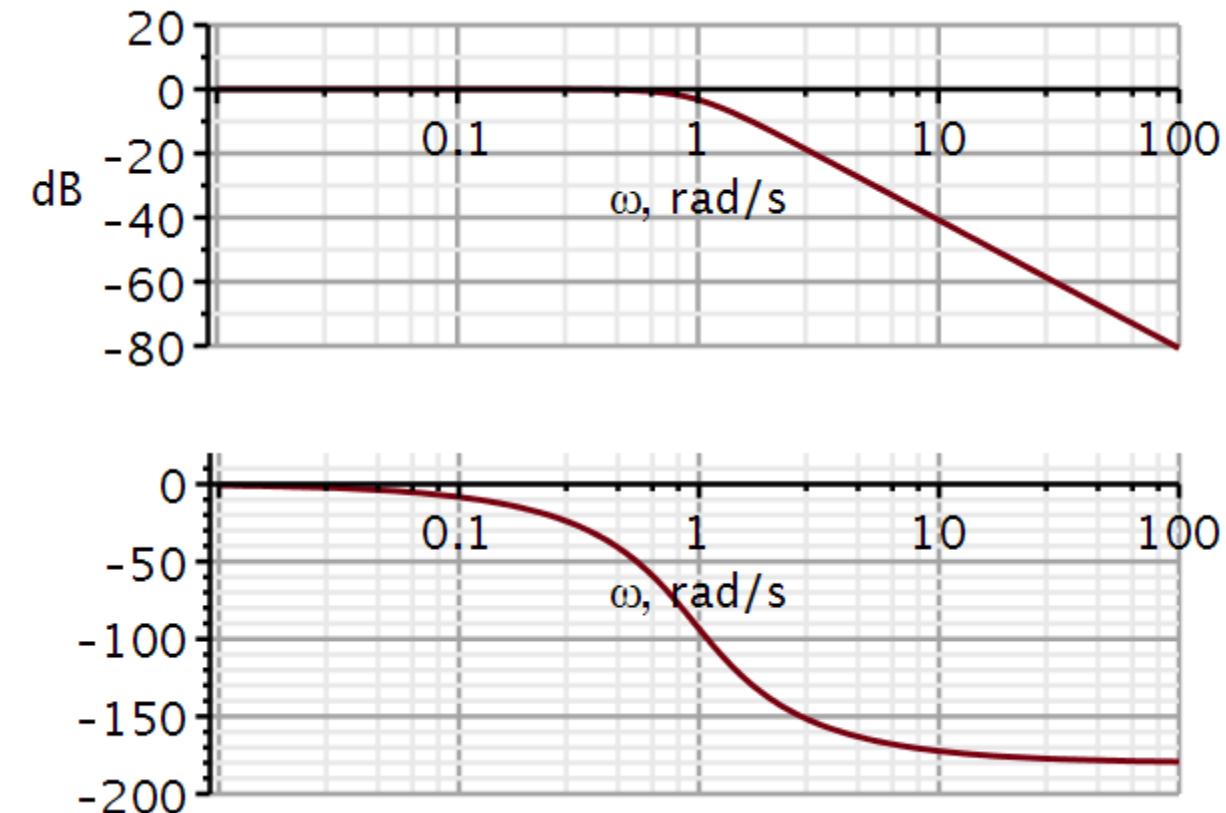
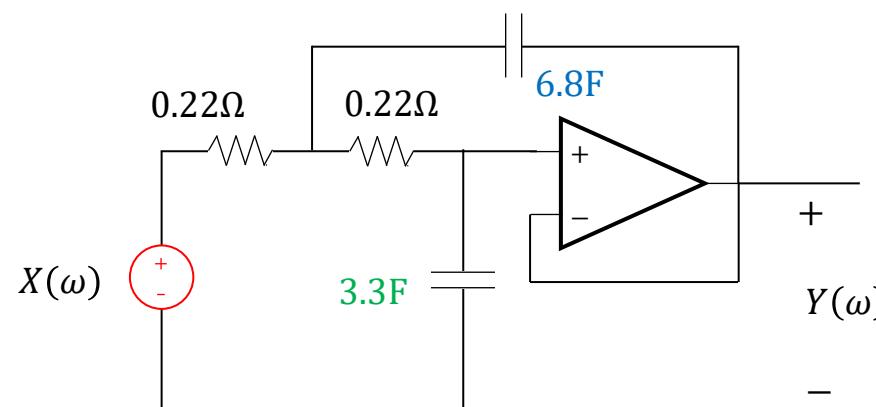
$$\hat{R}_1 \cdot \hat{C}_1 = \frac{a_1}{2a_0} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} = 0.7071$$



R1*C1	0,1	0,12	0,15	0,18	0,22	0,27	0,33	0,39	0,47	0,56	0,68	0,82	1	1,2	1,5	1,8	2,2	2,7	3,3	3,9	4,7	5,6	6,8	8,2	10
0,1	0,010	0,012	0,015	0,018	0,022	0,027	0,033	0,039	0,047	0,056	0,068	0,082	0,100	0,120	0,150	0,180	0,220	0,270	0,330	0,390	0,470	0,560	0,680	0,820	1,000
0,12	0,012	0,014	0,018	0,022	0,026	0,032	0,040	0,047	0,056	0,067	0,082	0,098	0,120	0,144	0,180	0,216	0,264	0,324	0,396	0,468	0,564	0,672	0,816	0,984	1,200
0,15	0,015	0,018	0,023	0,027	0,033	0,041	0,050	0,059	0,071	0,084	0,102	0,123	0,150	0,180	0,225	0,270	0,330	0,405	0,495	0,585	0,705	0,840	1,020	1,230	1,500
0,18	0,018	0,022	0,027	0,032	0,040	0,049	0,059	0,070	0,085	0,101	0,122	0,148	0,180	0,216	0,270	0,324	0,396	0,486	0,594	0,702	0,846	1,008	1,224	1,476	1,800
0,22	0,022	0,026	0,033	0,040	0,048	0,059	0,073	0,086	0,103	0,123	0,150	0,180	0,220	0,264	0,330	0,396	0,484	0,594	0,726	0,858	1,034	1,232	1,496	1,804	2,200
0,27	0,027	0,032	0,041	0,049	0,059	0,073	0,089	0,105	0,127	0,151	0,184	0,221	0,270	0,324	0,405	0,486	0,594	0,729	0,891	1,053	1,269	1,512	1,836	2,214	2,700
0,33	0,033	0,040	0,050	0,059	0,073	0,089	0,109	0,129	0,155	0,185	0,224	0,271	0,330	0,396	0,495	0,594	0,726	0,891	1,089	1,287	1,551	1,848	2,244	2,706	3,300
0,39	0,039	0,047	0,059	0,070	0,086	0,105	0,129	0,152	0,183	0,218	0,265	0,320	0,390	0,468	0,585	0,702	0,858	1,053	1,287	1,521	1,833	2,184	2,652	3,198	3,900
0,47	0,047	0,056	0,071	0,085	0,103	0,127	0,155	0,183	0,221	0,263	0,320	0,385	0,470	0,564	0,705	0,846	1,034	1,269	1,551	1,833	2,209	2,632	3,196	3,854	4,700
0,56	0,056	0,067	0,084	0,101	0,123	0,151	0,185	0,218	0,263	0,314	0,381	0,459	0,560	0,672	0,840	1,008	1,232	1,512	1,848	2,184	2,632	3,136	3,808	4,592	5,600
0,68	0,068	0,082	0,102	0,122	0,150	0,184	0,224	0,265	0,320	0,381	0,462	0,558	0,680	0,816	1,020	1,224	1,496	1,836	2,244	2,652	3,196	3,808	4,624	5,576	6,800
0,82	0,082	0,098	0,123	0,148	0,180	0,221	0,271	0,320	0,385	0,459	0,558	0,672	0,820	0,984	1,230	1,476	1,804	2,214	2,706	3,198	3,854	4,592	5,576	6,724	8,200
1	0,100	0,120	0,150	0,180	0,220	0,270	0,330	0,390	0,470	0,560	0,680	0,820	1,000	1,200	1,500	1,800	2,200	2,700	3,300	3,900	4,700	5,600	6,800	8,200	10,000
1,2	0,120	0,144	0,180	0,216	0,264	0,324	0,396	0,468	0,564	0,672	0,816	0,984	1,200	1,440	1,800	2,160	2,640	3,240	3,960	4,680	5,640	6,720	8,160	9,840	12,000
1,5	0,150	0,180	0,225	0,270	0,330	0,405	0,495	0,585	0,705	0,840	1,020	1,230	1,500	1,800	2,250	2,700	3,300	4,050	4,950	5,850	7,050	8,400	10,200	12,300	15,000
1,8	0,180	0,216	0,270	0,324	0,396	0,486	0,594	0,702	0,846	1,008	1,224	1,476	1,800	2,160	2,700	3,240	3,960	4,860	5,940	7,020	8,460	10,080	12,240	14,760	18,000
2,2	0,220	0,264	0,330	0,396	0,484	0,594	0,726	0,858	1,034	1,232	1,496	1,804	2,200	2,640	3,300	3,960	4,840	5,940	7,260	8,580	10,340	12,320	14,960	18,040	22,000
2,7	0,270	0,324	0,405	0,486	0,594	0,729	0,891	1,053	1,269	1,512	1,836	2,214	2,700	3,240	4,050	4,860	5,940	7,290	8,910	10,530	12,690	15,120	18,360	22,140	27,000
3,3	0,330	0,396	0,495	0,594	0,726	0,891	1,089	1,287	1,551	1,848	2,244	2,706	3,300	3,960	4,950	5,940	7,260	8,910	10,890	12,870	15,510	18,480	22,440	27,060	33,000
3,9	0,390	0,468	0,585	0,702	0,858	1,053	1,287	1,521	1,833	2,184	2,652	3,198	3,900	4,680	5,850	7,020	8,580	10,530	12,870	15,210	18,330	21,840	26,520	31,980	39,000
4,7	0,470	0,564	0,705	0,846	1,034	1,269	1,551	1,833	2,209	2,632	3,196	3,854	4,700	5,640	7,050	8,460	10,340	12,690	15,510	18,330	22,090	26,320	31,960	38,540	47,000
5,6	0,560	0,672	0,840	1,008	1,232	1,512	1,848	2,184	2,632	3,136	3,808	4,592	5,600	6,720	8,400	10,080	12,320	15,120	18,480	21,840	26,320	31,360	38,080	45,920	56,000
6,8	0,680	0,816	1,020	1,224	1,496	1,836	2,244	2,652	3,196	3,808	4,624	5,576	6,800	8,160	10,200	12,240	14,960	18,360	22,440	26,520	31,960	38,080	46,240	55,760	68,000
8,2	0,820	0,984	1,230	1,476	1,804	2,214	2,706	3,198	3,854	4,592	5,576	6,724	8,200	9,840	12,300	14,760	18,040	22,140	27,060	31,980	38,540	45,920	55,760	67,240	82,000
10	1,000	1,200	1,500	1,800	2,200	2,700	3,300	3,900	4,700	5,600	6,800	8,200	10,000	12,000	15,000	18,000	22,000	27,000	33,000	39,000	47,000	56,000	68,000	82,000	100,000

Design procedure for low sensitivity

$$\hat{C}_1 = 3.3F, \hat{C}_2 = 6.8F, \hat{R}_1 = 0.22\Omega, \hat{R}_2 = 0.22\Omega$$



The normalized filter looks agreeable. Next, we need to scale the cut-off frequency to 1000 rad/s.

The frequency-normalized filter:

Multiplying in the numerator
and denominator by a
frequency scaling factor K_F^2 :

Substituting new variables:

$$\omega \stackrel{\text{def}}{=} K_F \hat{\omega} \quad C' \stackrel{\text{def}}{=} \frac{C}{K_F} \quad R' \stackrel{\text{def}}{=} \hat{R}$$

We multiply the frequency by the same factor as we divide the capacitor. Hence the impedance of the capacitor is unchanged after frequency scaling.

$$H_{LP}(j\hat{\omega}) = \frac{\frac{K}{\hat{R}_1 \hat{R}_2 \hat{C}_1 \hat{C}_2}}{(j\hat{\omega})^2 + j\hat{\omega} \left(\frac{1}{\hat{R}_1 \hat{C}_2} + \frac{1}{\hat{R}_2 \hat{C}_2} + \frac{(1-K)}{\hat{R}_2 \hat{C}_1} \right) + \frac{1}{\hat{R}_1 \hat{R}_2 \hat{C}_1 \hat{C}_2}}$$

$$H_{LP}(j\hat{\omega}) = \frac{\frac{K \cdot K_F^2}{\hat{R}_1 \hat{R}_2 \hat{C}_1 \hat{C}_2}}{(jK_F \hat{\omega})^2 + jK_F \hat{\omega} \left(\frac{K_F}{\hat{R}_1 \hat{C}_2} + \frac{K_F}{\hat{R}_2 \hat{C}_2} + \frac{K_F(1-K)}{\hat{R}_2 \hat{C}_1} \right) + \frac{K_F^2}{\hat{R}_1 \hat{R}_2 \hat{C}_1 \hat{C}_2}}$$

$$H_{LP}(j\omega) = \frac{\frac{K}{R'_1 R'_2 C'_1 C'_2}}{(j\omega)^2 + j\omega \left(\frac{1}{R'_1 C'_2} + \frac{1}{R'_2 C'_2} + \frac{(1-K)}{R'_2 C'_1} \right) + \frac{1}{R'_1 R'_2 C'_1 C'_2}}$$

$$\omega C' = \hat{\omega} \hat{C} \Rightarrow \frac{1}{j\omega C'} = \frac{1}{j\hat{\omega} \hat{C}} \Rightarrow \frac{1}{j\omega_c C'} = \frac{1}{j\hat{\omega}_c \hat{C}}$$

Frequency scaling

Now let us move the cut-off frequency to 1000 rad/s.

The frequency scaling needed:

$$K_F = \frac{\omega_c}{\hat{\omega}_c} = \frac{10^3 \text{ rad/s}}{1 \text{ rad/s}} = 1000$$

The frequency scaling of capacitors:

$$C' = \frac{\hat{C}}{K_F}$$

$$C'_1 = \frac{3.3 \text{ F}}{1000} = 3.3 \text{ mF}$$

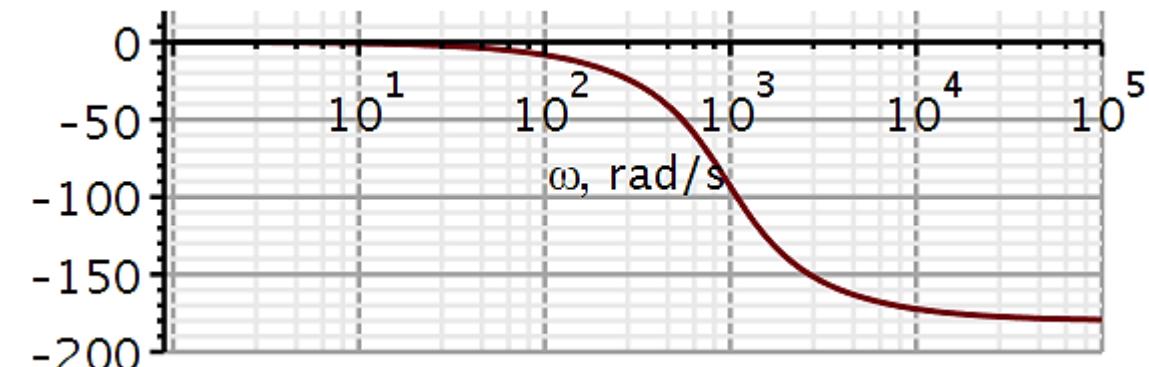
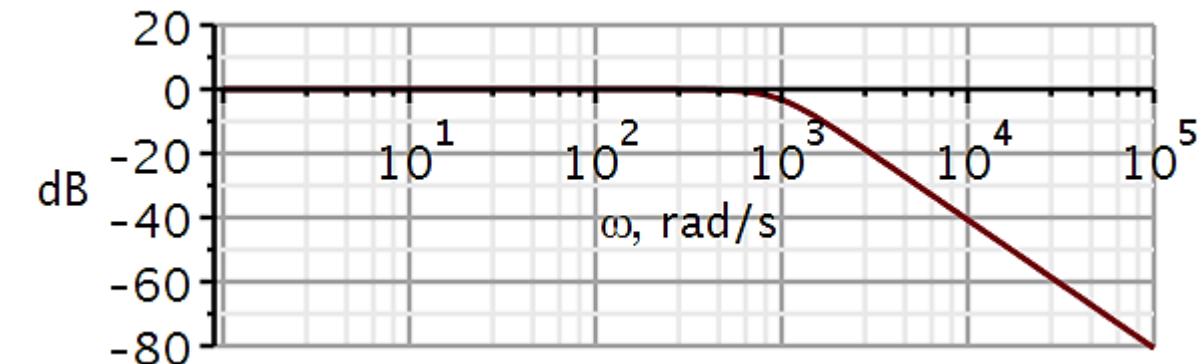
$$C'_2 = 6.8 \text{ mF}$$

Resistors are not frequency scaled.

$$R' = \hat{R}$$

Plotting the frequency characteristics after frequency scaling confirms that the cut-off frequency has moved to the desired value, but nothing else has changed.

Frequency scaling:



Impedance scaling

The capacitor values are far too big. We would like them to be in the nF range.

We are free to scale the components, as long as we do not change the time constants. Hence, we can multiply the resistors with a scale factor, provided we divide the capacitors by the same scale factor.

$$R \stackrel{\text{def}}{=} K_z R' \quad C \stackrel{\text{def}}{=} \frac{C'}{K_z}$$

$$\tau = RC = R'C' = \tau'$$

$$H_{LP}(j\omega) = \frac{\frac{K}{R'_1 R'_2 C'_1 C'_2}}{(j\omega)^2 + j\omega \left(\frac{1}{R'_1 C'_2} + \frac{1}{R'_2 C'_2} + \frac{(1-K)}{R'_2 C'_1} \right) + \frac{1}{R'_1 R'_2 C'_1 C'_2}}$$

$$H_{LP}(j\omega) = \frac{\frac{K}{K_z R'_1 K_z R'_2 \frac{C'_1 C'_2}{K_z K_z}}}{(j\omega)^2 + j\omega \left(\frac{1}{K_z R'_1 \frac{C'_2}{K_z}} + \frac{1}{K_z R'_2 \frac{C'_2}{K_z}} + \frac{(1-K)}{K_z R'_2 \frac{C'_1}{K_z}} \right) + \frac{1}{K_z R'_1 K_z R'_2 \frac{C'_1 C'_2}{K_z K_z}}}$$

$$H_{LP}(j\omega) = \frac{\frac{K}{R_1 R_2 C_1 C_2}}{(j\omega)^2 + j\omega \left(\frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} + \frac{(1-K)}{R_2 C_1} \right) + \frac{1}{R_1 R_2 C_1 C_2}}$$

Design procedure for low sensitivity

Impedance scaling

We choose an impedance scaling which preserves the E6/E12 values.

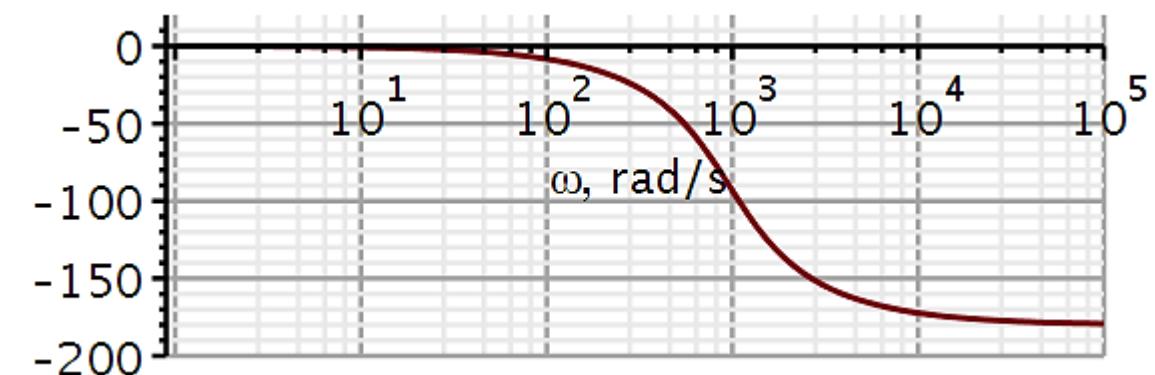
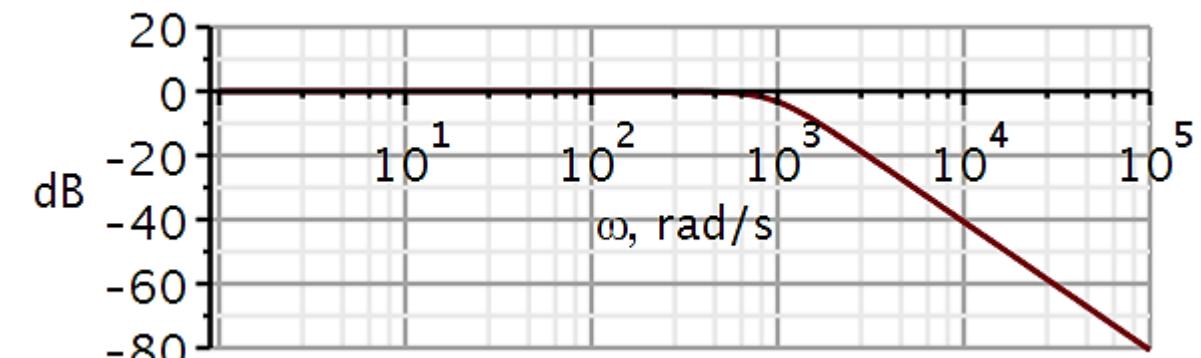
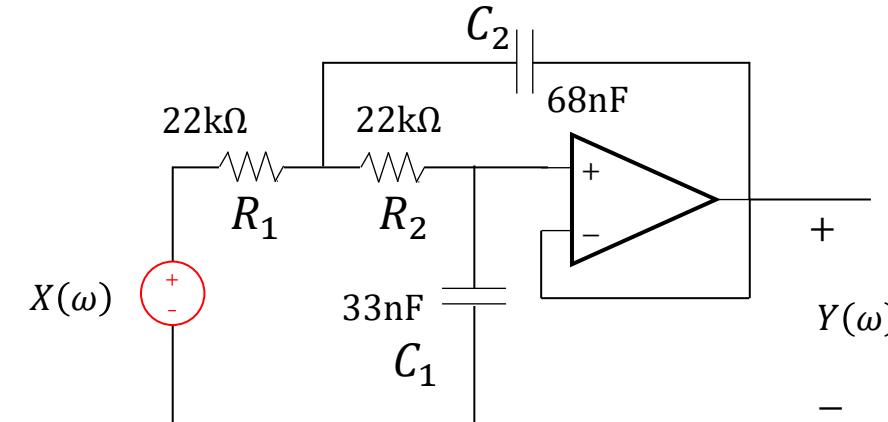
$$C_1 \stackrel{\text{def}}{=} 33\text{nF}$$

$$K_z = \frac{C'_1}{C_1} = \frac{3.3\text{mF}}{33\text{nF}} = 10^5$$

$$C_2 = \frac{C'_2}{K_z} = \frac{6.8\text{mF}}{10^5} = 68\text{nF}$$

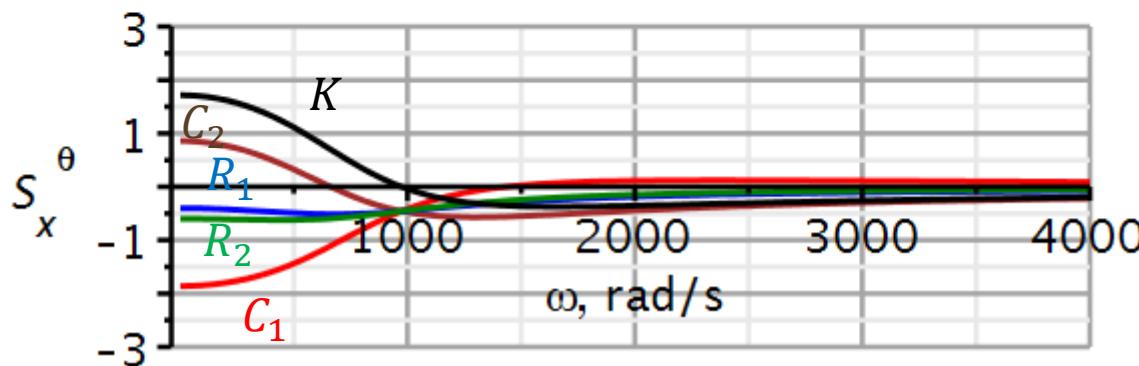
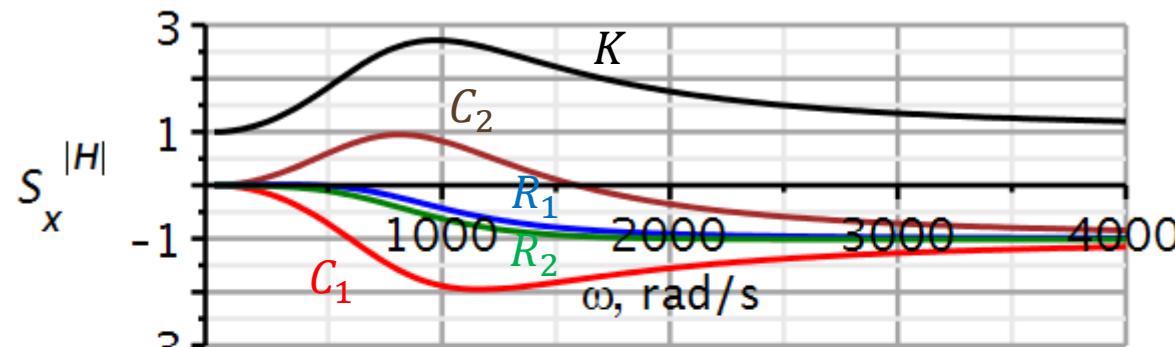
$$R_1 = K_z \cdot R'_1 = 10^5 \cdot 0.22\Omega = 22\text{k}\Omega$$

$$R_2 = K_z \cdot R'_2 = 10^5 \cdot 0.22\Omega = 22\text{k}\Omega$$

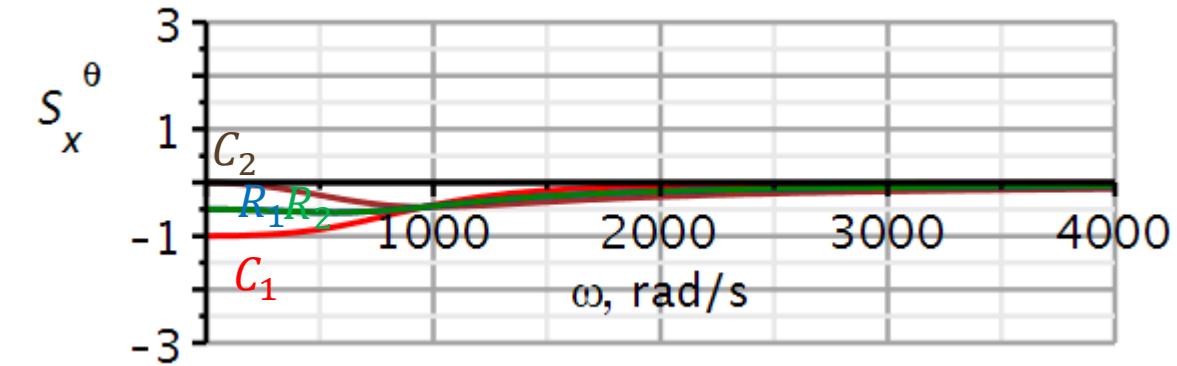
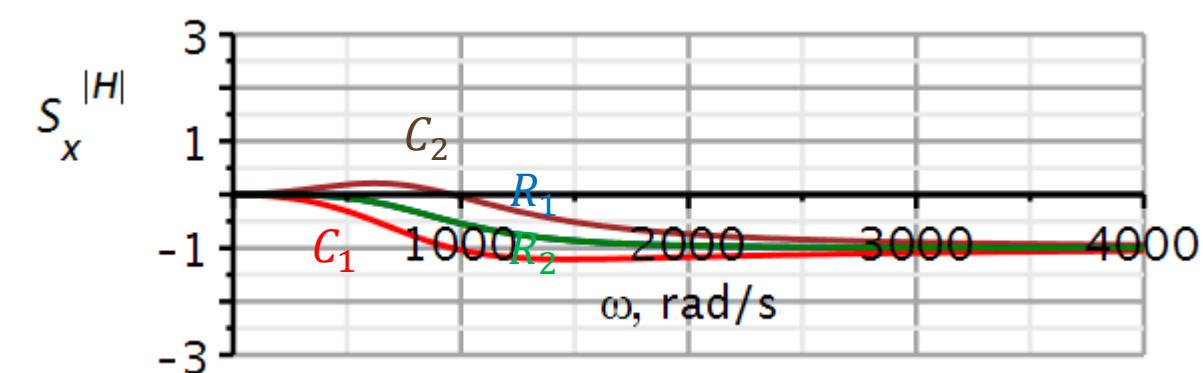


Design procedure for low sensitivity

Sensitivity for original design



Sensitivity for improved design

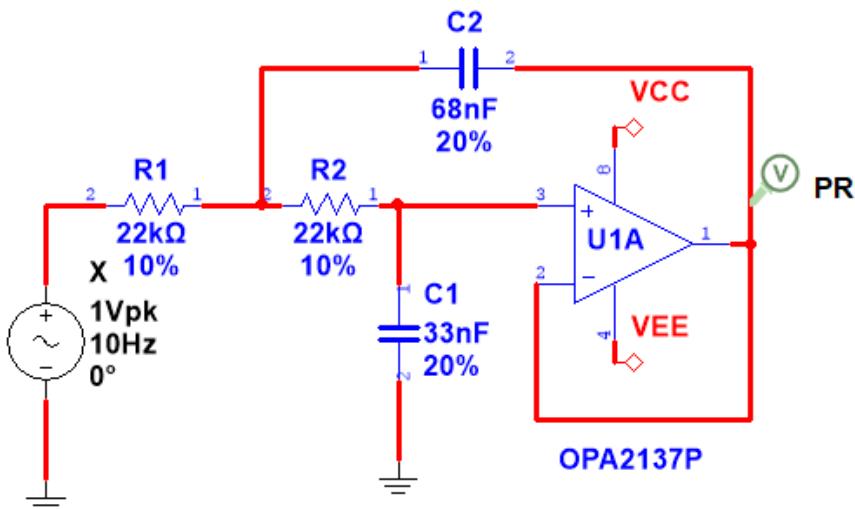


Our new design procedure has reduced sensitivities significantly.

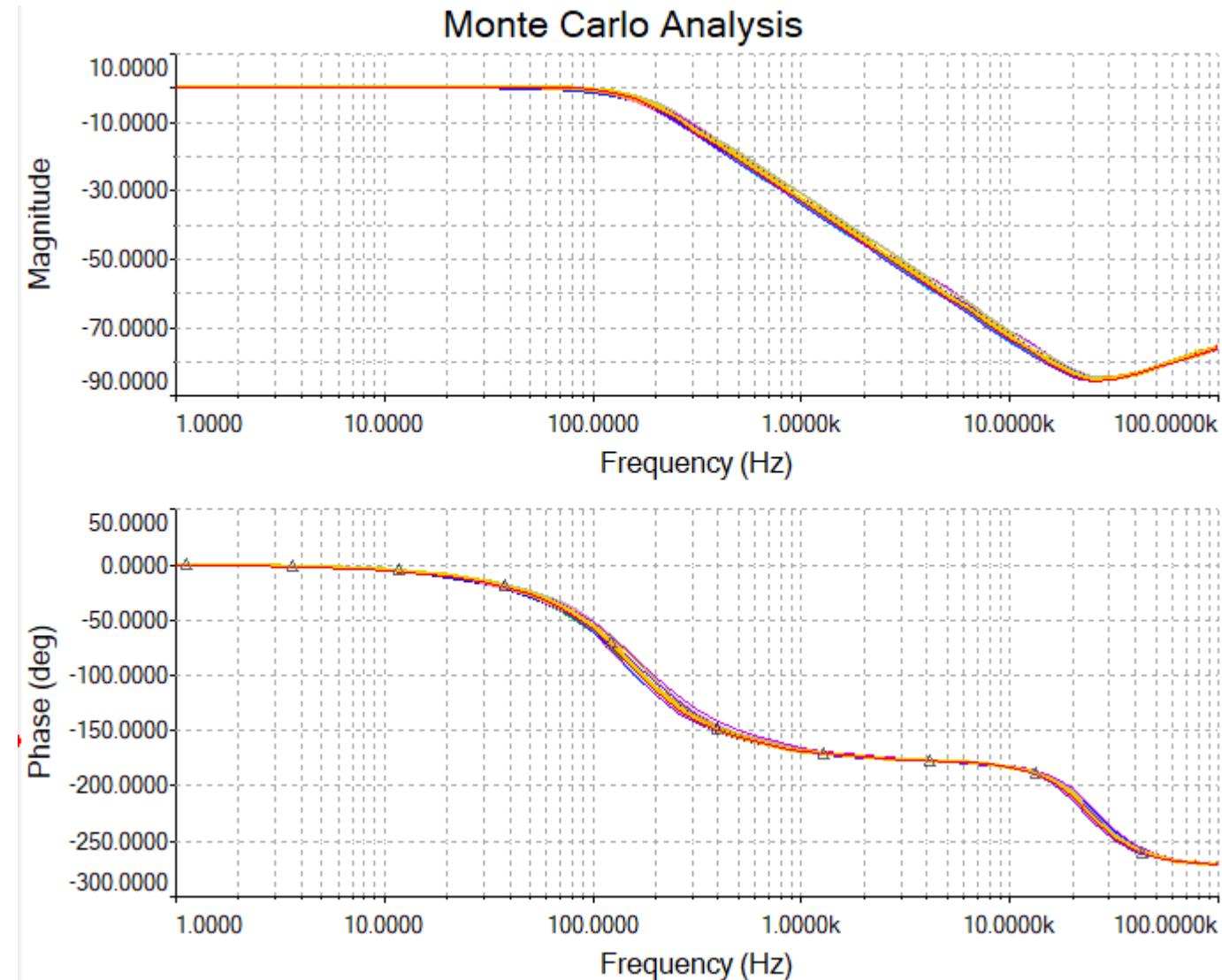
Monte Carlo simulation of low sensitivity filter

Not available in KiCad

Monte Carlo simulations are used in some circuit simulators to visualize the sensitivity to component change. The simulator changes component values using a specified probability distribution and component range.



It gives the **most likely variation**, not the worst-case variation.

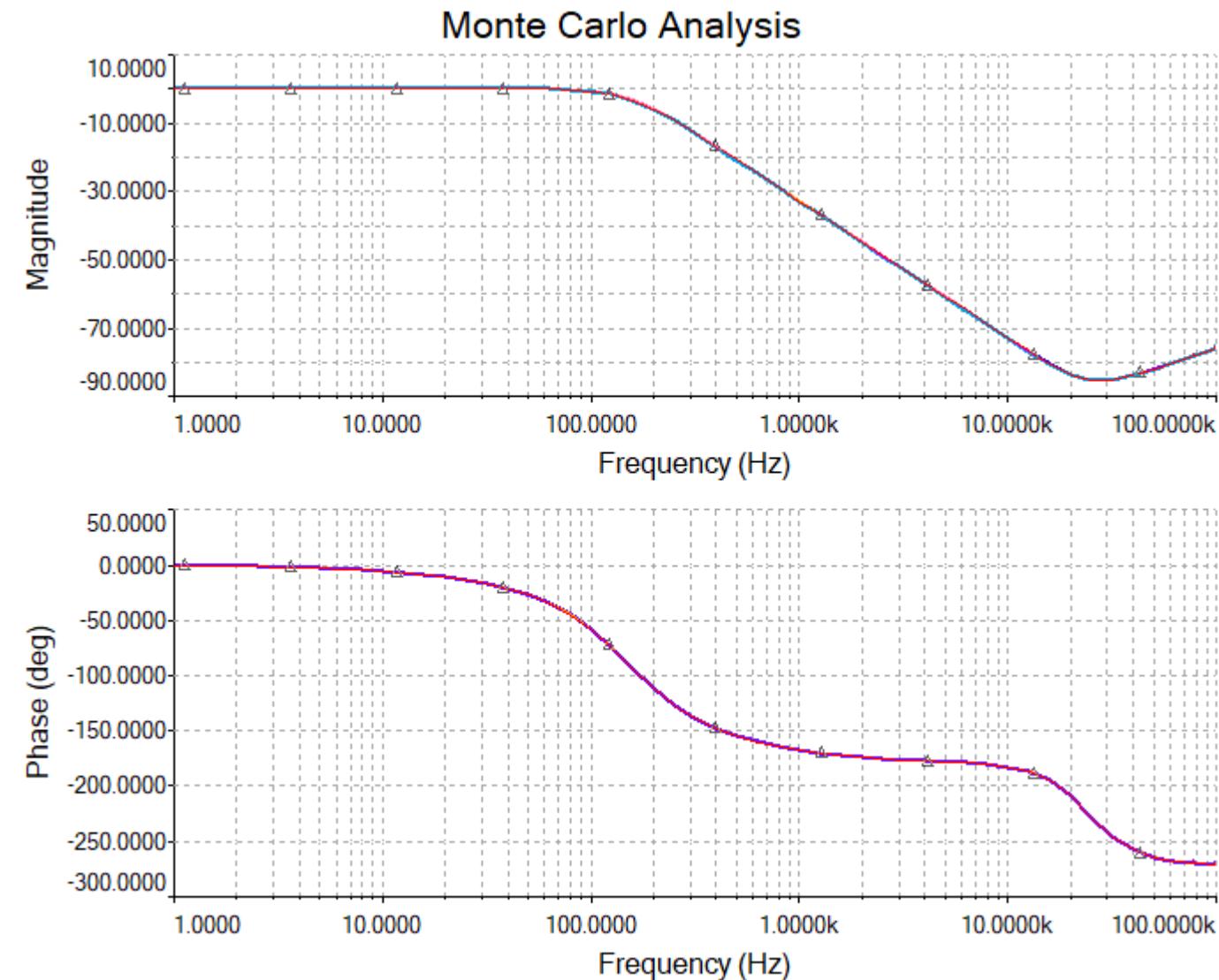
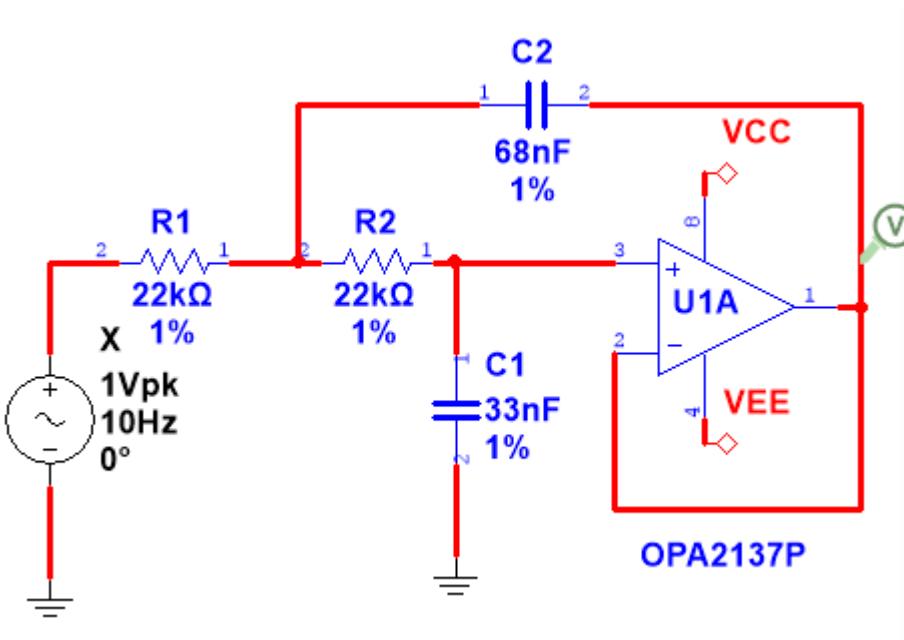


Monte Carlo simulation of low sensitivity filter

Not available in KiCad

Lowering the tolerance on the components, lowers the variability.

By lowering the tolerance on components one by one, it is possible to single out the components which benefits the most from low tolerance.



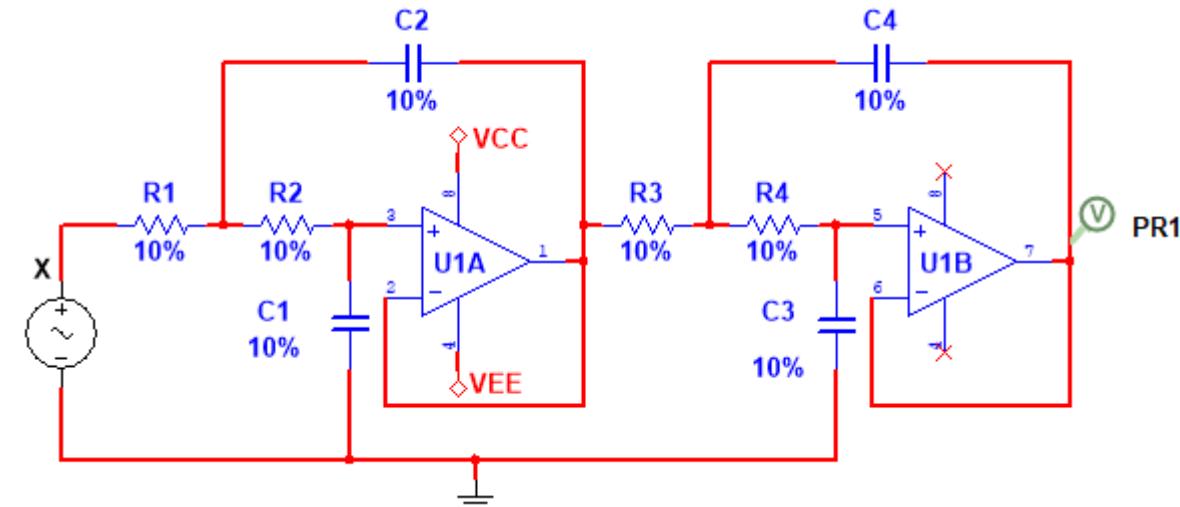
Problems

Problem 1 – EMG lowpass filter

Design a 4th order Butterworth lowpass filter with unity gain and a cut-off frequency of 500 Hz.

- Use two Sallen-Key stages in cascade. The filter should be very insensitive to component drift.
- Draw the amplitude and phase spectra in Maple/Python for the frequency-normalized filter as well as for the final design. Draw the spectra for each stage and for the combination.
- Use resistors from the E12 series and capacitors from the E6 series as much as possible, but not at all costs.

Stage 1



Stage 2

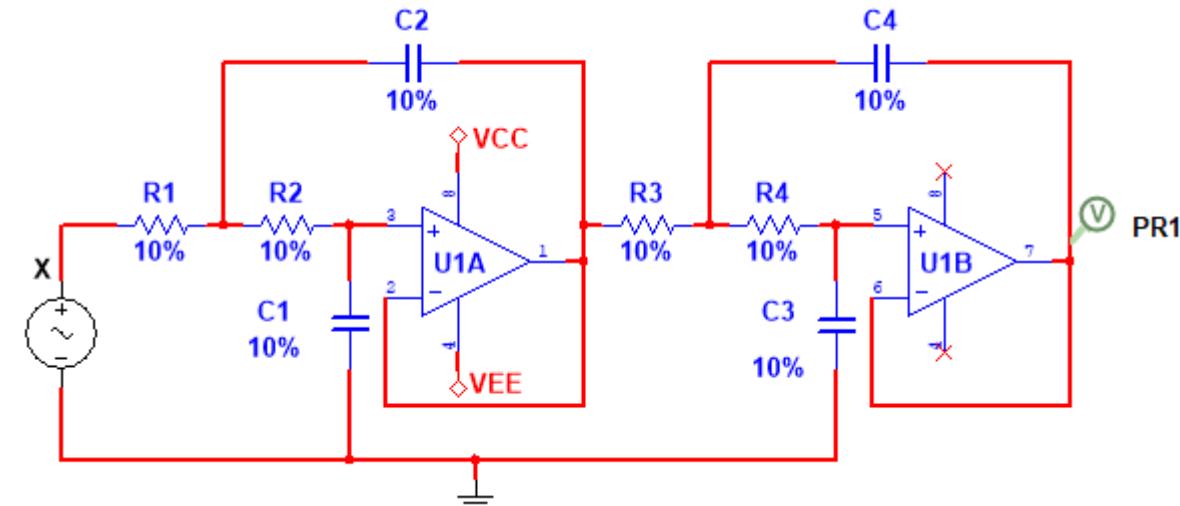
Solutions

Problem 1 – EMG lowpass filter

Design a 4th order Butterworth lowpass filter with unity gain and a cut-off frequency of 500 Hz.

- Use two Sallen-Key stages in cascade. The filter should be very insensitive to component drift.
- Draw the amplitude and phase spectra in Maple for the frequency-normalized filter as well as for the final design. Draw the spectra for each stage and for the combination.
- Use resistors from the E12 series and capacitors from the E6 series as much as possible, but not at all costs.

Stage 1



Stage 2

Problem 1 – EMG lowpass filter (sol)

Stage 1:

Looking up in the table of Butterworth coefficients:

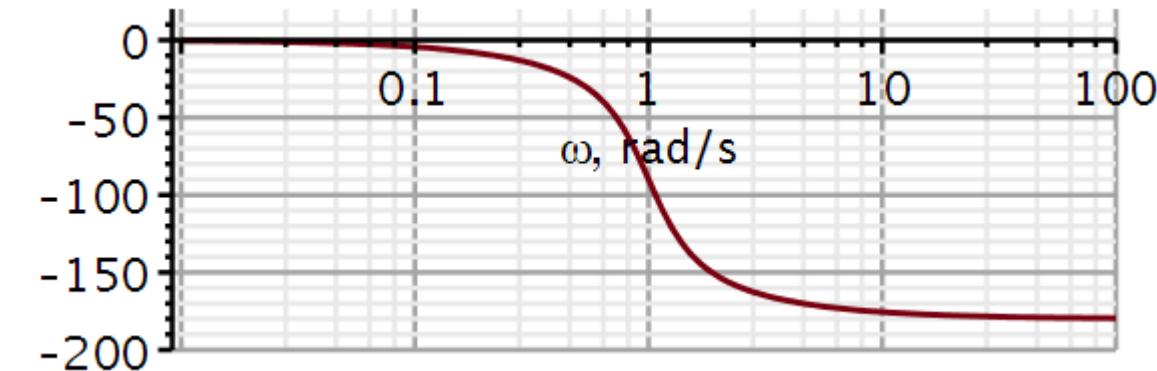
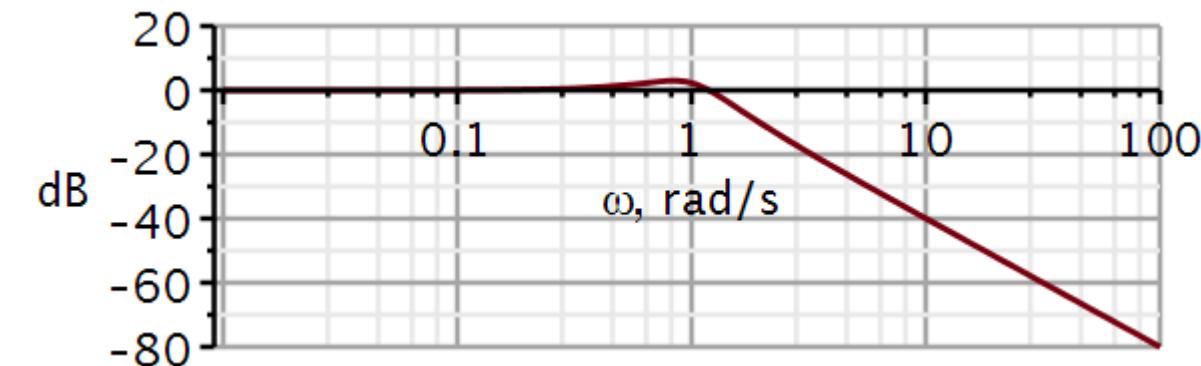
$$a_1 = 0.765367$$

$$a_0 = 1$$

Using the guidelines from sensitivity analysis:

$$\hat{C}_2 = \frac{4a_0}{a_1^2} = \frac{4}{(0.765367)^2} = 6.82842 \approx \frac{56F}{8.2F} = 6.82927$$

$$\hat{R}_1 = \hat{R}_2 = \frac{a_1}{2a_0\hat{C}_1} = 0.0466687\Omega$$



Problem 1 – EMG lowpass filter (sol)

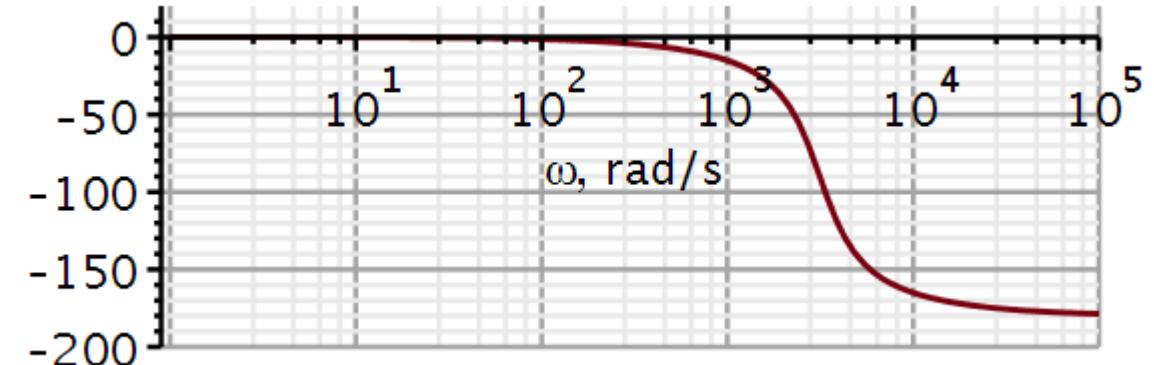
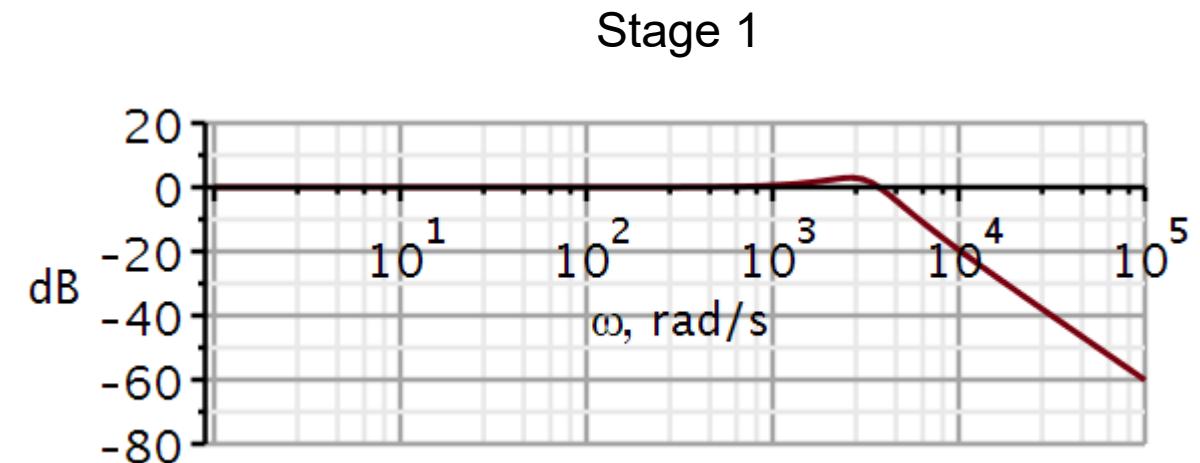
Frequency scaling

$$K_F = 2\pi \cdot 500 = 3141.592$$

$$C'_1 = \frac{\hat{C}_1}{K_F} = \frac{8.2\text{F}}{3141.592} = 2.610\text{mF}$$

$$C'_2 = \frac{\hat{C}_2}{K_F} = \frac{56\text{F}}{3141.592} = 17.83\text{mF}$$

$$R'_1 = R'_2 = \hat{R}_1 = \hat{R}_2 = 0.0466687\Omega$$



Problem 1 – EMG lowpass filter (sol)

In our first attempt at Impedance scaling, we seek to maintain the ratio of 56/8.2 for the capacitors.

Hence, we determine an impedance scaling factor, such that $C_1 = 8.2\text{nF}$ and $C_2 = 56\text{nF}$.

The resistors then scale to $15\text{k}\Omega$.

This works well, but 8.2nF is not in the E-6 series. On the next slide we seek an E-6 value.

$$C'_1 = 2.610\text{mF}$$

$$C'_2 = 17.83\text{mF}$$

$$R'_1 = R'_2 = 0.0466687\Omega$$

$$K_z = \frac{C'_1}{8.2 \cdot 10^{-9}\text{F}} = \frac{2.610\text{mF}}{8.2 \cdot 10^{-9}\text{F}} = 318292$$

$$C_2 = \frac{C'_2}{K_z} = \frac{17.83\text{mF}}{318292} = 56\text{nF}$$

$$R_1 = R_2 = 318292 \times 0.0466687\Omega = 14.85\text{k}\Omega \approx 15\text{k}\Omega$$

Problem 1 – EMG lowpass filter (sol)

Impedance scaling.

After frequency scaling, we have this product:

$$R'_1 C'_1 = 1.218 \cdot 10^{-4}$$

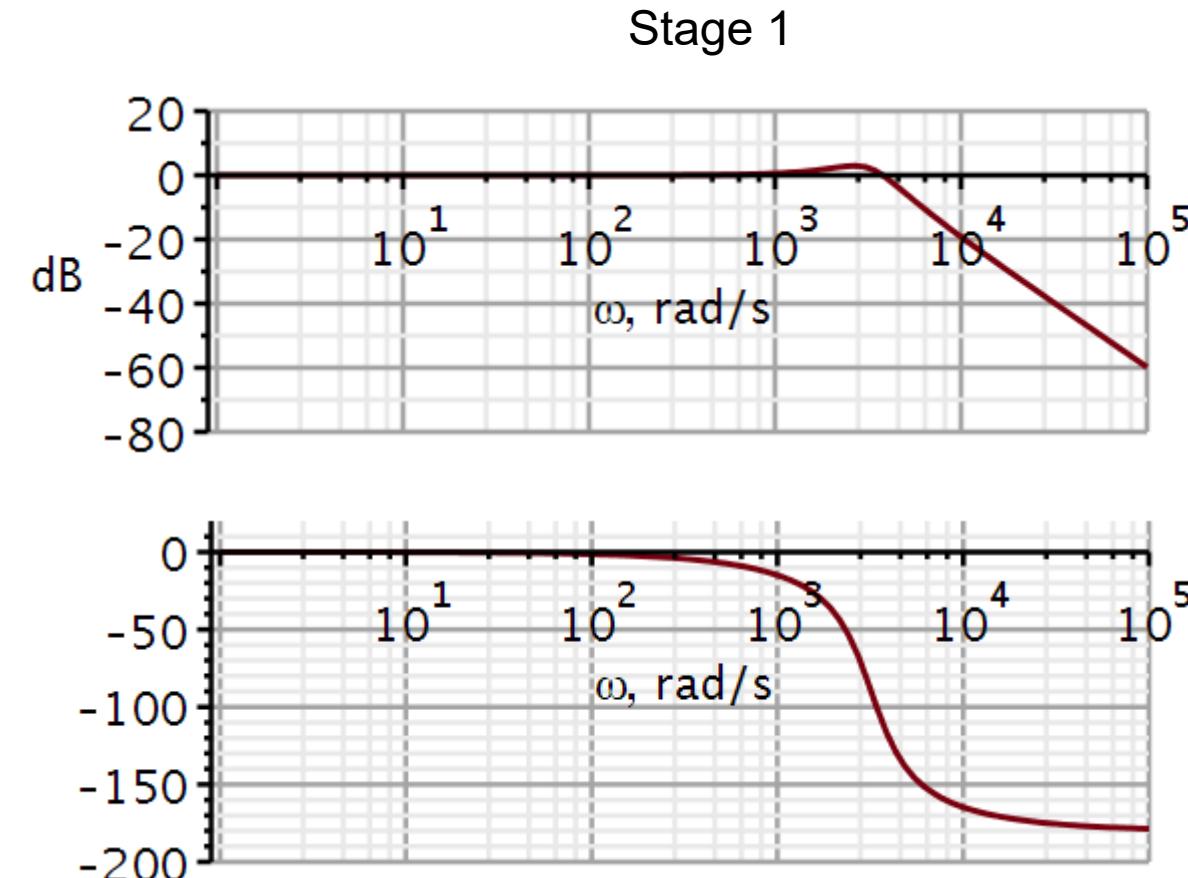
By looking up in the $R \cdot C$ look-up table earlier in the slide set, we can pick a base value for the resistor of 1.2 and a base value for the capacitor of 1.0 (in the E-6 series). Now we just need to find the correct magnitude for each.

$$K_z = \frac{C'_1}{10 \cdot 10^{-9} F} = \frac{2.610 \text{mF}}{10 \cdot 10^{-9} F} = 261014$$

$$C_1 = \frac{C'_1}{K_z} = 10 \text{nF} \quad C_2 = \frac{C'_2}{K_z} = 68.29 \text{nF} \approx 68 \text{nF}$$

$$R_1 = R_2 = K_z R'_1 = 261014 \times 0.0466687 \Omega = 12181 \Omega \approx 12 \text{k}\Omega$$

10nF and 68nF are both E-6 values, and 12k Ω is an E-12 value.



Stage 2:

Looking up in the table of Butterworth coefficients:

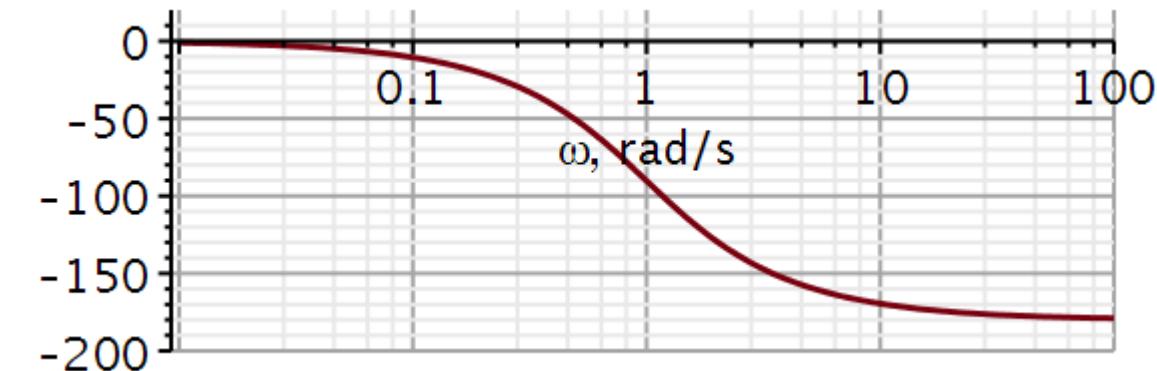
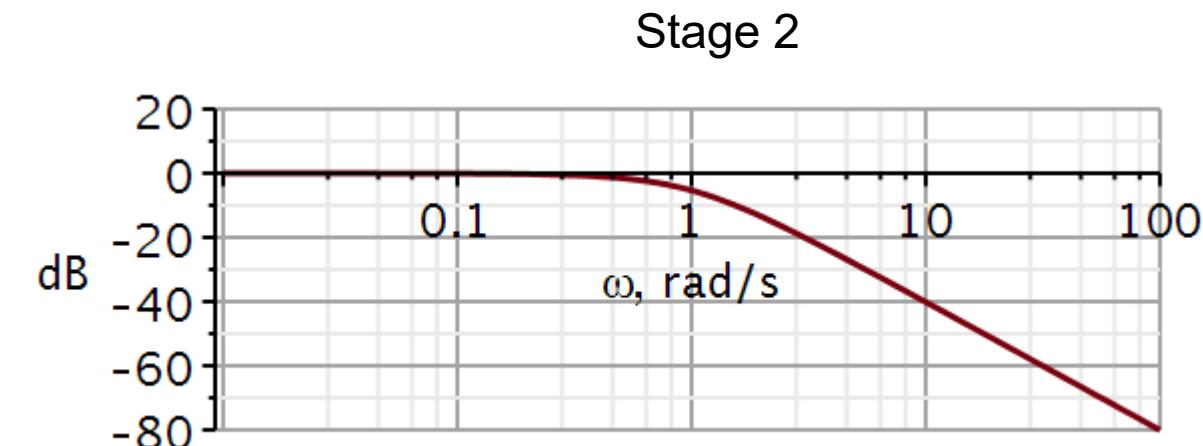
$$a_1 = 1.84776$$

$$a_0 = 1$$

Using the guidelines from sensitivity analysis:

$$\frac{\hat{C}_4}{\hat{C}_3} = \frac{4a_0}{a_1^2} = \frac{4}{(1.84776)^2} = 1.17157 \approx \frac{3.9F}{3.3F} = 1.18182$$

$$\hat{R}_3 = \hat{R}_4 = \frac{a_1}{2a_0\hat{C}_3} = 0.27996\Omega$$



Problem 1 – EMG lowpass filter (sol)

Frequency scaling

$$\omega_n = \frac{1}{\sqrt{\hat{R}_3 \hat{R}_4 \hat{C}_3 \hat{C}_4}} = 0.99566$$

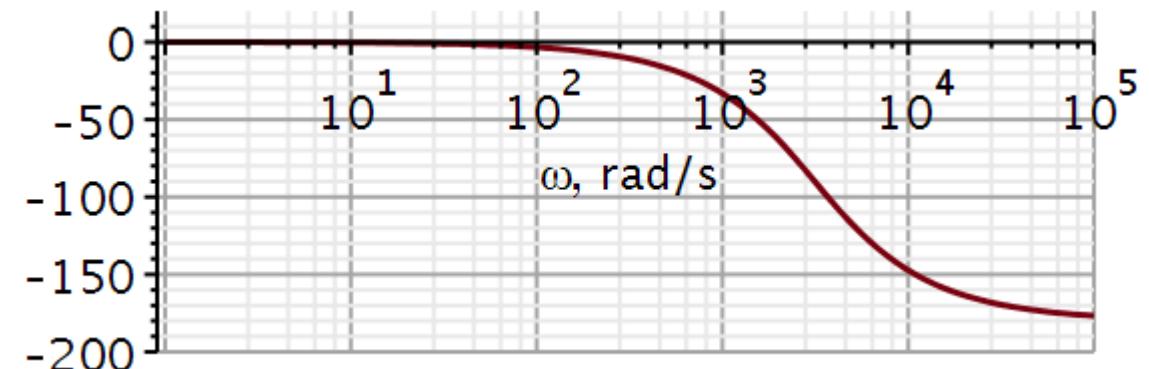
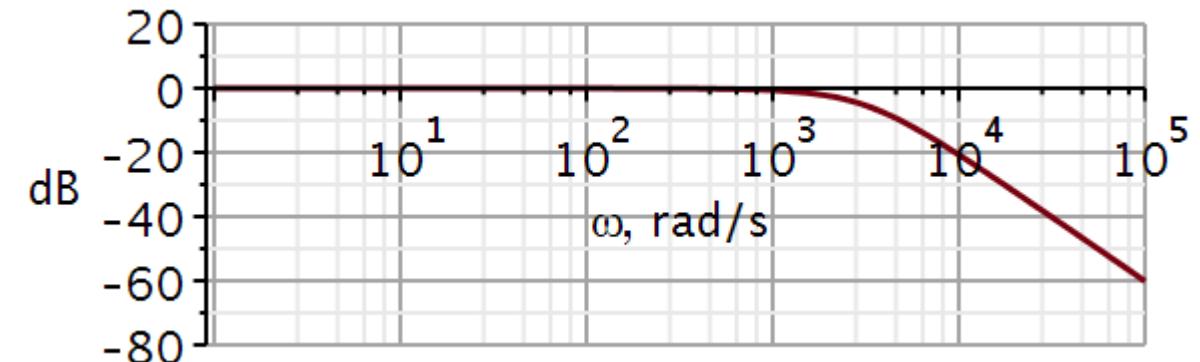
$$K_F = \frac{2\pi \cdot 500}{\omega_n} = 3155.3$$

$$C'_3 = \frac{\hat{C}_3}{K_F} = \frac{3.3\text{F}}{3155.3} = 1.046\text{mF}$$

$$C'_4 = \frac{\hat{C}_4}{K_F} = \frac{3.9\text{F}}{3155.3} = 1.236\text{mF}$$

$$R'_3 = R'_4 = \hat{R}_3 = \hat{R}_4 = 0.27996\Omega$$

Stage 2



Problem 1 – EMG lowpass filter (sol)

We have:

$$C'_3 = 1.046\text{mF}$$

$$C'_4 = 1.236\text{mF}$$

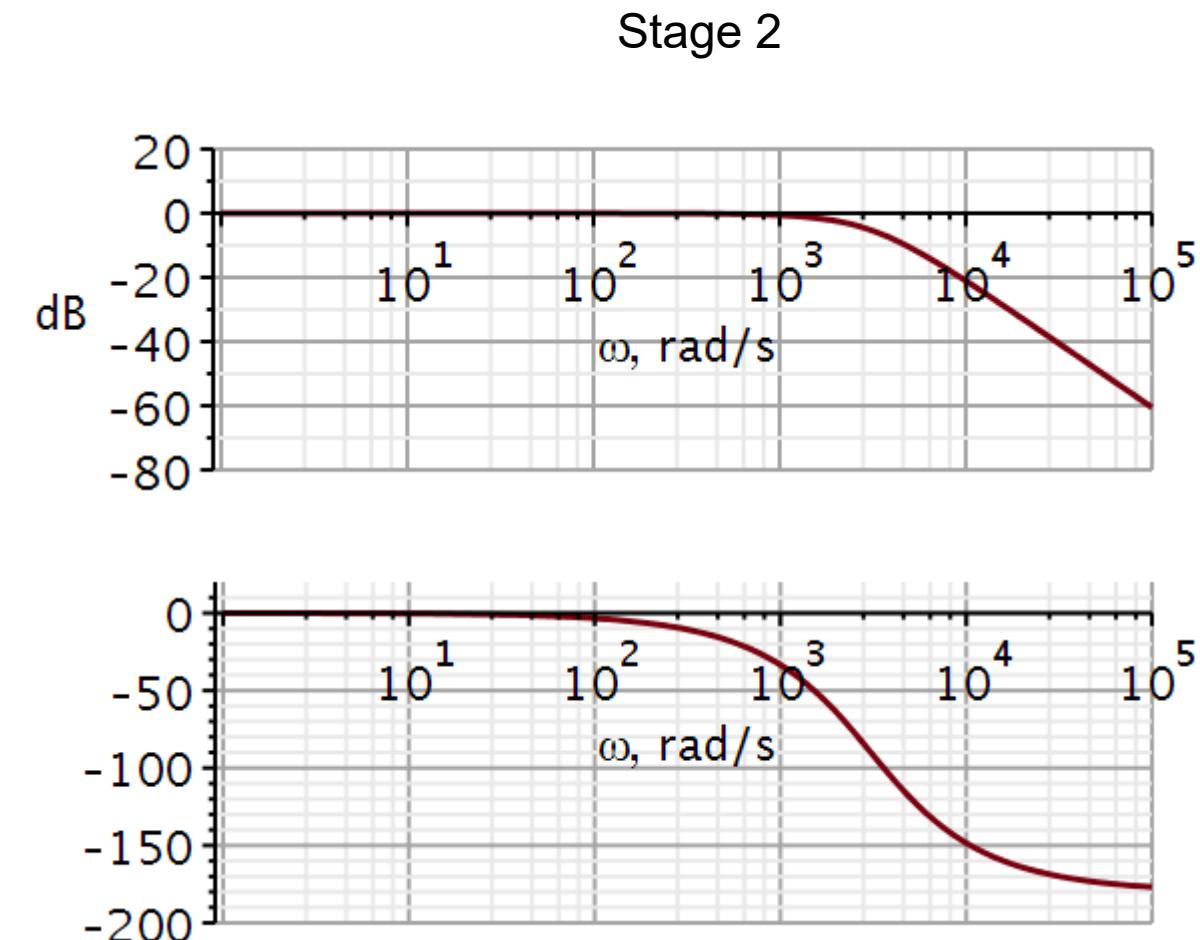
$$R'_3 = R'_4 = 0.27996\Omega$$

1.2 is not an E-6 value. Because all three values are close to E-12 values, we will use values from E-12.

$$K_z = \frac{C'_3}{10\text{nF}} = \frac{1.046\text{mF}}{10\text{nF}} = 104600.$$

$$C_4 = \frac{C'_4}{K_z} = \frac{1.236\text{mF}}{104600} = 11.8\text{nF} \approx 12\text{nF}$$

$$R_3 = R_4 = 104600 \times 0.27996\Omega = 29.2\text{k}\Omega \approx 27\text{k}\Omega$$

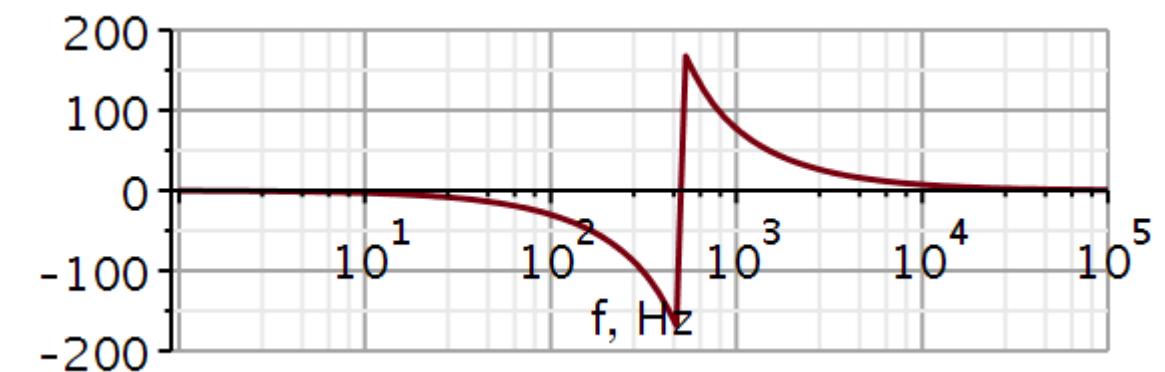
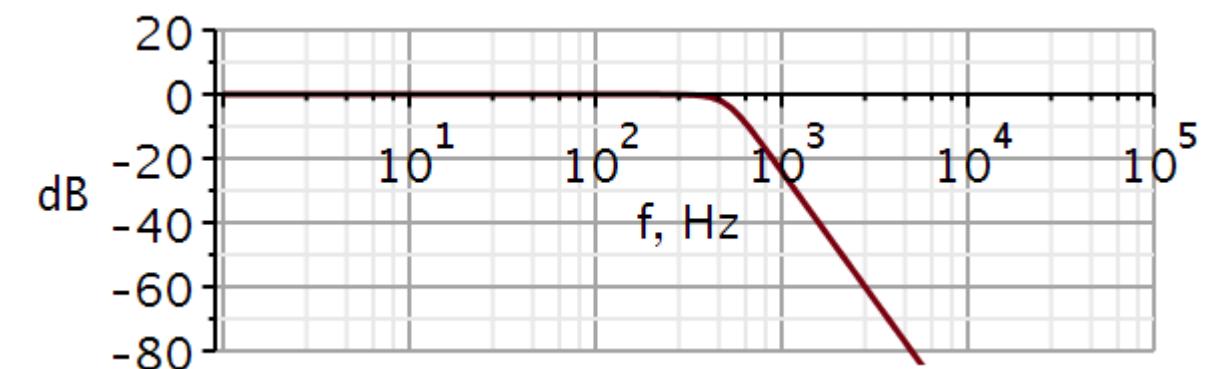


Stage 2

Problem 1 – EMG lowpass filter (sol)

$$H(\omega) = H_1(\omega) \cdot H_2(\omega)$$

In the final plots the frequency axis have been converted to Hz.



Problem 1 – EMG lowpass filter (sol)

We would like to compare the magnitude of the two individual stages with the combined system.

We observe a small peak in the magnitude of stage 1.

$$\begin{aligned} a_1 &= 0.765367 \\ a_0 &= 1 \end{aligned}$$

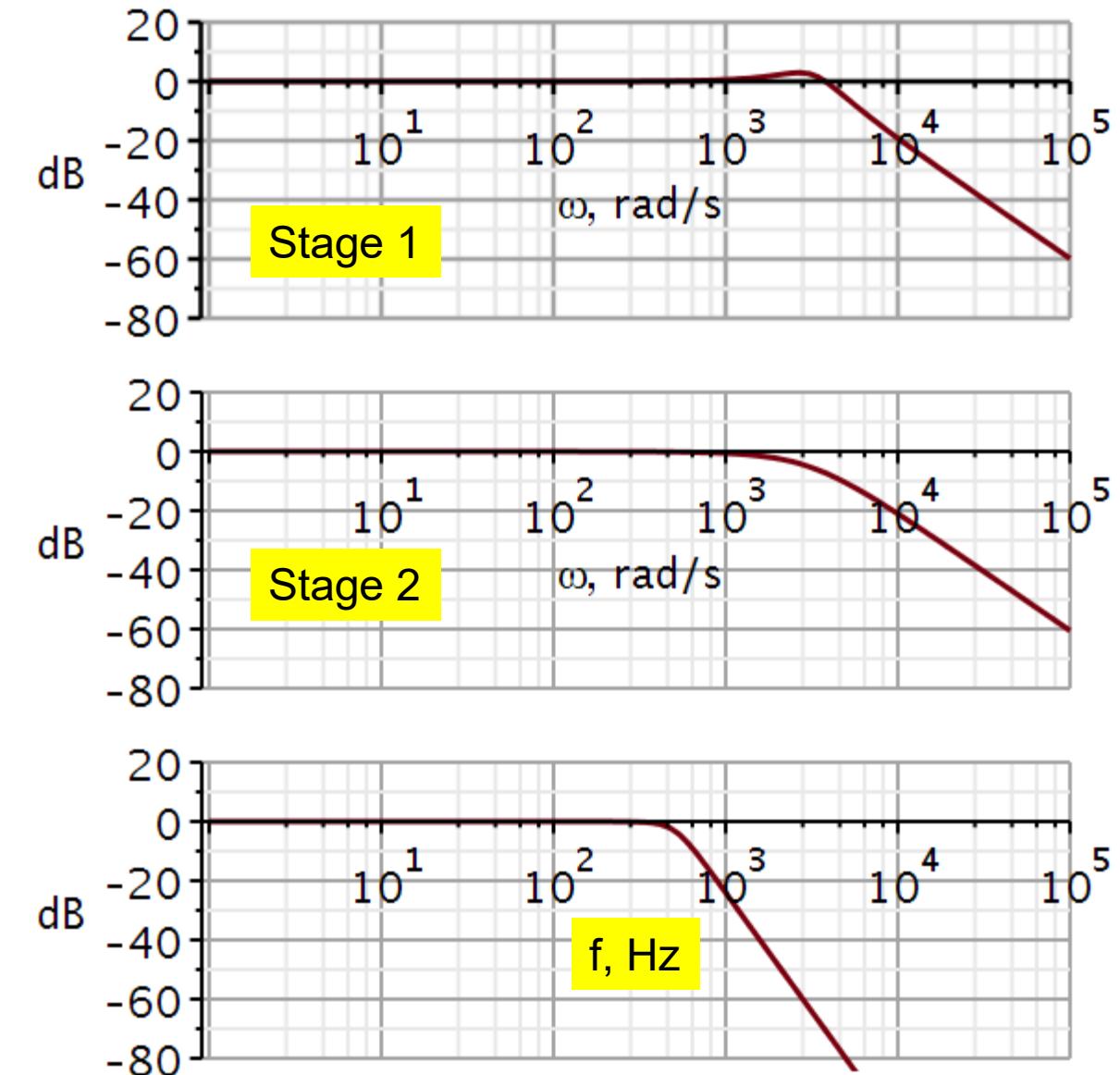
$$\zeta_1 = \frac{a_1}{2\sqrt{a_0}} = \frac{0.765}{2} = 0.38$$

For stage 2:

$$\begin{aligned} a_1 &= 1.84776 \\ a_0 &= 1 \end{aligned}$$

$$\zeta_2 = \frac{a_1}{2\sqrt{a_0}} = \frac{1.848}{2} = 0.92$$

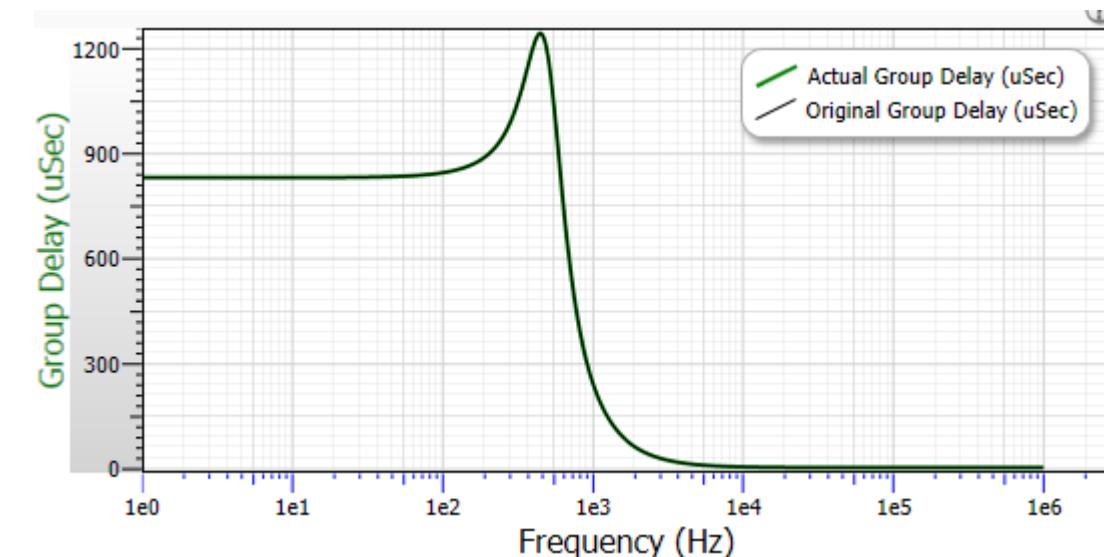
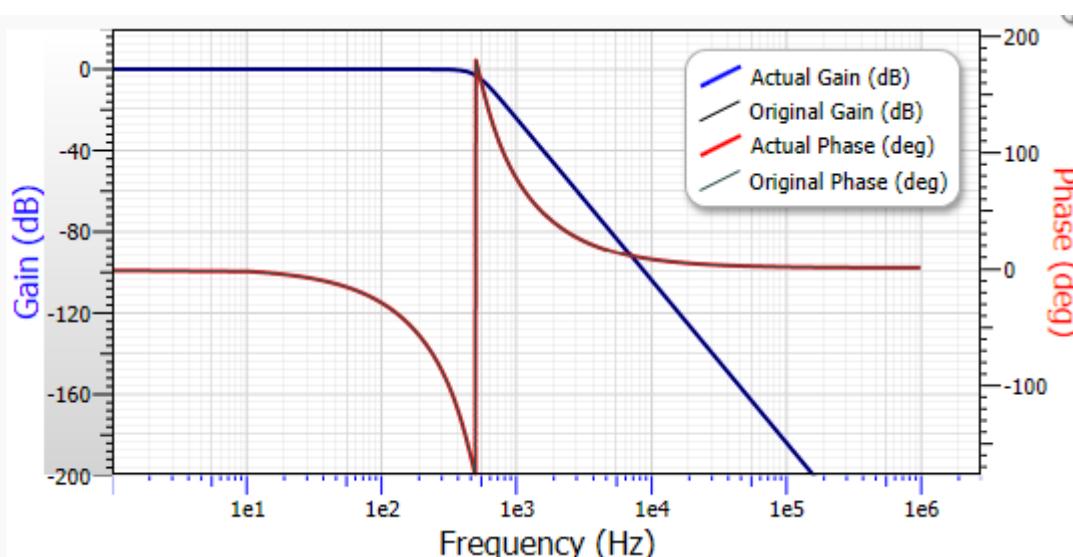
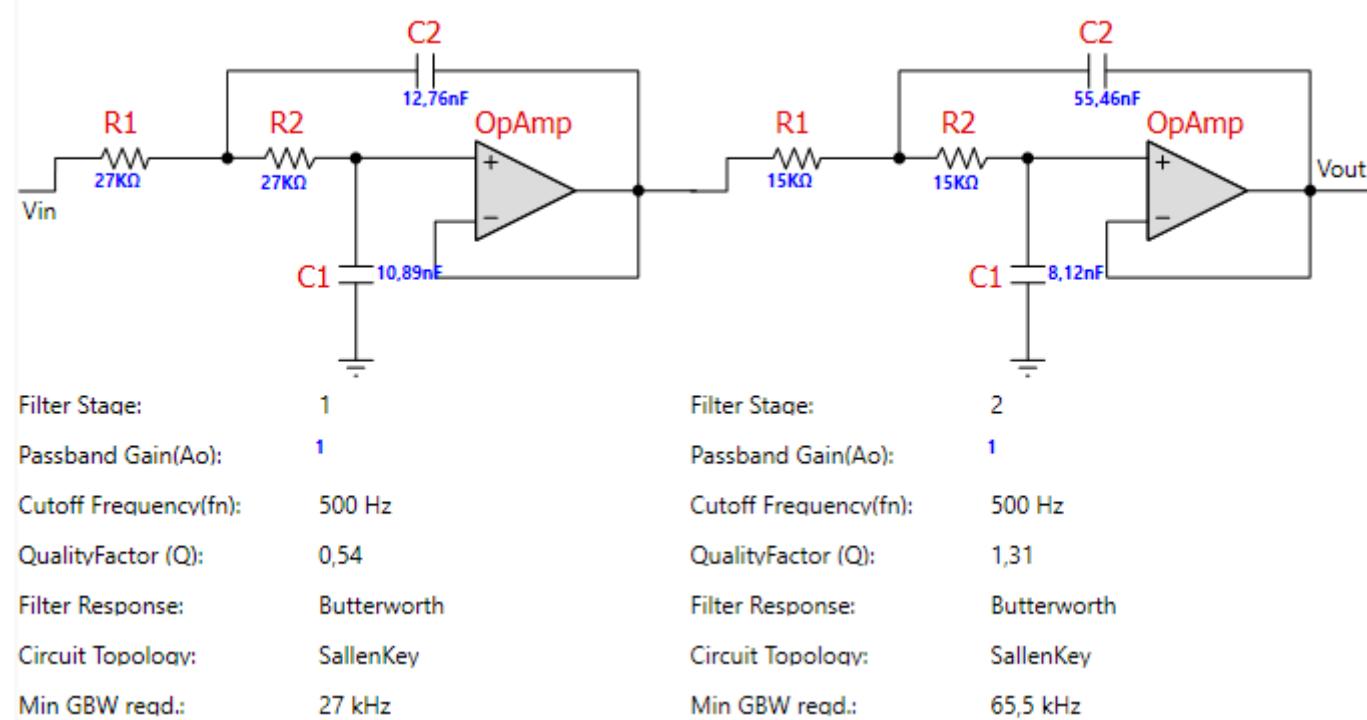
As expected of a 4th order filter. One pole pair lies below and the other pair above the 45° degree line in the complex plane. **Which is which?**



Problem 1 – EMG lowpass filter (sol)

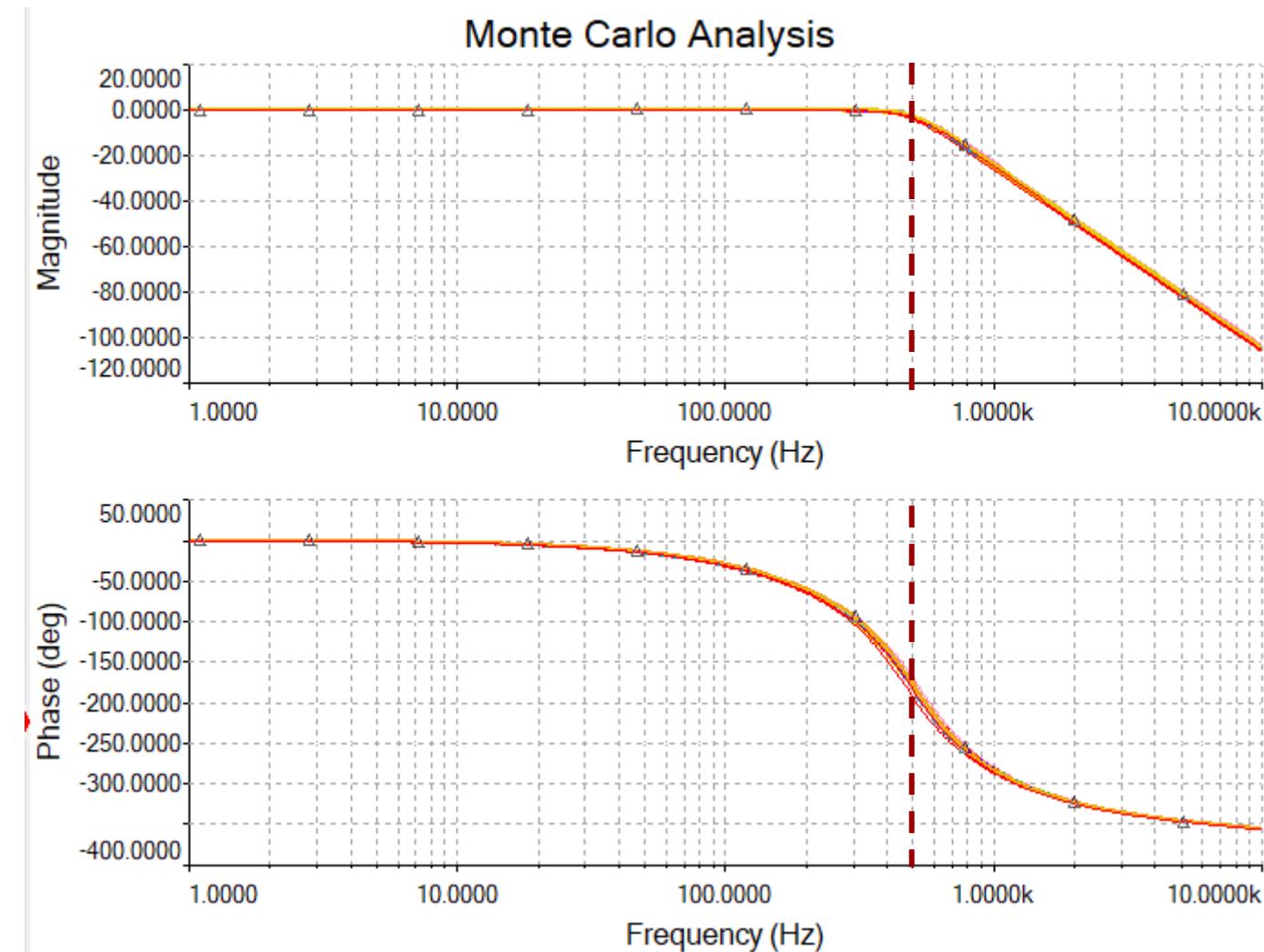
Circuit components calculated in FilterPro (Texas Instruments).

Here the resistors in FilterPro have been changed to my own values. The capacitor values should then also change to match my values. And they do. They have not yet been rounded to E12 values.



Problem 1 – EMG lowpass filter (sol)

Monte Carlo analysis in Multisim
using 10% tolerance on all
components.



22050 Continuous-Time Signals and Linear Systems

Kaj-Åge Henneberg

L12

Butterworth filter design – part 2
Instrumentation amplifier

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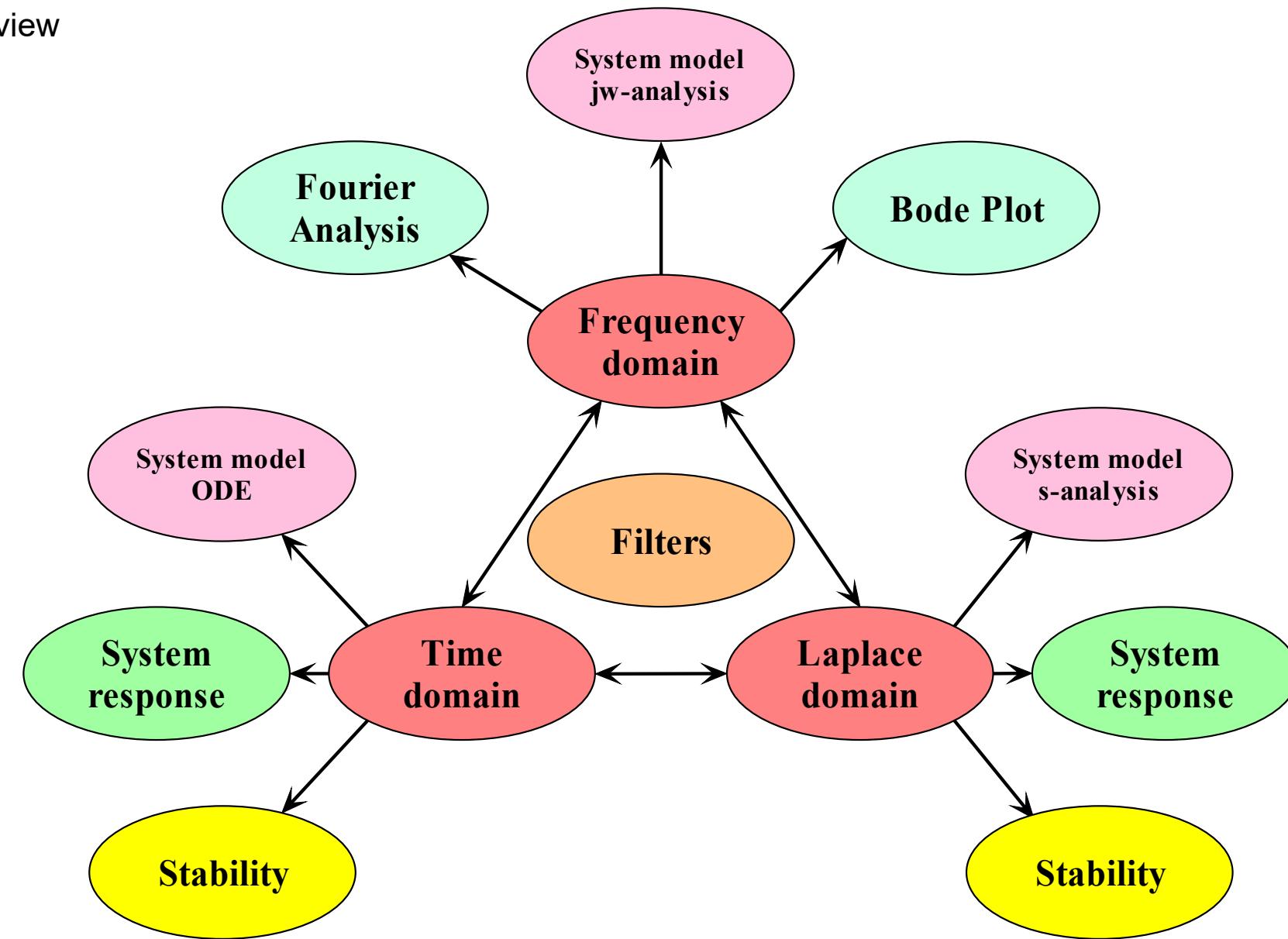
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- Frequency transformation from lowpass to highpass – Lathi 7.7
 - Filter specification
 - Matching Butterworth Transfer function to Sallen-Key filter circuit
 - Sensitivity-constrained filter design
- Instrumentation amplifier (in-amp) – Not in Lathi
 - Applications of instrumentation amplifiers
 - The difference amplifier
 - Input impedances of difference amplifier
 - Adding unity gain buffers to the difference amplifier
 - AC coupled instrumentation amplifier
 - Design of in-amp Highpass filter
 - Common mode rejection ratio (CMRR)

From Lowpass to Highpass Filter specifications

Video

Lowpass to Highpass Transformation

Transformation of a prototype lowpass filter into the final highpass is accomplished by mapping of the frequency axis:

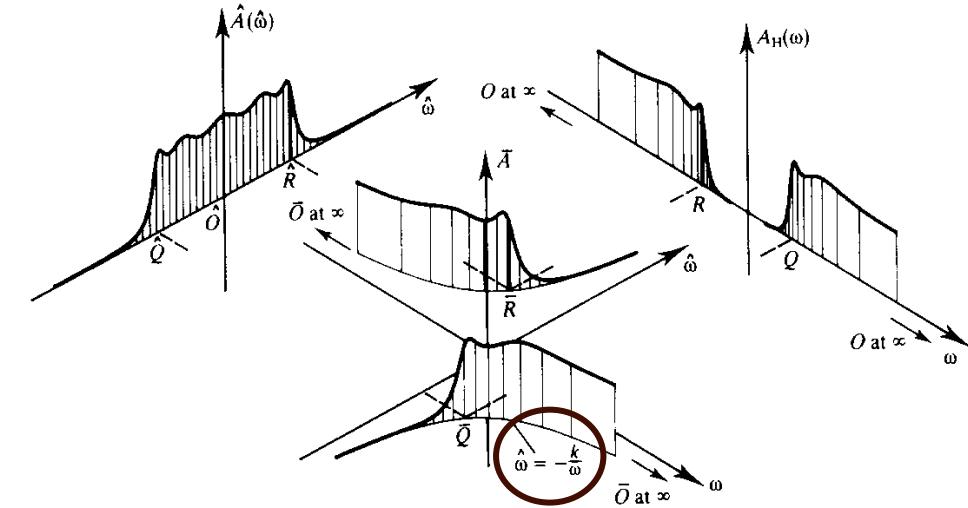
$$\hat{s}_{LP} \leftarrow \frac{\omega_p}{s_{HP}}$$

$$j\hat{\omega}_{LP} \leftarrow \frac{\omega_p}{j\omega_{HP}} = -j \frac{\omega_p}{\omega_{HP}}$$

$$\hat{\omega}_{LP} \leftarrow -\frac{\omega_p}{\omega_{HP}}$$

$$\hat{\omega} \leftarrow -\frac{k}{\omega}$$

lowpass \longrightarrow highpass



Lowpass prototype $\hat{A}(\hat{\omega})$

Highpass filter $H_H(\omega)$

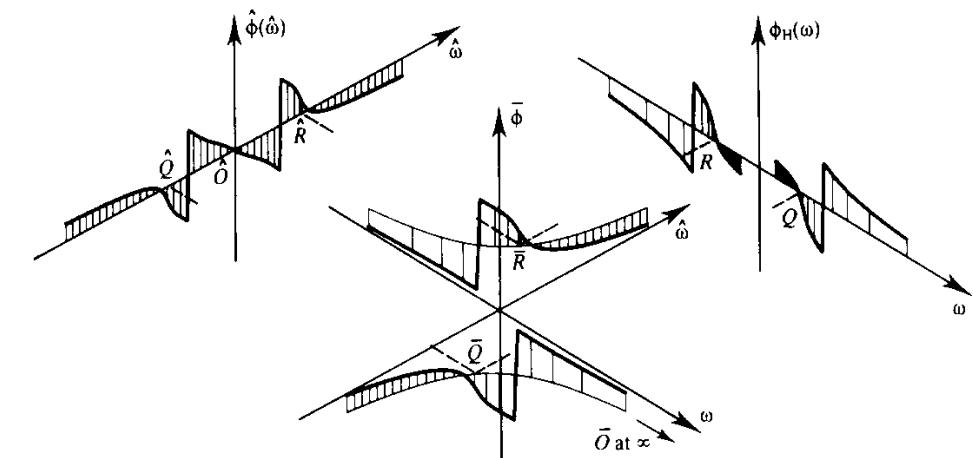


Figure 10.34 Lowpass-to-highpass frequency mapping: $\hat{\omega} = -k/\omega$.

P. Kraniauskas, Transforms in Signals and Systems, Addison Wesley.

Using frequency transformation, we can rewrite a lowpass filter characteristic into one for a highpass filter.

Consequently, if we start out with a frequency-normalized lowpass filter, we can transform it into a frequency-normalized highpass filter.

For this reason, we will rarely see filter coefficient tables for highpass, bandpass and bandstop filters.

Frequency-normalized highpass filter:

$$H_{LP}(s) = \frac{b_0}{s^2 + a_1 s + a_0} = \frac{\frac{b_0}{a_0}}{\left(\frac{s}{\sqrt{a_0}}\right)^2 + \frac{a_1}{\sqrt{a_0}} \frac{s}{\sqrt{a_0}} + 1}$$

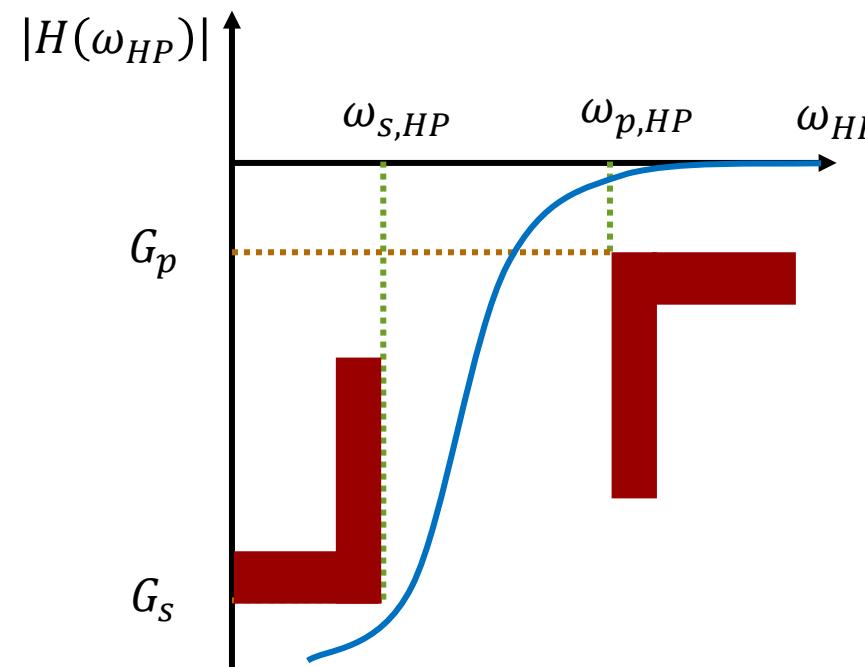
$$H_{LP}(\hat{s}_{LP}) = \frac{\frac{b_0}{a_0}}{\hat{s}_{LP}^2 + \frac{a_1}{\sqrt{a_0}} \hat{s}_{LP} + 1} \quad \begin{aligned} \hat{s}_{LP} &\stackrel{\text{def}}{=} \frac{s_{LP}}{\sqrt{a_0}} \\ \frac{1}{\hat{s}_{HP}} &\rightarrow \hat{s}_{LP} \end{aligned}$$

$$H_{HP}(\hat{s}_{HP}) = \frac{\frac{b_0}{a_0}}{\left(\frac{1}{\hat{s}_{HP}}\right)^2 + \frac{a_1}{\sqrt{a_0}} \frac{1}{\hat{s}_{HP}} + 1} \quad \hat{s}_{HP} \stackrel{\text{def}}{=} \frac{s_{HP}}{\omega_p}$$

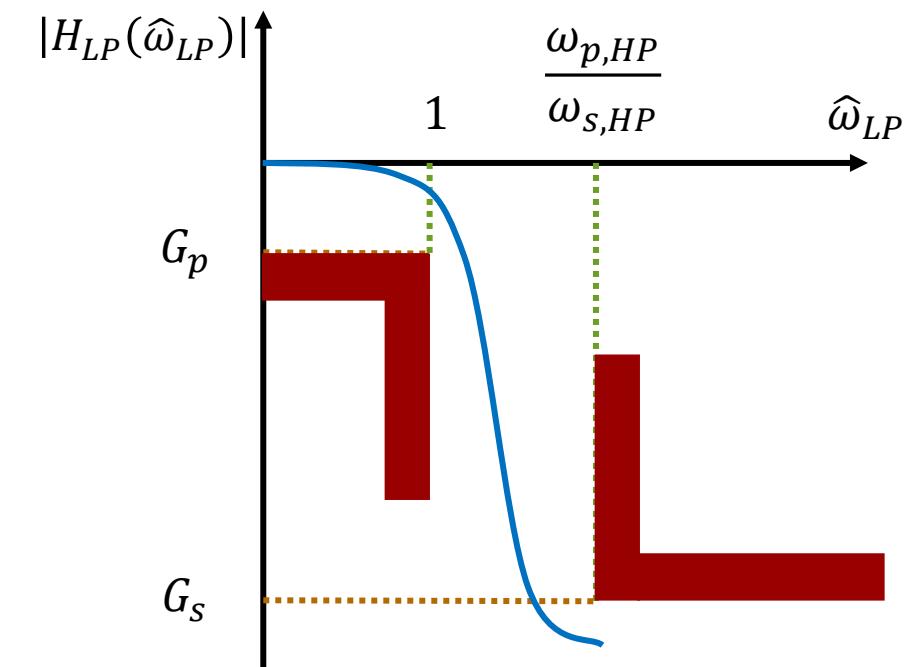
$$H_{HP}(\hat{s}_{HP}) = \frac{b_2 \hat{s}_{HP}^2}{\hat{s}_{HP}^2 + \frac{a_1}{\sqrt{a_0}} \hat{s}_{HP} + 1} \quad b_2 = \frac{b_0}{a_0}$$

Butterworth Highpass Filter Specifications

Filter performance specifications for a highpass filter are formulated for the highpass filter and then transformed to the frequency-normalized lowpass filter, so that we can determine the required filter order.



$$\hat{\omega}_{LP} \leftarrow -\frac{\omega_{p,HP}}{\omega_{HP}}$$



$$\hat{\omega}_{p,LP} \leftarrow -\frac{\omega_{p,HP}}{\omega_{p,HP}} = -1$$

$$\hat{\omega}_{s,LP} \leftarrow -\frac{\omega_{p,HP}}{\omega_{s,HP}}$$

$$n = \frac{1}{2\log_{10}(\hat{\omega}_{s,LP}/\hat{\omega}_{p,LP})} \log_{10} \left(\frac{10^{-\frac{G_{s,dB}}{10}} - 1}{10^{-\frac{G_{p,dB}}{10}} - 1} \right) = \frac{1}{2\log_{10}(\omega_{p,HP}/\omega_{s,HP})} \log_{10} \left(\frac{10^{-\frac{G_{s,dB}}{10}} - 1}{10^{-\frac{G_{p,dB}}{10}} - 1} \right)$$

Example: Highpass Butterworth Filter

Find the transfer function order for the filter satisfying the following specifications:

$$\hat{\omega}_{p,LP} \leftarrow -\frac{\omega_{p,HP}}{\omega_{p,HP}} = -\frac{0.05}{0.05} = -1$$

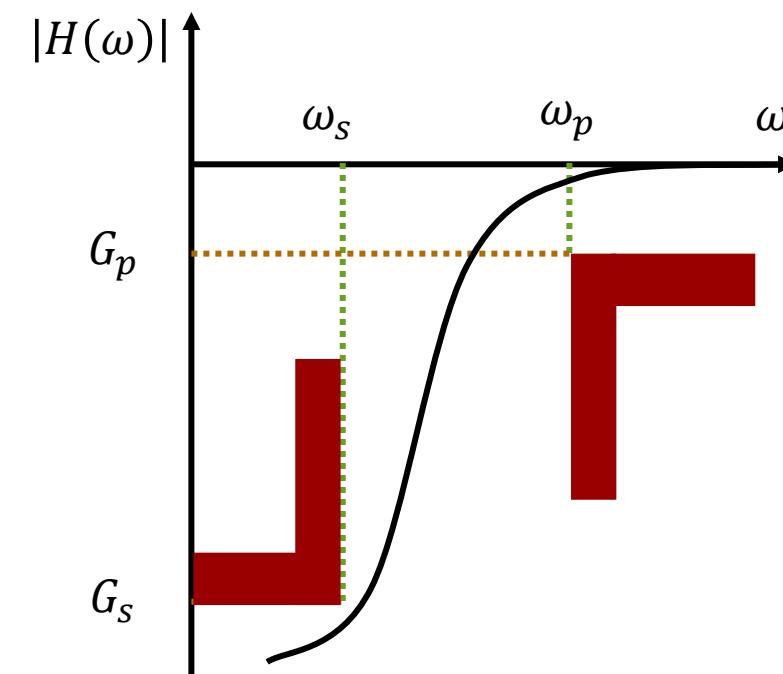
$$\hat{\omega}_{s,LP} \leftarrow -\frac{\omega_{p,HP}}{\omega_{s,HP}} = -\frac{0.05}{0.01} = -5$$

$$n = \frac{1}{2\log_{10}(\hat{\omega}_{s,LP}/\hat{\omega}_{p,LP})} \log_{10} \left(\frac{10^{-\frac{G_{s,dB}}{10}} - 1}{10^{-\frac{G_{p,dB}}{10}} - 1} \right)$$

$$= \frac{1}{2\log_{10}(\omega_{p,HP}/\omega_{s,HP})} \log_{10} \left(\frac{10^{-\frac{G_{s,dB}}{10}} - 1}{10^{-\frac{G_{p,dB}}{10}} - 1} \right)$$

$$n = 1.861 \rightarrow 2$$

$$\begin{aligned} G_s &\leq -26 \text{ dB}, \quad \omega_s = 2\pi 0.01 \text{ rad/s} \\ G_p &= -3 \text{ dB}, \quad \omega_p = 2\pi 0.05 \text{ rad/s} \end{aligned}$$



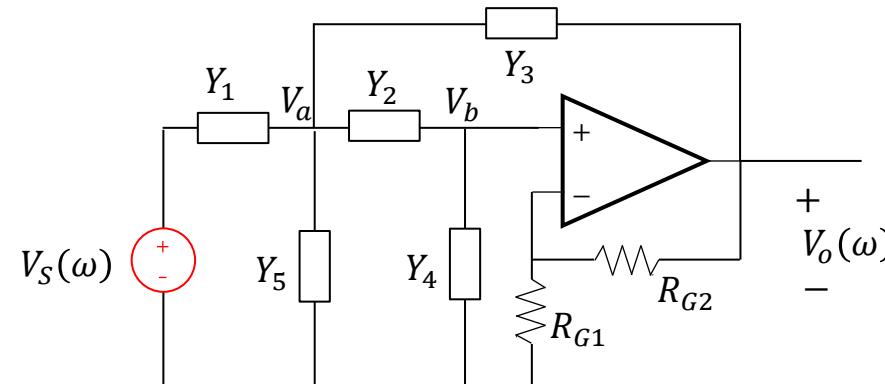
From Lowpass to Highpass Filter circuit

Video

2nd order Sallen-Key high pass filter

High pass filter:

$$H_{HP}(j\omega) = \frac{K(j\omega)^2}{(j\omega)^2 + a_1(j\omega) + a_0}$$



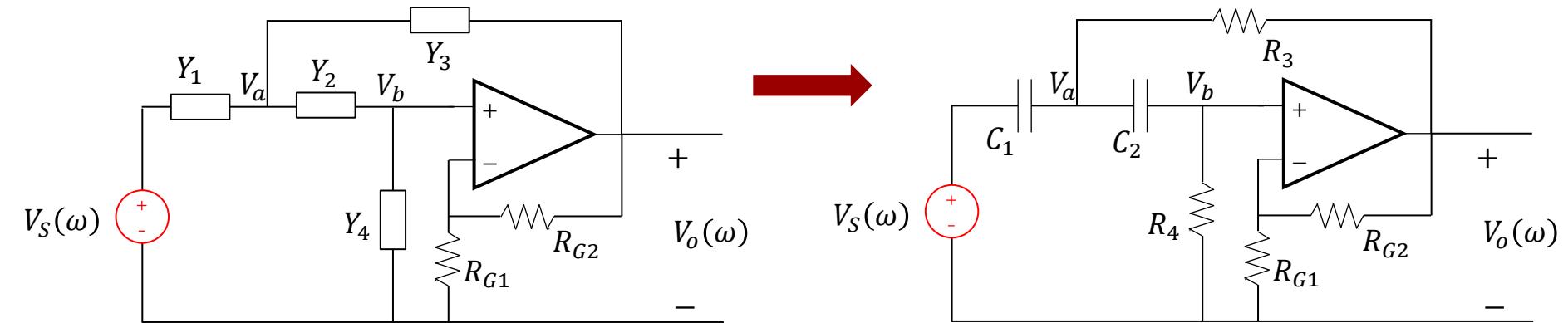
$$H(j\omega) = \frac{KY_1Y_2}{Y_1Y_2 + Y_1Y_4 + Y_2Y_4 + Y_3Y_4 + Y_2Y_5 + Y_4Y_5 + Y_2Y_3(1 - K)}$$

Matching coefficients we can decide, where the resistors and capacitors should be placed.

- Either Y_2 or Y_4 must be a resistor to allow DC bias current through the positive input terminal.
- In the high pass filter, the numerator is a quadratic term, hence both Y_1 and Y_2 must be capacitors.
- Therefore Y_4 must be a resistor.
- It follows that Y_1Y_2 is also the quadratic term in the denominator.
- Furthermore, the linear term is: $Y_1Y_4 + Y_2Y_4 + Y_2Y_5 + Y_2Y_3(1 - K)$.
- Because Y_1 and Y_2 are capacitors, Y_3 , Y_4 , and Y_5 must be resistors.
- The constant term is either Y_3Y_4 or Y_4Y_5 .
- We take Y_3Y_4 , because we need Y_3 in the linear term. Hence, we don't need Y_5 .
- We set $Y_5 = 0$.

2nd order Sallen-Key high pass filter

Replacing the
impedance blocks
with components:



Inserting the impedance for
each components:

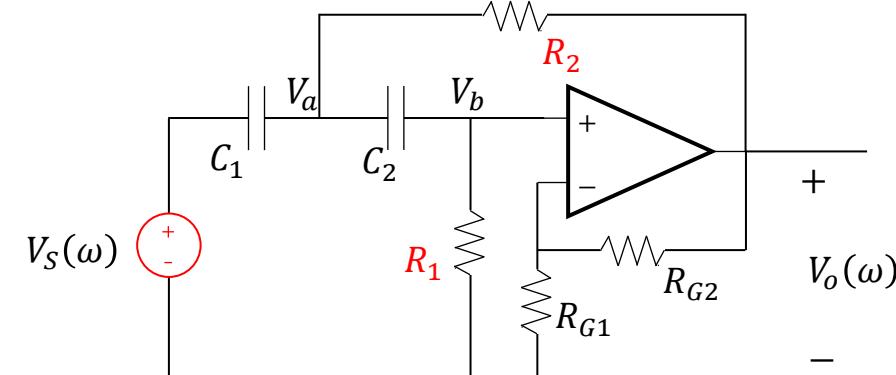
$$H_{HP}(j\omega) = \frac{KY_1 Y_2}{Y_1 Y_2 + Y_1 Y_4 + Y_2 Y_4 + Y_3 Y_4 + Y_2 Y_3(1 - K)}$$

$$H_{HP}(j\omega) = \frac{K(j\omega)^2}{(j\omega)^2 + j\omega \left(\frac{1}{R_4 C_1} + \frac{1}{R_4 C_2} + \frac{(1 - K)}{R_3 C_1} \right) + \frac{1}{R_3 R_4 C_1 C_2}}$$

2nd order Sallen-Key high pass filter

Renumbering the components:

To determine the component values, we must match the circuit equation with the performance equation.



$$H_{HP}(j\omega) = \frac{K(j\omega)^2}{(j\omega)^2 + j\omega \left(\frac{1}{R_1 C_1} + \frac{1}{R_1 C_2} + \frac{(1-K)}{R_2 C_1} \right) + \frac{1}{R_1 R_2 C_1 C_2}}$$

The performance equation for the **frequency-normalized** Butterworth highpass filter:

$$H_{HP}(\hat{\omega}_{HP}) = \frac{b_2 (j\hat{\omega}_{HP})^2}{(j\hat{\omega}_{HP})^2 + \frac{a_1}{\sqrt{a_0}} (j\hat{\omega}_{HP}) + 1}$$

From Lowpass to Highpass Filter design

Video

2nd order Sallen-Key high pass filter - Sensitivity

Sensitivity analysis on coefficients.

$$\frac{1}{Q} \stackrel{\text{def}}{=} 2\zeta \Rightarrow Q = \frac{1}{2\zeta} = \frac{1}{2} \frac{2\sqrt{a_0}}{a_1} = \frac{\sqrt{a_0}}{a_1}$$

We will set the gain to unity, $K = 1$.

$$H_{HP}(j\omega) = \frac{K(j\omega)^2}{(j\omega)^2 + j\omega \left(\frac{1}{R_1 C_1} + \frac{1}{R_1 C_2} + \frac{(1-K)}{R_2 C_1} \right) + \frac{1}{R_1 R_2 C_1 C_2}}$$

$$\omega_n = \sqrt{\frac{1}{R_1 R_2 C_1 C_2}}$$

$$Q = \frac{\sqrt{R_1 R_2 C_1 C_2}}{(C_1 + C_2)R_2 + R_1 C_2(1 - K)}$$

$$Q = \frac{\sqrt{R_1 R_2 C_1 C_2}}{(C_1 + C_2)R_2} = \sqrt{\frac{R_1}{R_2} \cdot \frac{\sqrt{C_1 C_2}}{C_1 + C_2}}$$

$$S_x^y \stackrel{\text{def}}{=} \frac{x}{y} \frac{\partial y}{\partial x}$$

$$S_x^{\omega_n} = -\frac{1}{2}$$

$$S_{R_1}^Q = \frac{1}{2} \quad S_{R_2}^Q = -\frac{1}{2}$$

$$S_{C_1}^Q = \frac{C_2 - C_1}{2(C_1 + C_2)}$$

$$S_{C_2}^Q = \frac{C_1 - C_2}{2(C_1 + C_2)}$$

Observation:

For a Sallen-Key highpass filters we obtain low sensitivity for Q if we match the capacitors.

2nd order Sallen-Key high pass filter

By matching coefficients in the above two equations, we obtain two equations:

This time we match capacitors.

$$K \stackrel{\text{def}}{=} 1, \hat{C}_1 \stackrel{\text{def}}{=} \hat{C}_2 \stackrel{\text{def}}{=} \hat{C}$$

We can obtain a ratio as we did for the lowpass filter:

$$H_{HP}(j\hat{\omega}) = \frac{K(j\omega)^2}{(j\hat{\omega})^2 + j\hat{\omega} \left(\frac{1}{\hat{R}_1 \hat{C}_1} + \frac{1}{\hat{R}_1 \hat{C}_2} + \frac{(1-K)}{\hat{R}_2 \hat{C}_1} \right) + \frac{1}{\hat{R}_1 \hat{R}_2 \hat{C}_1 \hat{C}_2}}$$

$$H_{HP}(j\hat{\omega}) = \frac{K(j\hat{\omega})^2}{(j\hat{\omega})^2 + a_1(j\hat{\omega}) + a_0}$$

$$\frac{2}{\hat{R}_1 \hat{C}} = a_1 \quad \hat{R}_1 \hat{C} = \frac{2}{a_1}$$

$$\hat{R}_1 \hat{R}_2 \hat{C}^2 = \frac{1}{a_0} \quad \hat{R}_2 \hat{C} \frac{2}{a_1} = \frac{1}{a_0} \quad \hat{R}_2 \hat{C} = \frac{a_1}{2a_0}$$

$$\frac{\hat{R}_1}{\hat{R}_2} = \frac{2}{a_1} \frac{2a_0}{a_1} = \frac{4a_0}{a_1^2} = \frac{4 \cdot 1}{(\sqrt{2})^2} = 2 \approx \frac{68}{33}$$

2nd order Sallen-Key high pass filter

We have several options.

We can use a look-up table for time constants and in one and the same column (same C value) find the two best time constants.

But there might be more than one useful column (C value).

If we decide to stick to a ratio of ~2 for the resistors, we can obtain a suggestion for the best capacitor value:

$$\hat{\tau}_1 = \hat{R}_1 \hat{C} = \frac{2}{a_1} = 1.414 \Rightarrow \hat{C} = \frac{1.414s}{0.68\Omega} = 2.079F$$

$$\hat{\tau}_2 = \hat{R}_2 \hat{C} = \frac{a_1}{2a_0} = 0.707 \Rightarrow \hat{C} = \frac{0.707s}{0.33\Omega} = 2.142F$$

Good options are 1.2, 1.5, and 2.2F

$$\hat{R}_1 \hat{R}_2 \hat{C}^2 = \frac{1}{a_0} \Rightarrow \hat{C}^2 = \frac{1}{a_0 \hat{R}_1 \hat{R}_2}$$

$$\hat{C} = \frac{1}{\sqrt{a_0 \hat{R}_1 \hat{R}_2}} = \frac{1}{\sqrt{0.68 \times 0.33}} = 2.11 \approx 2.2$$

$$\hat{R}_1 \hat{R}_2 \hat{C}^2 = \frac{1}{a_0} = 1 \Rightarrow \tau_1 \tau_2 = 1$$

We see that the product of the two time constants is 1.
This is best obtained for $C = 1.2F, R1 = 1.2\Omega, R2 = 0.56\Omega$

2nd order Sallen-Key high pass filter - Butterworth

$$\hat{\tau}_1 = \hat{R}_1 \hat{C} = \frac{2}{a_1} = \frac{2}{\sqrt{2}} = \sqrt{2} = 1.414$$

$$\hat{\tau}_2 = \hat{R}_2 \hat{C} = \frac{a_1}{2a_0} = \frac{\sqrt{2}}{2} = 0.707$$

R1*C1	0,1	0,12	0,15	0,18	0,22	0,27	0,33	0,39	0,47	0,56	0,68	0,82	1	1,2	1,5	1,8	2,2	2,7	3,3	3,9	4,7	5,6	6,8	8,2	10
0,1	0,010	0,012	0,015	0,018	0,022	0,027	0,033	0,039	0,047	0,056	0,068	0,082	0,100	0,120	0,150	0,180	0,220	0,270	0,330	0,390	0,470	0,560	0,680	0,820	1,000
0,12	0,012	0,014	0,018	0,022	0,026	0,032	0,040	0,047	0,056	0,067	0,082	0,098	0,120	0,144	0,180	0,216	0,264	0,324	0,396	0,468	0,564	0,672	0,816	0,984	1,200
0,15	0,015	0,018	0,023	0,027	0,033	0,041	0,050	0,059	0,071	0,084	0,102	0,123	0,150	0,180	0,225	0,270	0,330	0,405	0,495	0,585	0,705	0,840	1,020	1,230	1,500
0,18	0,018	0,022	0,027	0,032	0,040	0,049	0,059	0,070	0,085	0,101	0,122	0,148	0,180	0,216	0,270	0,324	0,396	0,486	0,594	0,702	0,846	1,008	1,224	1,476	1,800
0,22	0,022	0,026	0,033	0,040	0,048	0,059	0,073	0,086	0,103	0,123	0,150	0,180	0,220	0,264	0,330	0,396	0,484	0,594	0,726	0,858	1,034	1,232	1,496	1,804	2,200
0,27	0,027	0,032	0,041	0,049	0,059	0,073	0,089	0,105	0,127	0,151	0,184	0,221	0,270	0,324	0,405	0,486	0,594	0,729	0,891	1,053	1,269	1,512	1,836	2,214	2,700
0,33	0,033	0,040	0,050	0,059	0,073	0,089	0,109	0,129	0,155	0,185	0,224	0,271	0,330	0,396	0,495	0,594	0,726	0,891	1,089	1,287	1,551	1,848	2,244	2,706	3,300
0,39	0,039	0,047	0,059	0,070	0,086	0,105	0,129	0,152	0,183	0,218	0,265	0,320	0,390	0,468	0,585	0,702	0,858	1,053	1,287	1,521	1,833	2,184	2,652	3,198	3,900
0,47	0,047	0,056	0,071	0,085	0,103	0,127	0,155	0,183	0,221	0,263	0,320	0,385	0,470	0,564	0,705	0,846	1,034	1,269	1,551	1,833	2,209	2,632	3,196	3,854	4,700
0,56	0,056	0,067	0,084	0,101	0,123	0,151	0,185	0,218	0,263	0,314	0,381	0,459	0,560	0,672	0,840	1,008	1,232	1,512	1,848	2,184	2,632	3,136	3,808	4,592	5,600
0,68	0,068	0,082	0,102	0,122	0,150	0,184	0,224	0,265	0,320	0,381	0,462	0,558	0,680	0,816	1,020	1,224	1,496	1,836	2,244	2,652	3,196	3,808	4,624	5,576	6,800
0,82	0,082	0,098	0,123	0,148	0,180	0,221	0,271	0,320	0,385	0,459	0,558	0,672	0,820	0,984	1,230	1,476	1,804	2,214	2,706	3,198	3,854	4,592	5,576	6,724	8,200
1	0,100	0,120	0,150	0,180	0,220	0,270	0,330	0,390	0,470	0,560	0,680	0,820	1,000	1,200	1,500	1,800	2,200	2,700	3,300	3,900	4,700	5,600	6,800	8,200	10,000
1,2	0,120	0,144	0,180	0,216	0,264	0,324	0,396	0,468	0,564	0,672	0,816	0,984	1,200	1,440	1,800	2,160	2,640	3,240	3,960	4,680	5,640	6,720	8,160	9,840	12,000
1,5	0,150	0,180	0,225	0,270	0,330	0,405	0,495	0,585	0,705	0,840	1,020	1,230	1,500	1,800	2,250	2,700	3,300	4,050	4,950	5,850	7,050	8,400	10,200	12,300	15,000
1,8	0,180	0,216	0,270	0,324	0,396	0,486	0,594	0,702	0,846	1,008	1,224	1,476	1,800	2,160	2,700	3,240	3,960	4,860	5,940	7,020	8,460	10,080	12,240	14,760	18,000
2,2	0,220	0,264	0,330	0,396	0,484	0,594	0,726	0,858	1,034	1,232	1,496	1,804	2,200	2,640	3,300	3,960	4,840	5,940	7,260	8,580	10,340	12,320	14,960	18,040	22,000
2,7	0,270	0,324	0,405	0,486	0,594	0,729	0,891	1,053	1,269	1,512	1,836	2,214	2,700	3,240	4,050	4,860	5,940	7,290	8,910	10,530	12,690	15,120	18,360	22,140	27,000
3,3	0,330	0,396	0,495	0,594	0,726	0,891	1,089	1,287	1,551	1,848	2,244	2,706	3,300	3,960	4,950	5,940	7,260	8,910	10,890	12,870	15,510	18,480	22,440	27,060	33,000
3,9	0,390	0,468	0,585	0,702	0,858	1,053	1,287	1,521	1,833	2,184	2,652	3,198	3,900	4,680	5,850	7,020	8,580	10,530	12,870	15,210	18,330	21,840	26,520	31,980	39,000
4,7	0,470	0,564	0,705	0,846	1,034	1,269	1,551	1,833	2,209	2,632	3,196	3,854	4,700	5,640	7,050	8,460	10,340	12,690	15,510	18,330	22,090	26,320	31,960	38,540	47,000
5,6	0,560	0,672	0,840	1,008	1,232	1,512	1,848	2,184	2,632	3,136	3,808	4,592	5,600	6,720	8,400	10,080	12,320	15,120	18,480	21,840	26,320	31,360	38,080	45,920	56,000
6,8	0,680	0,816	1,020	1,224	1,496	1,836	2,244	2,652	3,196	3,808	4,624	5,576	6,800	8,160	10,200	12,240	14,960	18,360	22,440	26,520	31,960	38,080	46,240	55,760	68,000
8,2	0,820	0,984	1,230	1,476	1,804	2,214	2,706	3,198	3,854	4,592	5,576	6,724	8,200	9,840	12,300	14,760	18,040	22,140	27,060	31,980	38,540	45,920	55,760	67,240	82,000
10	1,000	1,200	1,500	1,800	2,200	2,700	3,300	3,900	4,700	5,600	6,800	8,200	10,000	12,000	15,000	18,000	22,000	27,000	33,000	39,000	47,000	56,000	68,000	82,000	100,000

$$\tau_1 \tau_2 = 0.968 \quad 1.0575 \quad 1.086$$

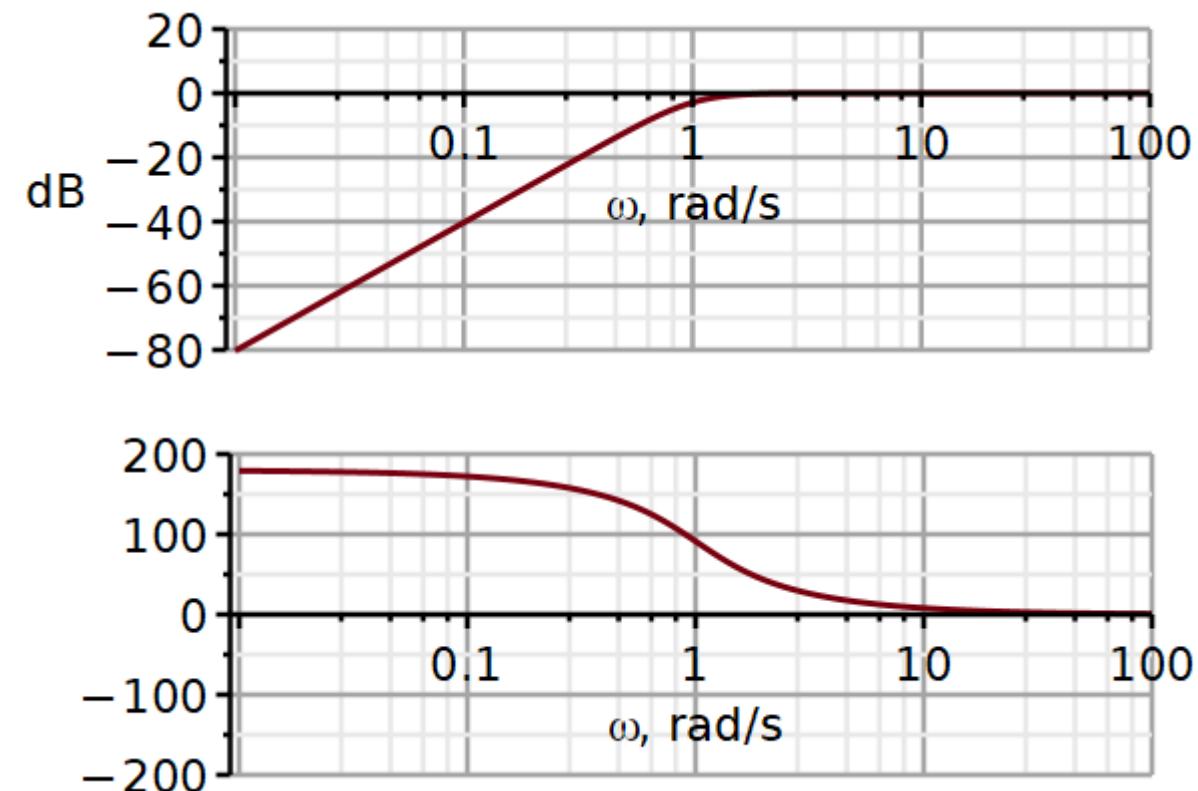
2nd order Sallen-Key high pass filter - Butterworth

In the RC look-up table on the previous slide we see that there are several options. Here we have chosen this combination:

$$\hat{R}_1 \approx 1.2\Omega$$

$$\hat{R}_2 \approx 0.56\Omega$$

$$\hat{C} \approx 1.2F$$



The low-frequency asymptote has a slope of +40 dB/decade, the cut-off is at 1 rad/s and the phase is 90 degrees at the cut-off. So, it looks promising.

The frequency characteristic for the frequency-normalized highpass filter. The hats indicate that the components have the value for a cut-off frequency of 1 rad/s.

We can multiply the normalized frequency variable $\hat{\omega}$ by a factor K_F if we also divide all capacitors by the same factor.

$$\omega \stackrel{\text{def}}{=} K_F \hat{\omega}$$

$$H_{HP}(j\hat{\omega}) = \frac{K(j\hat{\omega})^2}{(j\hat{\omega})^2 + j\hat{\omega} \left(\frac{1}{\hat{R}_1 \hat{C}_1} + \frac{1}{\hat{R}_1 \hat{C}_2} + \frac{(1-K)}{\hat{R}_2 \hat{C}_1} \right) + \frac{1}{\hat{R}_1 \hat{R}_2 \hat{C}_1 \hat{C}_2}}$$

$$H_{HP}(j\hat{\omega}) = \frac{K(j\hat{\omega})^2 \hat{C}_1 \hat{C}_2}{(j\hat{\omega})^2 \hat{C}_1 \hat{C}_2 + j\hat{\omega} \left(\frac{\hat{C}_2}{\hat{R}_1} + \frac{\hat{C}_1}{\hat{R}_1} + \frac{\hat{C}_2(1-K)}{\hat{R}_2} \right) + \frac{1}{\hat{R}_1 \hat{R}_2}}$$

$$H_{HP}(j\hat{\omega}) = \frac{K(jK_F \hat{\omega})^2 \frac{\hat{C}_1}{K_F} \frac{\hat{C}_2}{K_F}}{(jK_F \hat{\omega})^2 \frac{\hat{C}_1}{K_F} \frac{\hat{C}_2}{K_F} + jK_F \hat{\omega} \left(\frac{\hat{C}_2}{K_F \hat{R}_1} + \frac{\hat{C}_1}{K_F \hat{R}_1} + \frac{\hat{C}_2(1-K)}{K_F \hat{R}_2} \right) + \frac{1}{\hat{R}_1 \hat{R}_2}}$$

$$H_{HP}(j\omega) = \frac{K(j\omega)^2 \frac{\hat{C}_1}{K_F} \frac{\hat{C}_2}{K_F}}{(j\omega)^2 \frac{\hat{C}_1}{K_F} \frac{\hat{C}_2}{K_F} + j\omega \left(\frac{\hat{C}_2}{K_F \hat{R}_1} + \frac{\hat{C}_1}{K_F \hat{R}_1} + \frac{\hat{C}_2(1-K)}{K_F \hat{R}_2} \right) + \frac{1}{\hat{R}_1 \hat{R}_2}}$$

By frequency scaling, we scale the normalized frequency axis while also downscaling all capacitors.

The shape of the magnitude and phase curves are determined by the resistance values and the values of the capacitor impedances. At the normalized cut-off frequency $\hat{\omega}_c$, the resistors and the capacitor impedances have certain values.

The frequency scaling changes the frequency scale and the capacitor values without changing the capacitor impedance. So, we now have the same capacitor impedance, but at a new cut-off frequency ω_c .

$$H_{HP}(j\omega) = \frac{K(j\omega)^2 \frac{\hat{C}_1}{K_F} \frac{\hat{C}_2}{K_F}}{(j\omega)^2 \frac{\hat{C}_1}{K_F} \frac{\hat{C}_2}{K_F} + j\omega \left(\frac{\hat{C}_2}{K_F \hat{R}_1} + \frac{\hat{C}_1}{K_F \hat{R}_1} + \frac{\hat{C}_2(1-K)}{K_F \hat{R}_2} \right) + \frac{1}{\hat{R}_1 \hat{R}_2}}$$

$$H_{HP}(j\omega) = \frac{K(j\omega)^2 C'_1 C'_2}{(j\omega)^2 C'_1 C'_2 + j\omega \left(\frac{C'_2}{R'_1} + \frac{C'_1}{R'_1} + \frac{C'_2(1-K)}{R'_2} \right) + \frac{1}{R'_1 R'_2}}$$

$$R' = \hat{R} \quad C' \stackrel{\text{def}}{=} \frac{\hat{C}}{K_F}$$

$$Z_C = \frac{1}{j\hat{\omega}_c \hat{C}} = \frac{1}{j(K_F \hat{\omega}_c) \left(\frac{\hat{C}}{K_F} \right)} = \frac{1}{j\omega_c C'}$$

2nd order Sallen-Key high pass filter - Butterworth

Frequency scaling

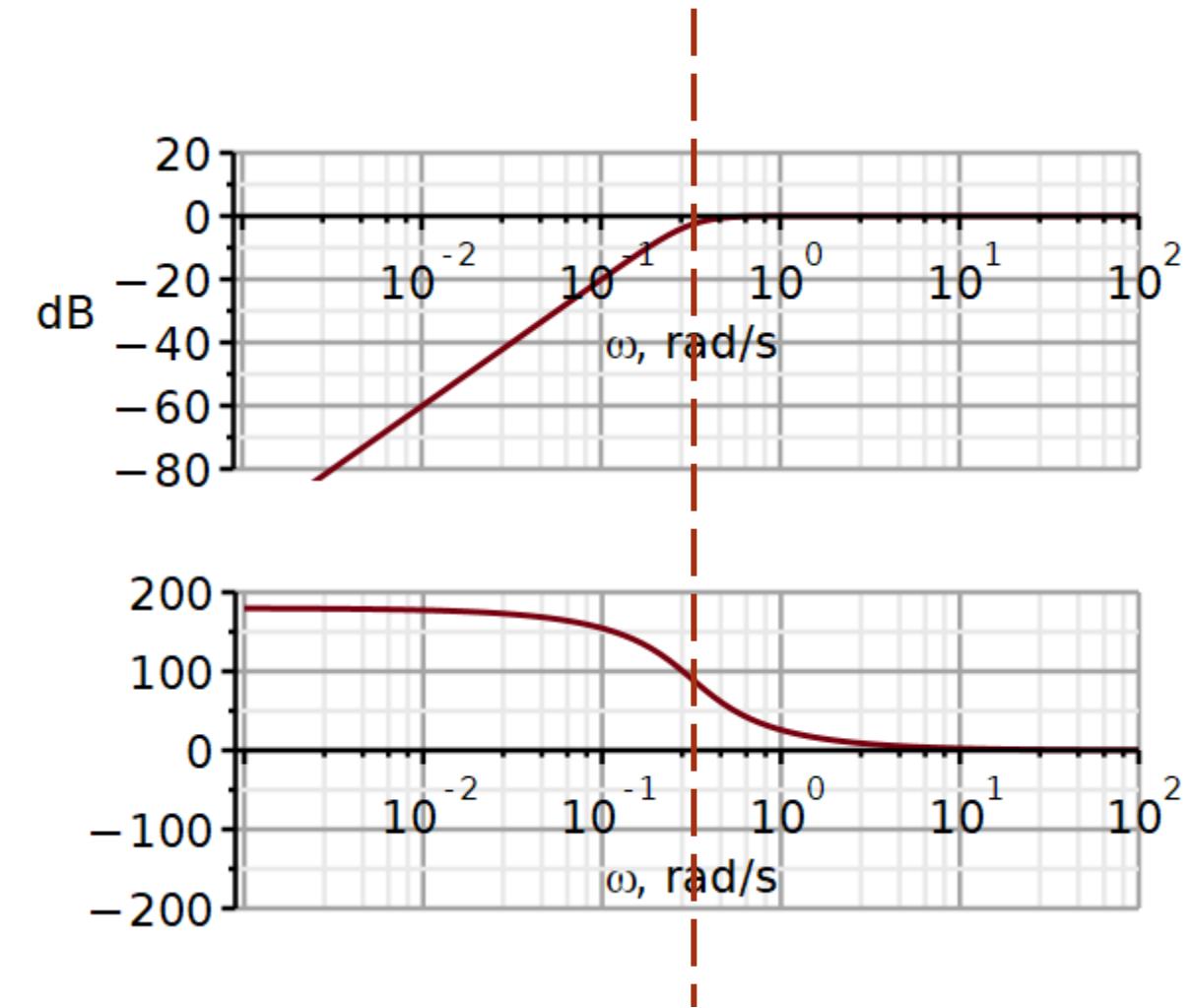
$$K_F = 2\pi \cdot 0.05 = 0.314159$$

$$C'_1 = \frac{\hat{C}_1}{K_F} = \frac{1.2F}{0.314159} = 3.82F \approx 3.9 F$$

$$C'_2 = \frac{\hat{C}_1}{K_F} = 3.9 F$$

$$R'_1 = \hat{R}_1 = 1.2\Omega$$

$$R'_2 = \hat{R}_2 = 0.56\Omega$$



2nd order Sallen-Key high pass filter - Butterworth

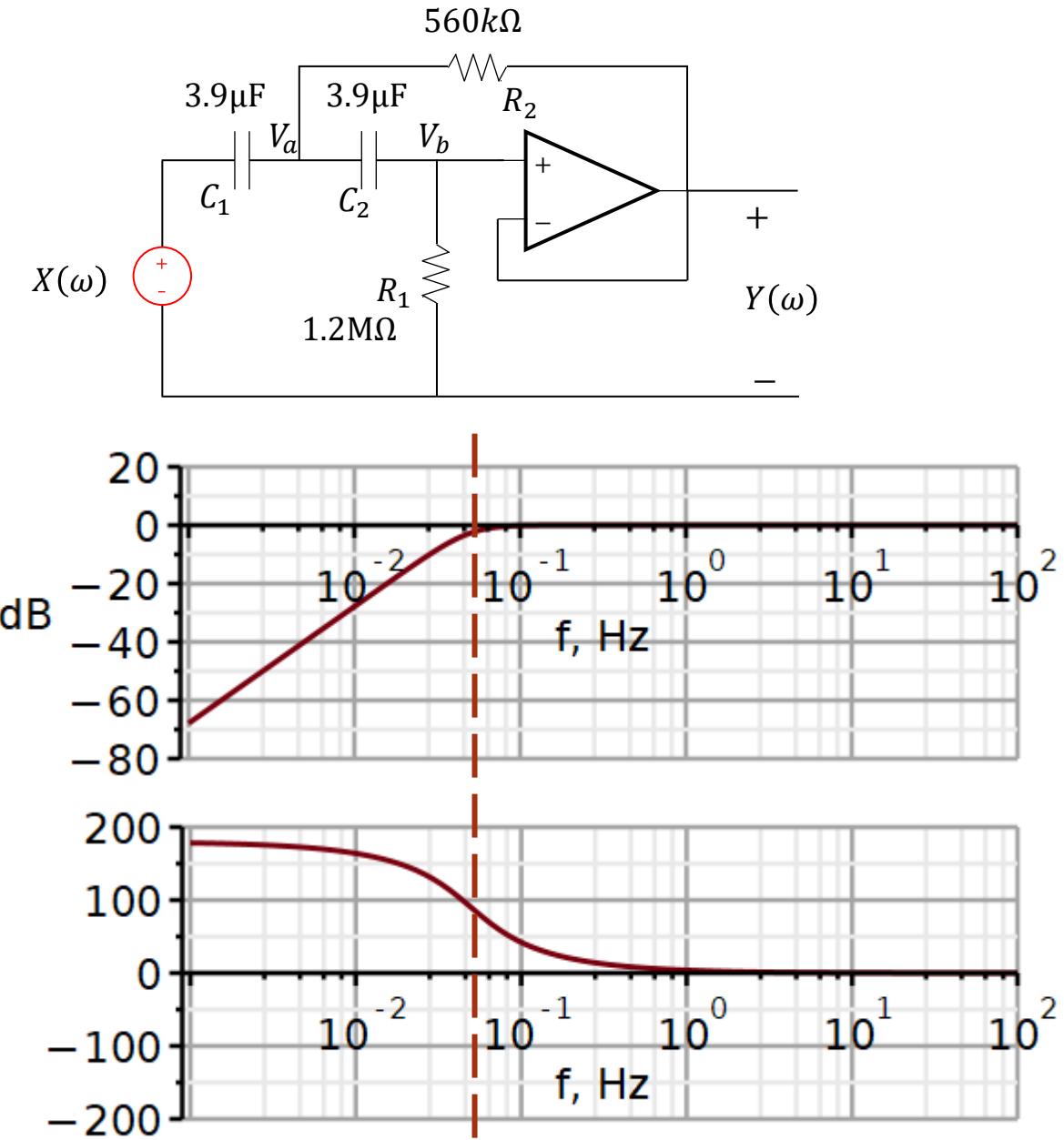
Impedance scaling:

$$C_1 = C_2 = 3.9\mu F \quad \text{← desired capacitor value}$$

$$K_z = \frac{C'_1}{3.9 \cdot 10^{-6} F} = \frac{3.9 F}{3.9 \cdot 10^{-6} F} = 1 \times 10^6$$

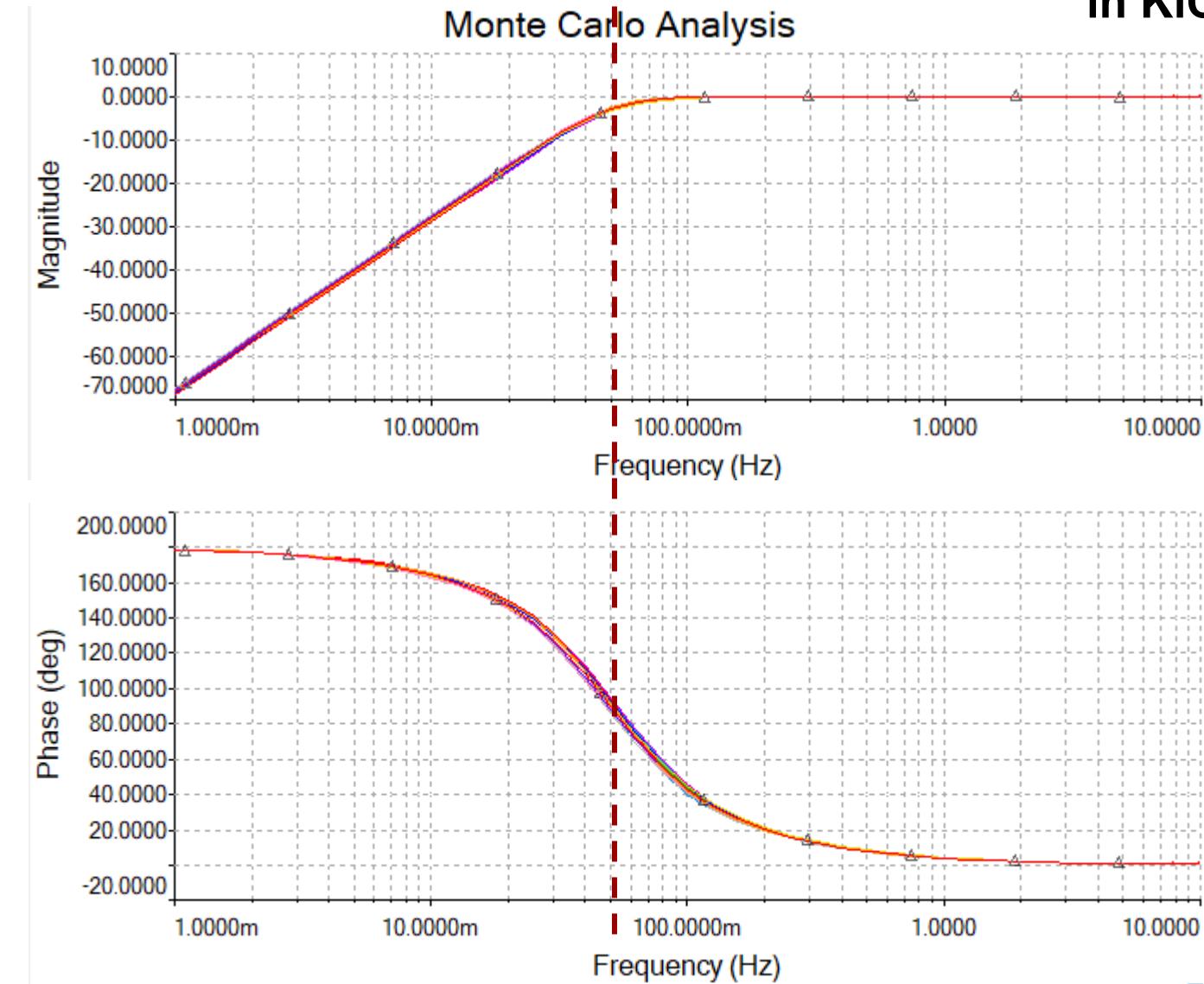
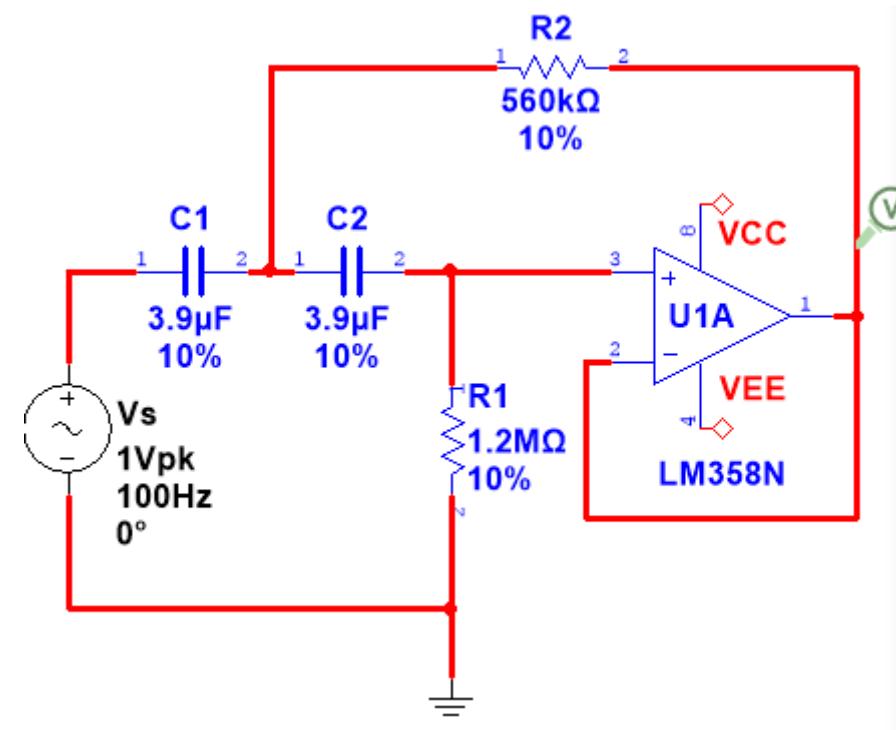
$$R_1 = K_z \cdot R'_1 = K_z \cdot 1.2\Omega = 1.2M\Omega$$

$$R_2 = K_z \cdot R'_2 = K_z \cdot 0.56\Omega = 560k\Omega$$



Monte Carlo analysis of original Highpass filter

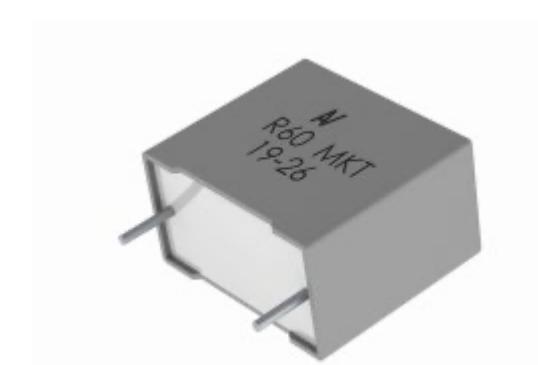
Not available in KiCad



Size of capacitors

Capacitors for filter circuits can be quite large.

This one will not fit in a Smartwatch.



VDC	VAC	Capacitance Value (μF)	Dimensions in mm			Lead Spacing
			B	H	L	
63	40	0.68	5.0	11.0	18.0	15.0
63	40	1.0	5.0	11.0	18.0	15.0
63	40	1.5	5.0	11.0	18.0	15.0
63	40	2.2	6.0	12.0	18.0	15.0
63	40	3.3	7.5	13.5	18.0	15.0
63	40	3.3	9.0	12.5	18.0	15.0
63	40	4.7	8.5	14.5	18.0	15.0
63	40	4.7	13.0	12.0	18.0	15.0
63	40	6.8	10.0	16.0	18.0	15.0
63	40	3.3	6.0	15.0	26.5	22.5
63	40	4.7	7.0	16.0	26.5	22.5
63	40	6.8	7.0	16.0	26.5	22.5
63	40	10.0	8.5	17.0	26.5	22.5
63	40	15.0	11.0	20.0	26.5	22.5
63	40	4.7	13.0	12.0	32.0	27.5
63	40	6.8	13.0	12.0	32.0	27.5
63	40	10.0	9.0	17.0	32.0	27.5
63	40	15.0	11.0	20.0	32.0	27.5
63	40	15.0	24.0	15.0	32.0	27.5
63	40	22.0	13.0	22.0	32.0	27.5
63	40	22.0	24.0	15.0	32.0	27.5
63	40	22.0	11.0	20.0	32.0	27.5
63	40	33.0	18.0	33.0	32.0	27.5
63	40	47.0	18.0	33.0	32.0	27.5
63	40	47.0	22.0	37.0	32.0	27.5
63	40	68.0	22.0	37.0	32.0	27.5
63	40	22.0	11.0	22.0	41.5	37.5
63	40	22.0	24.0	15.0	41.5	37.5
63	40	33.0	16.0	28.5	41.5	37.5
63	40	33.0	24.0	15.0	41.5	37.5
63	40	47.0	16.0	28.5	41.5	37.5
63	40	47.0	24.0	19.0	41.5	37.5
63	40	68.0	19.0	32.0	41.5	37.5
63	40	100.0	20.0	40.0	41.5	37.5
63	40	100.0	24.0	44.0	41.5	37.5
100	63	0.33	5.0	11.0	18.0	15.0
100	63	0.47	5.0	11.0	18.0	15.0
VDC	VAC	Capacitance Value (μF)	B (mm)	H (mm)	L (mm)	Lead Spacing

<https://docs.rs-online.com/74f3/0900766b8171c883.pdf>

Instrumentation amplifier

Video

Applications of instrumentation amplifiers

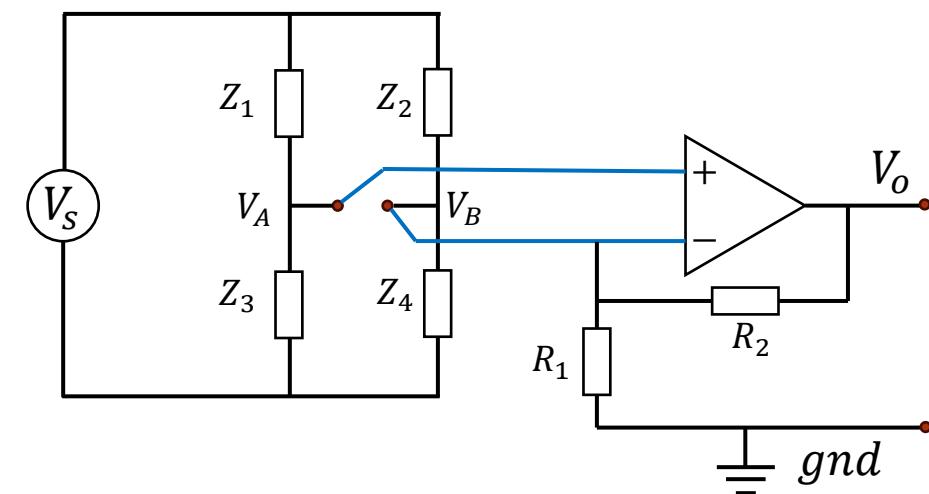
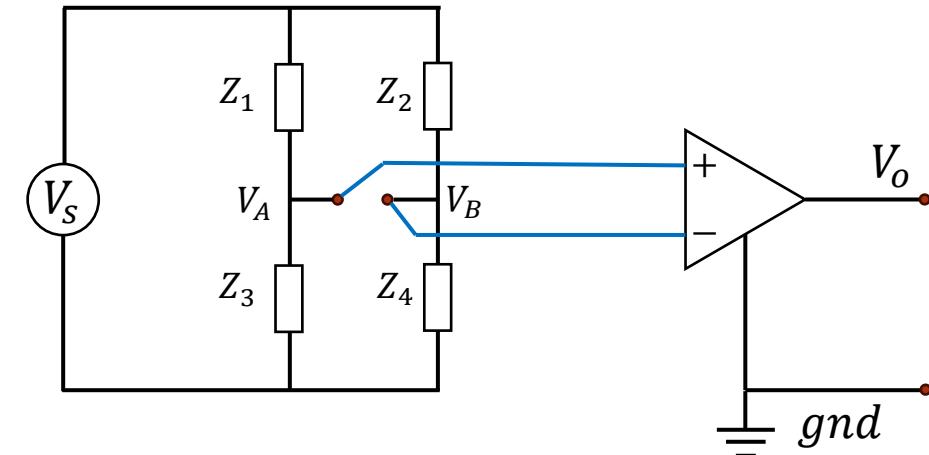
On the right is shown a Wheatstone bridge with four impedances and an AC voltage supply V_s .

An amplifier is to amplify the voltage difference $V_A - V_B$. We have not been specific about the type of amplifier.

Let us try with one of our favorites, the non-inverting op-amp:

The nodal point A is connected to the positive terminal of the op-amp. No current will run toward the op-amp to the positive terminal and V_A will not be perturbed by the amplifier.

Current will run toward R_1 and R_2 , hence V_B will certainly change when the amplifier is connected. Since the amplifier is not allowed to perturb the system to be amplified, we cannot use this type of amplifier.



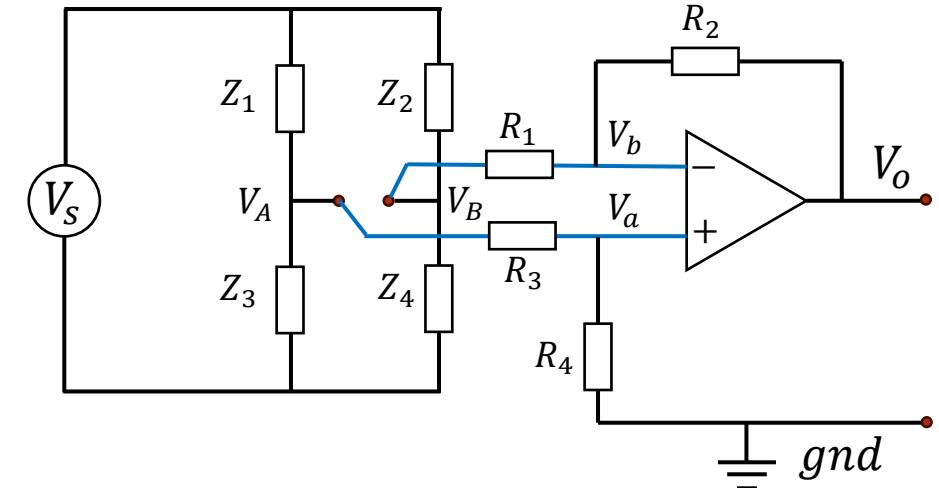
Applications of instrumentation amplifiers

Now an op-amp is used in a **differential amplifier** circuit and connected to the Wheatstone bridge.

We need to derive an equation for the input-output relationship for this differential amplifier.

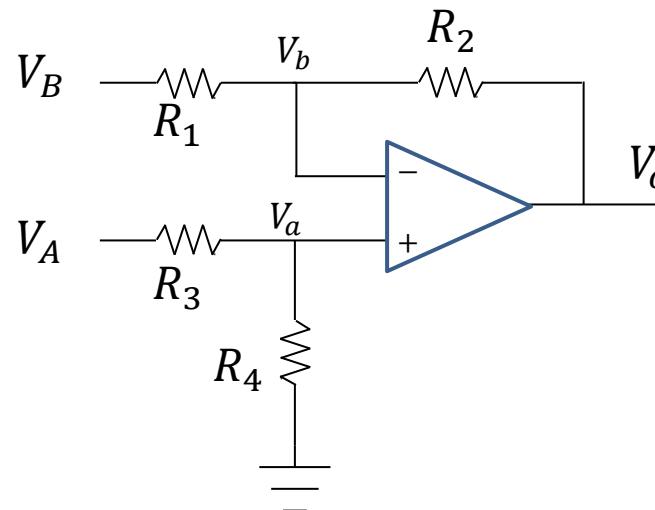
We see that there is still a **potential loading** issue where the amplifier draws current out of the nodal points in the Wheatstone.

How much loading we will experience depends on the input resistance of the amplifier. Hence, we need to also derive the **input resistance**.



The difference amplifier

Using source superposition:



$$V_{o,A} \Big|_{V_B=0} = \left(1 + \frac{R_2}{R_1}\right) V_a = \left(1 + \frac{R_2}{R_1}\right) \left(\frac{R_4}{R_3 + R_4}\right) V_A$$

$$V_{o,B} \Big|_{V_A=0} = -\frac{R_2}{R_1} V_B$$

$$V_o = V_{o,A} \Big|_{V_B=0} + V_{o,B} \Big|_{V_A=0} = \left(\frac{R_1 + R_2}{R_1}\right) \left(\frac{R_4}{R_3 + R_4}\right) V_A - \frac{R_2}{R_1} V_B$$

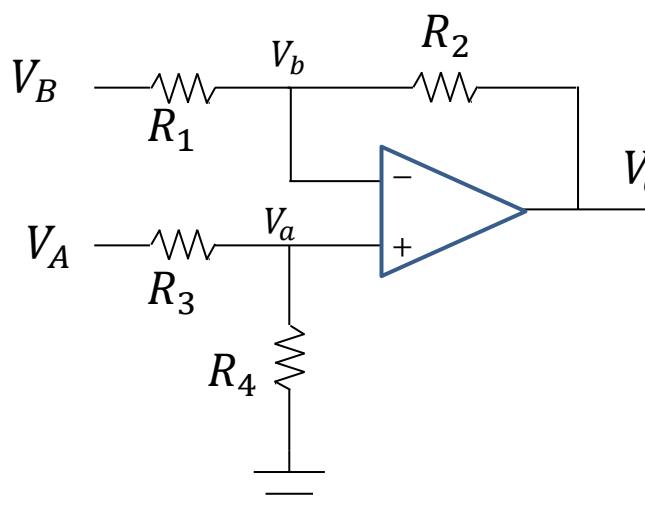
By matching resistance values, we can simplify the circuit equation.

$$V_o = \frac{R_4}{R_3} (V_A - V_B) \quad R_1 = R_3 \quad R_2 = R_4$$

The input impedance

A weak feature of this circuit is its input impedances.

We use this opportunity to learn about input impedance.



$$Z_{in} \stackrel{\text{def}}{=} \left. \frac{V_{in}}{I_{in}} \right|_{V_o=0}$$

$$Z_{in+} \stackrel{\text{def}}{=} \left. \frac{V_A}{I_{in}} \right|_{V_o=0} = R_3 + R_4$$

$$Z_{in-} \stackrel{\text{def}}{=} \left. \frac{V_B}{I_{in}} \right|_{V_o=0} = \left. \frac{V_B}{\frac{V_B - V_b}{R_1}} \right|_{V_o=0}$$

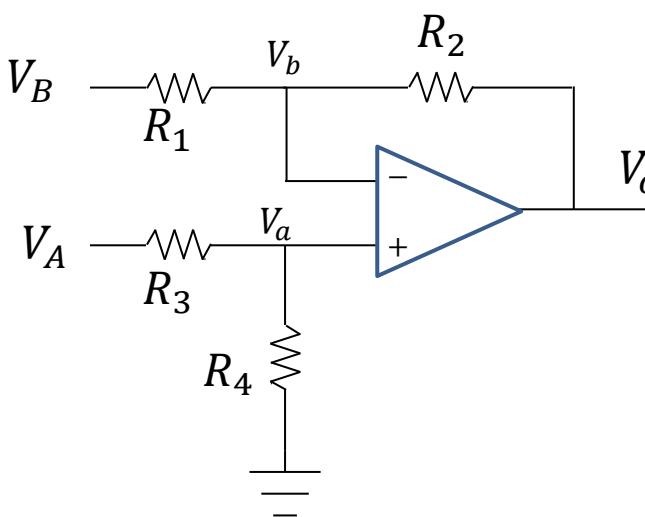
$$V_b = V_a = \frac{R_4}{R_3 + R_4} V_A$$

$$Z_{in-} = R_1 \left. \frac{V_B}{V_B - \frac{R_4}{R_3 + R_4} V_A} \right|_{V_o=0}$$

$$Z_{in-} = R_1 \left. \frac{1}{1 - \frac{R_4}{R_3 + R_4} \frac{V_A}{V_B}} \right|_{V_o=0}$$

The input impedance

Example values:



$$Z_{in+} \stackrel{\text{def}}{=} \left. \frac{V_A}{I_{in}} \right|_{V_o=0} = R_3 + R_4$$

$$Z_{in-} = R_1 \left. \frac{1}{1 - \frac{R_4}{R_3 + R_4} \frac{V_A}{V_B}} \right|_{V_o=0}$$

$$V_A = V_B; R_1 = R_2 = R_3 = R_4 = 10\text{k}\Omega;$$

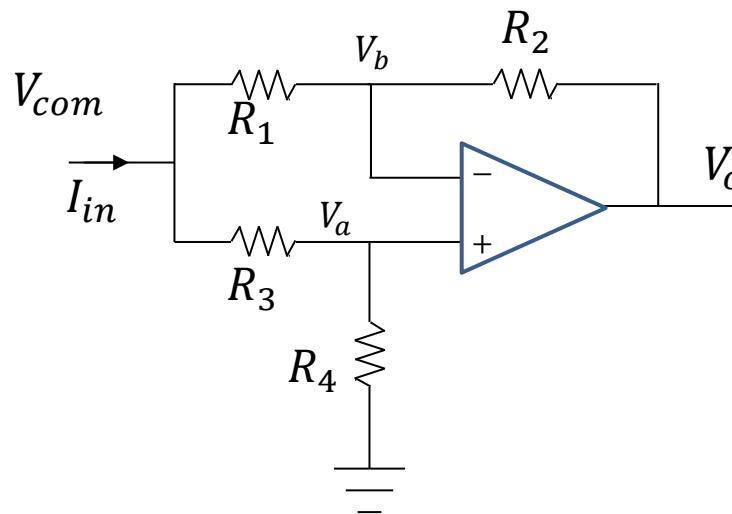
$$Z_{in+} = 20\text{k}\Omega \quad Z_{in-} = 20\text{k}\Omega$$

$$V_A = -V_B; R_1 = R_2 = R_3 = R_4 = 10\text{k}\Omega;$$

$$Z_{in+} = 20\text{k}\Omega \quad Z_{in-} = 6.66\text{k}\Omega$$

The input impedance

A third input impedance is the **common mode input impedance**. It is the impedance that a common mode signal is facing.



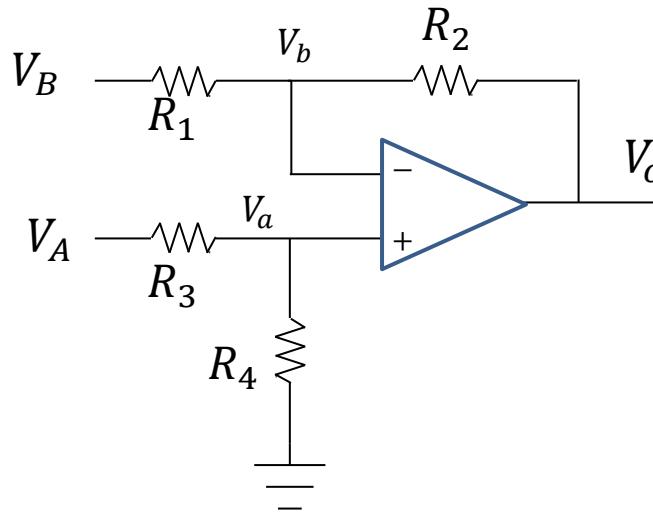
$$Z_{in,com} \stackrel{\text{def}}{=} \left. \frac{V_{com}}{I_{in}} \right|_{V_o=0} = Z_{in-} || Z_{in+}$$

$$Z_{in+} = 20\text{k}\Omega \quad Z_{in-} = 20\text{k}\Omega$$

$$Z_{in,com} = \frac{20\text{k}\Omega \cdot 20\text{k}\Omega}{20\text{k}\Omega + 20\text{k}\Omega} = 10\text{k}\Omega$$

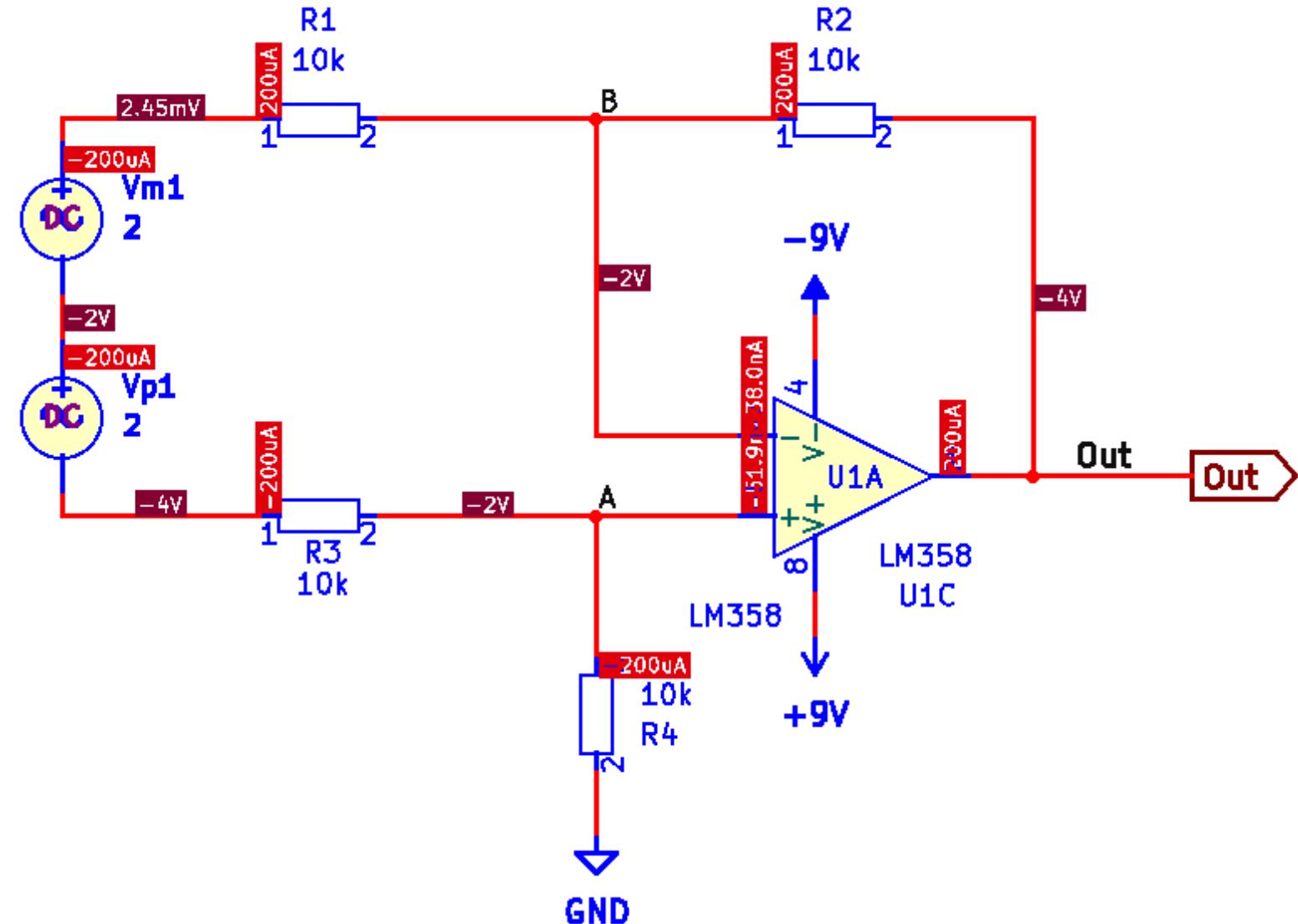
The input impedance

Differential mode input impedance:



$$Z_{in,dif} \stackrel{\text{def}}{=} \left. \frac{V_A - V_B}{I_{in}} \right|_{V_o=0} = R_1 + R_3 = 20k\Omega$$

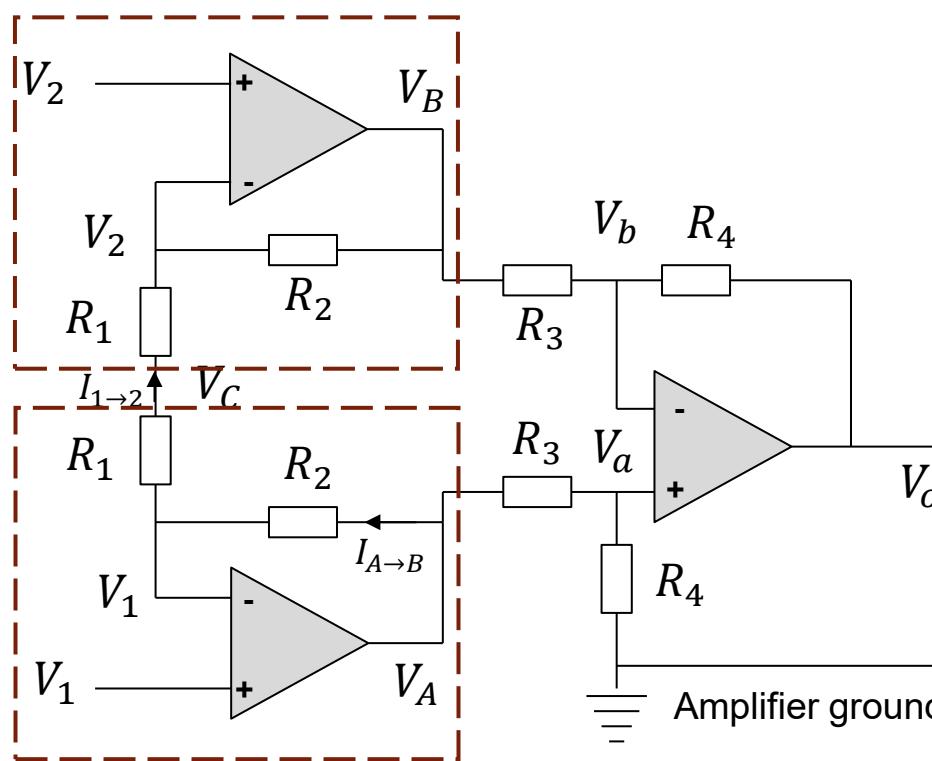
Remember that $V_a \approx V_b$ so it is as if R_1 and R_3 are in series.



$$Z_{in,diff} = \frac{4V}{0.2mA} = 20k\Omega$$

The instrumentation amplifier

To increase the input impedance, we have now **added non-inverting amplifier** in front of V_A and V_B .



$$I_{A \rightarrow B} = I_{1 \rightarrow 2}$$

$$\frac{V_A - V_B}{2(R_1 + R_2)} = \underbrace{\frac{V_1 - V_2}{2R_1}}_{I_{1 \rightarrow 2}}$$

$$V_A - V_B = \frac{2(R_1 + R_2)}{2R_1} (V_1 - V_2)$$

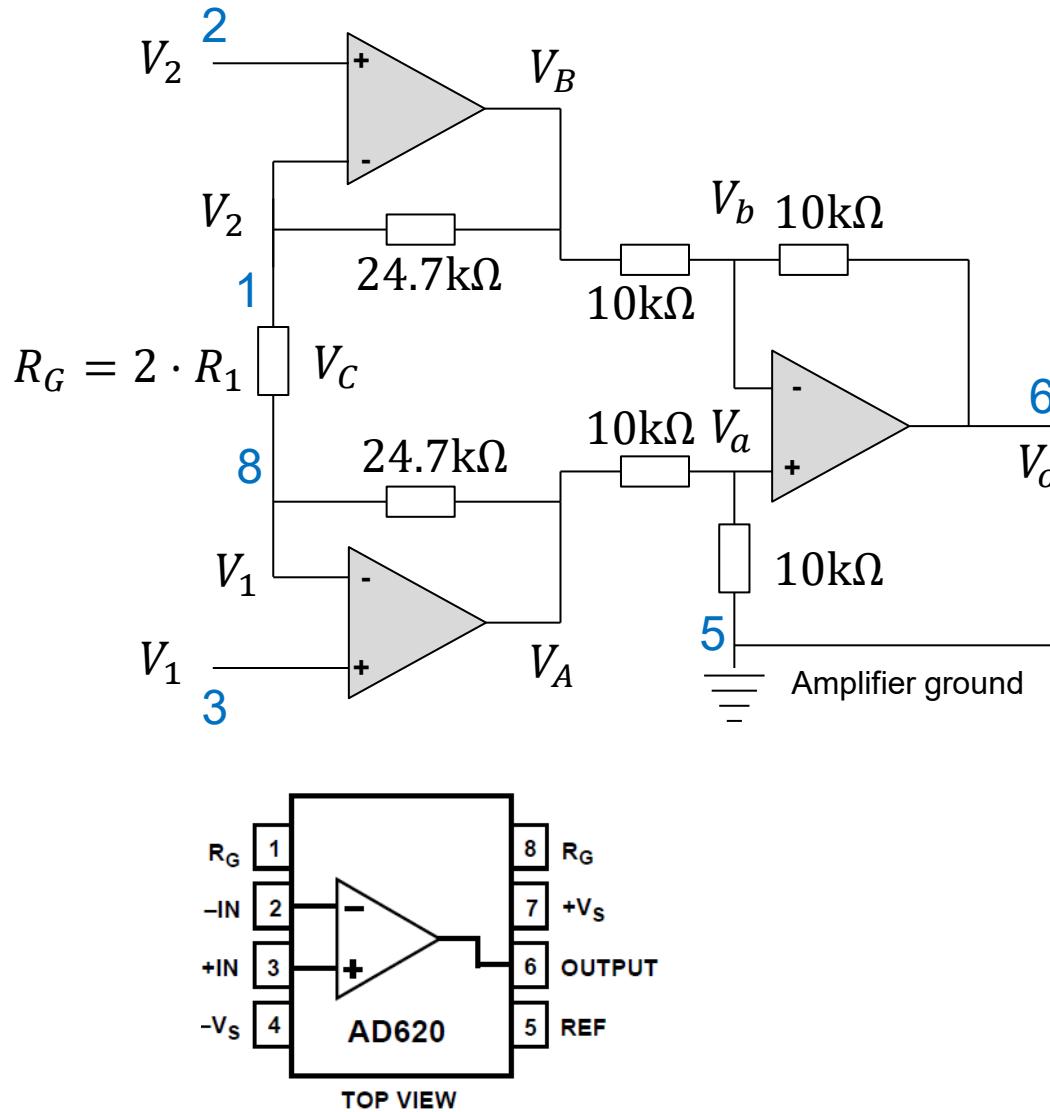
$$V_A - V_B = \left(1 + \frac{R_2}{R_1}\right) (V_1 - V_2)$$

Inserting into:

$$V_o = \frac{R_4}{R_3} (V_A - V_B)$$

$$V_o = \frac{R_4}{R_3} \left(1 + \frac{R_2}{R_1}\right) (V_1 - V_2)$$

The AD620 Instrumentation amplifier



$$V_o = \frac{R_4}{R_3} \left(1 + \frac{R_2}{R_G/2} \right) (V_1 - V_2)$$

$$V_o = \frac{10\text{k}\Omega}{10\text{k}\Omega} \left(1 + \frac{24.7\text{k}\Omega}{R_G/2} \right) (V_1 - V_2)$$

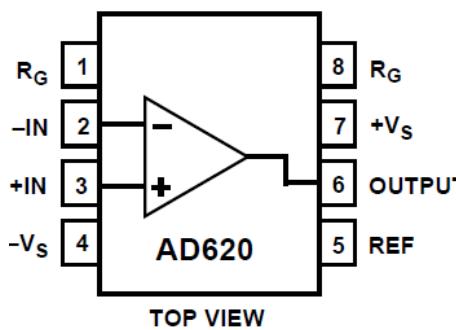
$$V_o = \left(1 + \frac{49.4\text{k}\Omega}{R_G} \right) (V_1 - V_2)$$

Gain

$$G = 1 + \frac{49.4\text{k}\Omega}{R_G}$$

Gain resistor

$$R_G = \frac{49.4\text{k}\Omega}{G - 1}$$



AC coupled instrumentation amplifier

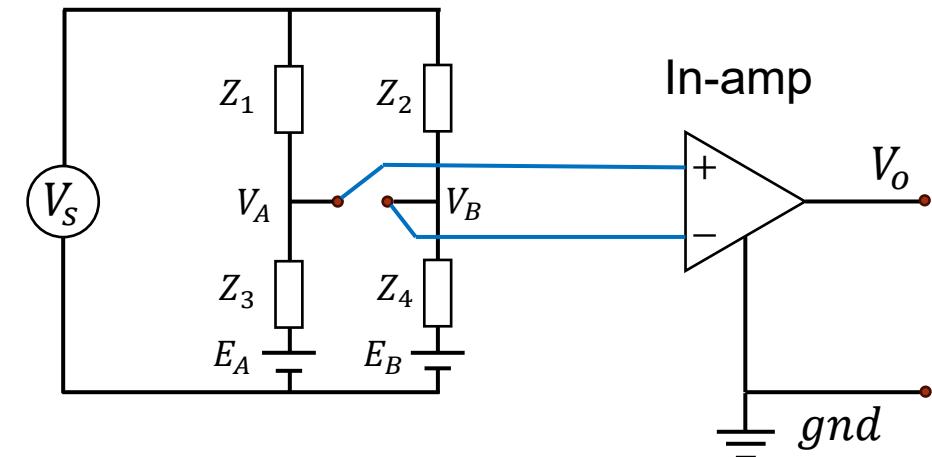
Video

DC components in source

Now we will add a DC source in series with Z_3 and Z_4 . This may challenge the amplifier.

We need to derive an expression for $V_A - V_B$.

Using the technique of current superposition:



$$\text{Node A: } \frac{V_A}{Z_A} = \frac{V_s}{Z_1} + \frac{E_A}{Z_3} \quad Z_A \stackrel{\text{def}}{=} \frac{Z_1 Z_3}{Z_1 + Z_3}$$

$$V_A = \frac{Z_A}{Z_1} V_s + \frac{Z_A}{Z_3} E_A = \frac{Z_3}{Z_1 + Z_3} V_s + \frac{Z_1}{Z_1 + Z_3} E_A$$

$$\text{Node B: } \frac{V_B}{Z_B} = \frac{V_s}{Z_2} + \frac{E_B}{Z_4} \quad Z_B \stackrel{\text{def}}{=} \frac{Z_2 Z_4}{Z_2 + Z_4}$$

$$V_B = \frac{Z_B}{Z_2} V_s + \frac{Z_B}{Z_4} E_B = \frac{Z_4}{Z_2 + Z_4} V_s + \frac{Z_2}{Z_2 + Z_4} E_B$$

As expected, the amplifier now must amplify both an AC and a DC signal:

$$V_A - V_B = \underbrace{\left(\frac{Z_3}{Z_1 + Z_3} - \frac{Z_4}{Z_2 + Z_4} \right) V_s}_{\text{AC signal}} + \underbrace{\left(\frac{Z_1}{Z_1 + Z_3} E_A - \frac{Z_2}{Z_2 + Z_4} E_B \right)}_{\text{DC signal}}$$

Limited gain due to DC component

We have not been specific about the context, so we do not know anything about the magnitudes of the impedances, nor the magnitude of the AC and DC sources.

We will assume that all impedances have very similar values, but that there is a small detectable difference ΔZ between Z_3 and Z_4 .

$$Z_1 = Z_2$$

$$Z_3 = Z_4 + \Delta Z$$

$$\Delta Z \ll \min(Z_1, Z_2, Z_3, Z_4)$$

$$V_A - V_B = \underbrace{\left(\frac{Z_3}{Z_1 + Z_3} - \frac{Z_4}{Z_2 + Z_4} \right) V_s}_{AC\ signal} + \underbrace{\left(\frac{Z_1}{Z_1 + Z_3} E_A - \frac{Z_2}{Z_2 + Z_4} E_B \right)}_{DC\ signal}$$

$$V_{AC} = \left(\frac{Z_3}{Z_1 + Z_3} - \frac{Z_4}{Z_2 + Z_4} \right) V_s = \left(\frac{Z_4 + \Delta Z}{Z_1 + Z_4 + \Delta Z} - \frac{Z_4}{Z_1 + Z_4} \right) V_s$$

$$\approx \left(\frac{Z_4 + \Delta Z}{Z_1 + Z_4} - \frac{Z_4}{Z_1 + Z_4} \right) V_s \approx \frac{\Delta Z}{Z_1 + Z_4} V_s$$

$$V_{DC} = \frac{Z_1}{Z_1 + Z_3} E_A - \frac{Z_2}{Z_2 + Z_4} E_B = \frac{Z_1}{Z_1 + Z_4 + \Delta Z} E_A - \frac{Z_1}{Z_1 + Z_4} E_B$$

$$\approx \frac{Z_1}{Z_1 + Z_4} (E_A - E_B)$$

$$V_A - V_B = V_{AC} + V_{DC}$$

Limited gain due to DC component

For the sake of illustration we will assume that the AC signal is in the range of $\pm 2.5\text{mV}$:

We will assume that the DC sources are half-cell voltages originating from within the sensors Z_3 and Z_4 . These DC sources can be hundreds of millivolts thus their difference can also be hundreds of millivolts. Here we will assume a DC voltage of 100mV .

We don't care about the DC signal. However, we want to amplify the AC signal to the full range of a 5V ADC.

To amplify the AC signal adequately, we need a gain of:

This gain will saturate the amplifier because the DC voltage on the output would have to be 100V .

If we are to obtain a gain of 1000 for the AC signal, we must remove the DC signal with a highpass filter.

$$V_{AC} = \underbrace{\frac{\Delta Z}{Z_1 + Z_4}}_{\ll 1} V_s \approx \pm 2.5\text{mV}$$

$$V_{DC} = \underbrace{\frac{Z_1}{Z_1 + Z_4}}_{\approx 1/2} (E_A - E_B) \approx 100\text{mV}$$

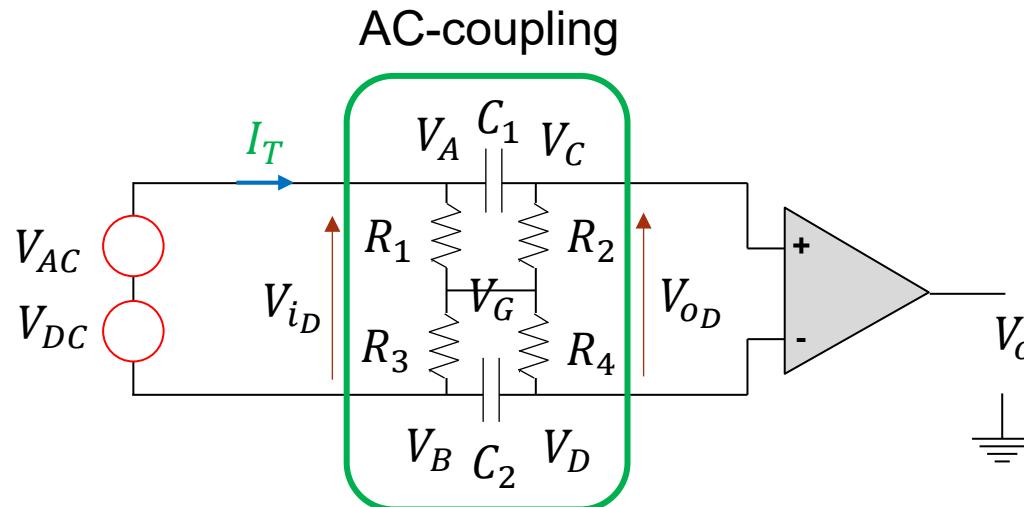
$$V_o = G_d \cdot V_{AC} + G_d \cdot V_{DC}$$

$$G_d = \frac{V_o}{V_{AC}} = \frac{5\text{V}}{5\text{mV}} = 1000$$

$$V_{o,DC} = 1000 \cdot 100\text{mV} = 100\text{V}$$

AC coupled In-amp at input

We need to block the DC potential before the instrumentation amplifier. Hence, we need an AC-coupled instrumentation amplifier.



V_{iD} : Differential input signal

V_{oD} : Differential output signal

Starting at the filter output:

$$V_{oD} = R_2 \textcolor{red}{I}_{R_2} + R_4 \textcolor{blue}{I}_{R_4}$$

Current dividers:

$$\textcolor{red}{I}_{R_2} = \textcolor{green}{I}_T \cdot \frac{R_1}{R_1 + R_2 + Z_{C_1}}$$

$$\textcolor{blue}{I}_{R_4} = \textcolor{green}{I}_T \cdot \frac{R_3}{R_3 + R_4 + Z_{C_2}}$$

$$\textcolor{green}{I}_T = \frac{V_{iD}}{\textcolor{violet}{Z}_T}$$

$$\textcolor{violet}{Z}_T = R_1 \parallel (R_2 + Z_{C_1}) + R_3 \parallel (R_4 + Z_{C_2})$$

AC coupled In-amp at input

From Maple:

$$\frac{V_{o_D}}{V_{i_D}} = \frac{\left(((R_2 + R_4)R_3 + R_2(R_4 + Z_{C_2}))R_1 + R_3R_4(R_2 + Z_{C_1}) \right)}{\left((R_2 + R_4 + Z_{C_1} + Z_{C_2})R_3 + (R_4 + Z_{C_2})(R_2 + Z_{C_1}) \right)R_1 + R_3(R_4 + Z_{C_2})(R_2 + Z_{C_1})}$$

To obtain symmetry we set:

$$R_3 = R_1 \quad R_4 = R_2 \quad C_2 = C_1$$

Letting Maple simplify again:

$$\frac{V_{o_D}}{V_{i_D}} = \frac{R_2}{R_2 + Z_{C_1}}$$

If all components are matched perfectly:

$$R_1 = R_2 = R_3 = R_4 \quad C_2 = C_1$$

$$\frac{V_{o_D}(j\omega)}{V_{i_D}(j\omega)} = \frac{(j\omega) R_1 C_1}{(j\omega) R_1 C_1 + 1}$$

We obtain a classic first order highpass filter with cut-off frequency:

$$f_c = \frac{1}{2\pi R_1 C_1} \quad C_1 = \frac{1}{2\pi R_1 f_c}$$

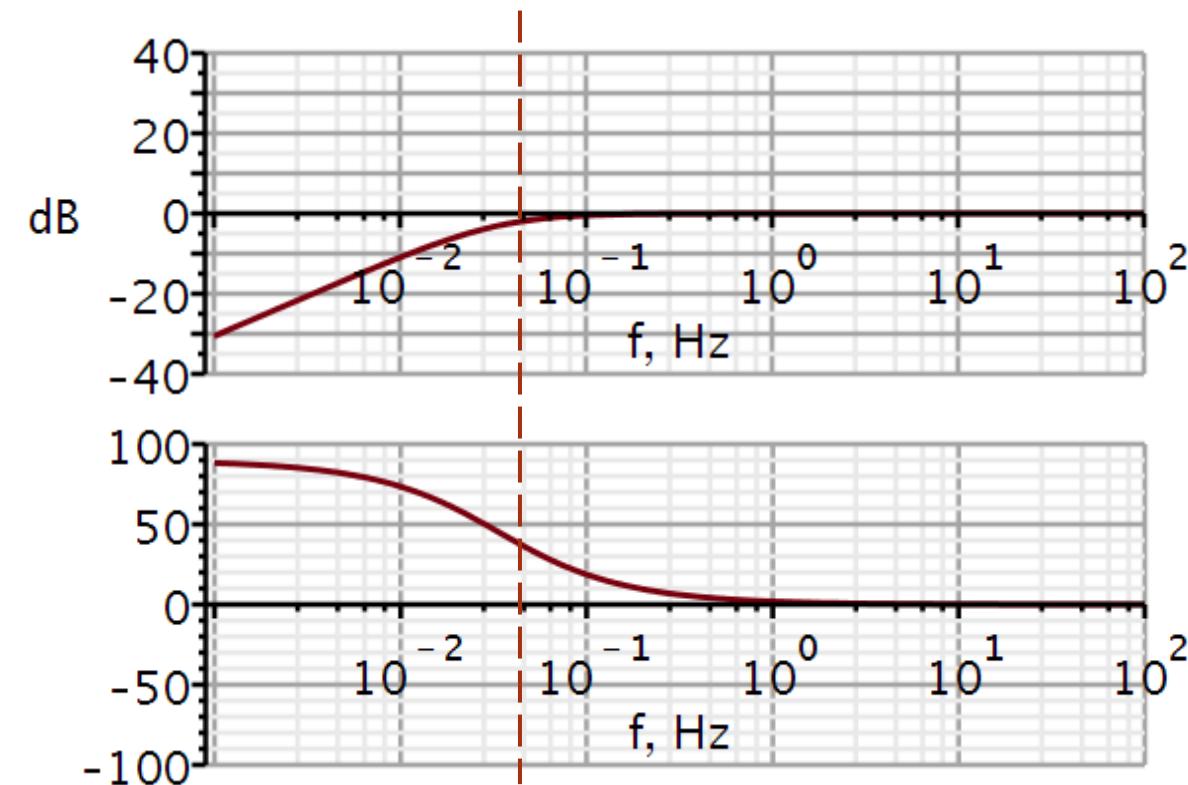
AC coupled In-amp at input

We need a high input impedance, hence R_1 and R_2 must be chosen in the $M\Omega$ range.

$$R_1 = R_2 = 4.7M\Omega$$

$$f_C = 0.03 \text{ Hz} \Rightarrow C_1 = C_2 = \frac{1}{2\pi 4.7M\Omega 0.03\text{Hz}} = 1.12\mu\text{F} \approx 1\mu\text{F}$$

$$f_C = 20 \text{ Hz} \Rightarrow C_1 = C_2 = \frac{1}{2\pi 4.7M\Omega 20\text{Hz}} = 1.7\text{nF} \approx 1.5\text{nF}$$



Instrumentation amplifier Common mode rejection ratio

Video

Common mode rejection ratio

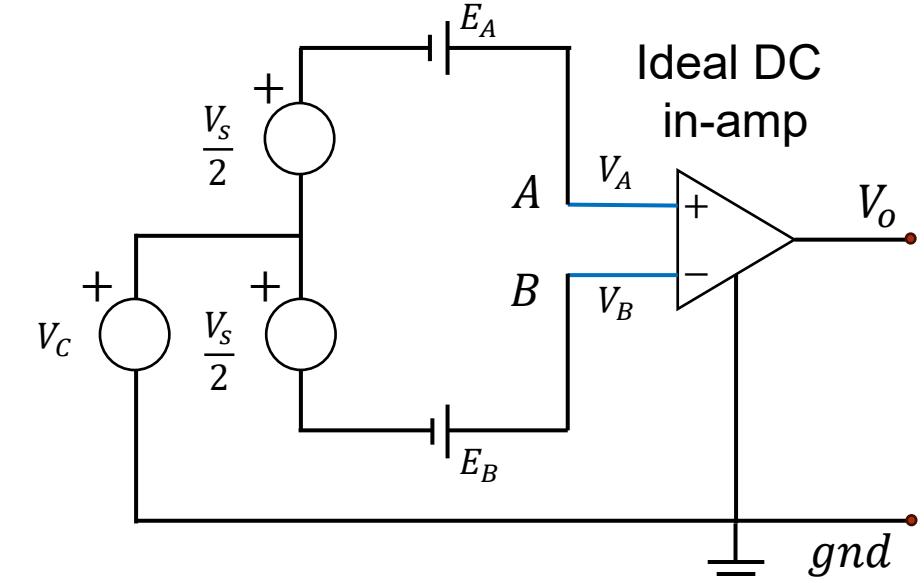
An important feature of the instrumentation amplifier is to reject common mode signal and only to amplify differential signals.

To the right, a combination of sources are now feeding signals to an in-amp. We would like to set up equations to determine the output signal.

A new source element is the common mode signal V_c . It can be either a DC or AC signal. A typical common mode signal is 50Hz noise from power lines.

For the purpose of symmetry, the differential source V_s has been split into two sources of equal magnitude.

If we assume an ideal in-amp with differential gain G_d , then output will be:



$$V_A = V_c + \frac{V_s}{2} + E_A$$

$$V_B = V_c - \frac{V_s}{2} + E_B$$

$$V_A - V_B = V_s + E_A - E_B$$

$$V_o = G_d \cdot V_s + G_d \cdot (E_A - E_B)$$

Again, a differential DC signal will be amplified. However, we can remove this with an AC-coupled in-amp.

Common mode rejection ratio

Now we will assume that the in-amp is **non-ideal**. This implies that some small common mode signal will be present at the output.

$$V_o = \underbrace{G_d \cdot V_{id}}_{V_{od}} + \underbrace{G_c \cdot V_{ic}}_{V_{oc}}$$

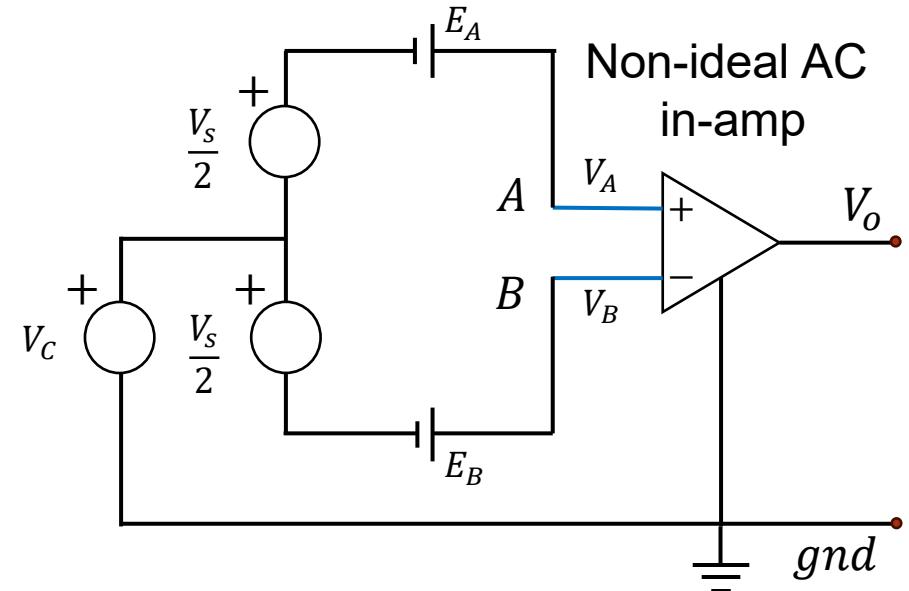
Differential mode gain

$$G_d \stackrel{\text{def}}{=} \frac{V_{od}}{V_{id}}$$

Common mode gain

$$G_c \stackrel{\text{def}}{=} \frac{V_{oc}}{V_{ic}}$$

For a high performance in-amp, G_d will be large ($G_d \gg 1$), and G_c will be small ($G_c \ll 1$).



$$V_A = V_C + \frac{V_s}{2} + E_A$$

$$V_B = V_C - \frac{V_s}{2} + E_B$$

$$V_{id} = V_A - V_B = V_s + E_A - E_B$$

$$V_{ic} = \frac{V_A + V_B}{2} = V_C + \frac{E_A + E_B}{2}$$

$$V_o = \underbrace{G_d \cdot V_{id}}_{V_{od}} + \underbrace{G_c \cdot V_{ic}}_{V_{oc}}$$

Common mode rejection ratio

A measure of how well an in-amp rejects common mode signal is referred to as the common mode rejection ratio, CMRR.

$$CMRR \stackrel{\text{def}}{=} \frac{G_d}{G_c}$$

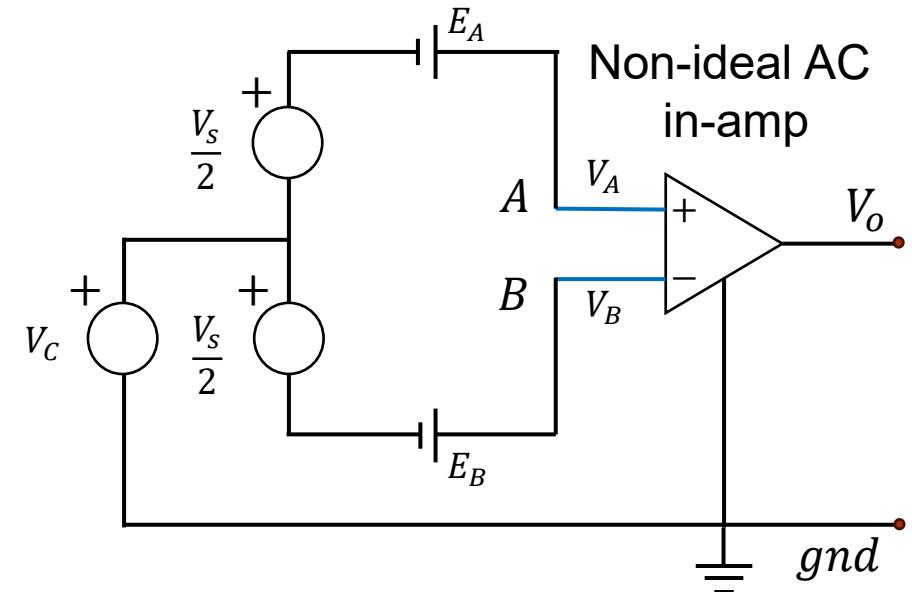
If is often presented in decibel:

$$CMRR_{dB} \stackrel{\text{def}}{=} 20 \log \frac{G_d}{G_c}$$

$$V_o = \underbrace{G_d \cdot V_{id}}_{V_{od}} + \underbrace{G_c \cdot V_{ic}}_{V_{oc}}$$

$$G_d \stackrel{\text{def}}{=} \frac{V_{od}}{V_{id}}$$

$$G_c \stackrel{\text{def}}{=} \frac{V_{oc}}{V_{ic}}$$



$$V_o = \underbrace{G_d \cdot V_{id}}_{V_{od}} + \underbrace{\frac{G_d}{CMRR} \cdot V_{ic}}_{V_{oc}}$$

Common mode rejection ratio

Example:

$$V_o = \frac{G_d \cdot V_{id}}{V_{od}} + \frac{G_c \cdot V_{ic}}{V_{oc}}$$

Calculate CMRR

$$CMRR \stackrel{\text{def}}{=} \frac{G_d}{G_c}$$

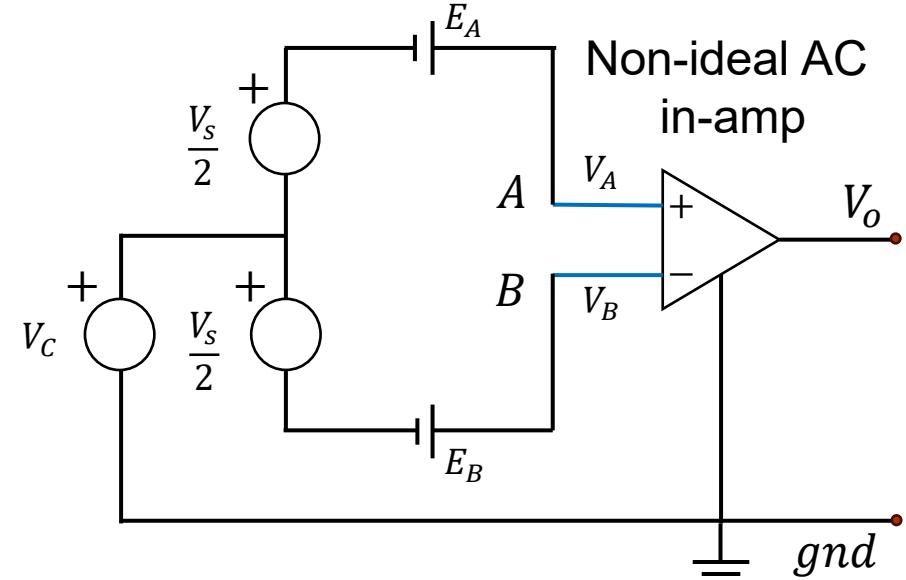
$$V_{id} = 5\text{mV}$$

$$V_{od} = 5\text{V}$$

$$V_{ic} = 500\text{mV}$$

$$V_{oc} = 50\text{mV}$$

The CMRR is typically determined by first determining G_d and G_c experimentally.
From this we get: $CMRR = G_d/G_c$



$$G_d = \frac{V_{od}}{V_{id}} = \frac{5\text{V}}{5\text{mV}} = 1000$$

$$G_c = \frac{V_{oc}}{V_{ic}} = \frac{50\text{mV}}{500\text{mV}} = 0.1$$

$$CMRR = \frac{G_d}{G_c} = \frac{1000}{0.1} = 10^4 = 80\text{dB}$$

Common mode rejection ratio

Example continued:

What is the improvement in SNR?

$$V_{id} = 5\text{mV}$$

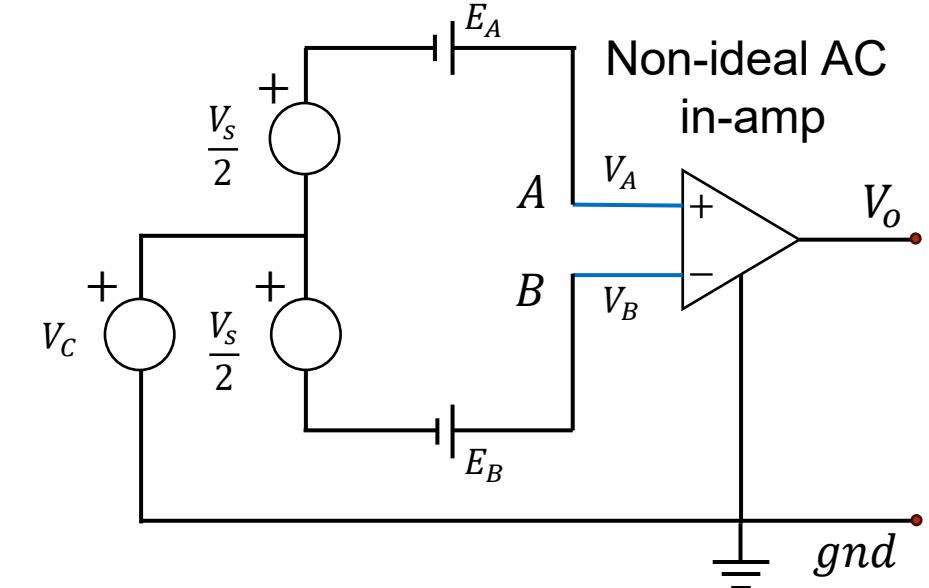
$$V_{od} = 5\text{V}$$

$$V_{ic} = 500\text{mV}$$

$$V_{oc} = 50\text{mV}$$

Input SNR

Output SNR



$$\frac{V_{id}}{V_{ic}} = \frac{5\text{mV}}{500\text{mV}} = 10^{-2} \sim -40\text{dB}$$

$$\frac{V_{od}}{V_{oc}} = \frac{5\text{V}}{0.05\text{V}} = 10^2 \sim 40\text{dB}$$

Observation:

The CMRR is the improvement in SNR from input to output.

$$\left(\frac{V_{od}}{V_{oc}}\right)_{dB} = \left(\frac{V_{id}}{V_{ic}}\right)_{dB} + CMRR_{dB}$$

AC performance

This circuit model includes a source model and an AC-coupled in-amp.

The source model has the following sources:

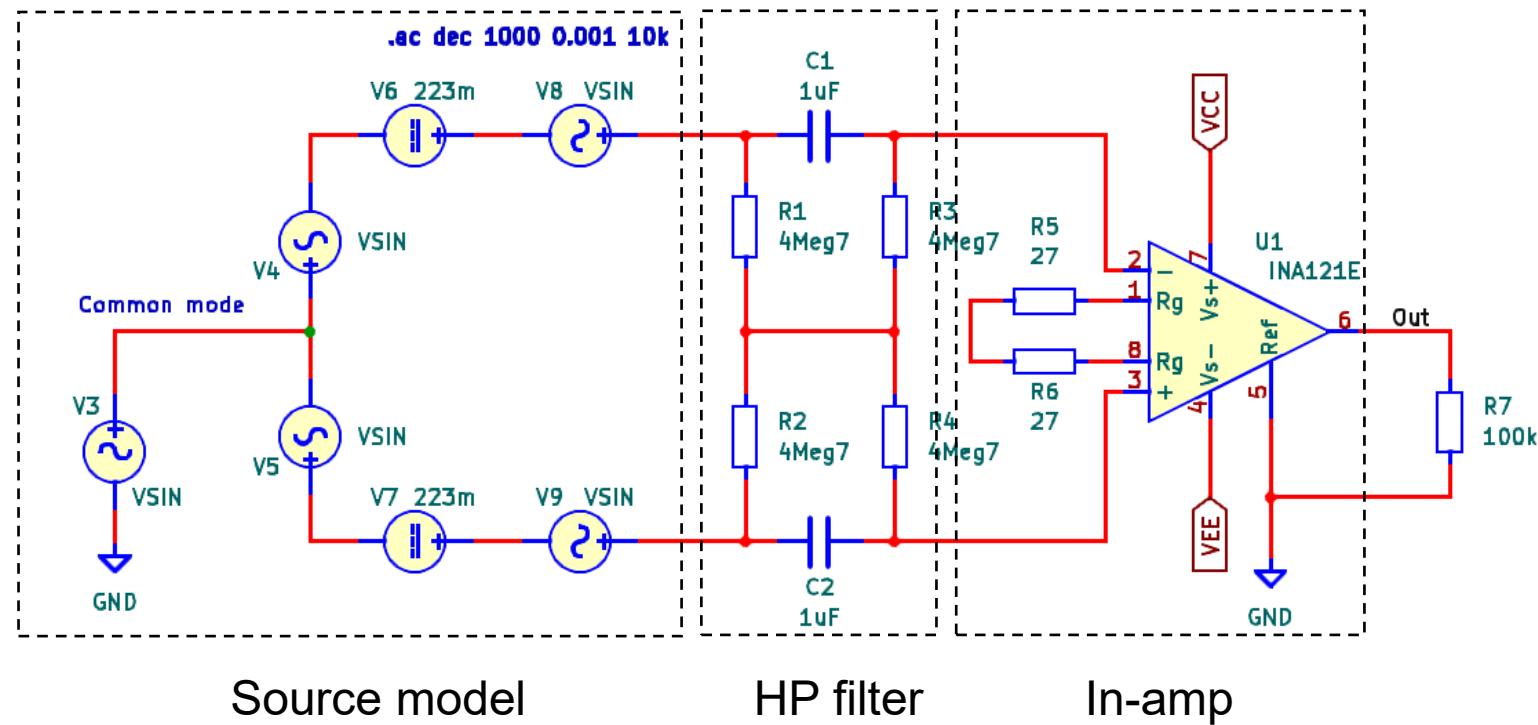
V_3 : common mode

$V_{4,5}$: differential sources

$V_{6,7}$: half-cell DC sources

V_8 : high-frequency noise

V_9 : low-frequency noise.



Source model

HP filter

In-amp

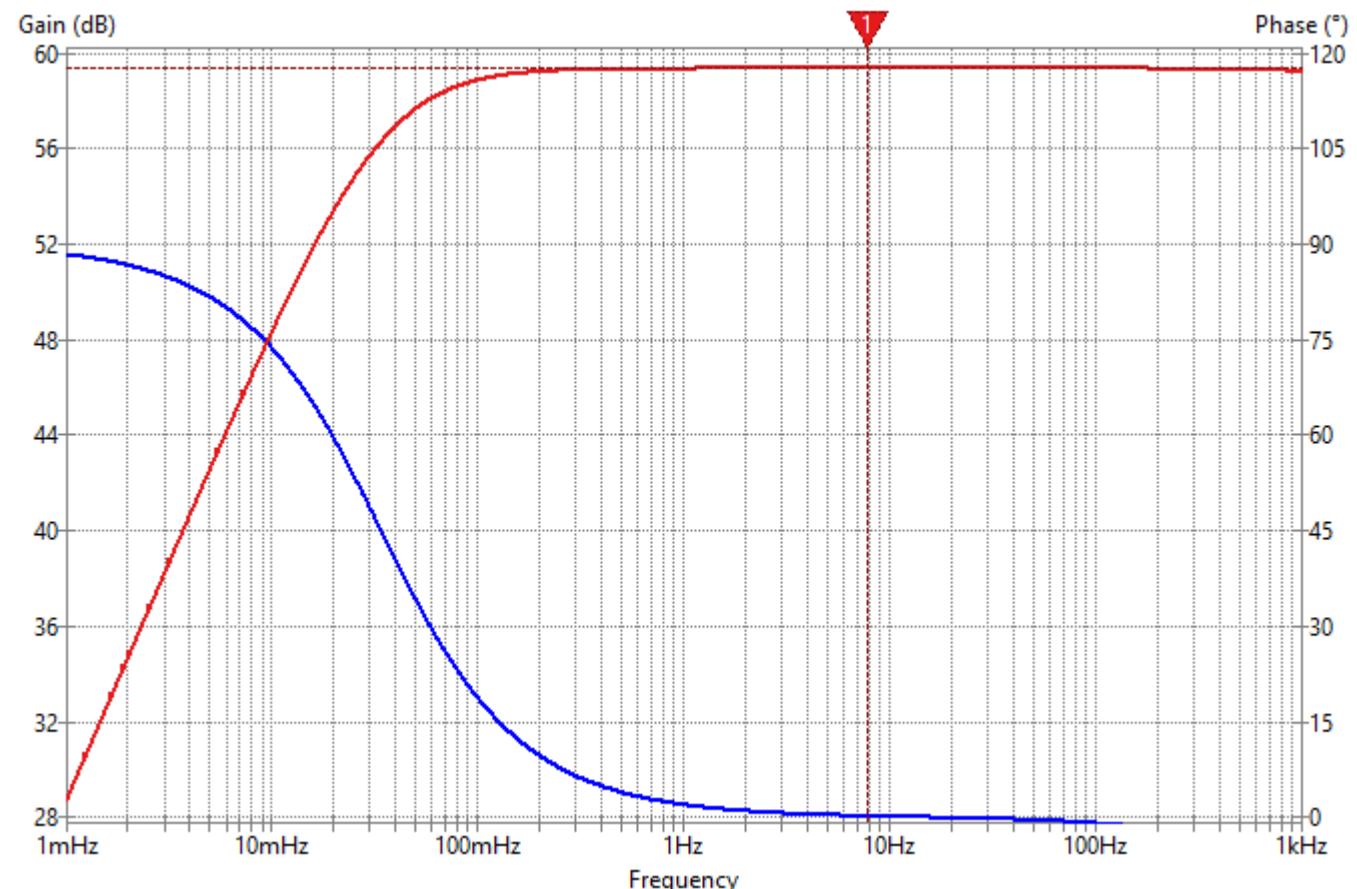
AC sweep of differential signal

In this AC sweep the two differential sources had an amplitude of 0.5V each. The common mode source had an amplitude of 0V. The differential gain $G_d \sim 1000$.

The red curve is the output from the in-amp.

Observations:

The output signal is 59.3dB (922.6).



AC sweep of common mode signal

Here is shown the output when **only a 1V common mode signal** is present on the input.

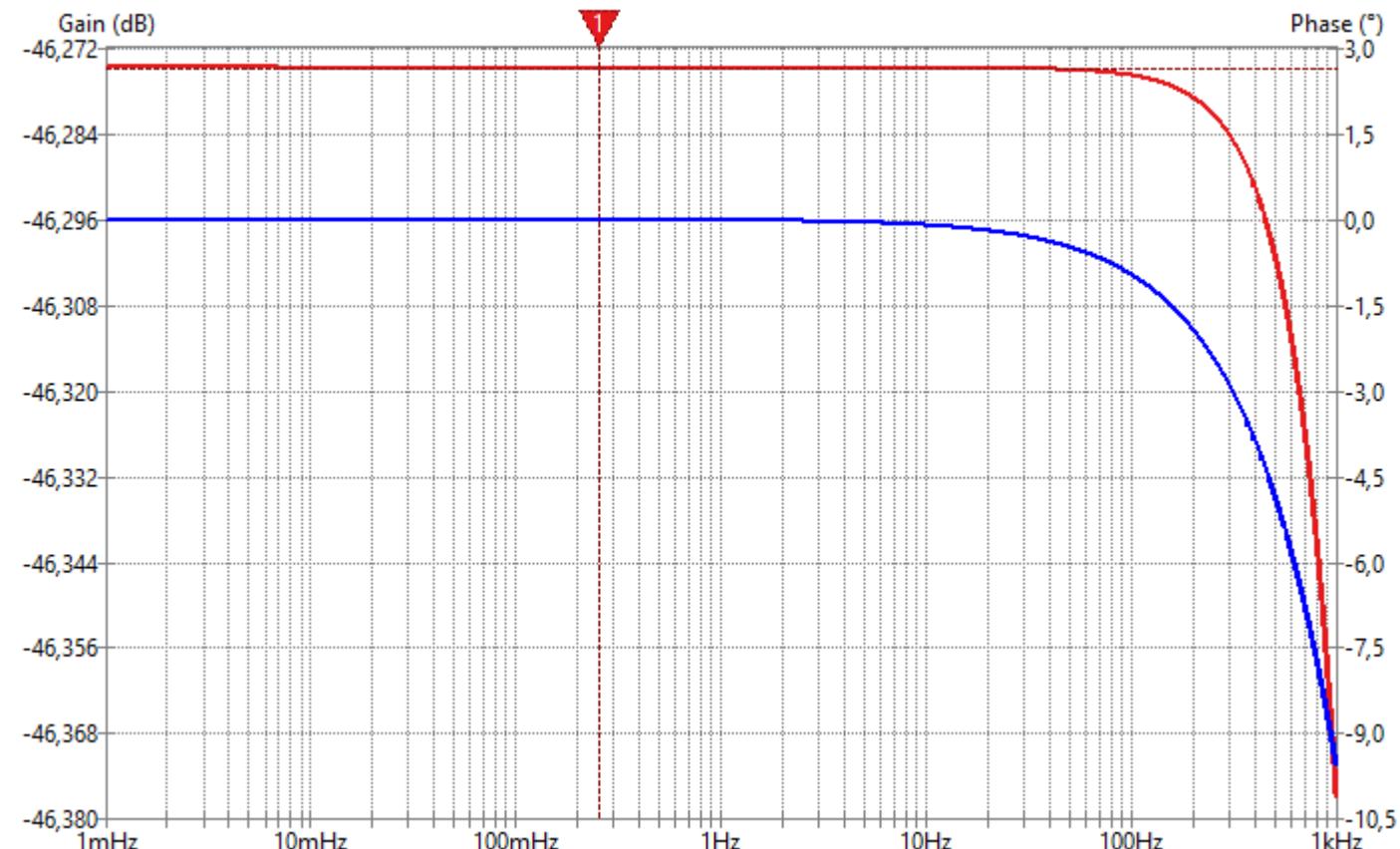
Within the passband of the lowpass filter, the common mode signal is attenuated at least 46.3dB.

$$G_c = 0.0048$$

The differential gain $G_d = 922.6$.

The CMRR is:

$$CMRR = \frac{G_d}{G_c} = \frac{922.6}{0.0048} = 1.92 \times 10^5 = 105.7\text{dB} = 59.3\text{dB} - (-46.3\text{dB})$$



Problems

Problem 1 – EMG highpass filter

Design a Butterworth highpass filter with the following specifications:

$$G_{\infty} = 0\text{dB}$$

$$\omega_c = \omega_p = 2\pi \cdot 2\text{Hz}$$

$$G_p = -3\text{dB}$$

$$\omega_s = 2\pi \cdot 0.02\text{Hz}$$

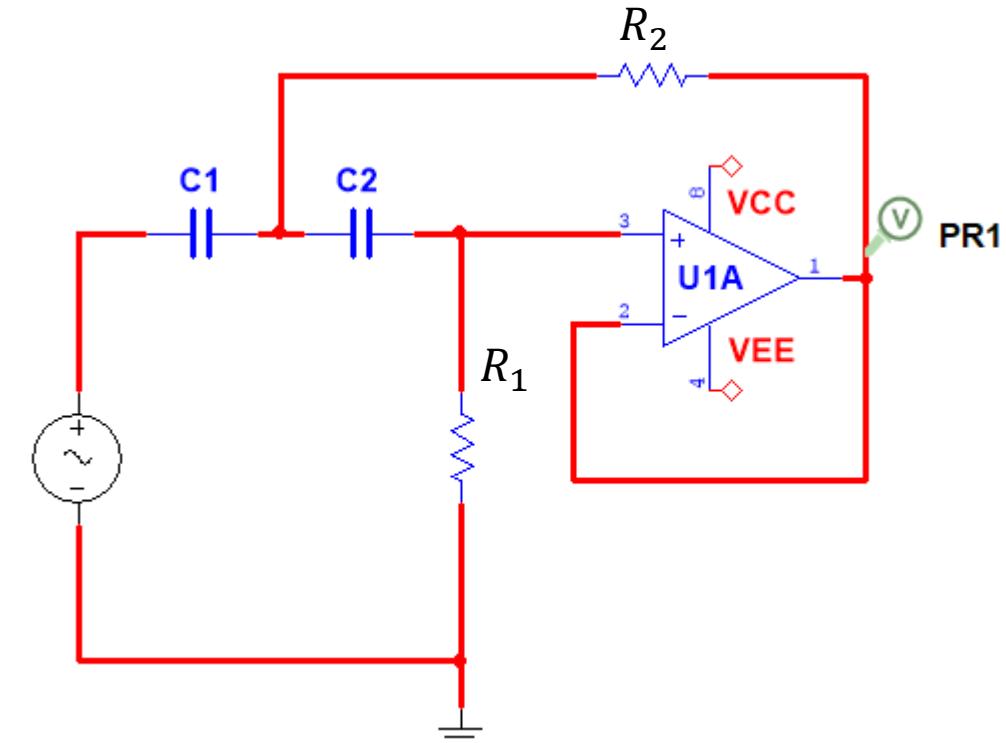
$$G_s = -60\text{dB}$$

Design constraints:

- Use a Sallen-Key circuit of lowest possible order.
- The filter should be very insensitive to component drift.
- Use resistors from the E12 series and capacitors from the E6 series as much as possible, but not at all costs.

Documentation:

- Draw the amplitude and phase spectra in Maple/Python for the frequency-normalized filter as well as for the final design.



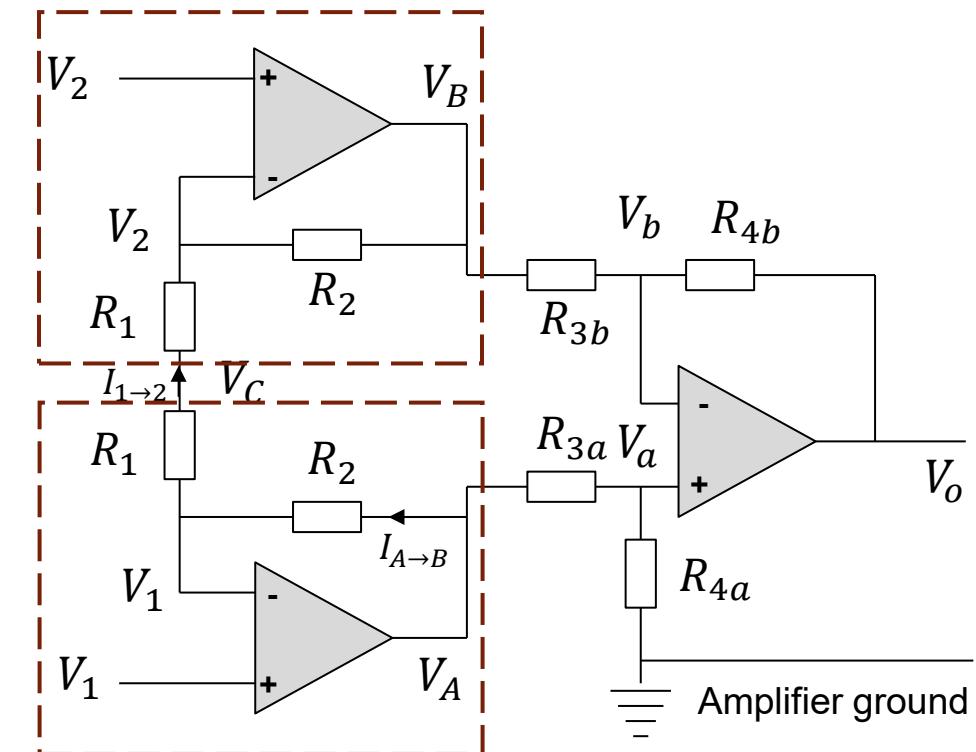
Hint: Do not start from scratch. Start with the frequency-normalized HP filter designed in the lecture and scale it to the design-specifications.

Problem 2 Imbalanced In-amp

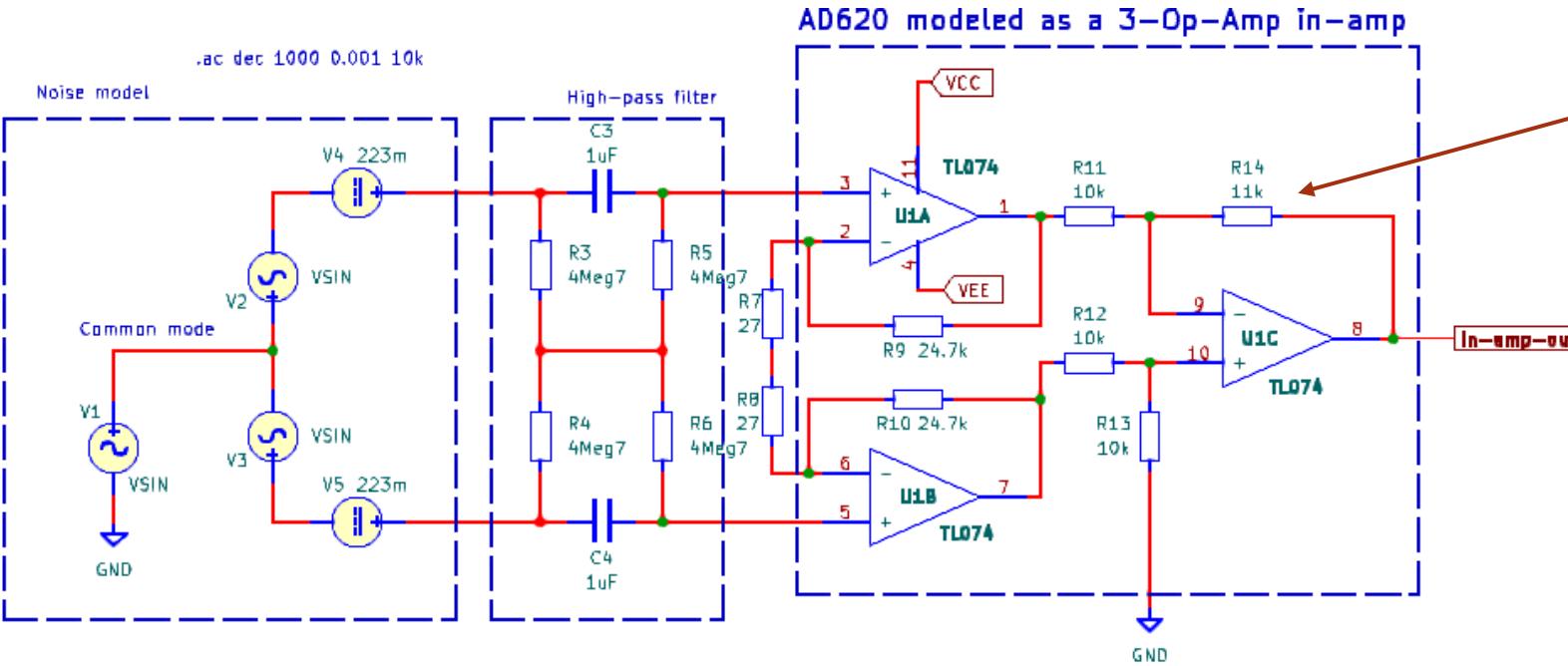
The ability of the in-amp to reject common mode signals depends on the degree to which resistance values are matched. In a One-chip in-amp these resistors are finely tuned to obtain a high CMRR.

In this problem you will experience the deterioration of the CMRR when R_{4b} deviates from R_{4a}.

To simulate this in KiCad/LTspice, we need to model the in-amp as a 3-op-amp instrumentation amplifier. A KiCad template file is available for download from DTU Learn.

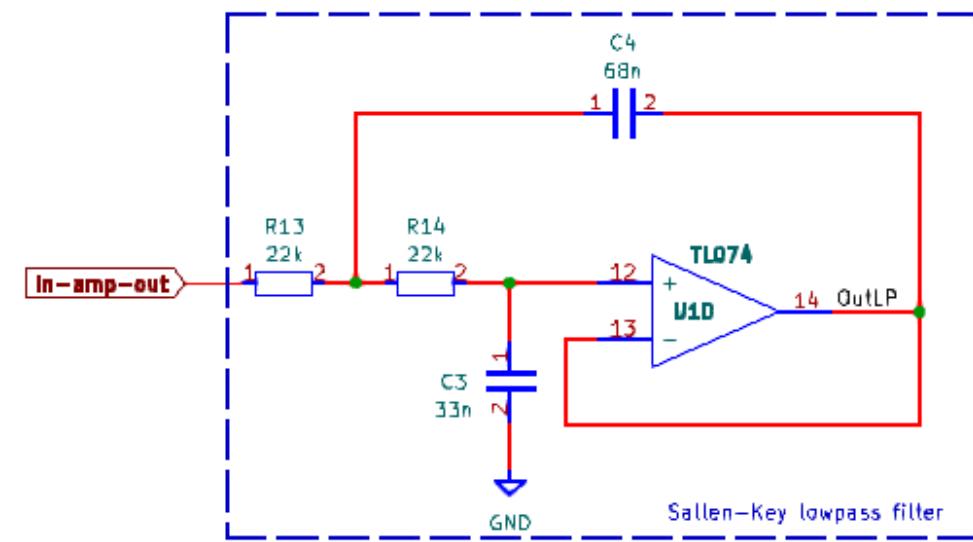


Problem 2 Imbalanced In-amp



Imbalance in R11. Ideally
R11 = R12.

- V1: common mode source
- V2: Half of a differential source
- V3: Half of a differential source
- V4: DC source in negative input
- V5: DC source in positive input.



Simulation 1

Balanced in-amp.

$R_{11} = R_{12} = 10\text{k}\Omega$.

$V_1 = 0\text{V AC}$.

$V_2 = V_3 = 0.5\text{V AC}$.

Simulate an AC sweep from 1mHz – 10kHz.

The differential gain should be $1000 = 60 \text{ dB}$.

Simulation 2

Balanced in-amp.

$R_{11} = R_{12} = 10\text{k}\Omega$.

$V_1 = 1\text{V AC}$.

$V_2 = V_3 = 0.0\text{V AC}$.

Simulate an AC sweep from 1mHz – 10kHz.

Read the common mode gain.

What is the CMRR?

Simulation 3

Imbalanced in-amp.

$R_{11} = 11\text{kOhm}$, $R_{12} = 10\text{kOhm}$.

$V_1 = 1\text{V AC}$.

$V_2 = V_3 = 0.0\text{V AC}$.

Simulate an AC sweep from 1mHz – 10kHz.

Read the common mode gain.

What is the CMRR?

Solutions

Problem 1 – EMG highpass filter

Design a Butterworth highpass filter with the following specifications:

$$G_{\infty} = 0\text{dB}$$

$$\omega_c = \omega_p = 2\pi \cdot 2\text{Hz}$$

$$G_p = -3\text{dB}$$

$$\omega_s = 2\pi \cdot 0.02\text{Hz}$$

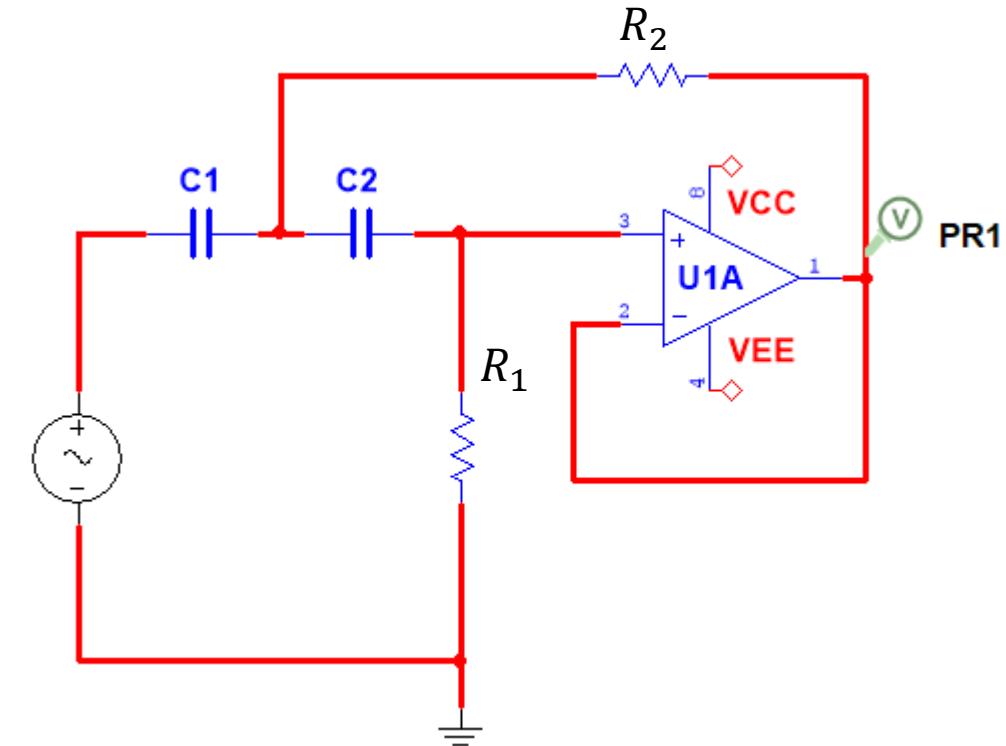
$$G_s = -60\text{dB}$$

Design constraints:

- Use a Sallen-Key circuit of lowest possible order.
- The filter should be very insensitive to component drift.
- Use resistors from the E12 series and capacitors from the E6 series as much as possible, but not at all costs.

Documentation:

- Draw the amplitude and phase spectra in Maple/Python for the frequency-normalized filter as well as for the final design.



Hint: Do not start from scratch. Start with the frequency-normalized HP filter designed in the lecture and scale it to the design-specifications.

Problem 1 – EMG highpass filter

Determination of filter order:

$$G_\infty = 0 \text{dB}$$

$$\omega_c = \omega_p = 2\pi \cdot 2 \text{Hz}$$

$$G_p = -3 \text{dB}$$

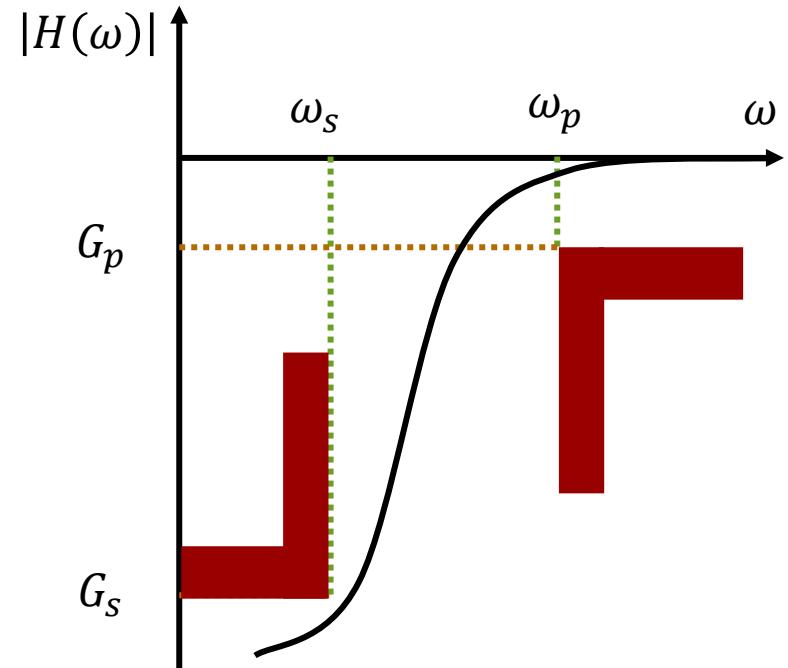
$$\omega_s = 2\pi \cdot 0.02 \text{Hz}$$

$$G_s = -60 \text{dB}$$

$$n_{HP} = \frac{1}{2\log_{10}(\omega_{p,HP}/\omega_{s,HP})} \log_{10} \left(\frac{10^{-\frac{G_{s,dB}}{10}} - 1}{10^{-\frac{G_{p,dB}}{10}} - 1} \right)$$

$$n_{HP} = \frac{1}{2\log_{10}(2/0.02)} \log_{10} \left(\frac{10^{-\frac{-60}{10}} - 1}{10^{-\frac{-3}{10}} - 1} \right) = 1.5$$

We need a 2nd order filter.



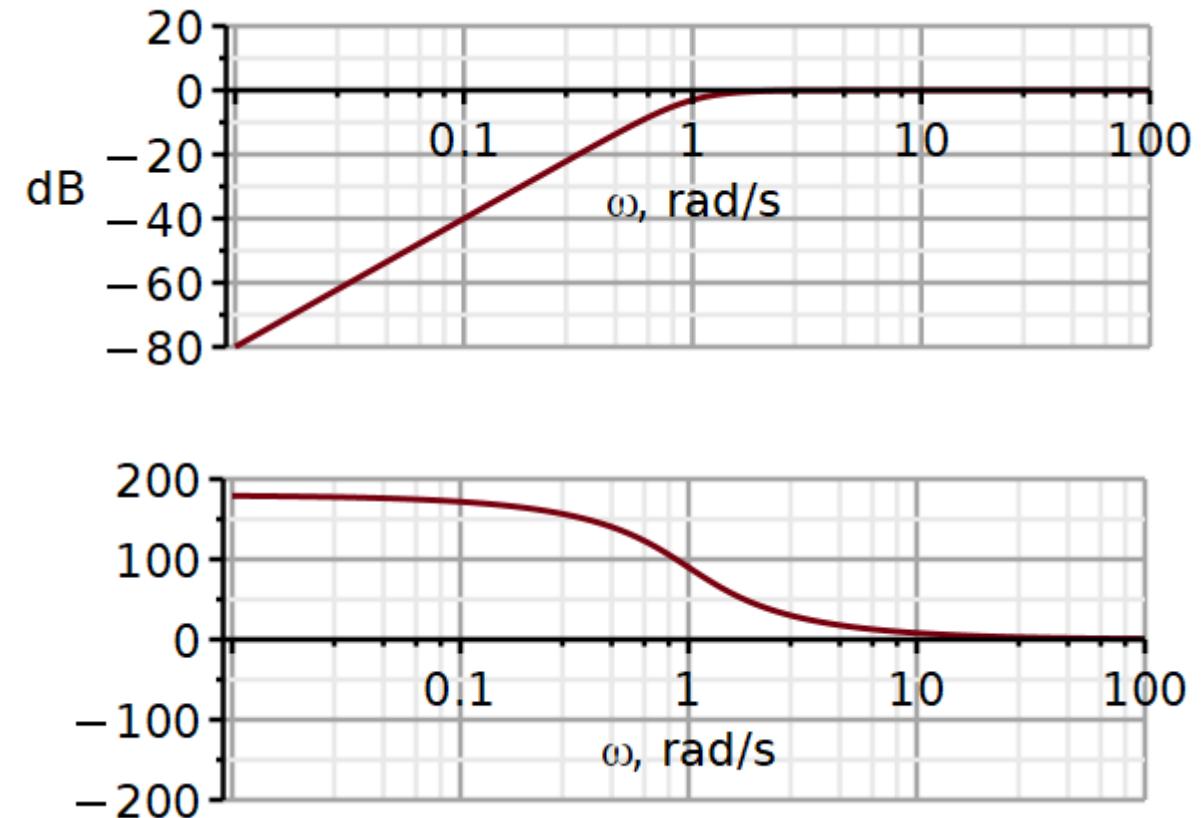
Problem 1 – EMG highpass filter (sol)

We use the component values from the lecture slides for the frequency-normalized filter:

$$\hat{R}_1 \approx 0.68\Omega$$

$$\hat{R}_2 \approx 0.33\Omega$$

$$\hat{C} \approx 2.2F$$



The low-frequency asymptote has a slope of +40dB/decade, the cut-off is at 1 rad/s and the phase is 90 degrees at the cut-off. So, it looks promising.

Frequency scaling

$$K_F = 2\pi \cdot 2 = 12.5663$$

$$C'_1 = \frac{\hat{C}_1}{K_F} = \frac{2.2\text{F}}{12.5663} = 175\text{mF}$$

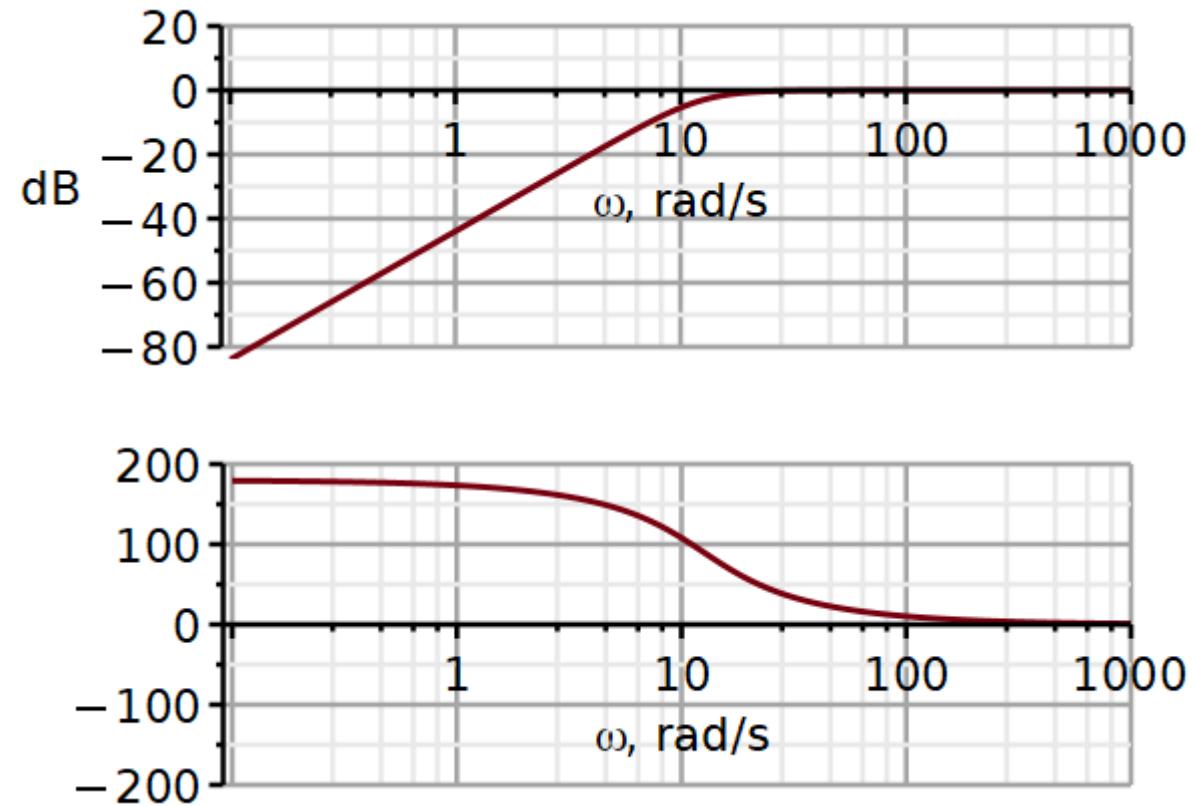
$$C'_2 = \frac{\hat{C}_2}{K_F} = 175\text{mF}$$

$$R'_1 = \hat{R}_1 = 0.68\Omega$$

$$R'_2 = \hat{R}_2 = 0.33\Omega$$

$$\tau_1 = R'_1 C'_1 = 0.119\text{s}$$

$$\tau_2 = R'_2 C'_2 = 0.05775\text{s}$$



Looking up the best component values in the time constant look-up table on the next slide, we find that a capacitor value of $1\mu\text{F}$ will work well.

A value of $1.2\mu\text{F}$ would also work well, but it belongs to the E12 series.

2nd order Sallen-Key high pass filter - Butterworth

$$\tau_1 = R_1 C = 0.119s$$

$$\tau_2 = R_2 C = 0.05775s$$

↓ E6 E12 E6 E12

R1*C1	0,1	0,12	0,15	0,18	0,22	0,27	0,33	0,39	0,47	0,56	0,68	0,82	1	1,2	1,5	1,8	2,2	2,7	3,3	3,9	4,7	5,6	6,8	8,2	10
0,1	0,010	0,012	0,015	0,018	0,022	0,027	0,033	0,039	0,047	0,056	0,068	0,082	0,100	0,120	0,150	0,180	0,220	0,270	0,330	0,390	0,470	0,560	0,680	0,820	1,000
0,12	0,012	0,014	0,018	0,022	0,026	0,032	0,040	0,047	0,056	0,067	0,082	0,098	0,120	0,144	0,180	0,216	0,264	0,324	0,396	0,468	0,564	0,672	0,816	0,984	1,200
0,15	0,015	0,018	0,023	0,027	0,033	0,041	0,050	0,059	0,071	0,084	0,102	0,123	0,150	0,180	0,225	0,270	0,330	0,405	0,495	0,585	0,705	0,840	1,020	1,230	1,500
0,18	0,018	0,022	0,027	0,032	0,040	0,049	0,059	0,070	0,085	0,101	0,122	0,148	0,180	0,216	0,270	0,324	0,396	0,486	0,594	0,702	0,846	1,008	1,224	1,476	1,800
0,22	0,022	0,026	0,033	0,040	0,048	0,059	0,073	0,086	0,103	0,123	0,150	0,180	0,220	0,264	0,330	0,396	0,484	0,594	0,726	0,858	1,034	1,232	1,496	1,804	2,200
0,27	0,027	0,032	0,041	0,049	0,059	0,073	0,089	0,105	0,127	0,151	0,184	0,221	0,270	0,324	0,405	0,486	0,594	0,729	0,891	1,053	1,269	1,512	1,836	2,214	2,700
0,33	0,033	0,040	0,050	0,059	0,073	0,089	0,109	0,129	0,155	0,185	0,224	0,271	0,330	0,396	0,495	0,594	0,726	0,891	1,089	1,287	1,551	1,848	2,244	2,706	3,300
0,39	0,039	0,047	0,059	0,070	0,086	0,105	0,129	0,152	0,183	0,218	0,265	0,320	0,390	0,468	0,585	0,702	0,858	1,053	1,287	1,521	1,833	2,184	2,652	3,198	3,900
0,47	0,047	0,056	0,071	0,085	0,103	0,127	0,155	0,183	0,221	0,263	0,320	0,385	0,470	0,564	0,705	0,846	1,034	1,269	1,551	1,833	2,209	2,632	3,196	3,854	4,700
0,56	0,056	0,067	0,084	0,101	0,123	0,151	0,185	0,218	0,263	0,314	0,381	0,459	0,560	0,672	0,840	1,008	1,232	1,512	1,848	2,184	2,632	3,136	3,808	4,592	5,600
0,68	0,068	0,082	0,102	0,122	0,150	0,184	0,224	0,265	0,320	0,381	0,462	0,558	0,680	0,816	1,020	1,224	1,496	1,836	2,244	2,652	3,196	3,808	4,624	5,576	6,800
0,82	0,082	0,098	0,123	0,148	0,180	0,221	0,271	0,320	0,385	0,459	0,558	0,672	0,820	0,984	1,230	1,476	1,804	2,214	2,706	3,198	3,854	4,592	5,576	6,724	8,200
1	0,100	0,120	0,150	0,180	0,220	0,270	0,330	0,390	0,470	0,560	0,680	0,820	1,000	1,200	1,500	1,800	2,200	2,700	3,300	3,900	4,700	5,600	6,800	8,200	10,000
1,2	0,120	0,144	0,180	0,216	0,264	0,324	0,396	0,468	0,564	0,672	0,816	0,984	1,200	1,440	1,800	2,160	2,640	3,240	3,960	4,680	5,640	6,720	8,160	9,840	12,000
1,5	0,150	0,180	0,225	0,270	0,330	0,405	0,495	0,585	0,705	0,840	1,020	1,230	1,500	1,800	2,250	2,700	3,300	4,050	4,950	5,850	7,050	8,400	10,200	12,300	15,000
1,8	0,180	0,216	0,270	0,324	0,396	0,486	0,594	0,702	0,846	1,008	1,224	1,476	1,800	2,160	2,700	3,240	3,960	4,860	5,940	7,020	8,460	10,080	12,240	14,760	18,000
2,2	0,220	0,264	0,330	0,396	0,484	0,594	0,726	0,858	1,034	1,232	1,496	1,804	2,200	2,640	3,300	3,960	4,840	5,940	7,260	8,580	10,340	12,320	14,960	18,040	22,000
2,7	0,270	0,324	0,405	0,486	0,594	0,729	0,891	1,053	1,269	1,512	1,836	2,214	2,700	3,240	4,050	4,860	5,940	7,290	8,910	10,530	12,690	15,120	18,360	22,140	27,000
3,3	0,330	0,396	0,495	0,594	0,726	0,891	1,089	1,287	1,551	1,848	2,244	2,706	3,300	3,960	4,950	5,940	7,260	8,910	10,890	12,870	15,510	18,480	22,440	27,060	33,000
3,9	0,390	0,468	0,585	0,702	0,858	1,053	1,287	1,521	1,833	2,184	2,652	3,198	3,900	4,680	5,850	7,020	8,580	10,530	12,870	15,210	18,330	21,840	26,520	31,980	39,000
4,7	0,470	0,564	0,705	0,846	1,034	1,269	1,551	1,833	2,209	2,632	3,196	3,854	4,700	5,640	7,050	8,460	10,340	12,690	15,510	18,330	22,090	26,320	31,960	38,540	47,000
5,6	0,560	0,672	0,840	1,008	1,232	1,512	1,848	2,184	2,632	3,136	3,808	4,592	5,600	6,720	8,400	10,080	12,320	15,120	18,480	21,840	26,320	31,360	38,080	45,920	56,000
6,8	0,680	0,816	1,020	1,224	1,496	1,836	2,244	2,652	3,196	3,808	4,624	5,576	6,800	8,160	10,200	12,240	14,960	18,360	22,440	26,520	31,960	38,080	46,240	55,760	68,000
8,2	0,820	0,984	1,230	1,476	1,804	2,214	2,706	3,198	3,854	4,592	5,576	6,724	8,200	9,840	12,300	14,760	18,040	22,140	27,060	31,980	38,540	45,920	55,760	67,240	82,000
10	1,000	1,200	1,500	1,800	2,200	2,700	3,300	3,900	4,700	5,600	6,800	8,200	10,000	12,000	15,000	18,000	22,000	27,000	33,000	39,000	47,000	56,000	68,000	82,000	100,000

Problem 1 – EMG highpass filter (sol)

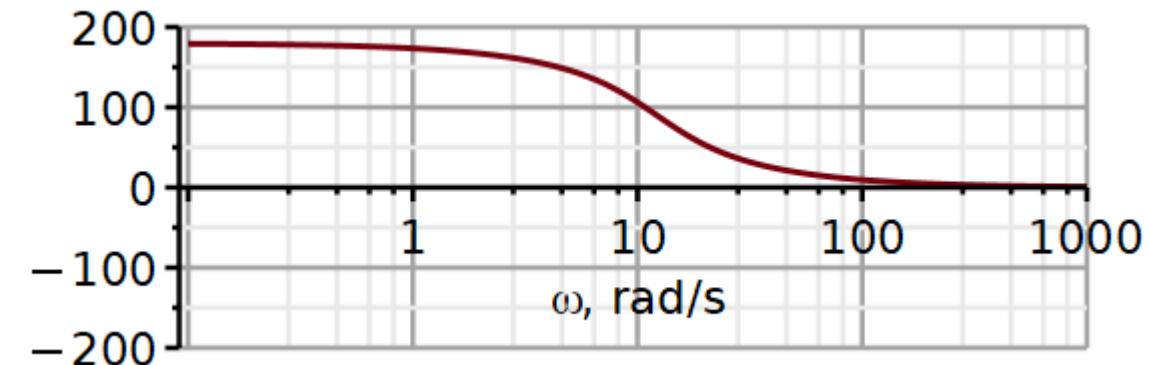
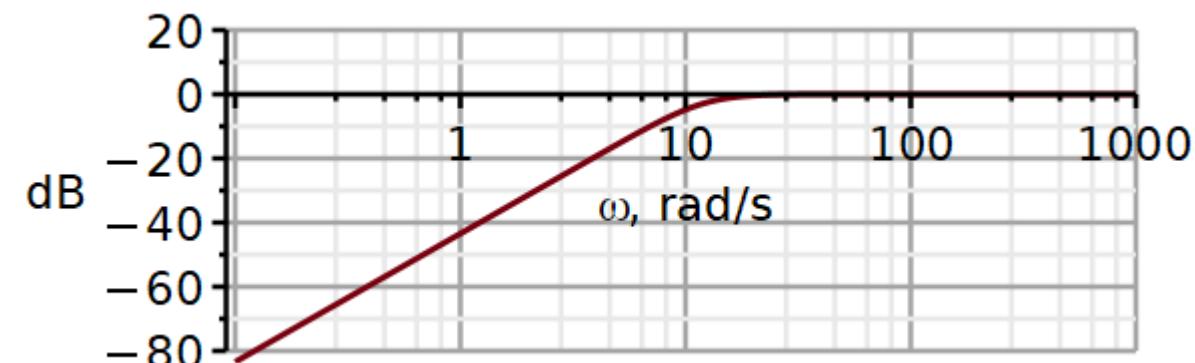
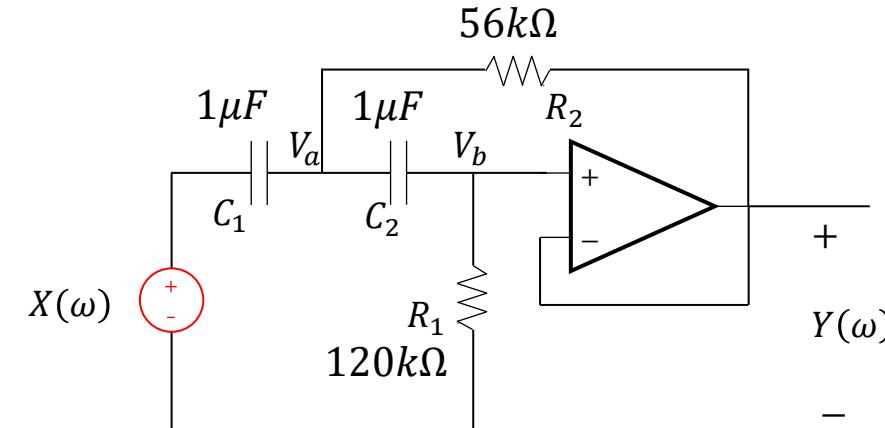
Impedance scaling:

$$C_1 = C_2 = 1\mu F \quad \text{← desired capacitor value}$$

$$K_z = \frac{C'_1}{1 \cdot 10^{-6} F} = \frac{175mF}{1 \cdot 10^{-6} F} = 175000$$

$$R_1 = K_z \cdot 0.68\Omega = 119.0k\Omega \approx 120k\Omega$$

$$R_2 = K_z \cdot 0.33\Omega = 57.77k\Omega \approx 56k\Omega$$



Problem 1 – EMG highpass filter (sol)

Design a Butterworth highpass filter with the following specifications:

$$G_{\infty} = 0 \text{dB}$$

$$\omega_c = \omega_p = 2\pi \cdot 2 \text{Hz}$$

$$G_p = -3 \text{dB}$$

$$\omega_s = 2\pi \cdot 0.02 \text{Hz}$$

$$G_s = -60 \text{dB}$$

Validation:

$$C1 := 1 \cdot 10^{-6}; C2 := 1 \cdot 10^{-6}; R1 := 120 \cdot 10^3; R2 := 56 \cdot 10^3;$$

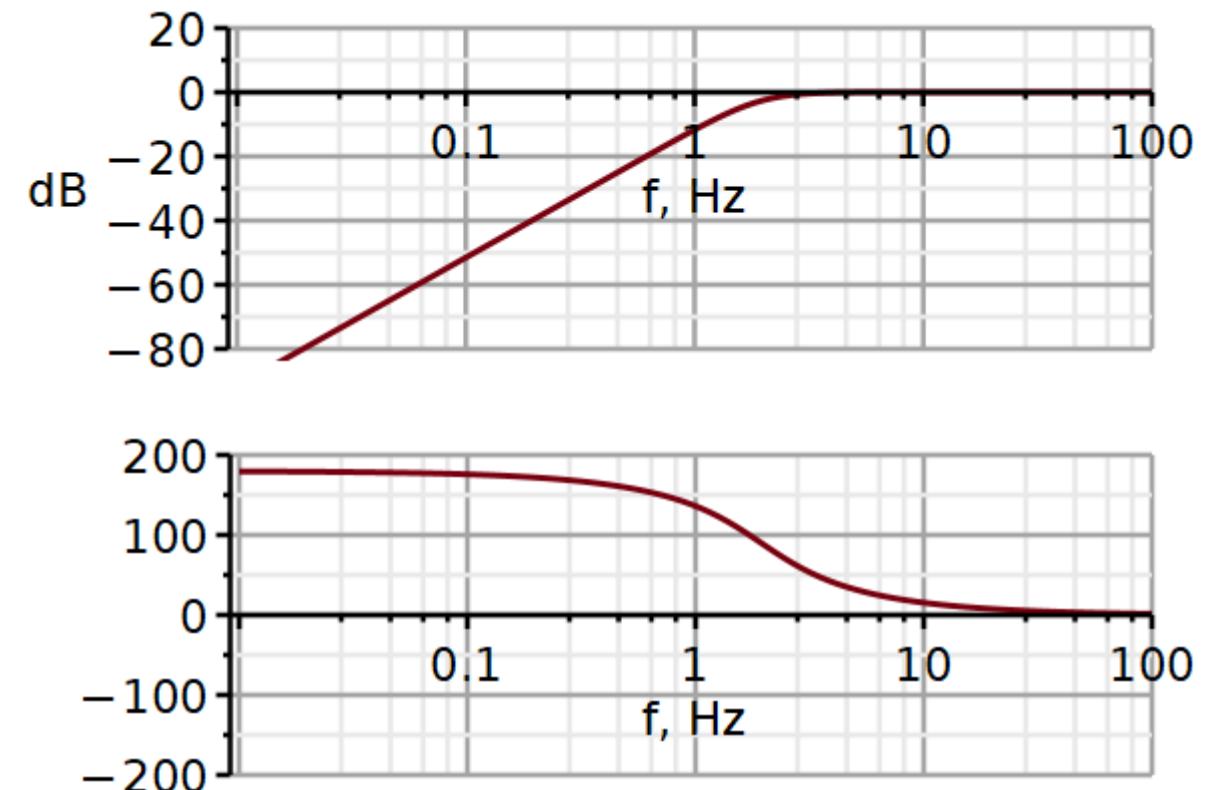
Test:

$$\text{evalf}(dB(H(2 \cdot \pi \cdot 2)))$$

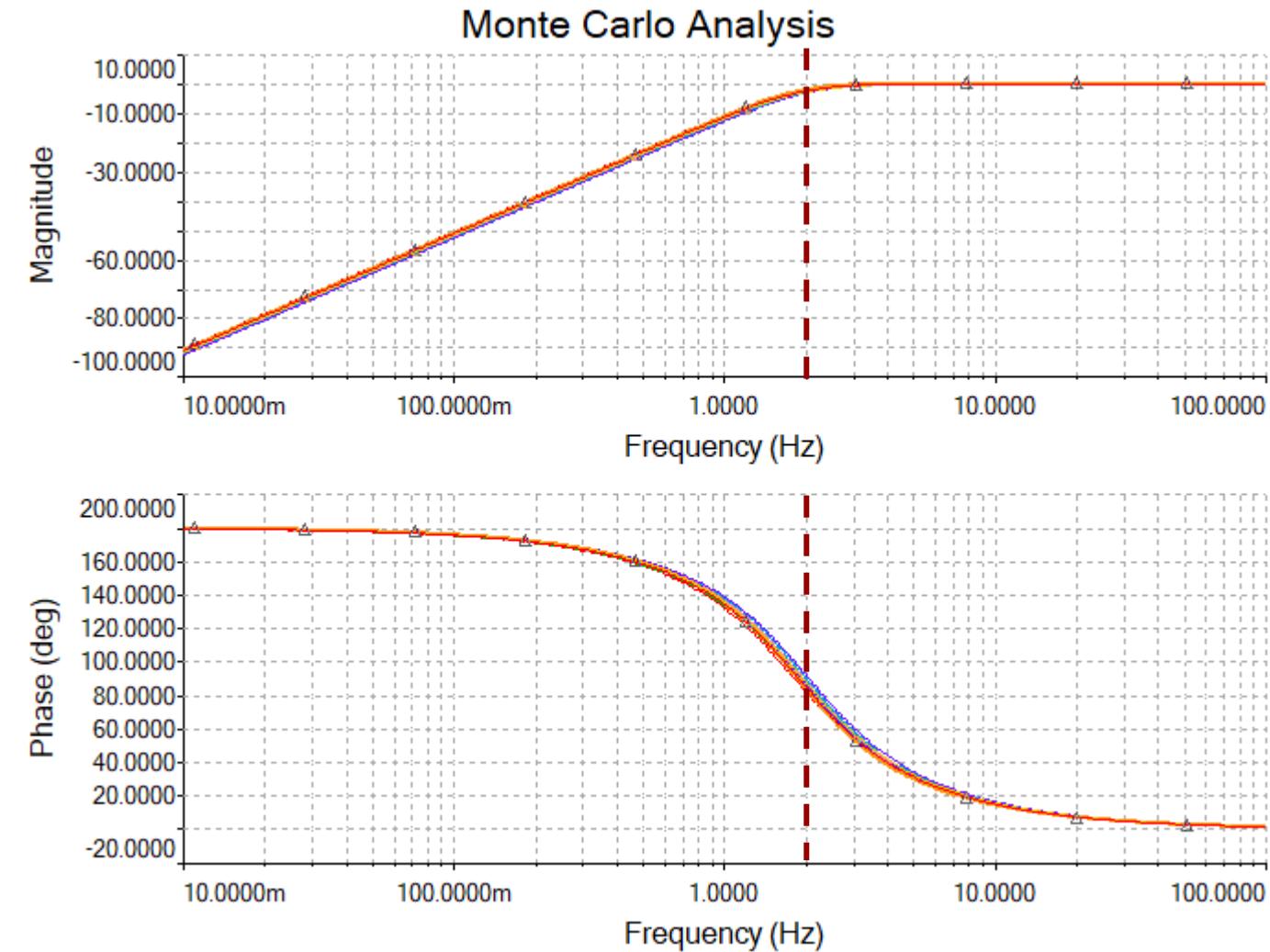
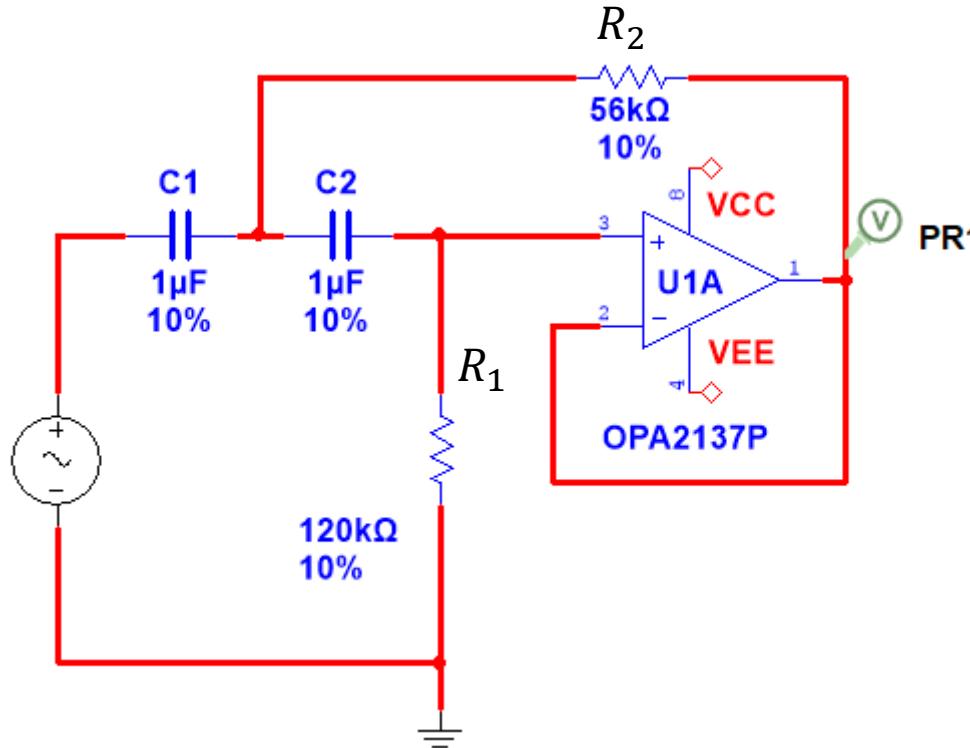
$$-2.460976232$$

$$\text{evalf}(dB(H(2 \cdot \pi \cdot 0.02)))$$

$$-79.48415858$$



Problem 1 – EMG highpass filter (sol)

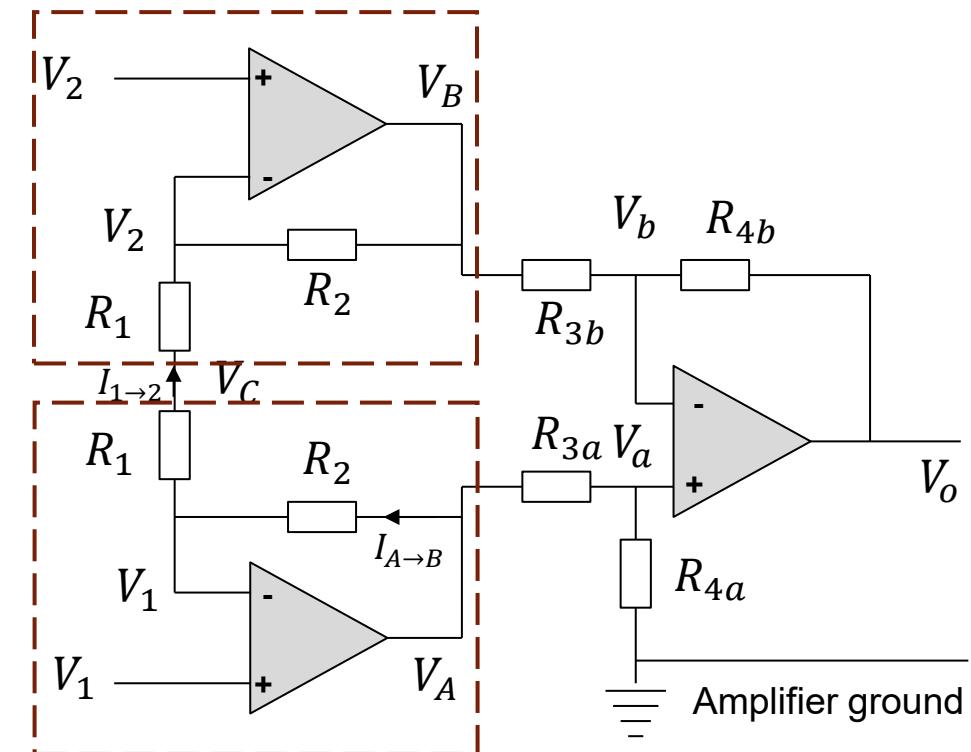


Problem 2 Imbalanced In-amp

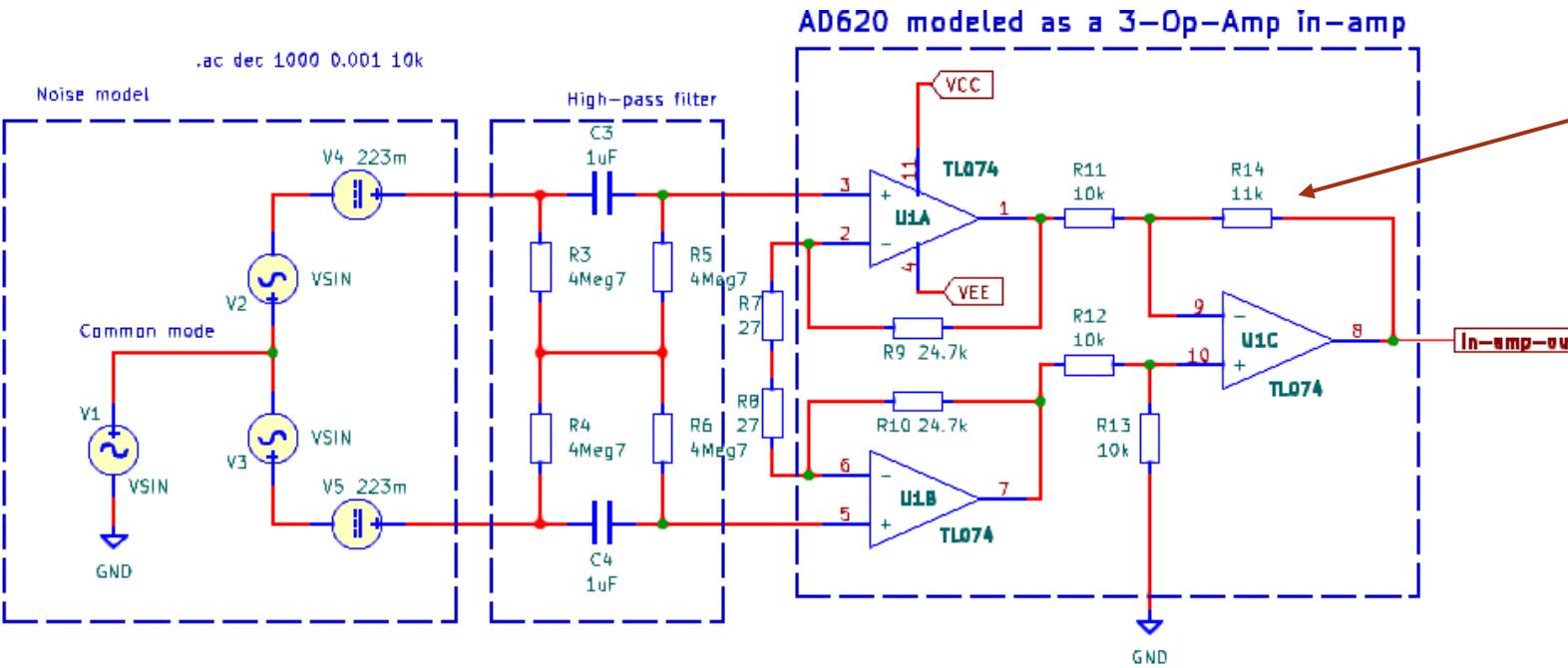
The ability of the in-amp to reject common mode signals depends on the degree to which resistance values are matched. In a One-chip in-amp these resistors are finely tuned to obtain a high CMRR.

In this problem you will experience the deterioration of the CMRR when R_{4b} deviates from R_{4a}.

To simulate this in KiCad/LTspice, we need to model the in-amp as a 3-op-amp instrumentation amplifier. A KiCad template file is available for download from DTU Learn.

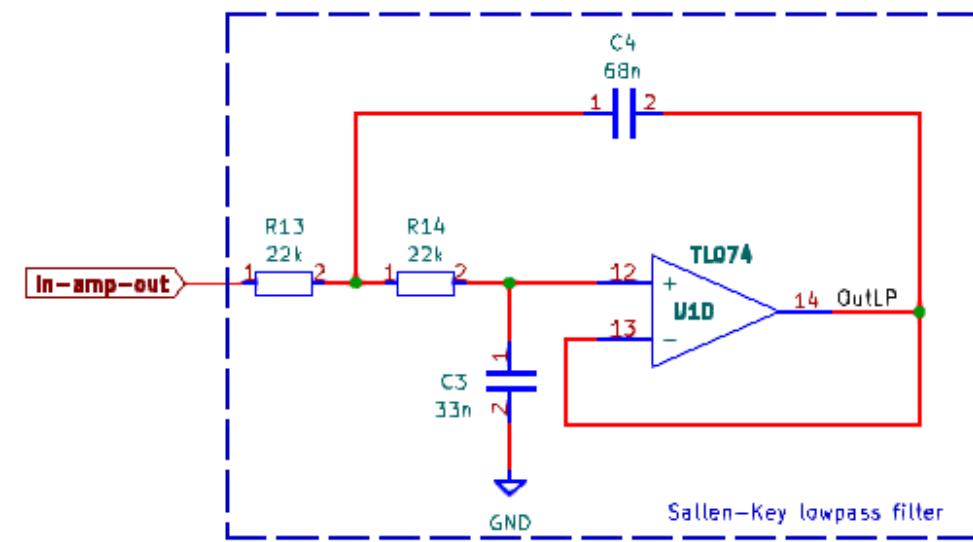


Problem 2 Imbalanced In-amp



Imbalance in R11. Ideally
R11 = R12.

- V1: common mode source
- V2: Half of a differential source
- V3: Half of a differential source
- V4: DC source in negative input
- V5: DC source in positive input.



Problem 2 Imbalanced In-amp (sol)

Simulation 1

Balanced in-amp.

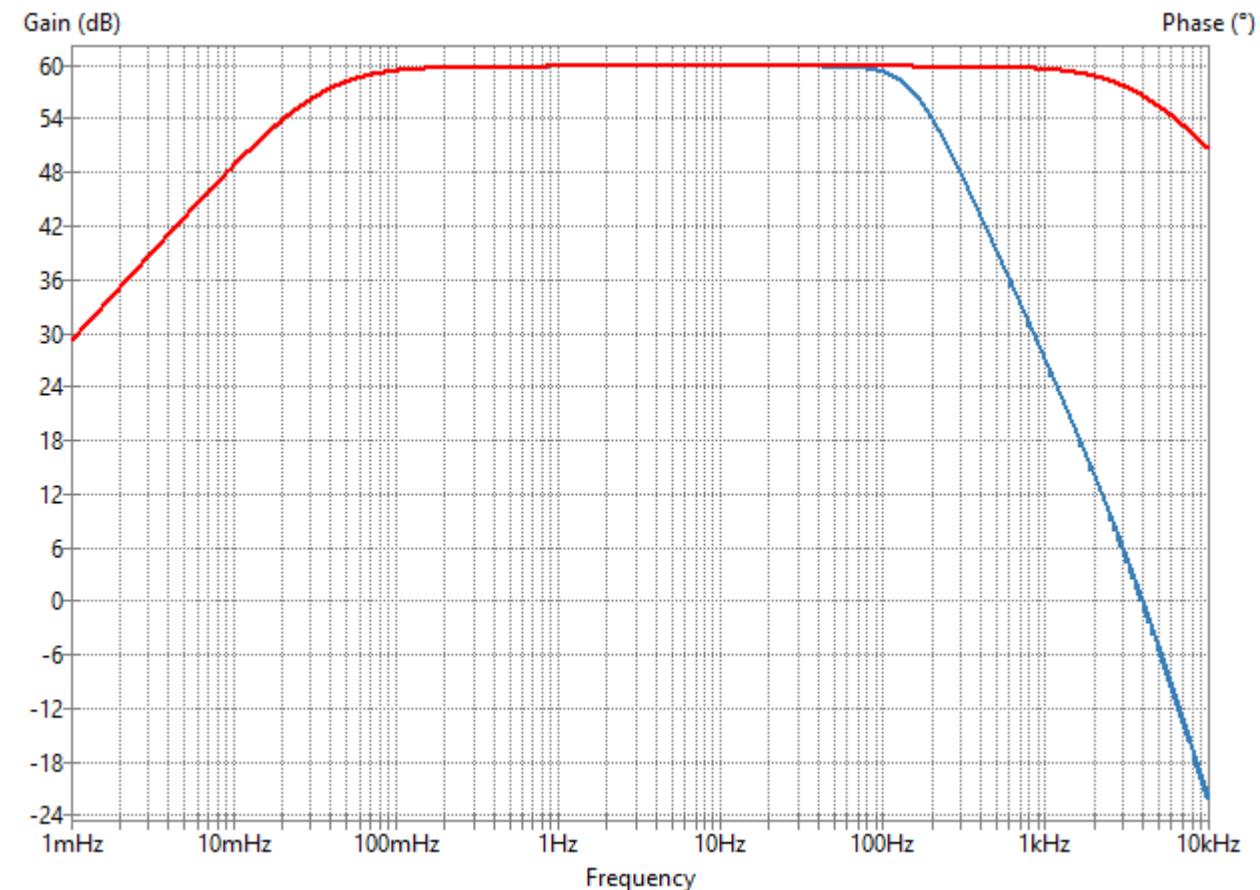
$R_{11} = R_{12} = 10\text{k}\Omega$.

$V_1 = 0\text{V AC}$.

$V_2 = V_3 = 0.5\text{V AC}$.

Simulate an AC sweep from $1\text{mHz} - 10\text{kHz}$.

The differential gain should be $1000 = 60 \text{ dB}$.



Simulation 2

Balanced in-amp.

$R_{11} = R_{12} = 10\text{k}\Omega$.

$V_1 = 1\text{V AC}$.

$V_2 = V_3 = 0.0\text{V AC}$.

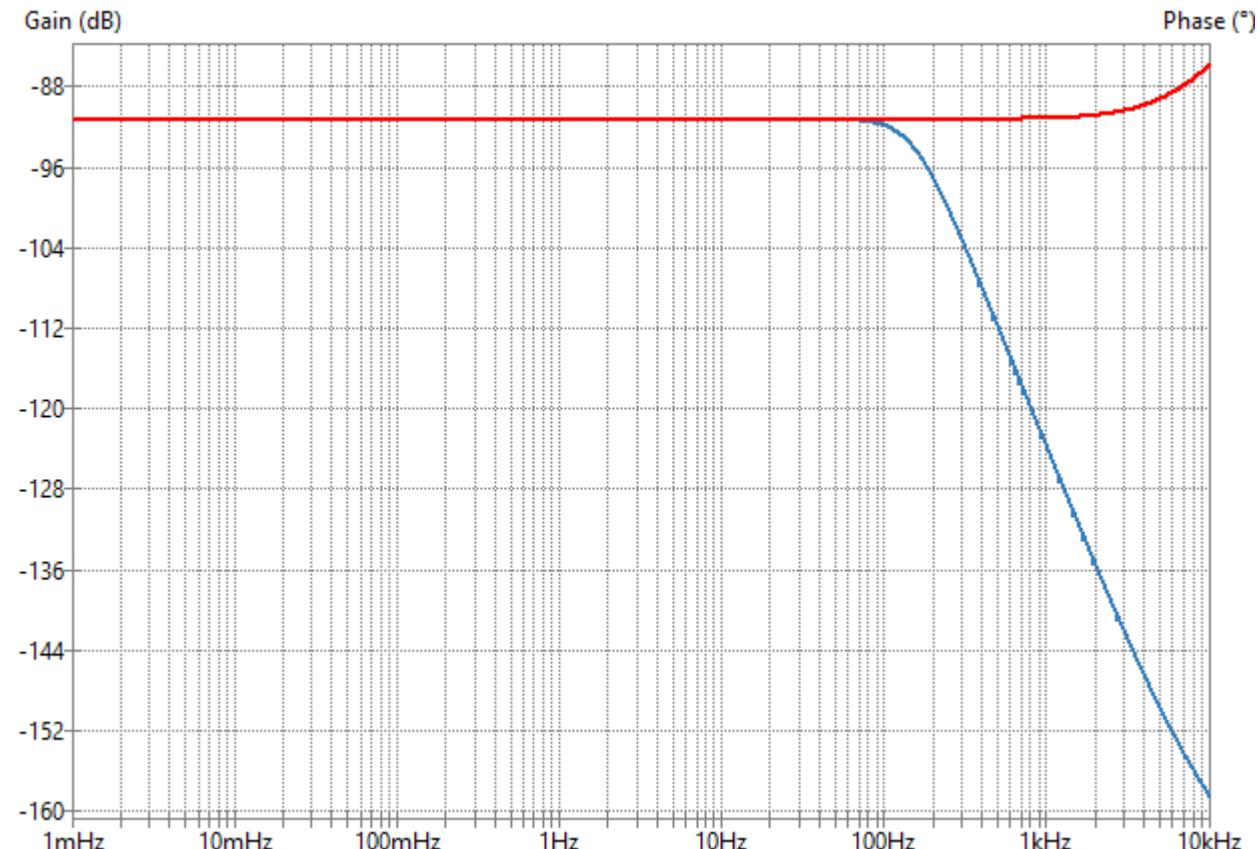
Simulate an AC sweep from $1\text{mHz} - 10\text{kHz}$.

Read the common mode gain.

What is the CMRR?

The solution shows a common mode gain of -91.3dB within the passband.

$$\text{CMRR} = 60\text{dB} - (-91.3\text{dB}) = 151.3\text{dB}$$



Problem 2 Imbalanced In-amp (sol)

Simulation 3

Imbalanced in-amp.

$R_{11} = 11\text{k}\Omega$, $R_{12} = 10\text{k}\Omega$.

$V_1 = 1\text{V AC}$.

$V_2 = V_3 = 0.0\text{V AC}$.

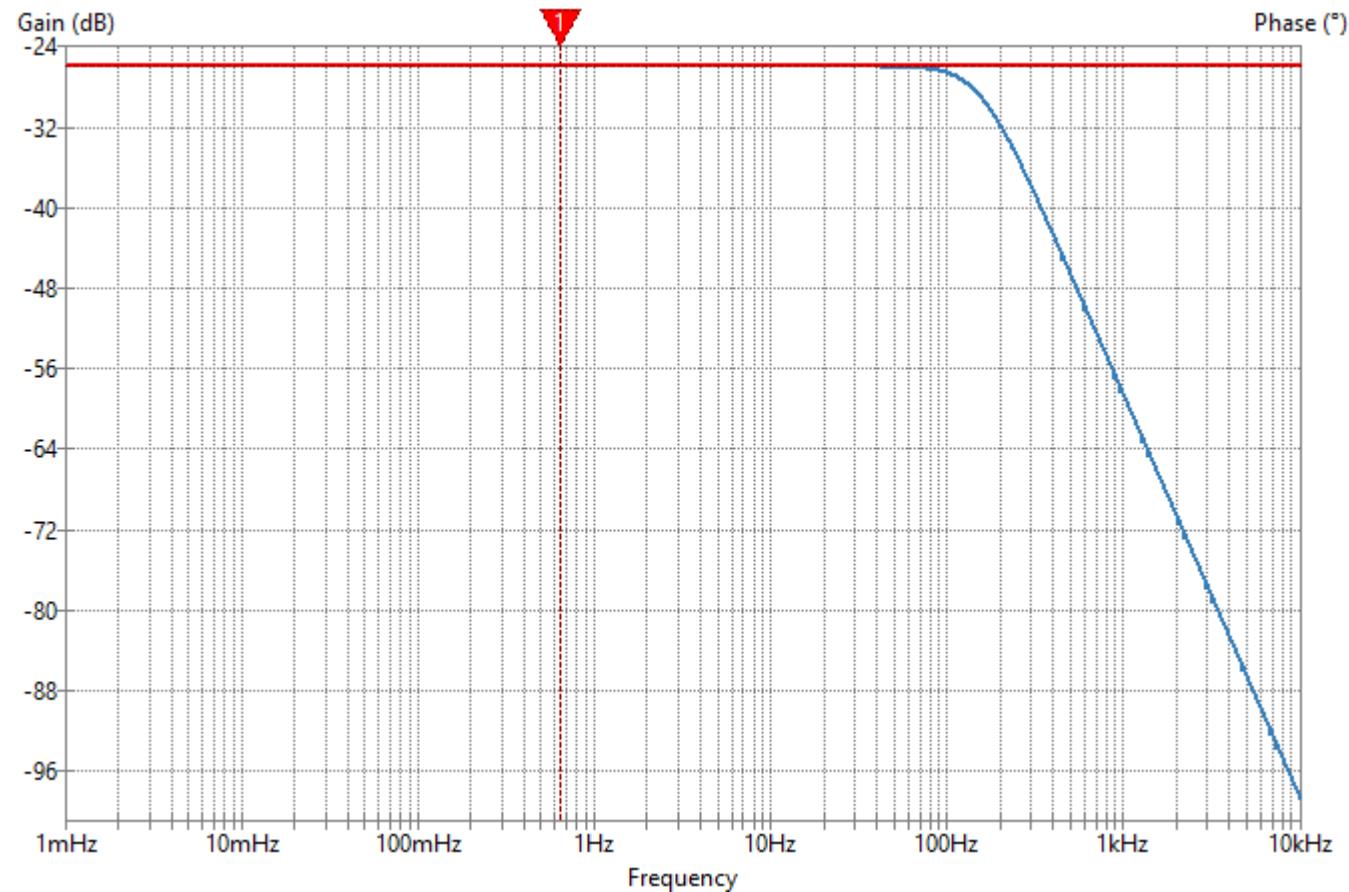
Simulate an AC sweep from $1\text{mHz} - 10\text{kHz}$.

Read the common mode gain.

What is the CMRR?

The solution shows a common mode gain of -26dB within the passband.

$$\text{CMRR} = 60\text{dB} - (-26\text{dB}) = 86\text{dB}$$



22050 Continuous-Time Signals and Linear Systems

Kaj-Åge Henneberg

L13

ADC performance

Sampling

Aliasing

Amplitude quantization

Quantization noise

Oversampling and averaging

Matching anti-aliasing filter to ADC

The reader of this document is granted permission to:

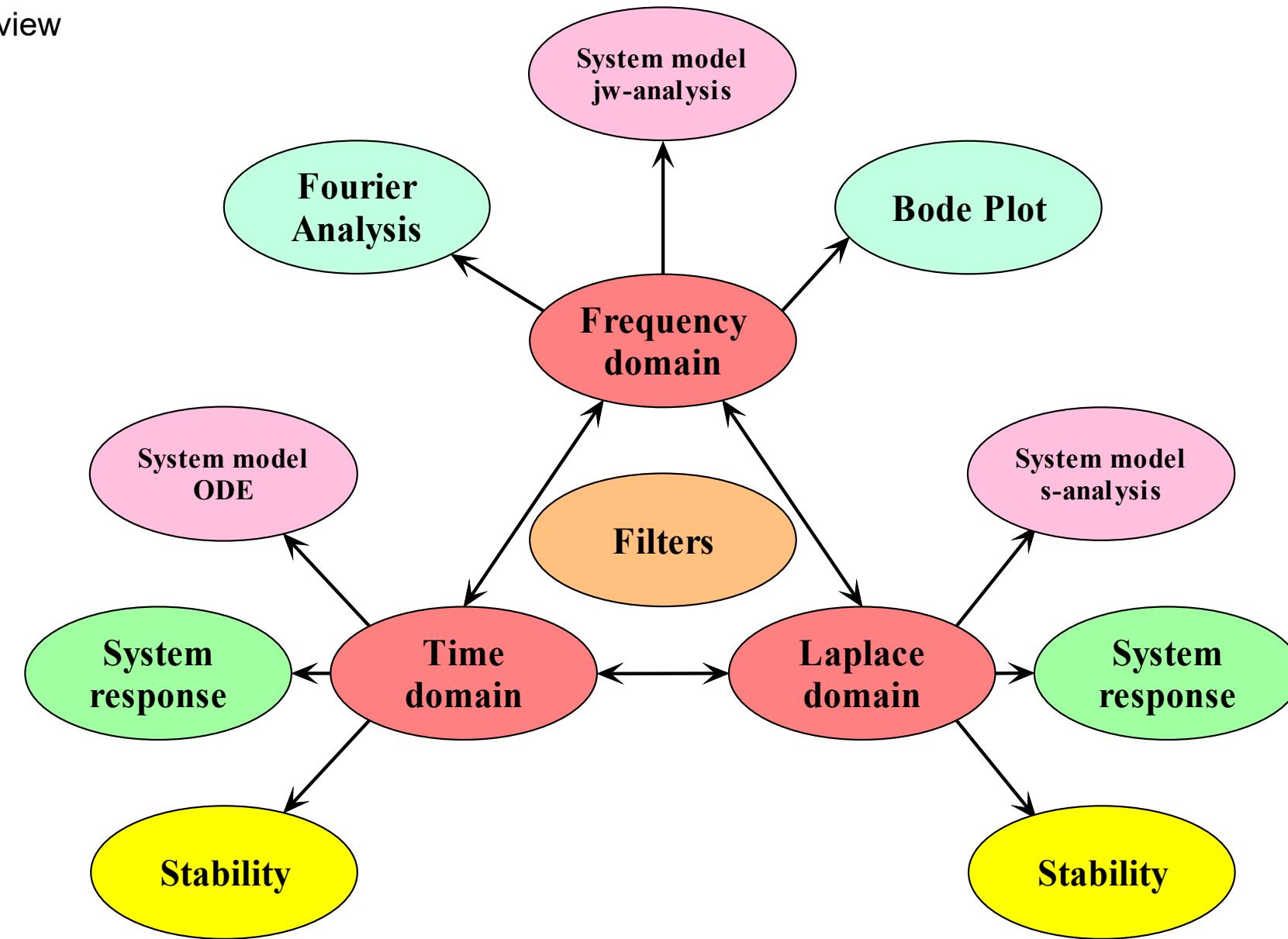
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- In course assignments, the reader is permitted to write the same equations and draw the same diagrams, but should do this using their own tools.



- Analog-digital conversion
 - Sampling
 - Aliasing
 - Amplitude quantization
 - Truncating ADC
 - Rounding ADC
 - Quantization noise
 - RMS value of signal with non-zero mean
 - Effects of averaging on mean and variance of a random variable
 - RMS value of quantization noise in truncating ADC
 - RMS value of quantization noise in rounding ADC
 - Signal to noise ratio
 - Truncating ADC
 - Rounding ADC
 - Effect of oversampling and averaging
 - Inadequate amplification
 - Effective number of bits
 - Matching anti-aliasing filter to ADC
 - Determining design specifications for anti-aliasing filter

Analog-Digital Conversion Aliasing

Video

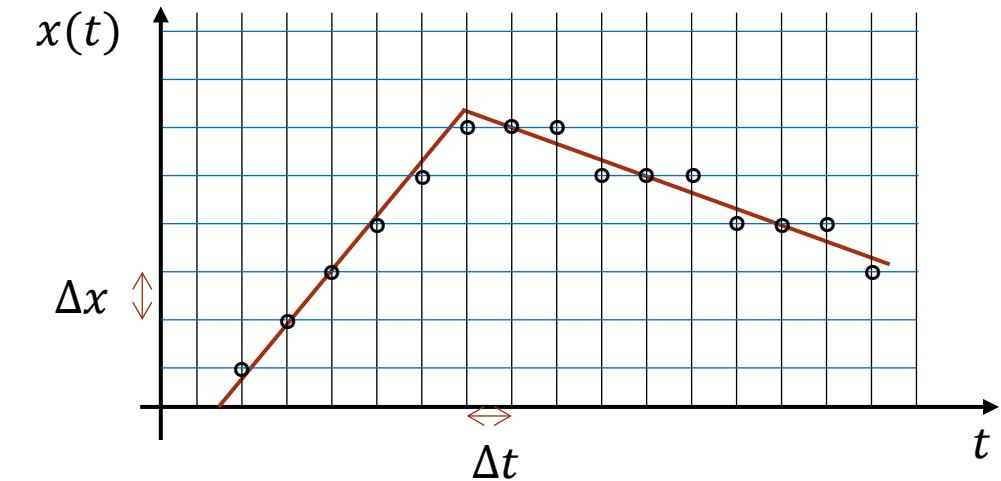
The fundamental steps in Analog-to-Digital conversion

Signals originating from physical processes are typically continuous in both time and amplitude. Such signals are said to be analogue. Examples of analogue signals are the ambient temperature, blood pressure and sound pressure from speech. An example of a signal that is not continuous in time is the heart rate interval. It only has a new value after each heartbeat, but no value in between heart beats. Such a signal is a discrete-time (D-T) signal.

The red curve in the graph is an analogue signal. We can make sample recordings at equidistant time intervals Δt marked by the thin vertical lines. This process is called **SAMPLING**.

If we are to store the amplitude of each sample using binary numbers, we only have a finite number of amplitude values, that we can store, here marked by the thin horizontal lines. Finding that binary value representable in the microprocessor which is closest to the amplitude of the analogue signal is called **QUANTIZATION**. The black circles indicate the result of **rounding/truncating** the amplitude of each sample to its nearest binary value. The vertical distance between the circle (binary value) and the red curve (analogue value) is called **quantization error**.

We will start the discussion of Analogue-Digital conversion by investigating the significance of Δt and Δx .

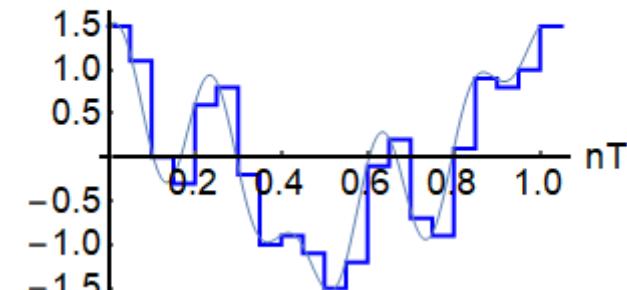


Sampling and Quantization

Quantization:

Signal amplitudes are rounded to nearest possible number

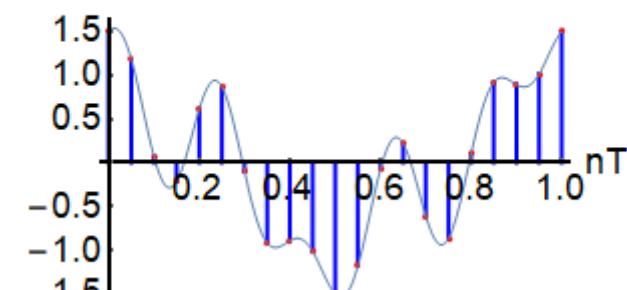
Continuous time/Discrete amplitude



Discrete time/Continuous amplitude

Sampling:

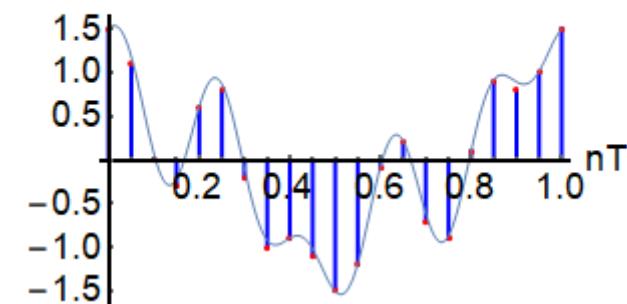
Signal amplitudes are sampled for a brief moment and the value held in a **Sample-&-Hold** circuit until conversion is complete.



Discrete time/Discrete amplitude

Sampling and quantization

The ADC rounds the amplitude of the time-discrete samples to nearest possible number.



Sampling

Sampling saves discrete values of a signal:

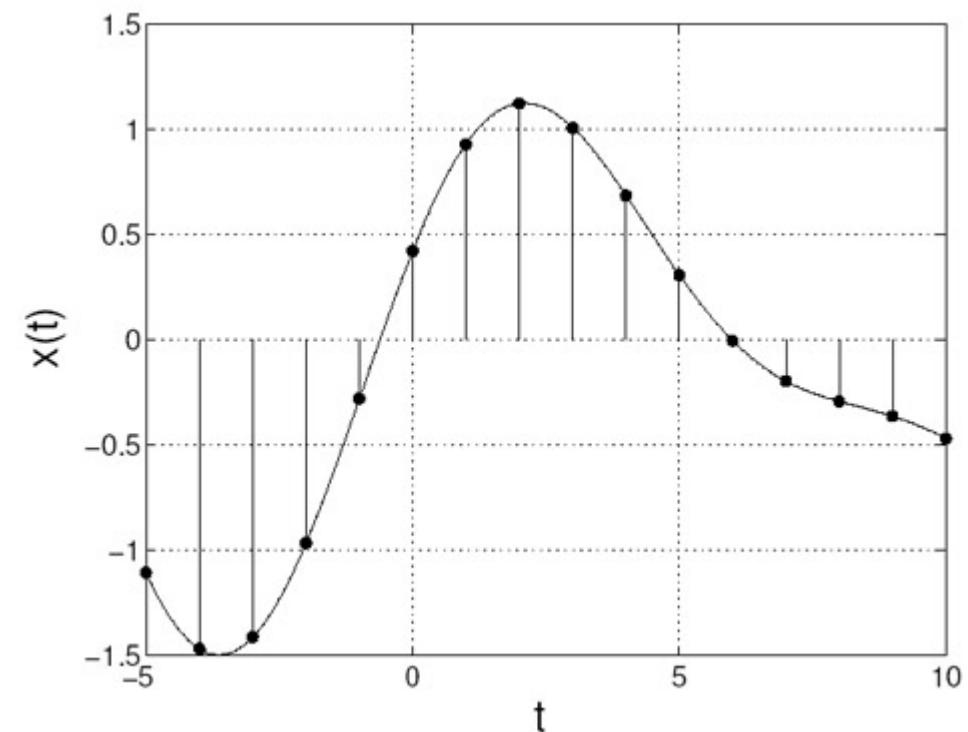
$$x(t) \rightarrow x(n \cdot \Delta t)$$

The time interval between sampling is the sampling interval, and its reciprocal is the **sampling frequency**:

$$F_S = \frac{1}{\Delta t}$$

Nyquist sampling theorem:
A unique reconstruction of a sinusoidal signal from its samples requires more than two samples in each period:

$$\Delta t < T/2 \Leftrightarrow F_S > 2f_{\max}$$



We know that the cos function is an even function:

$$x(t) = A \cos(2\pi f \cdot t) = A \cos(2\pi |f| \cdot t)$$

We make a sampled copy of the signal:

$$\Delta t = \frac{1}{f_s}$$

$$x_k(n\Delta t) = A \cos(2\pi \cdot |f_k| \cdot n\Delta t)$$

$$x_k(n\Delta t) = A \cos\left(2\pi \cdot |f_k| \cdot \frac{n}{f_s}\right)$$

$$x_k(n\Delta t) = A \cos\left(2\pi \cdot n \cdot \frac{|f_k|}{f_s}\right)$$

We can add a multiple of 2π :

We observe that if we add a multiple of the sampling frequency f_s to the signal frequency f_k , we will get the exact same sample values.

The signal frequency f_k is said to have **aliases** in $f_k + m \cdot f_s$

$$x_k(n\Delta t) = A \cos\left(2\pi \cdot n \cdot \left(\frac{|f_k|}{f_s} + m \frac{f_s}{f_s}\right)\right)$$

$$x_k(n\Delta t) = A \cos\left(2\pi \cdot n \cdot \left(\frac{|f_k| + m f_s}{f_s}\right)\right)$$

An example:

A 1 Hz cosine sampled at 6 Hz:

$$x_1(n\Delta t) = A \cos\left(2\pi \cdot n \cdot \frac{1}{6}\right) = A \cos\left(2\pi \cdot n \cdot \frac{-1}{6}\right)$$

A 5 Hz cosine sampled at 6 Hz:

$$x_5(n\Delta t) = A \cos\left(2\pi \cdot n \cdot \frac{5}{6}\right) = A \cos\left(2\pi \cdot n \cdot \left(\frac{-1 + 6}{6}\right)\right)$$

A 7 Hz cosine sampled at 6 Hz:

$$x_5(n\Delta t) = A \cos\left(2\pi \cdot n \cdot \left(\frac{-1}{6}\right) + 2\pi n\right) = x_1(n\Delta t)$$

$$x_7(n\Delta t) = A \cos\left(2\pi \cdot n \cdot \frac{7}{6}\right) = A \cos\left(2\pi \cdot n \cdot \left(\frac{1 + 6}{6}\right)\right)$$

$$x_7(n\Delta t) = A \cos\left(2\pi \cdot n \cdot \left(\frac{1}{6}\right) + 2\pi n\right) = x_1(n\Delta t)$$

Consequence of inadequate sampling frequency

Top: 1 Hz sinusoid sampled at a rate 6 Hz.

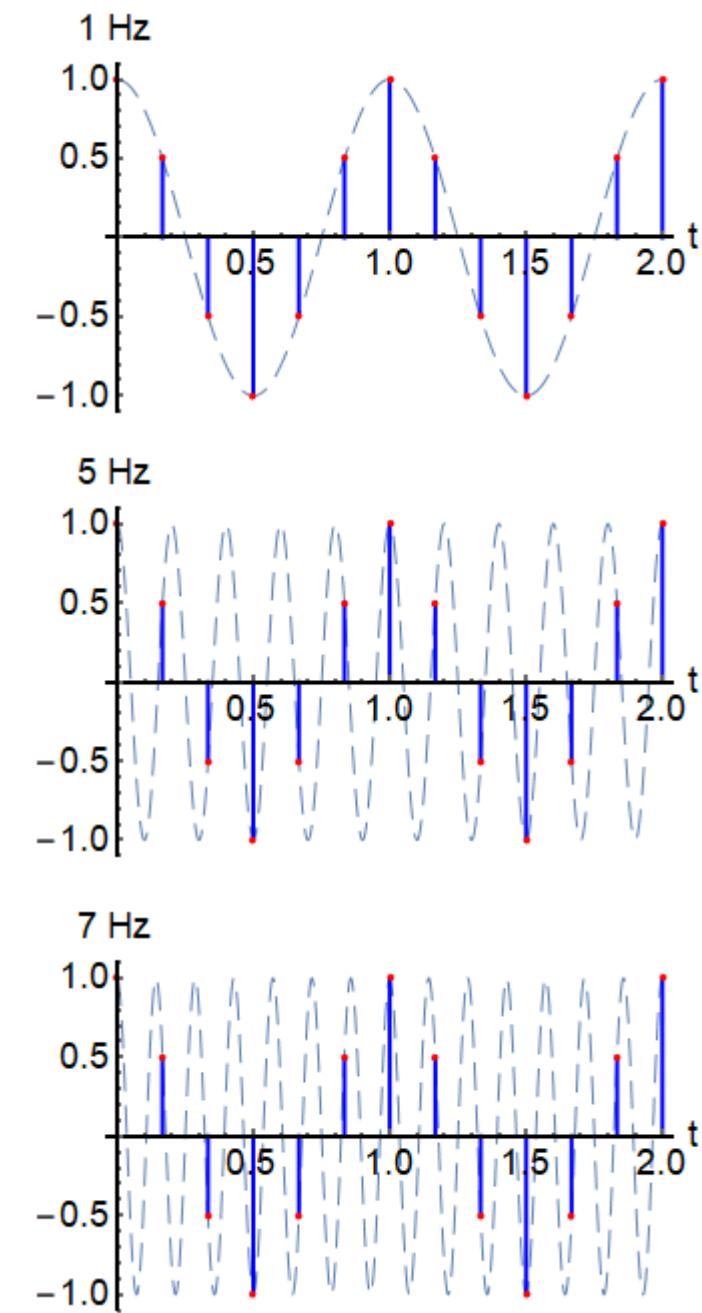
Center: 5 Hz sinusoid sampled at a rate of 6 Hz.

Bottom: 7 Hz sinusoid sampled at a rate of 6 Hz.

We see that even though we have sampled three different sinusoids, the samples come out identical in the three cases. If we were given the set of samples and were asked to draw the continuous time sinusoid through the samples, we would be faced with an ambiguous answer: should we draw a 1 Hz, a 5 Hz or a 7 Hz sinusoid?

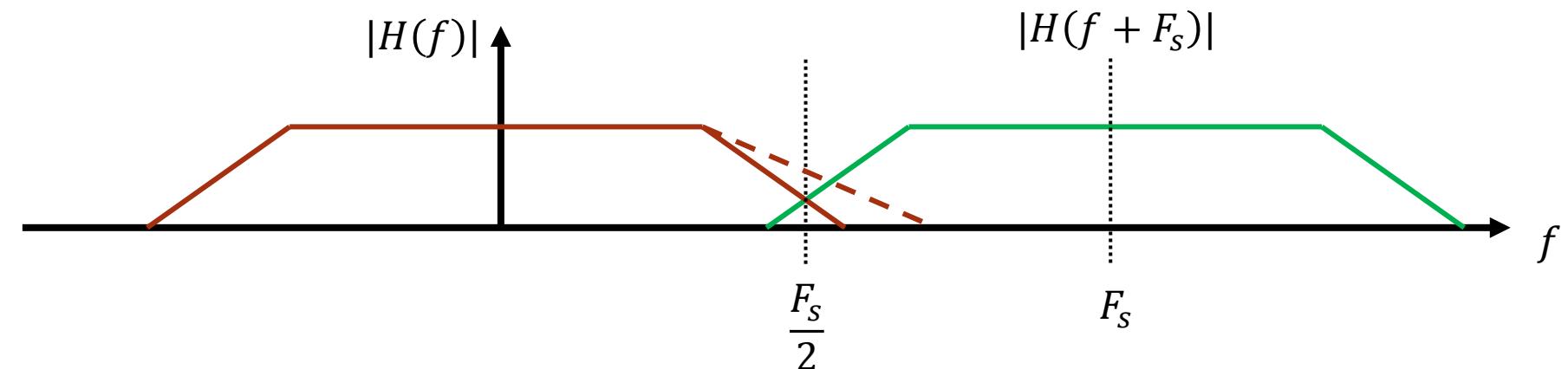
The three signals are said to be each other's alias and the issue is called **aliasing**.

If we assume that we have adhered to the Nyquist sampling theorem, we may conclude that the samples represent a 1 Hz sinusoid, not a 6 or 7 Hz sinusoid.

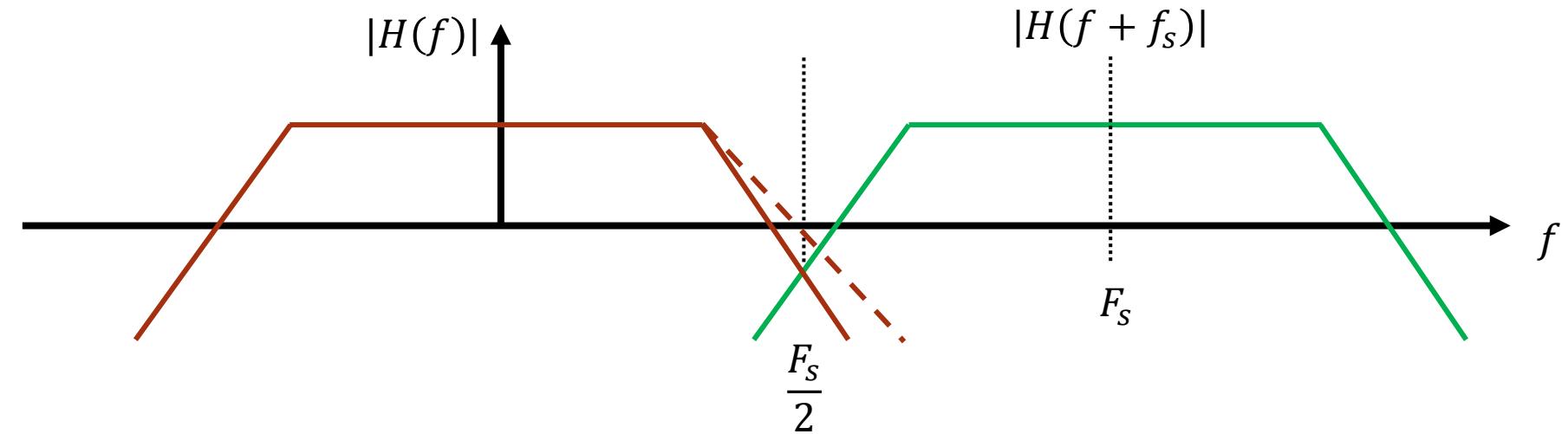


Reconstruction of Continuous-Time Signal

Aliasing increases energy around $F_s/2$.



Effect of aliasing reduced by lowpass filtering.



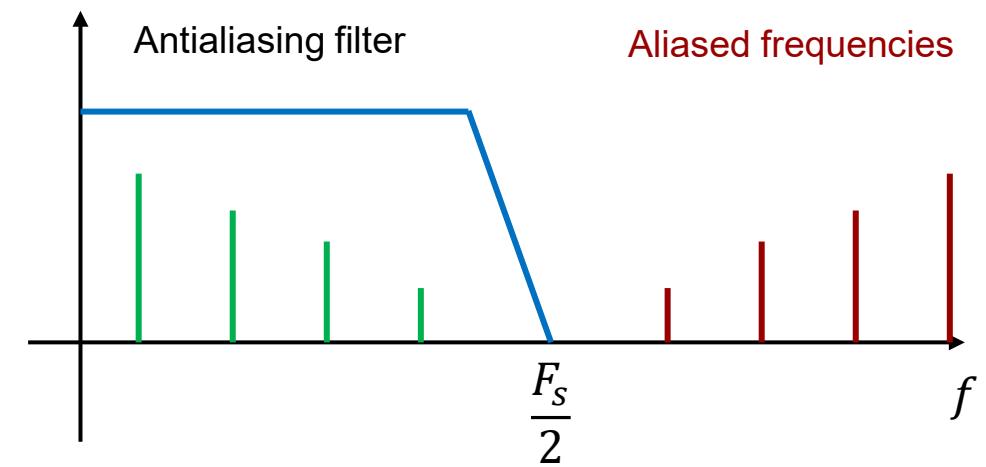
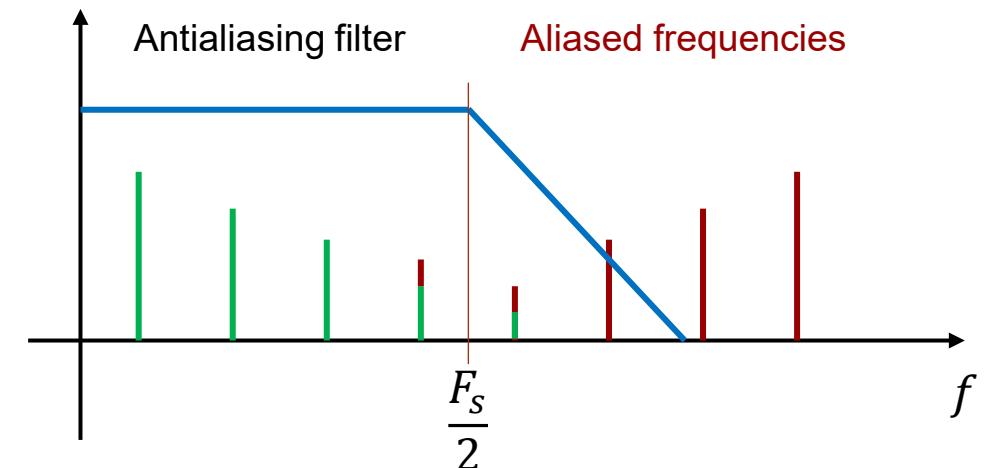
Lowpass filtering of aliased frequencies

The aliased frequencies will fold down and add to the fundamental frequencies (**green stems**) making their amplitudes higher than they are without aliasing.

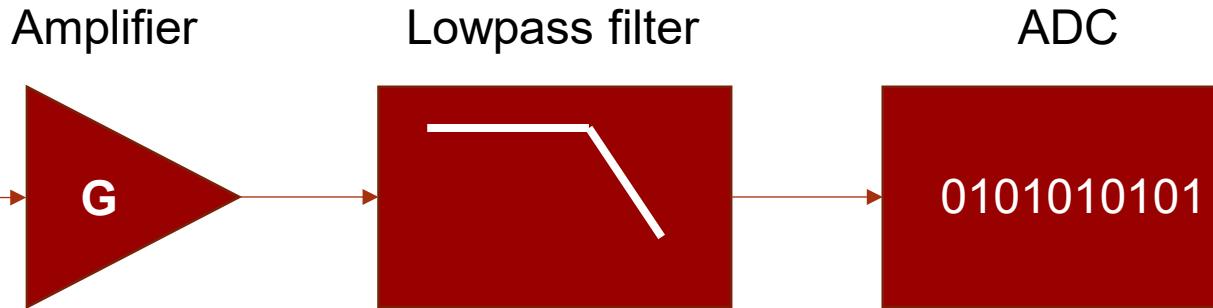
If we can dampen the magnitude of all frequencies above $F_s/2$, then we can reduce how much the aliased frequency add to the fundamental frequencies.

In the top figure we use a lowpass filter to dampen aliased frequencies, but the sampling frequency is too low compared to the cut-off frequency of the filter.

In the bottom figure, the sampling frequency has been increased, and the filter order has been increased as well.



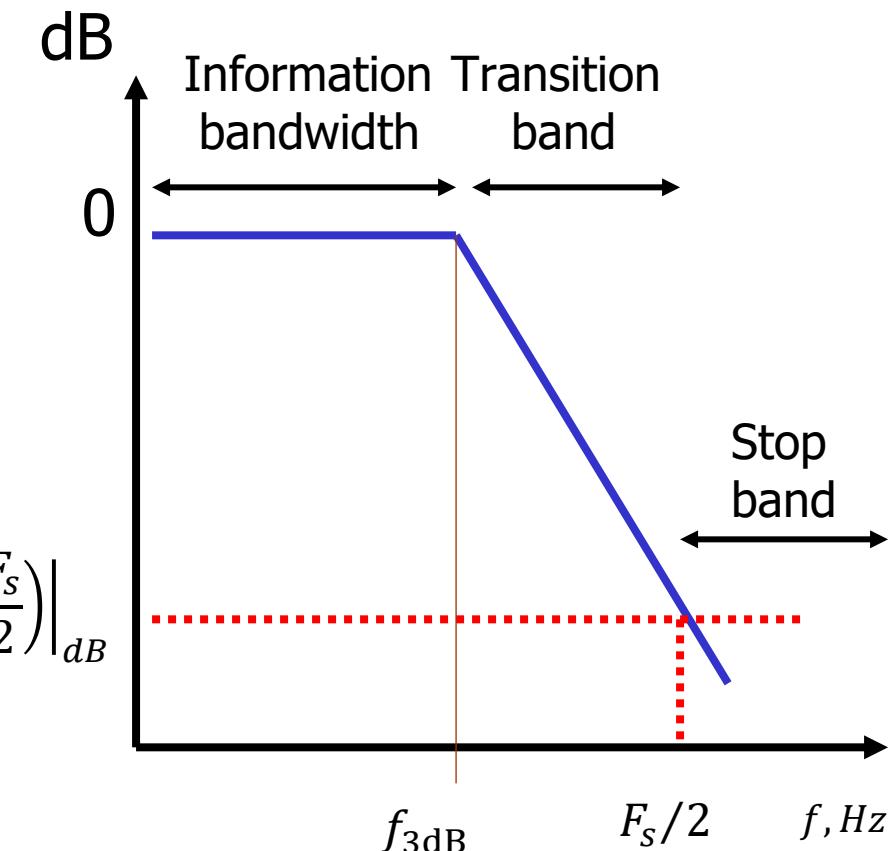
Anti-aliasing filter



Setting the sampling frequency to two times the 3dB cut-off frequency of the lowpass filter is not good enough because the signal is only attenuated about 3-6 dB at the cut-off frequency.

We could benefit from some guidance to setting an attenuation level for $F_s/2$. In other words, what level of damping should the anti-aliasing filter achieve at $F_s/2$?

The ADC has a finite number of bits. Hence there is a limit to how small signal amplitudes the ADC can “see”. We need to investigate, if the ADC can give some suggestions to how much the lowpass filter should attenuate the signal at $F_s/2$.



Analog-Digital Conversion Amplitude quantization

Video

4-bit ADC

To understand the conversion process, we will study an ADC with 4 bits.

The table on the right shows that with 4 bits we can represent $2^4 = 16$ values.

Here a reference voltage of 5V is used, hence the voltage range from 0V to 5V is divided into 16 intervals.

The voltage range of one interval is called the **code width**. We will denote this voltage interval ΔV_{LSB} . It is the voltage increase necessary to toggle the least significant bit (LSB).

$$\text{Code width: } \Delta V_{LSB} = \frac{V_{range}}{2^{\#bit}} = \frac{5V}{2^4} = 0.3125V$$

We must remember, that a certain bit pattern represents a **voltage range**, not a specific voltage. Since we in this case have 16 bit-patterns, we also get 16 voltage intervals.

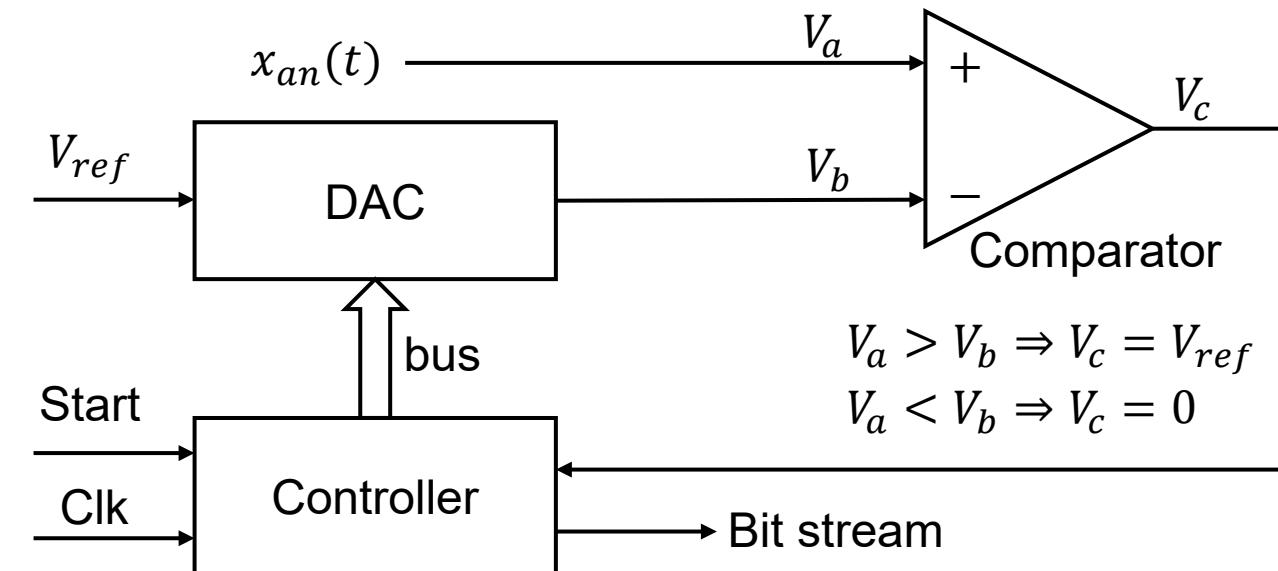
There are several different ADC principles for setting the bits. We will only consider the successive approximation register (SAR) ADC.

bit	3	2	1	0	Value
0	0	0	0	0	0,00000
1	0	0	0	1	0,31250
2	0	0	1	0	0,62500
3	0	0	1	1	0,93750
4	0	1	0	0	1,25000
5	0	1	0	1	1,56250
6	0	1	1	0	1,87500
7	0	1	1	1	2,18750
8	1	0	0	0	2,50000
9	1	0	0	1	2,81250
10	1	0	1	0	3,12500
11	1	0	1	1	3,43750
12	1	1	0	0	3,75000
13	1	1	0	1	4,06250
14	1	1	1	0	4,37500
15	1	1	1	1	4,68750
1	0	0	0	0	5,00000

Conversion principle of truncating SAR ADC

Successive approximation register (SAR)
ADC.

Starting from the most significant bit down to the least significant bit, the controller turns on each bit one at a time and generates an analogue signal (V_b), with the help of the digital-to-analogue converter (DAC), to be compared in the comparator with the original input analogue signal (V_a). Based on the result of the comparison (V_c), the controller changes or leaves the current bit and turns on the next most significant bit. The process continues until decisions are made for all available bits. The result is a bit pattern corresponding to 2.8125V, which is 0.1875V lower than the analogue value.



4-bit converter, $V_{ref} = 5V$

V_a (V)	V_b (V)	V_c (V)	MSB				LSB
			3	2	1	0	
3	2,500	5	1	0	0	0	
3	3,750	0	1	1	0	0	
3	3,1250	0	1	0	1	0	
3	2,8125	5	1	0	0	1	

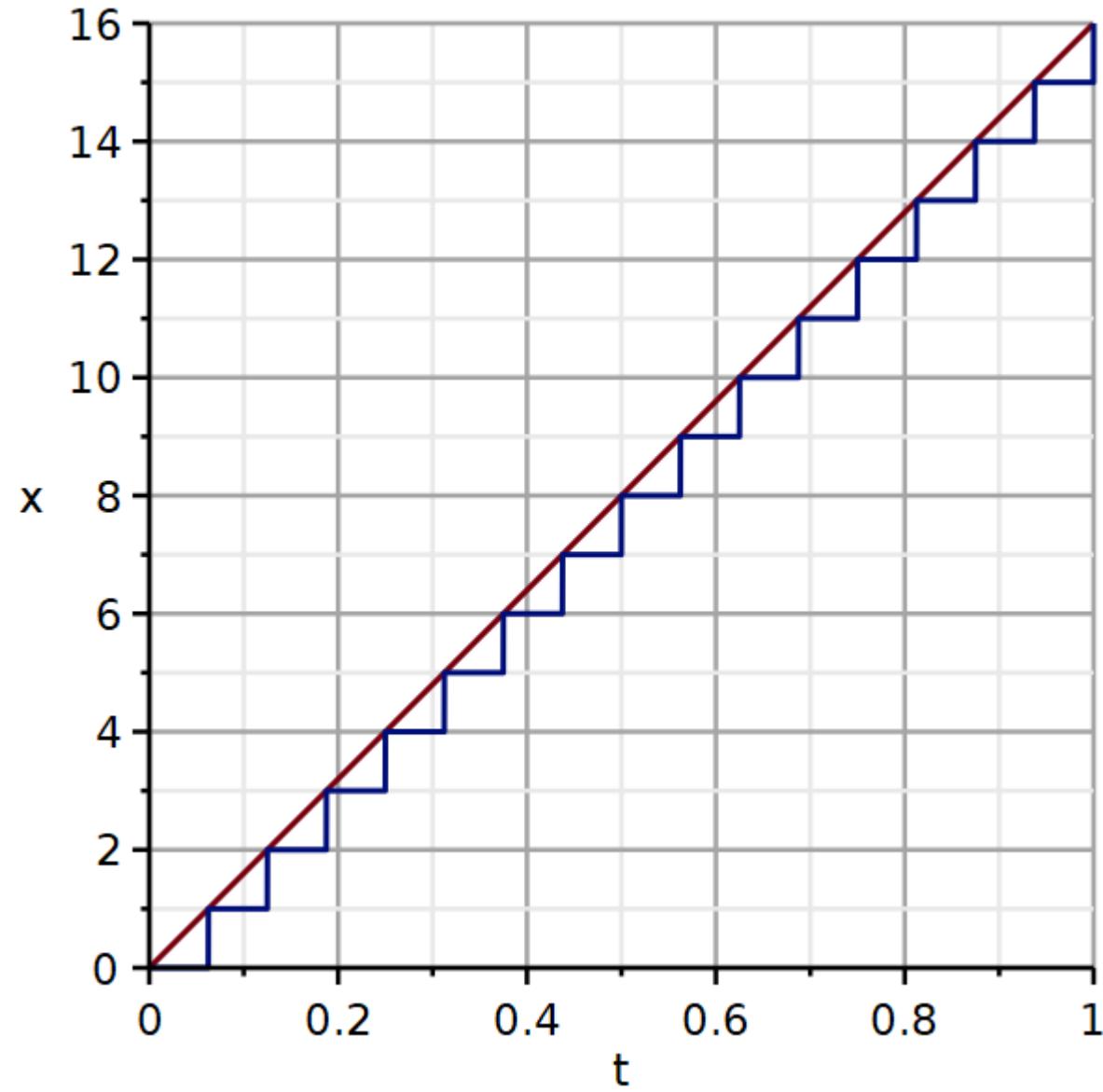
Observation: The result is always lower than the analogue value. We will call this a **truncating ADC**.

Amplitude quantization of truncating ADC

Here a ramp signal with an amplitude of 16V is digitized. The ADC has 4 bits and a 16V voltage range.

The red curve represents the analog signal $x_{an}(t)$. The blue curve represents the digitized signal $x_{dig}(t)$.

We observe that the digitized signal is consistently lower in amplitude than the analog signal. It is a truncating ADC.



Quantization noise of truncating ADC

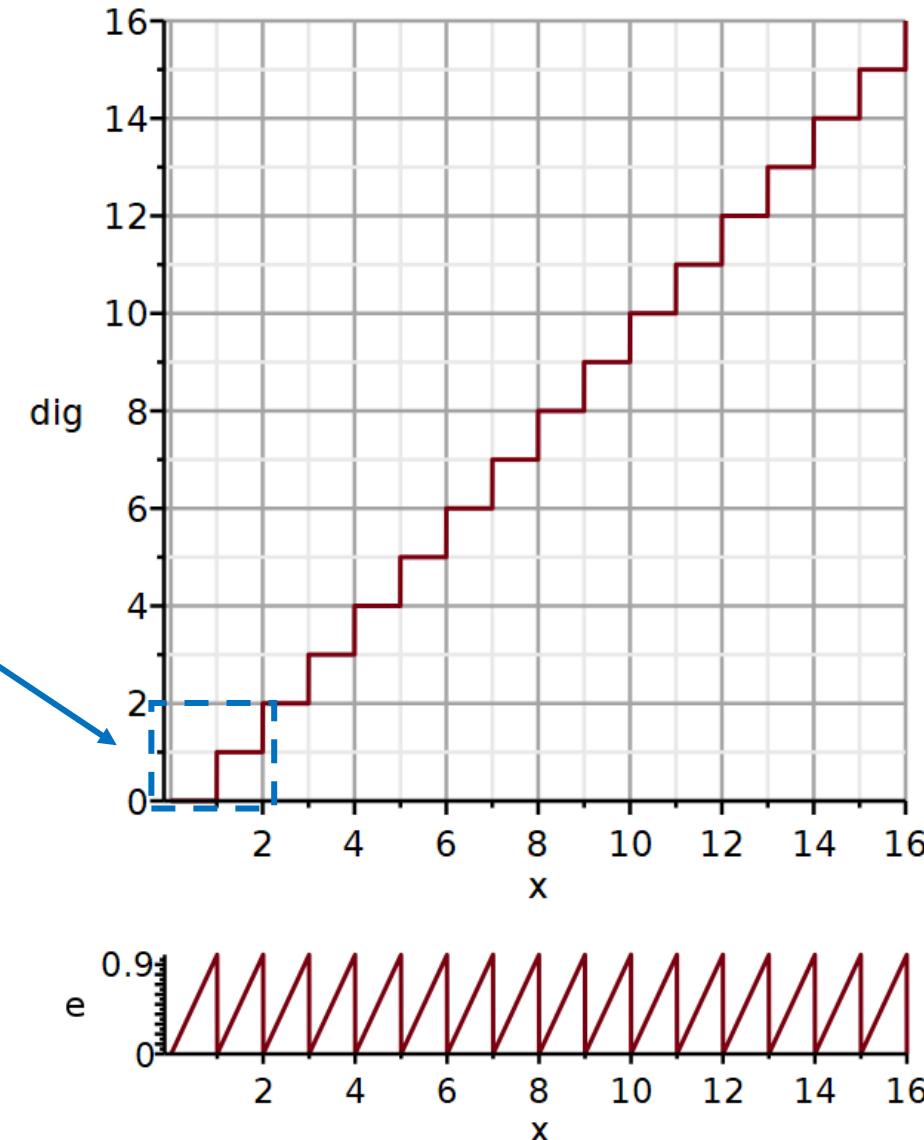
The top curve illustrates the input-output relationship of a truncating ADC.

When $x_{an}(t)$ climbs from 0V to 1V, the digital value $x_{dig}(t)$ remains 0V. We will define the difference as an error signal:

$$e(t) \stackrel{\text{def}}{=} x_{an}(t) - x_{dig}(t)$$

The lower graph illustrates the error signal $e(t)$.

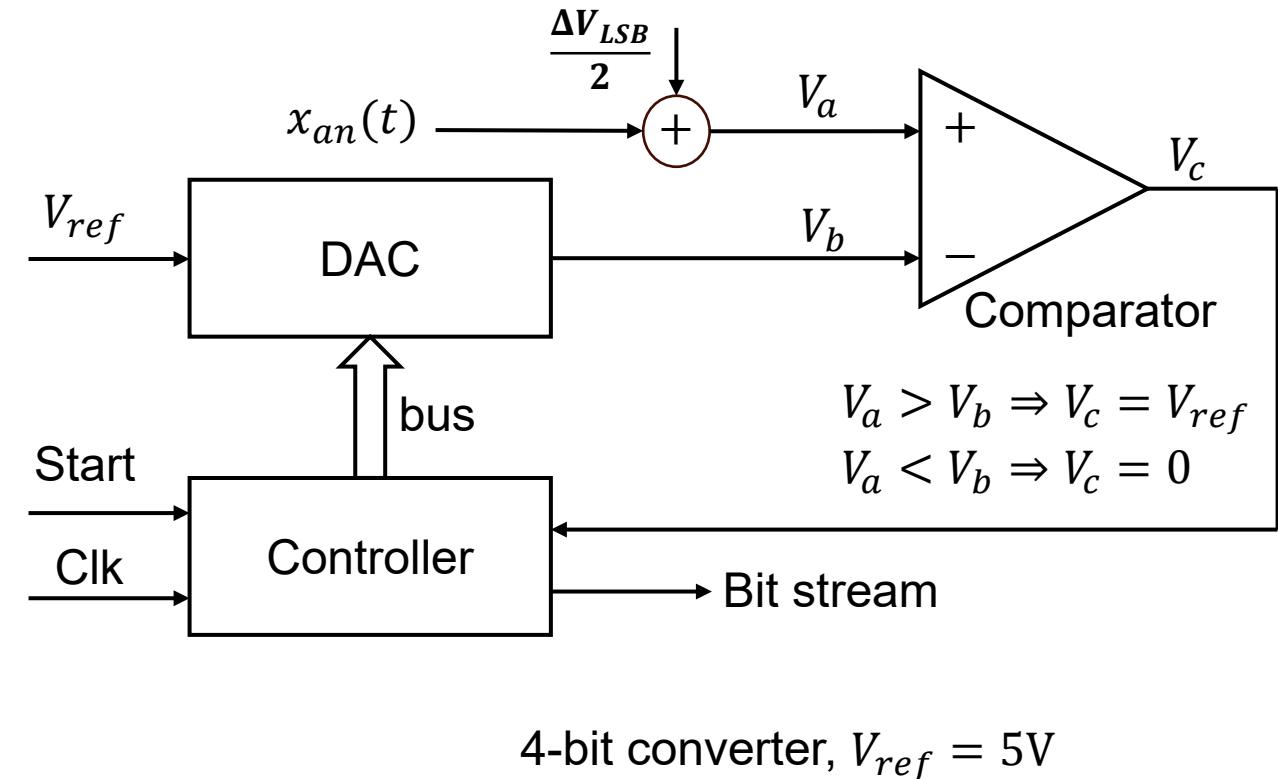
The mean value of the error signal is not zero. This error signal is referred to as quantization noise. For a truncating ADC, the **quantization noise is biased (non-zero mean)**.



Conversion principle of rounding ADC

Successive approximation, **rounding** ADC. $\frac{\Delta V_{LSB}}{2}$ is added to the analogue signal.

At the end of the conversion, a bit pattern is obtained corresponding to 3.125V. Hence the analogue signal of 3V is rounded up to the nearest binary value corresponding to 3.125V. The result is 0.125V higher than the analogue value.



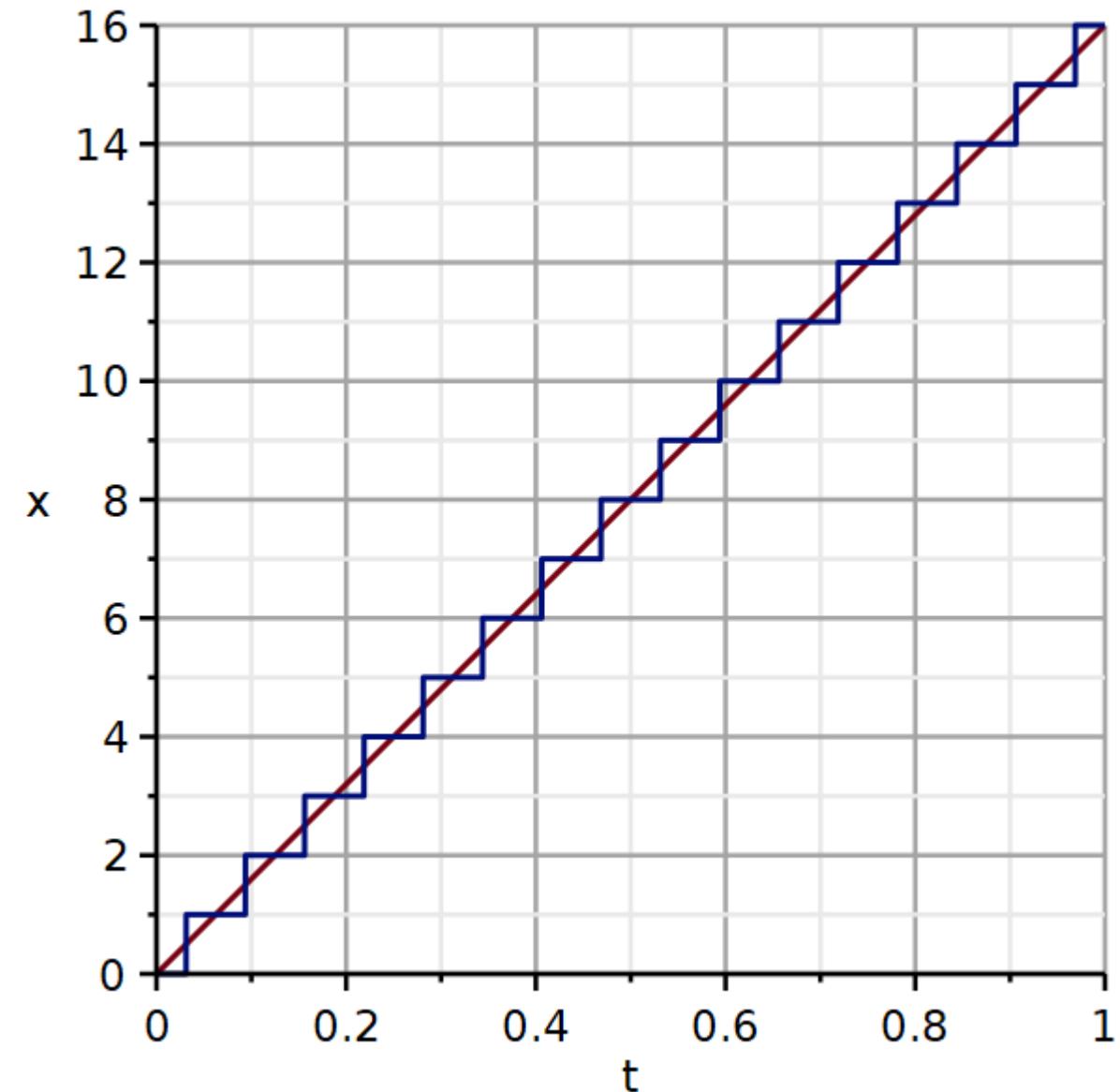
Observation: The result can be higher than the analogue value. We will call this a **rounding ADC**.

x (V)	V _a (V)	V _b (V)	V _c (V)	MSB 3	2	1	LSB 0
3	3.15625	2,500	5	1	0	0	0
3	3.15625	3,750	0	1	1	0	0
3	3.15625	3,1250	5	1	0	1	0
3	3.15625	3,4375	0	1	0	1	1
3	3.15625	3,1250	5	1	0	1	0

Amplitude quantization of rounding ADC

Here a ramp signal with an amplitude of 16V is digitized. The red curve represents the analog signal $x_{an}(t)$. The blue curve represents the digitized signal $x_{dig}(t)$.

We observe that the digitized signal is now rounded off to nearest binary value. This is the performance seen in a **rounding ADC**.



Quantization noise of rounding ADC

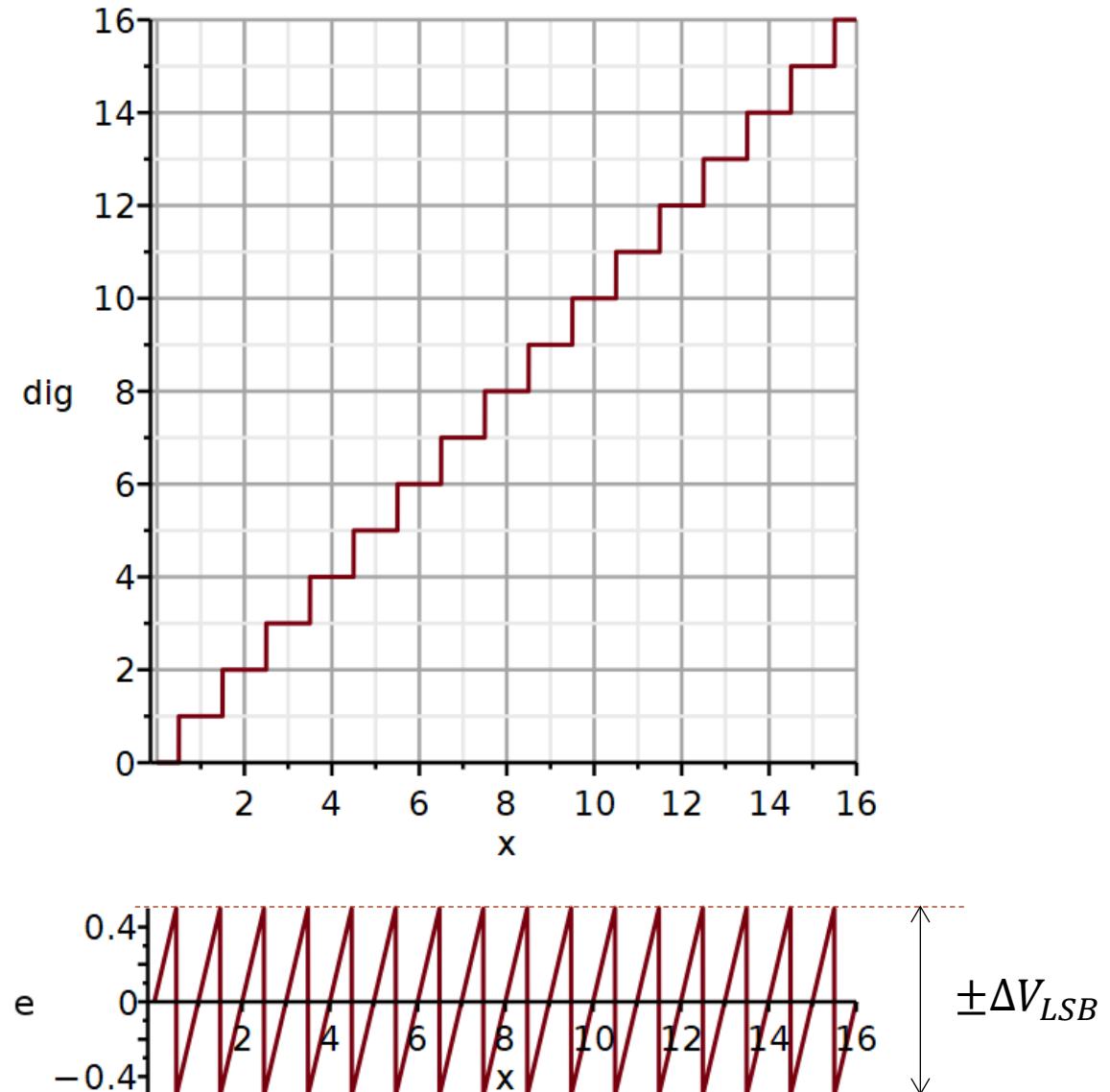
The top curve illustrates the input-output relationship of a rounding ADC.

When $x_{an}(t)$ climbs from 0V to 0.5V, the digital value $x_{dig}(t)$ remains 0V. When $x_{an}(t)$ climbs from 0.5V to 1V, the digital signal is 1V. We will define the difference as an error signal:

$$e(t) \stackrel{\text{def}}{=} x_{an}(t) - x_{dig}(t)$$

The lower graph illustrates the error signal $e(t)$ for a rounding ADC.

The mean value of the error signal is zero. For a rounding ADC, the **quantization noise is unbiased**.



Analog-Digital Conversion Quantization noise

Video

Signal to noise ratio for ADC - definition

We have seen that analogue-to-digital conversion results in quantization noise. An important performance parameter for an ADC is therefore its signal-to-noise ratio (SNR). SNR is usually listed in decibels. The SNR is calculated from the ratio of rms values of the signal and the noise.

The noise signal can be, besides quantization noise, also include other types of noise, e.g., thermal noise from resistors and 1/f noise from semiconductors.

In the present analysis, we will only consider the noise originating from quantization.

$$SNR = 20 \log \frac{x_{an,rms}}{n_{rms}}$$

The root mean square

The definition of the root mean square:

We separate the signal into a sum of its mean value and a signal with zero mean:

$$x_{an}(t) = \bar{x}_{an} + x_{an,0}(t)$$

The green integral is the mean value of a signal with zero mean.

The power of a signal is the sum of the power of the mean and the power of the zero-mean component. The latter is the variance of the signal.

$$x_{an,rms} \stackrel{\text{def}}{=} \sqrt{\frac{1}{T} \int_{-T/2}^{T/2} x_{an}^2(t) dt}$$

$$x_{an,rms}^2 = \frac{1}{T} \int_{-T/2}^{T/2} (\bar{x}_{an} + x_{an,0}(t))^2 dt$$

$$x_{an,rms}^2 = \frac{1}{T} \int_{-T/2}^{T/2} (\bar{x}_{an}^2 + 2\bar{x}_{an}x_{an,0}(t) + x_{an,0}^2(t)) dt$$

$$= \bar{x}_{an}^2 + 2\bar{x}_{an} \underbrace{\frac{1}{T} \int_{-T/2}^{T/2} (x_{an,0}(t)) dt}_{=0} + \frac{1}{T} \int_{-T/2}^{T/2} x_{an,0}^2(t) dt$$

$$= \bar{x}_{an}^2 + \sigma_{x_{an}}^2$$

$$x_{an,rms} = \sqrt{\bar{x}_{an}^2 + \sigma_{x_{an}}^2}$$

Signal to noise ratio for ADC – rms value of signal

For the sake of illustration we will assume that the analogue signal is a sinusoidal signal with a constant offset:

$$x_{an}(t) = \frac{A}{2} + \frac{A}{2} \sin \omega t$$

The full voltage range of the ADC is:

For a sinusoidal signal with a peak-to-peak amplitude A which is only a fraction α of the ADC voltage range, we have:

Its rms value is:

$$x_{an,rms} = \sqrt{\frac{1}{T} \int_{-T/2}^{T/2} x_{an}^2(t) dt}$$

$$x_{an,rms} = \sqrt{\frac{A^2}{4} + \frac{A^2}{8}} = \sqrt{\frac{3A^2}{8}} = A \sqrt{\frac{3}{8}} \approx 0.612A$$

$$Full\ scale = 2^N \times \Delta V_{LSB}$$

$$A = \alpha 2^N \times \Delta V_{LSB}$$

$$x_{an,rms} = \sqrt{\frac{3}{8}} \alpha 2^N \times \Delta V_{LSB}$$

Signal to noise ratio for rounding ADC – rms value of noise

To derive the rms value of the quantization noise we need to calculate the mean and variance of a random variable X .

Let X be a random variable with uniform probability distribution from a to b .

Then the expected mean value $\mu = E(X)$ is:

The expected mean of the square $E(X^2)$ is:

$$n_{rms} = \sqrt{\bar{n}^2 + \sigma_n^2}$$

$$\mu = E(X) = \frac{1}{b-a} \int_a^b X dX = \frac{1}{b-a} \left[\frac{X^2}{2} \right]_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

$$E(X^2) = \frac{1}{b-a} \int_a^b X^2 dX = \frac{1}{b-a} \left[\frac{X^3}{3} \right]_a^b = \frac{1}{3} \frac{1}{b-a} [b^3 - a^3]$$

$$= \frac{1}{3} \frac{1}{b-a} (b-a)(b^2 + ab + a^2)$$

$$= \frac{1}{3} (b^2 + ab + a^2)$$

Variance of uniform distribution with zero mean.

The variance σ^2 is:

$$\begin{aligned}\sigma^2 &\stackrel{\text{def}}{=} E(X^2) - \mu^2 = \frac{1}{3}(b^2 + ab + a^2) - \frac{(a+b)^2}{4} \\&= \frac{b^3}{3} + \frac{ab}{3} + \frac{a^2}{3} - \frac{a^2}{4} - \frac{2ab}{4} - \frac{b^2}{4} \\&= \frac{4b^3 + 4ab + 4a^2 - 3a^2 - 6ab - 3b^2}{12} = \frac{b^2 - 2ab + a^2}{12} \\&= \frac{(b-a)^2}{12}\end{aligned}$$

The rms value calculated using distributions:

$$X_{rms} = \sqrt{E(X^2)} = \sqrt{\left(\frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12}} = \sqrt{\mu^2 + \sigma^2}$$

The rms value calculated using time integrals:

$$x_{an,rms} = \sqrt{\bar{x}_{an}^2 + \sigma_{x_{an}}^2}$$

Are these calculations related?

chatGPT:

I can calculate the rms value of a signal by calculating a time integral. But I can also calculate the rms value from a distribution of a random variable. How are they connected?

1. RMS from a Time Integral

For a continuous-time signal $x(t)$ over a period T , the RMS value is given by:

$$x_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T x^2(t) dt}$$

This formula treats the signal as a deterministic function of time.

2. RMS from a Probability Distribution

For a random variable X with a probability density function (PDF) $p(x)$, the RMS value is the square root of the expected value of X^2 :

$$X_{\text{rms}} = \sqrt{\mathbb{E}[X^2]} = \sqrt{\int_{-\infty}^{\infty} x^2 p(x) dx}$$

This interpretation arises in stochastic processes, where X represents the amplitude of the signal at a random instant.

The Connection

The time-domain integral and the expectation integral are formally the same if we interpret the time integral as an expectation over time. If the signal $x(t)$ is a **stationary random process**, meaning its statistical properties don't change over time, then:

- The long-term time average of $x^2(t)$ converges to the statistical expectation of X^2 .
- The probability density function $p(x)$ describes how likely it is to observe a particular amplitude value.

In essence, when a signal is a **random process**, its RMS value can be computed using either method:

1. **Time-domain approach:** Taking a long enough observation period and averaging $x^2(t)$.
2. **Statistical approach:** Using the probability distribution of $x(t)$ and computing the expectation $\mathbb{E}[X^2]$.

Stationary random process: μ and σ^2 are time invariant.

Effects of averaging a random variable

For the rounding ADC the quantization noise will sometimes be positive, other times negative. If several ADC conversions are made in rapid succession, and the average value is calculated from the set of samples, then the analog signal may not have changed significantly in the short time duration, but the quantization noise may be reduced due to cancelation of noise samples with opposite signs.

Hence there is a possibility of improving the signal-to-noise ratio by using the average of a set of samples obtained at a higher sampling frequency. This technique is called **oversampling**.

To see the effect of **oversampling and averaging**, we need to study the effect of averaging samples of a random variable.

Averaging a random variable

We need to find out what effect averaging has on the mean and variance of a random variable.

Let us get help from AI.



Certainly! Let's derive the expressions for how averaging affects the **mean** and **variance** of a uniformly distributed random variable. We will consider a uniformly distributed random variable X with **mean** μ and **variance** σ^2 (which we denote as v here).

1. Definitions

- Let X be a uniformly distributed random variable with:
 - Mean: μ
 - Variance: v

2. Averaging n Samples

When we take n independent samples X_1, X_2, \dots, X_n from the distribution of X , we can calculate the average of these samples:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

3. Mean of the Average

The **mean** of the average \bar{X} can be found using the linearity of expectation:

$$E[\bar{X}] = E\left[\frac{1}{n} \sum_{i=1}^n X_i\right]$$

Using the linearity of expectation:

$$E[\bar{X}] = \frac{1}{n} \sum_{i=1}^n E[X_i] = \frac{1}{n} \sum_{i=1}^n \mu = \frac{n\mu}{n} = \mu$$

Conclusion for Mean

Thus, the **mean** of the average \bar{X} remains the same as the **mean** of the original distribution:

$$\text{Mean of } \bar{X} = \mu$$

We see that if the random variable has a nonzero mean value, averaging will not change the mean value.

Averaging a random variable

We also want to know what the effect of averaging is on the variance?

If we average M samples, the contribution to rms from the variance will be reduced, but not the contribution from the mean.

4. Variance of the Average

Next, we will derive the variance of the average \bar{X} . The variance of the average can be calculated as follows:

$$\text{Var}(\bar{X}) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right)$$

Using the property that for independent random variables, the variance of their sum is the sum of their variances:

$$\text{Var}(\bar{X}) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) = \frac{1}{n^2} \sum_{i=1}^n v = \frac{nv}{n^2} = \frac{v}{n}$$

Conclusion for Variance

Thus, the variance of the average \bar{X} is reduced by a factor of n :

$$\text{Variance of } \bar{X} = \frac{v}{n}$$

$$X_{rms,M} = \sqrt{\mu^2 + \frac{\sigma^2}{M}}$$

Summary of Results

- **Mean of the Average:** The mean of the average \bar{X} remains the same as the original mean :

$$\text{Mean of } \bar{X} = \mu$$

- **Variance of the Average:** The variance of the average \bar{X} decreases inversely with the number of samples n :

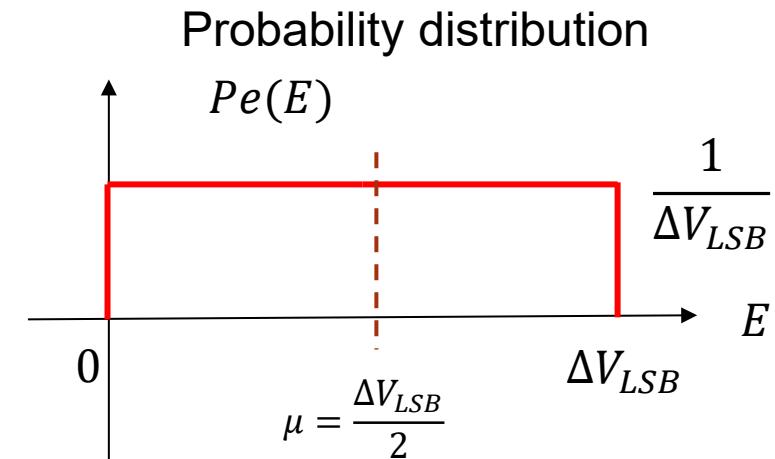
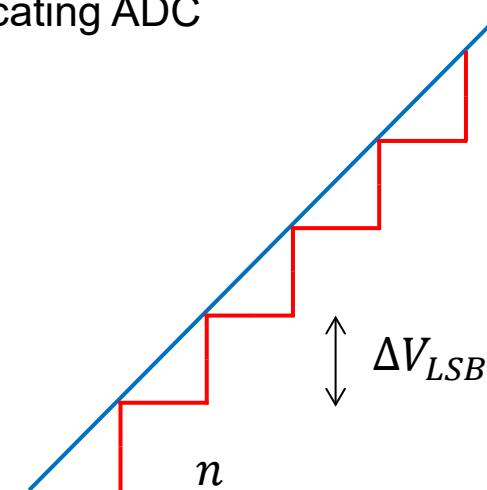
$$\text{Variance of } \bar{X} = \frac{v}{n}$$

Implications

- **Averaging** tends to keep the expected value (mean) unchanged while reducing the variability (variance) of the estimates. This makes averaging an effective technique for improving the reliability of measurements derived from random variables.

RMS value of quantization noise for truncating ADC

Using these results for the **truncating ADC**, we see that averaging has limited effect on reducing the noise due to the bias (non-zero mean).



$$X_{rms,M} = \sqrt{\mu^2 + \frac{\sigma^2}{M}}$$

$$\mu^2 = \left(\frac{\Delta V_{LSB}}{2}\right)^2$$

$$\sigma^2 = \frac{(b-a)^2}{12} = \frac{(\Delta V_{LSB} - 0)^2}{12} = \frac{\Delta V_{LSB}^2}{12}$$

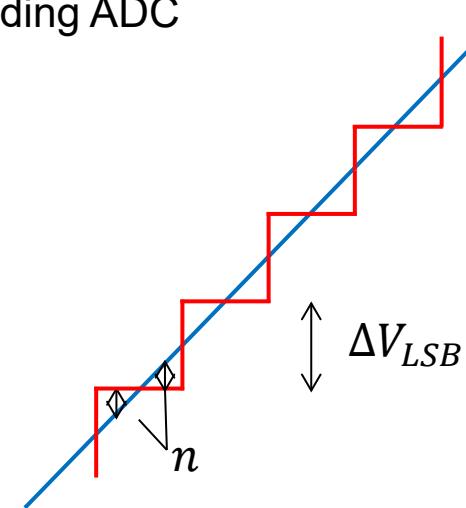
Observation:

Averaging samples from a truncating ADC reduces the variance, but not the mean.

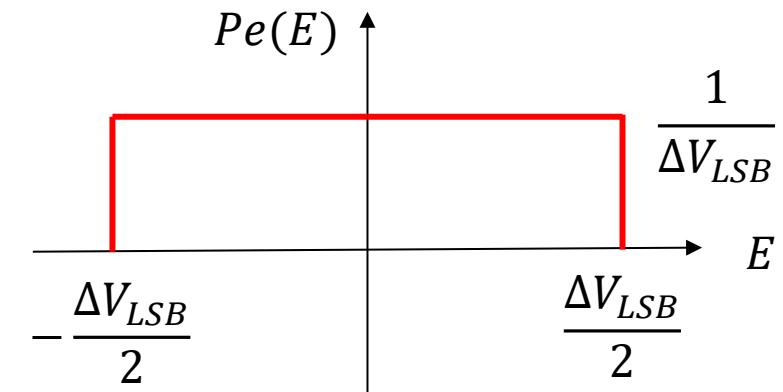
$$n_{rms,M} = \sqrt{\mu^2 + \frac{\sigma^2}{M}} = \sqrt{\left(\frac{\Delta V_{LSB}}{2}\right)^2 + \left(\frac{\Delta V_{LSB}}{12M}\right)^2}$$

RMS value of quantization noise for rounding ADC

The quantization noise from a **rounding ADC** has zero mean value, hence oversampling is more effective in reducing quantization noise.



Probability distribution



$$X_{rms,M} = \sqrt{\mu^2 + \frac{\sigma^2}{M}}$$

$$\mu^2 = 0$$

$$\sigma^2 = \frac{(b-a)^2}{12} = \frac{\left(\frac{\Delta V_{LSB}}{2} - \frac{-\Delta V_{LSB}}{2}\right)^2}{12} = \frac{\Delta V_{LSB}^2}{12}$$

Observation:

Averaging samples from a rounding ADC reduces the variance and thus the quantization noise.

$$n_{rms,M} = \sqrt{\frac{\Delta V_{LSB}^2}{12M}}$$

Signal to noise ratio for truncating ADC

SNR of a truncating ADC:

$$SNR = 20 \log \frac{x_{an,rms}}{n_{rms,M}} = 20 \log \frac{\sqrt{\frac{3}{8} \alpha 2^N \times \Delta V_{LSB}}}{\sqrt{\left(\frac{\Delta V_{LSB}}{2}\right)^2 + \left(\frac{\Delta V_{LSB}}{\sqrt{12M}}\right)^2}} = 20 \log \frac{\sqrt{\frac{3}{8} \alpha 2^N}}{\sqrt{\frac{1}{4} + \frac{1}{12M}}}$$

Separating out terms, we can better judge the significance of each factor.

$$\begin{aligned} SNR &= 20 \log(2^N) + 20 \log(\alpha) + 20 \log \left(\sqrt{\frac{3}{8}} \right) - 20 \log \sqrt{\frac{1}{4} + \frac{1}{12M}} \\ &= N20 \log(2) + 20 \log(\alpha) - 4.26 \text{dB} - 20 \log \sqrt{\frac{1}{4} + \frac{1}{12M}} \\ &\approx 6.02N + 20 \log(\alpha) - 4.26 \text{dB} - 20 \log \sqrt{\frac{1}{4} + \frac{1}{12M}} \end{aligned}$$

SNR of a rounding ADC:

$$SNR = 20 \log \frac{x_{an,rms}}{n_{rms,M}} = 20 \log \frac{\sqrt{\frac{3}{8}} \alpha 2^N \times \Delta V_{LSB}}{\sqrt{\left(\frac{\Delta V_{LSB}}{\sqrt{12M}}\right)^2}} = 20 \log \frac{\sqrt{\frac{3}{8}} \alpha 2^N}{\sqrt{\frac{1}{12M}}}$$

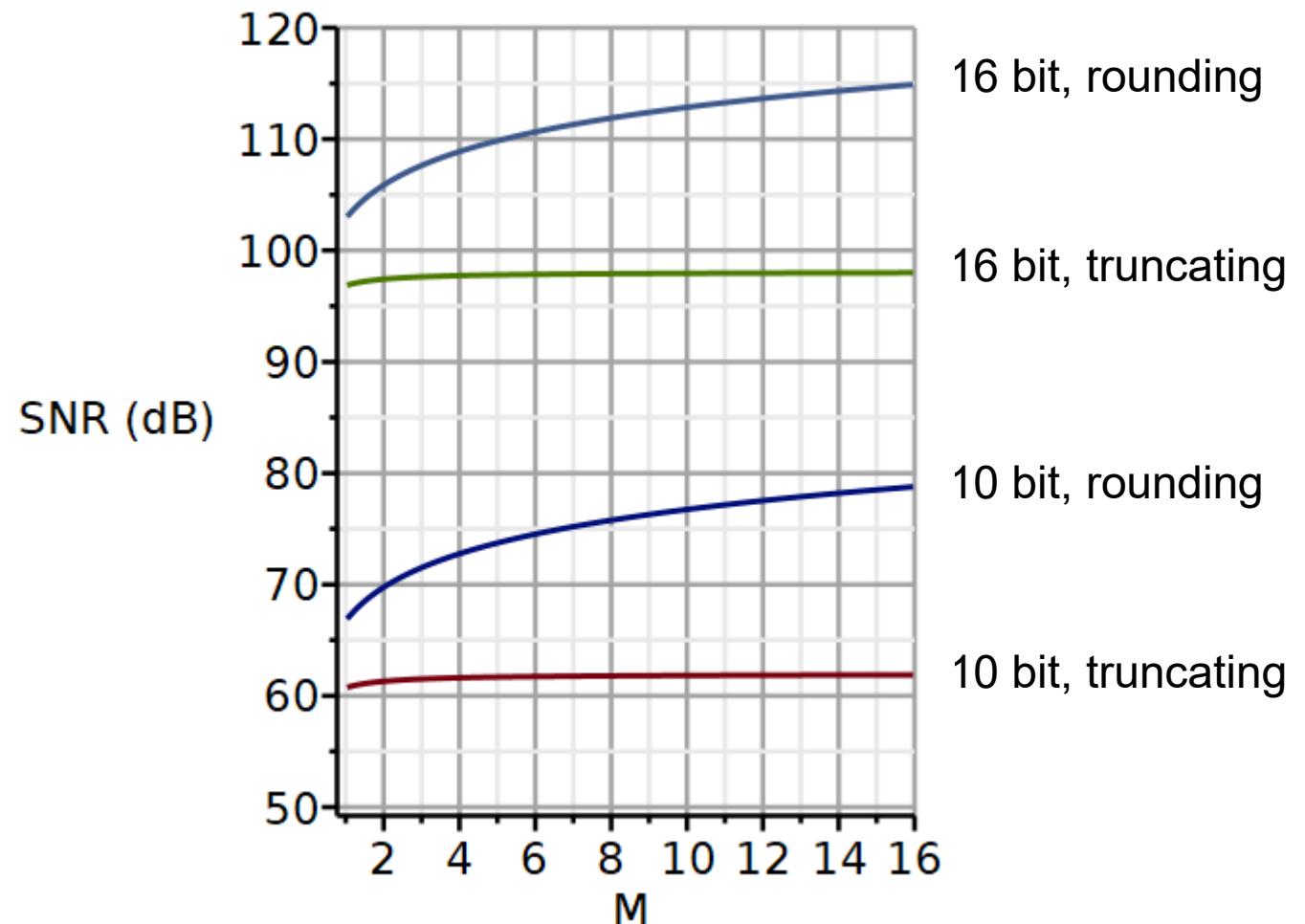
$$\begin{aligned} SNR &= 20 \log(2^N) + 20 \log(\alpha) + 20 \log \left(\sqrt{\frac{3}{8}} \right) - 20 \log \sqrt{\frac{1}{12M}} \\ &= N20 \log(2) + 20 \log(\alpha) - 4.26 \text{dB} + 20 \log \sqrt{12} + 20 \log \sqrt{M} \\ &\approx 6.02N + 20 \log(\alpha) + 20 \log \sqrt{M} - 4.26 \text{dB} + 10.79 \text{dB} \end{aligned}$$

SNR comparison for truncating and rounding ADC

Comparing SNR for truncating and rounding ADCs

Observation:

There is essentially no improvement in SNR due to quantization noise by oversampling, if the ADC is of the truncating type.



We see that the SNR increases 6dB per bit.

We can then figure out how many averagings we need to improve the SNR equivalent to one bit:

The number of averages performed can be expressed as:

If we have a 10-bit rounding ADC, but need a SNR equivalent of a 12-bit ADC, we must obtain a 2-bit improvement in SNR. Hence the number of averages required is:

$$SNR \approx 6.02N + 20 \log(\alpha) + 20 \log \sqrt{M} - 4.26\text{dB} + 10.79\text{dB}$$

$$20 \log \sqrt{M} = 6.02 \Rightarrow M = 4$$

$$M = 4^x \quad x: \text{The number of bits gained by averaging.}$$

$$SNR \approx 6.02N + 20 \log(\alpha) + 20 \log \sqrt{4^x} - 4.26\text{dB} + 10.79\text{dB}$$

$$SNR \approx 6.02N + 20 \log(\alpha) + 20 \log 4^{x/2} - 4.26\text{dB} + 10.79\text{dB}$$

$$SNR \approx 6.02N + 20 \log(\alpha) + x \times 20 \log 2 - 4.26\text{dB} + 10.79\text{dB}$$

$$SNR \approx 6.02N + 20 \log(\alpha) + x \times 6.02\text{dB} - 4.26\text{dB} + 10.79\text{dB}$$

$$M = 4^x = 4^2 = 16$$

Checking with the graphs on the previous slide, we confirm a 12 dB (2-bit improvement) by oversampling by 16.

Inadequate amplification of analogue signal

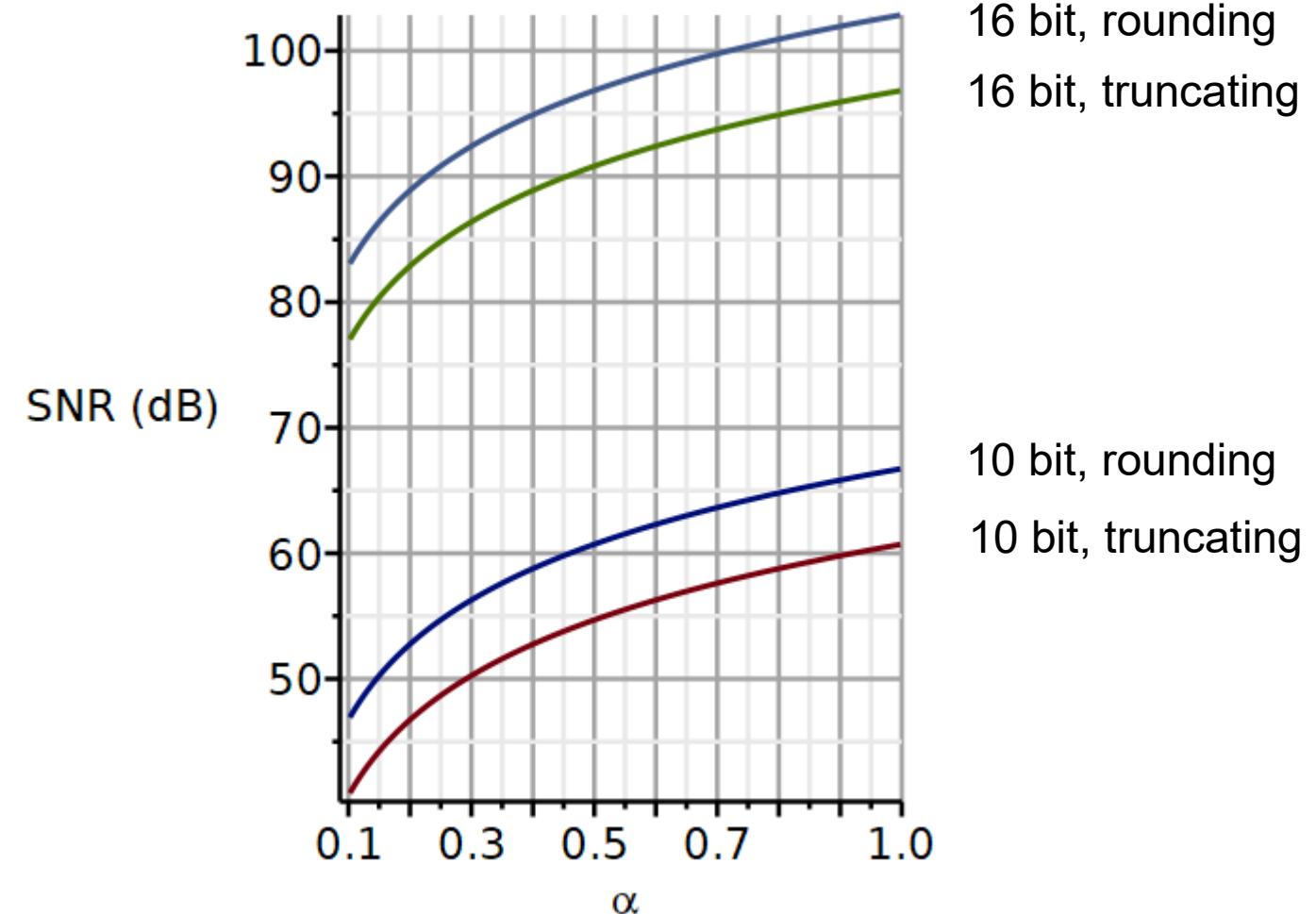
If the full voltage range of the ADC is not utilized, not all bits are put to use, and the SNR will be less than optimal.

Here are shown the dependence of SNR on the parameter α which represents the fractional utilization of the full voltage range.

No oversampling is used here.

The SNR improves by $20 \log(\alpha)$. Hence a doubling in amplification will improve the SNR by 6dB \sim 1bit.

A factor 10 increase in amplification will improve the SNR by 20dB \sim 3.3bits.



Consequence of inadequate amplification

We saw on the previous slide that an inadequate gain will reduce the utilization of the ADC voltage range and therefore also the utilization of some bits. We can express this mathematically:

If we only use 50% of max gain,
we lose the use of 1bit.

If we only use 10% of max gain,
we lose the use of 3.3bit.

$$\alpha = 2^{-K} \Rightarrow -K \log(2) = \log(\alpha)$$

$$K = \frac{-\log(\alpha)}{\log(2)} = \frac{\log(1/\alpha)}{\log(2)}$$

$$K = \frac{\log(2)}{\log(2)} = 1 \text{ bit}$$

$$K = \frac{\log(10)}{\log(2)} \approx 3.3 \text{ bit}$$

The fewer bits you have in your ADC, the more important it is to provide adequate amplification. On the other hand, too high amplification will cause the amplified signal to exceed the voltage range of the ADC. This will cause **clipping**, as the digital level cannot exceed the full range.

The effects of clipping are serious and renders the digitized data useless. If the maximum amplitude of a signal is hard to predict, it is necessary to allow some **head room** to avoid clipping. If we used a 16-bit ADC instead of a 10-bit ADC, we could use the 6 MSB as head room, while maintaining the same amplitude resolution.

The effective number of bits (ENOB)

In an ideal ADC the smallest amplitude that can be converted is:

In decibel the amplitude range covered by an ADC is called its **dynamic range**:

In a real ADC, the DR is slightly less than the ideal value. Hence one can calculate the **effective number of bits** (ENOB) that would correspond to the real-life DR.

Example:

A non-ideal 10-bit truncating ADC is said in the data sheet to have a DR = 55 dB. Assuming the full voltage scale is used, and no oversampling is done, what is the effective number of bits?
What is the SNR?

$$\Delta V_{LSB} = \frac{V_{range}}{2^N}$$

$$DR_{theory} = 6.02N \text{ dB}$$

$$ENOB = \frac{DR_{real}}{6.02}$$

$$SNR = 6.02N + 20 \log(\alpha) + 20 \log \sqrt{M} - 4.26 \text{ dB} + 10.79 \text{ dB}$$

$$SNR = DR_{real} + 20 \log(\alpha) + 20 \log \sqrt{M} - 4.26 \text{ dB} + 10.79 \text{ dB}$$

$$ENOB = \frac{DR_{real}}{6.02} = \frac{55}{6.02} = 9.14$$

$$SNR = 55 + 20 \log(1) + 20 \log \sqrt{1} - 4.26 \text{ dB} + 10.79 \text{ dB}$$

$$= 61.5 \text{ dB} \quad \text{A reduction from } 66.73 \text{ dB}$$

Matching antialiasing filter to ADC

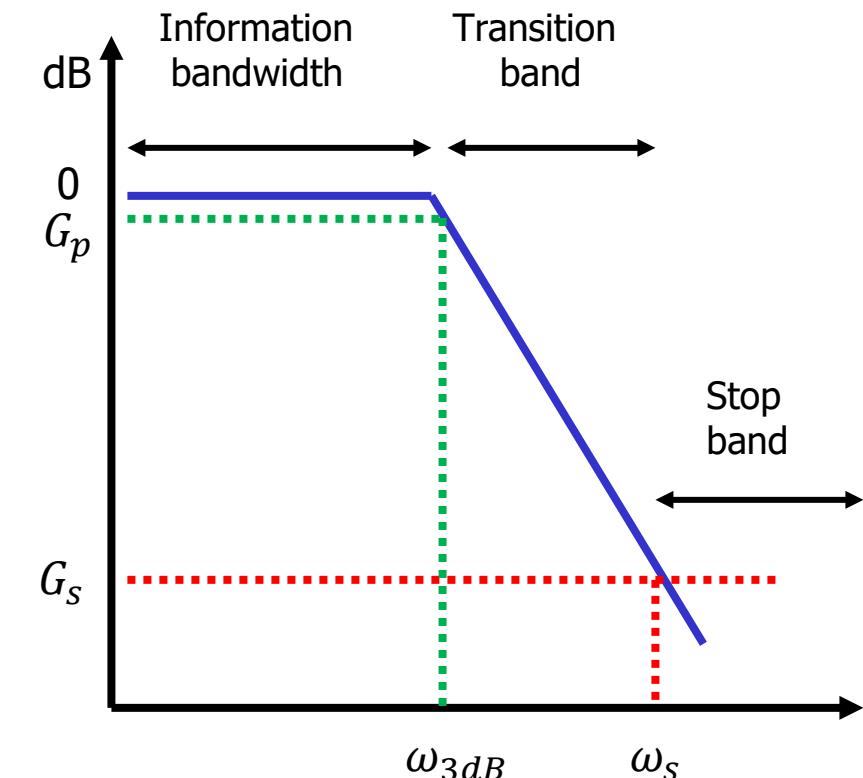
Video

Matching filter to ADC

From our analysis of the ADC performance, we have learned that

1. Inadequate sampling frequency will increase the negative consequences of aliasing.
2. Inadequate bits will increase quantization noise.

Because all practical signals have **finite duration**, their frequency spectrum has **infinite bandwidth**. No matter how high the sampling frequency is, some level of aliasing will occur.



An antialiasing lowpass filter can attenuate the energy in frequency ranges outside the information bandwidth. Adequate filter order will therefore attenuate the aliased frequency components to a level that can be considered insignificant.

In the figure shown here, the desired level of attenuation is denoted G_s , referring to the gain at the start of the stop band.

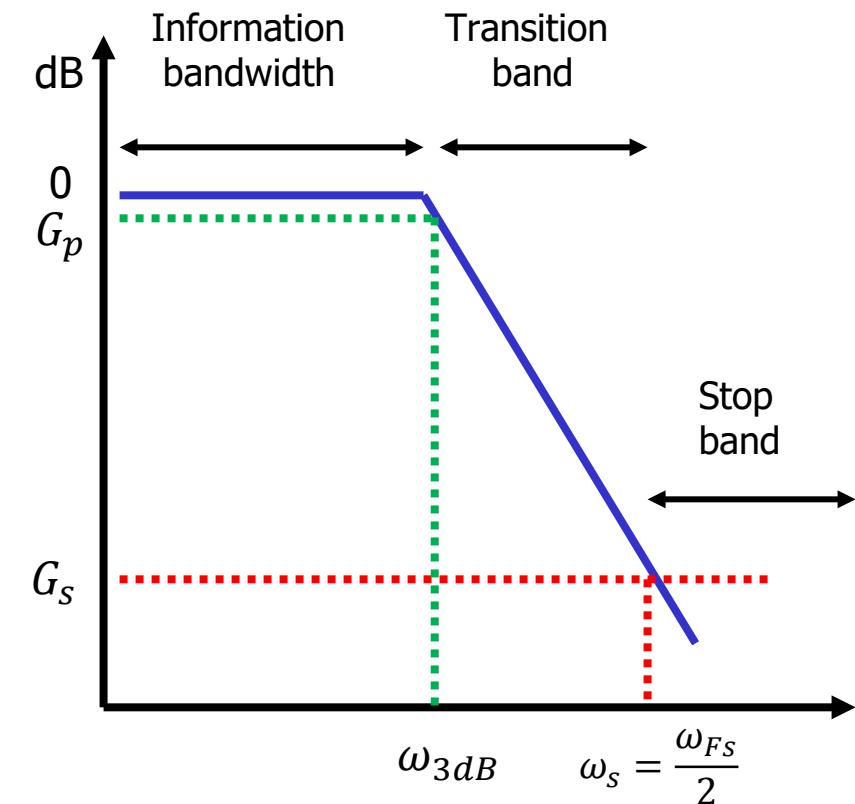
Matching filter to ADC

Nyquist sampling theorem states that the sampling frequency should be at least two times the highest frequency of significant magnitude.

We can then make the choice that the highest frequency of significant amplitude should define the start of the stop band of the antialiasing filter.

In other words, we set:

$$\omega_s = \frac{\omega_{FS}}{2}$$



Matching filter to ADC

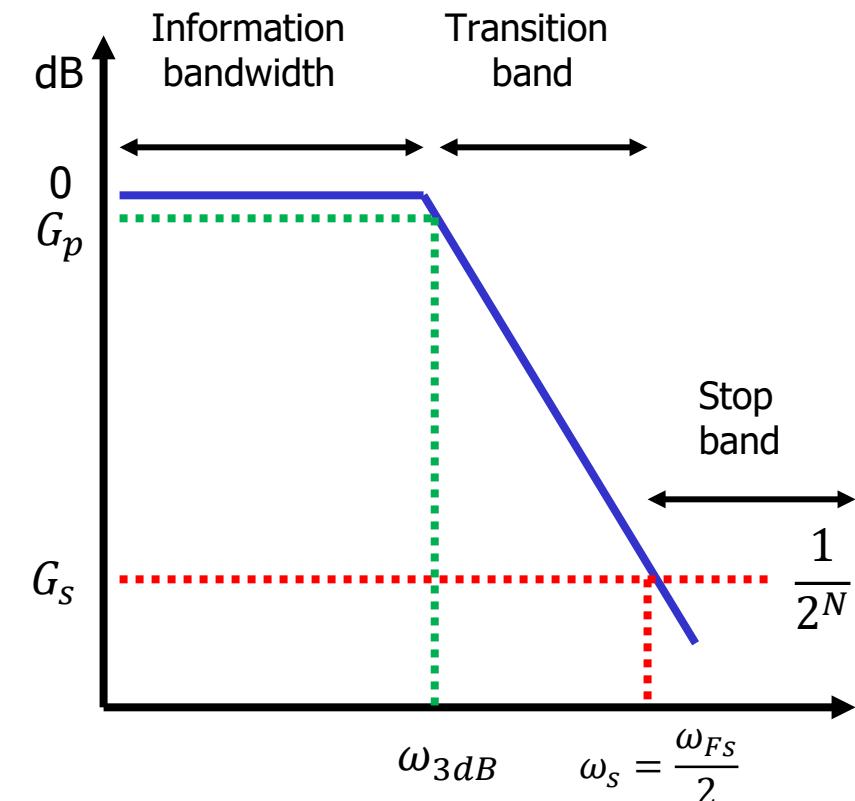
Another important decision is the level of attenuation G_s at $\omega = \omega_s$. This is the amplitude of the frequencies folded about $\omega_{FS}/2$.

We could also state that the aliased frequencies should be attenuated by at least 1000 (60 dB). For a given level of attenuation, we can ask how many bits can be toggled by the aliased frequencies.

A 12-bit ADC will toggle the LSB if the signal exceeds -72dB.

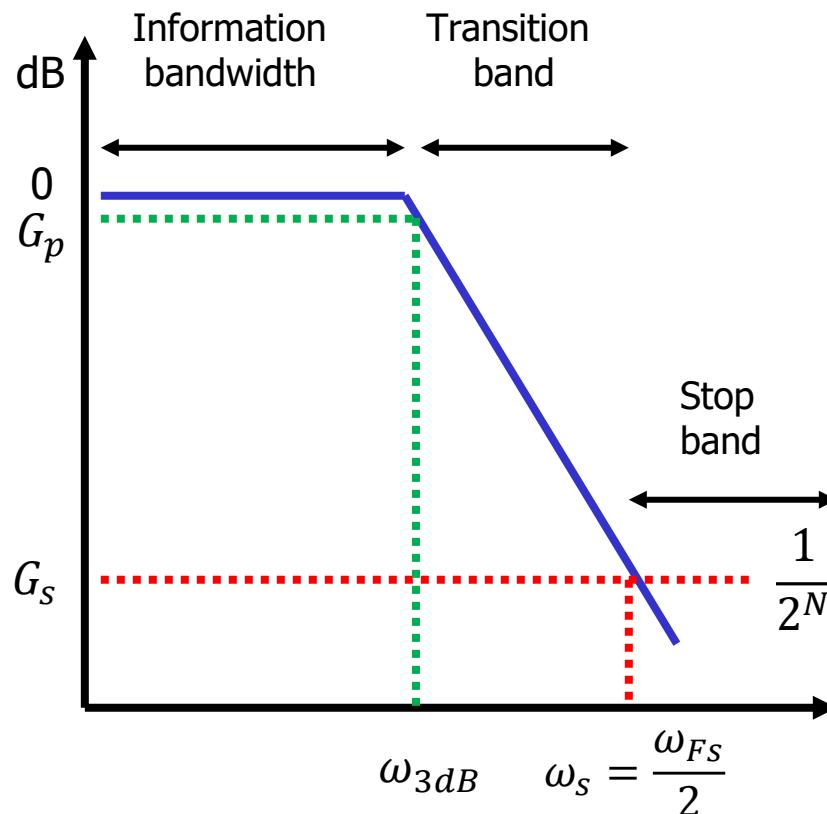
So, if the level of attenuation is 60dB at $\omega_{FS}/2$, we should expect that aliased frequency components will have enough amplitude to toggle the 2 LSB of the ADC.

Let us try and be extreme and insist that the filter attenuation equals ΔV_{LSB} :



$$G_s = 20 \log \frac{1}{2^N}$$

Ratio between 3dB cut-off and sampling frequency



Setting the stop band frequency equal to half the sampling frequency:

$$|H(j\omega_s)| \stackrel{\text{def}}{=} \frac{1}{\sqrt{1 + \left(\frac{\omega_s}{\omega_{3dB}}\right)^{2n}}} = \frac{1}{2^N} \quad \frac{1}{1 + \left(\frac{\omega_s}{\omega_{3dB}}\right)^{2n}} = \frac{1}{2^{2N}}$$

$$1 + \left(\frac{\omega_s}{\omega_{3dB}}\right)^{2n} = 2^{2N} \quad \left(\frac{\omega_s}{\omega_{3dB}}\right)^{2n} = 2^{2N} - 1$$

$$N \geq 8 \Rightarrow \text{error} \leq \frac{|(2^{16} - 1) - 2^{16}|}{(2^{16} - 1)} \approx 1.53 \times 10^{-5}$$

$$\left(\frac{\omega_s}{\omega_{3dB}}\right)^{2n} \approx 2^{2N} \Rightarrow \frac{\omega_s}{\omega_{3dB}} \approx 2^{\frac{N}{n}}$$

$$\omega_s = \frac{\omega_{FS}}{2} \Rightarrow$$

$$\boxed{\frac{\omega_{FS}}{\omega_{3dB}} = 2^{\frac{N}{n}+1}}$$

Ratios between 3dB cut-off and sampling frequency

$$\frac{\omega_{Fs}}{\omega_{3dB}} = 2^{\frac{N}{n}+1}$$

	N	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536	131072	262144 (2^bits)
n	1	4.00	8.00	16.00	32.00	64.00	128.00	256.00	512.00	1024.00	2048.00	4096.00	8192.00	16384.00	32768.00	65536.00	131072.00	262144.00	524288.00
2	2.83	4.00	5.66	8.00	11.31	16.00	22.63	32.00	45.25	64.00	90.51	128.00	181.02	256.00	362.04	512.00	724.08	1024.00	
3	2.52	3.17	4.00	5.04	6.35	8.00	10.08	12.70	16.00	20.16	25.40	32.00	40.32	50.80	64.00	80.63	101.59	128.00	
4	2.38	2.83	3.36	4.00	4.76	5.66	6.73	8.00	9.51	11.31	13.45	16.00	19.03	22.63	26.91	32.00	38.05	45.25	
5	2.30	2.64	3.03	3.48	4.00	4.59	5.28	6.06	6.96	8.00	9.19	10.56	12.13	13.93	16.00	18.38	21.11	24.25	
6	2.24	2.52	2.83	3.17	3.56	4.00	4.49	5.04	5.66	6.35	7.13	8.00	8.98	10.08	11.31	12.70	14.25	16.00	
7	2.21	2.44	2.69	2.97	3.28	3.62	4.00	4.42	4.88	5.38	5.94	6.56	7.25	8.00	8.83	9.75	10.77	11.89	
8	2.18	2.38	2.59	2.83	3.08	3.36	3.67	4.00	4.36	4.76	5.19	5.66	6.17	6.73	7.34	8.00	8.72	9.51	
9	2.16	2.33	2.52	2.72	2.94	3.17	3.43	3.70	4.00	4.32	4.67	5.04	5.44	5.88	6.35	6.86	7.41	8.00	
10	2.14	2.30	2.46	2.64	2.83	3.03	3.25	3.48	3.73	4.00	4.29	4.59	4.92	5.28	5.66	6.06	6.50	6.96	
order																			

In this table, the rows 1 – 10 is the filter order n . The columns represent the number of bits N in the ADC.

Example: Fourth order Butterworth lowpass filter, 10-bit ADC.

$\frac{\omega_{Fs}}{\omega_{3dB}} = 11.31$. Hence if the 3dB cut-off frequency is 100Hz, the sampling frequency needs to be 1131Hz.

At this sampling frequency, the aliased signal frequencies have been attenuated by a factor 1024 or more. This is just below the LSB amplitude level and will be non-detectable for a 10-bit ADC.

Ratios between 3dB cut-off and sampling frequency

$$\frac{\omega_{FS}}{\omega_{3dB}} = 2^{\frac{N}{n}+1}$$

	N	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536	131072	262144 (2^bits)
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18 bits	
1	4.00	8.00	16.00	32.00	64.00	128.00	256.00	512.00	1024.00	2048.00	4096.00	8192.00	16384.00	32768.00	65536.00	131072.00	262144.00	524288.00	
2	2.83	4.00	5.66	8.00	11.31	16.00	22.63	32.00	45.25	64.00	90.51	128.00	181.02	256.00	362.04	512.00	724.08	1024.00	
3	2.52	3.17	4.00	5.04	6.35	8.00	10.08	12.70	16.00	20.16	25.40	32.00	40.32	50.80	64.00	80.63	101.59	128.00	
4	2.38	2.83	3.38	4.00	4.70	5.00	5.73	8.00	9.51	11.31	13.45	16.00	19.03	22.63	26.91	32.00	38.05	45.25	
5	2.30	2.64	3.03	3.48	4.00	4.59	5.28	6.06	6.96	8.00	9.19	10.56	12.13	13.93	16.00	18.38	21.11	24.25	
6	2.24	2.52	2.83	3.17	3.56	4.00	4.49	5.04	5.66	6.35	7.13	8.00	8.98	10.08	11.31	12.70	14.25	16.00	
7	2.21	2.44	2.69	2.97	3.28	3.62	4.00	4.42	4.88	5.38	5.94	6.56	7.25	8.00	8.83	9.75	10.77	11.89	
8	2.18	2.38	2.59	2.83	3.08	3.36	3.67	4.00	4.36	4.76	5.19	5.66	6.17	6.73	7.34	8.00	8.72	9.51	
9	2.16	2.33	2.52	2.72	2.94	3.17	3.43	3.70	4.00	4.32	4.67	5.04	5.44	5.88	6.35	6.86	7.41	8.00	
10	2.14	2.30	2.46	2.64	2.83	3.03	3.25	3.48	3.73	4.00	4.29	4.59	4.92	5.28	5.66	6.06	6.50	6.96	
order																			

Example: Fourth order Butterworth lowpass filter, 10-bit ADC.

If we are willing to accept the signal level only to be reduced by a factor 256 (~8 bits), then we can sample with 800 Hz and the signal level will toggle the two LSB of a 10-bit ADC.

Ratio between 3dB cut-off and sampling frequency

$$\frac{\omega_{FS}}{\omega_{3dB}} = 2^{\frac{N}{n}+1}$$

	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536	131072	262144 (2^bits)
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18 bits
1	4.00	8.00	16.00	32.00	64.00	128.00	256.00	512.00	1024.00	2048.00	4096.00	8192.00	16384.00	32768.00	65536.00	131072.00	262144.00	524288.00
2	2.83	4.00	5.66	8.00	11.31	16.00	22.63	32.00	45.25	64.00	90.51	128.00	181.02	256.00	362.04	512.00	724.08	1024.00
3	2.52	3.17	4.00	5.04	6.35	8.00	10.08	12.70	16.00	20.16	25.40	32.00	40.32	50.80	64.00	80.63	101.59	128.00
4	2.38	2.83	3.36	4.00	4.76	5.66	6.73	8.00	9.51	11.31	13.45	16.00	19.03	22.63	26.91	32.00	38.05	45.25
5	2.30	2.64	3.03	3.48	4.00	4.59	5.28	6.06	6.96	8.00	9.19	10.56	12.13	13.93	16.00	18.38	21.11	24.25
6	2.24	2.52	2.83	3.17	3.56	4.00	4.49	5.04	5.66	6.35	7.13	8.00	8.98	10.08	11.31	12.70	14.25	16.00
7	2.21	2.44	2.69	2.97	3.28	3.62	4.00	4.42	4.88	5.38	5.94	6.56	7.25	8.00	8.83	9.75	10.77	11.89
8	2.18	2.38	2.59	2.83	3.08	3.36	3.67	4.00	4.36	4.76	5.19	5.66	6.17	6.73	7.34	8.00	8.72	9.51
9	2.16	2.33	2.52	2.72	2.94	3.17	3.43	3.70	4.00	4.32	4.67	5.04	5.44	5.88	6.35	6.86	7.41	8.00
10	2.14	2.30	2.46	2.64	2.83	3.03	3.25	3.48	3.73	4.00	4.29	4.59	4.92	5.28	5.66	6.06	6.50	6.96
order																		

Assume a 12-bit ADC and a 4th order lowpass filter.

We insist that: $\frac{\omega_{FS}}{\omega_{3dB}} \leq 4$

Now the filter will only attenuate the aliased frequency to the LSB of a 4-bit ADC, i.e., by a factor of 16.

Ratio between 3dB cut-off and sampling frequency

$$\frac{\omega_{FS}}{\omega_{3dB}} = 2^{\frac{N}{n}+1}$$

	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536	131072	262144 (2^bits)
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18 bits
1	4.00	8.00	16.00	32.00	64.00	128.00	256.00	512.00	1024.00	2048.00	4096.00	8192.00	16384.00	32768.00	65536.00	131072.00	262144.00	524288.00
2	2.83	4.00	5.66	8.00	11.31	16.00	22.63	32.00	45.25	64.00	90.51	128.00	181.02	256.00	362.04	512.00	724.08	1024.00
3	2.52	3.17	4.00	5.04	6.35	8.00	10.08	12.70	16.00	20.16	25.40	32.00	40.32	50.80	64.00	80.63	101.59	128.00
4	2.38	2.83	3.36	4.00	4.76	5.66	6.73	8.00	9.51	11.31	13.45	16.00	19.03	22.63	26.91	32.00	38.05	45.25
5	2.30	2.64	3.03	3.48	4.00	4.59	5.28	6.06	6.96	8.00	9.19	10.56	12.13	13.93	16.00	18.38	21.11	24.25
6	2.24	2.52	2.83	3.17	3.56	4.00	4.49	5.04	5.66	6.35	7.13	8.00	8.98	10.08	11.31	12.70	14.25	16.00
7	2.21	2.44	2.69	2.97	3.28	3.62	4.00	4.42	4.88	5.38	5.94	6.56	7.25	8.00	8.83	9.75	10.77	11.89
8	2.18	2.38	2.59	2.83	3.08	3.36	3.67	4.00	4.36	4.76	5.19	5.66	6.17	6.73	7.34	8.00	8.72	9.51
9	2.16	2.33	2.52	2.72	2.94	3.17	3.43	3.70	4.00	4.32	4.67	5.04	5.44	5.88	6.35	6.86	7.41	8.00
10	2.14	2.30	2.46	2.64	2.83	3.03	3.25	3.48	3.73	4.00	4.29	4.59	4.92	5.28	5.66	6.06	6.50	6.96
order																		

We insist that: $\frac{\omega_{FS}}{\omega_{3dB}} \leq 4$

If we want the attenuation factor to be 100 (40dB ~ 7 bits), then we may decide to use a 7th order filter.

We must conclude that the constraint of sampling at 4 times the 3dB cut-off frequency is quite severe. It either demands a high-order filter or forces us to accept a poor attenuation of aliased frequencies.

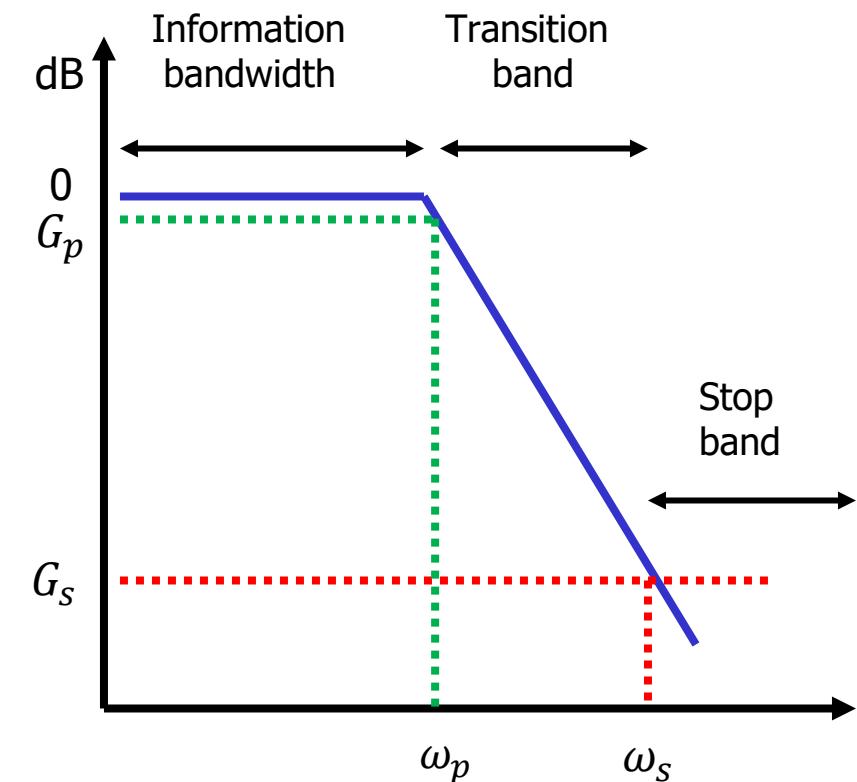
Determining the filter order, Lathi 7.5

Regardless of the criteria used to define the passband gain G_p and the stopband gain G_s , a suitable filter order n must be determined, to obtain the desired passband and stopband attenuations.

Inserting the passband and stopband frequencies into the amplitude spectrum of the Butterworth filter we get:

$$\begin{aligned} G_p &= 20 \log_{10} \frac{1}{\sqrt{1 + (\omega_p/\omega_c)^{2n}}} \\ &= -10 \log [1 + (\omega_p/\omega_c)^{2n}] : \text{Passband gain} \end{aligned}$$

$$\begin{aligned} G_s &= 20 \log_{10} \frac{1}{\sqrt{1 + (\omega_s/\omega_c)^{2n}}} \\ &= -10 \log_{10}[1 + (\omega_s/\omega_c)^{2n}] : \text{Stopband gain} \end{aligned}$$



Determining the filter order, Lathi 7.5

Solving for the filter order:

$$n = \frac{\log_{10} \left(\frac{10^{-\frac{G_s}{10}} - 1}{10^{-\frac{G_p}{10}} - 1} \right)}{2 \log_{10} \left(\frac{\omega_s}{\omega_p} \right)}$$

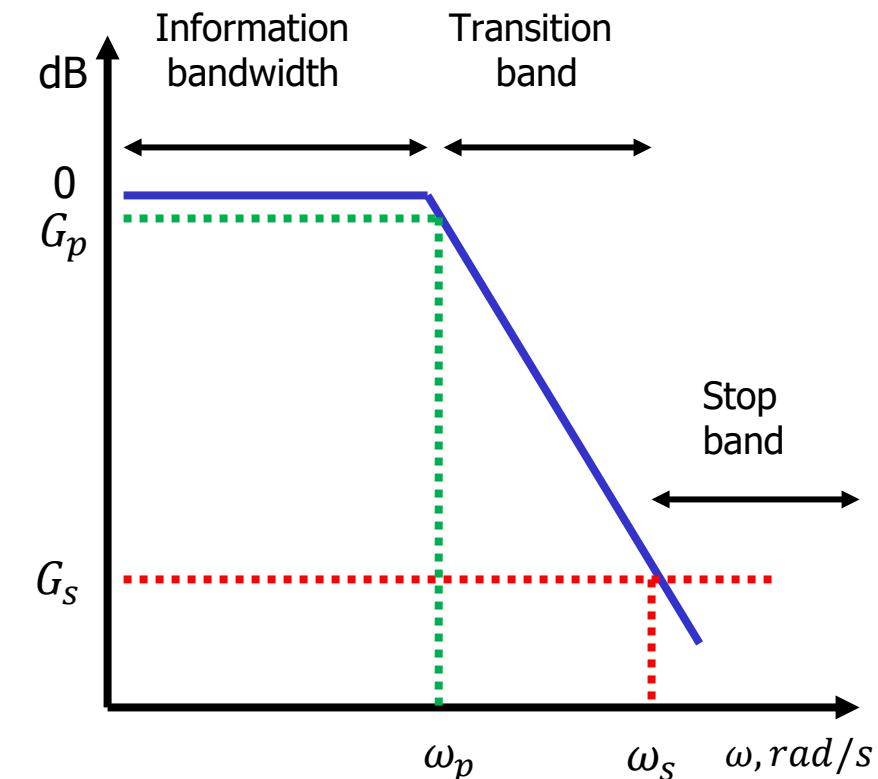
We also need to define ω_c when n is rounded off to an integer:

$$\omega_c = \frac{\omega_p}{[10^{-G_p/10} - 1]^{1/2n}}$$

$$G(s) = \frac{1}{1 + \left(\frac{s}{j\omega_c} \right)^{2n}}$$

After designing the frequency normalized filter of order n , a frequency scaling must be done:

$$K_F = \frac{\omega_c}{1}$$



Example:

A signal has information bandwidth from 0 – 150Hz. It will be sampled with a 16-bit ADC with a sampling frequency of 3kHz.

The anti-aliasing filter is a Butterworth lowpass filter.

At 150Hz the signal must not be attenuated more than 3dB.

The anti-aliasing filter should attenuate the signal to a level equal to or below LSB at half the sampling frequency.

Calculate the minimum filter order and the resulting cut-off frequency in radians.

First step is to define passband and stopband:

$$\text{Pass band: } G_p = -3\text{dB}$$

$$\omega_p = 942.5\text{rad/s}$$

$$\text{Stop band: } \omega_s = \frac{\omega_{FS}}{2} = 9425\text{rad/s}$$

$$G_s = -96\text{dB} \approx 16 \text{ bit} \times (-6\text{dB})$$

Determining the filter order

Example continued

Calculate the minimum filter order and the resulting cut-off frequency in radians:

$$\text{Pass band: } G_p = -3\text{dB}$$

$$\omega_p = 942.5 \text{ rad/s}$$

$$\text{Stop band: } G_s = -96\text{dB} \approx 16 \text{ bit} \times (-6\text{dB})$$

$$\omega_s = 9425 \text{ rad/s}$$

We would need a 5th order filter if we would like to attenuate by 96 dB within one decade.

The frequency normalized filter must be frequency scaled by a factor:

$$n = \frac{\log\left(\frac{10^{-\frac{G_s}{10}} - 1}{10^{-\frac{G_p}{10}} - 1}\right)}{2 \log\left(\frac{\omega_s}{\omega_p}\right)}$$

$$n = \frac{\log\left(\frac{10^{-\frac{-96}{10}} - 1}{10^{-\frac{-3}{10}} - 1}\right)}{2 \log\left(\frac{9425}{942.5}\right)} = \frac{9.602}{2} = 4.8 \approx 5$$

$$\omega_c = \frac{\omega_p}{[10^{-G_p/10} - 1]^{1/2n}}$$

$$\omega_c = \frac{942.5}{[10^{0.3} - 1]^{1/10}} = 942.9 \text{ rad/s}$$

$$K_F = \frac{\omega_c}{1 \text{ rad/s}} = 942.9$$

Successive Approximation Register (SAR) ADCs typically use truncation when converting an analog signal to a digital value. However, some SAR ADCs incorporate rounding mechanisms to improve accuracy, particularly in applications where minimizing quantization bias is important.

SAR ADCs with Rounding:

1. Analog Devices AD400x Series (e.g., AD4000, AD4003)

1. These ADCs use a rounding feature in their digital output coding.
2. They have an optional **dither and averaging mode**, which can reduce quantization errors.

2. Texas Instruments ADS89xx Series (e.g., ADS8900B, ADS8910B, ADS8920B)

1. These high-precision SAR ADCs support rounding instead of simple truncation.
2. The rounding feature helps minimize DC offset errors.

3. Maxim Integrated (now part of Analog Devices) MAX11156

1. This ADC has an internal averaging mechanism that effectively performs rounding.

Many SAR ADCs allow users to implement rounding externally in firmware by adjusting the Least Significant Bit (LSB) representation, but some include built-in rounding logic.

Most microcontrollers with built-in SAR ADCs use **truncation** rather than rounding by default. However, some MCUs offer features that allow rounding behavior through oversampling, digital filtering, or configurable data processing. Here are some microcontrollers that either directly support rounding or allow it through built-in features:

Microcontrollers with SAR ADCs that Support Rounding or Similar Features

1.Texas Instruments MSP430FRxxxx Series

1. Some MSP430 microcontrollers feature **oversampling and averaging modes**, which can approximate rounding by reducing quantization errors.
2. Example: **MSP430FR5994** (16-bit SAR ADC with configurable oversampling).

2.Microchip PIC32MZ and PIC24F Series

1. These MCUs feature **hardware averaging**, which helps achieve rounding-like behavior.
2. Example: **PIC32MZ EF series** has a 12-bit SAR ADC with an averaging mode.

3.STMicroelectronics STM32G4 and STM32L4 Series

1. The **STM32G4xx** and **STM32L4xx** have SAR ADCs with **hardware oversampling** that can reduce truncation effects.
2. Example: **STM32G474** (16-bit SAR ADC with hardware oversampling and digital post-processing).

4.NXP LPC55S69 (LPC5500 Series)

1. Features a **12-bit SAR ADC with accumulation and averaging**, which can approximate rounding.

Some microcontrollers support **true digital rounding** before output. These include:

- **Texas Instruments ADS89xx series (external SAR ADCs)**
- **STM32G4 / STM32L4 series** (oversampling with built-in rounding mode)
- **Delta-Sigma ADCs** (in many MCUs like MSP430 and dsPIC)

The **STM32L476RG** microcontroller from STMicroelectronics features a 12-bit SAR (Successive Approximation Register) ADC. By default, this ADC performs **truncation** during digital conversion, meaning it rounds down any fractional results.

However, the STM32L476RG also offers a **hardware oversampling** feature that can effectively increase resolution and reduce noise. This feature allows the ADC to average multiple samples, which can mitigate quantization errors associated with truncation. The oversampling ratio is configurable, enabling trade-offs between resolution and sampling time.

To implement rounding behavior in the STM32L476RG's ADC:

- 1. Enable Hardware Oversampling:** Configure the ADC to use hardware oversampling with the desired ratio.
- 2. Set the Right Shift Parameter:** Adjust the right shift parameter to align the averaged result correctly, effectively achieving rounding.

For detailed guidance on configuring these settings, refer to STMicroelectronics' application note on ADC hardware oversampling for STM32L4 series microcontrollers. [STMicroelectronics+1](#)

In summary, while the STM32L476RG's ADC defaults to truncation, utilizing its hardware oversampling feature allows for effective rounding, enhancing measurement accuracy and reducing noise.

The **STM32F429** microcontroller from STMicroelectronics features a 12-bit SAR (Successive Approximation Register) ADC. Unlike some newer STM32 series, such as the STM32L4, the STM32F429's ADC does **not** include a built-in hardware oversampling feature with automatic averaging and right-shifting capabilities. By default, the ADC in the STM32F429 performs **truncation** when converting analog signals to digital values, meaning any fractional part of the conversion result is discarded, potentially introducing a small negative bias in the measurements.

To achieve rounding behavior and reduce noise in the STM32F429, you can implement **software-based oversampling and averaging**. This involves taking multiple samples of the same input signal and computing their average in software, which can effectively increase resolution and mitigate quantization errors. However, it's important to note that this method increases CPU load and may reduce the overall sampling rate. [ST Community](#)

For detailed guidance on implementing software-based oversampling and averaging techniques, refer to STMicroelectronics' application note on optimizing ADC accuracy in STM32 microcontrollers. [STMicroelectronics+1](#)

In summary, while the STM32F429's ADC lacks hardware support for oversampling with rounding, implementing software-based oversampling and averaging can help achieve similar results, enhancing measurement accuracy and reducing noise.

Problems

Part 1:

A signal has information bandwidth from 0 – 500Hz. It will be sampled with a 12-bit ADC.

The anti-aliasing filter is a Butterworth lowpass filter.

At 500Hz the signal must not be attenuated more than 3dB.

The anti-aliasing filter should attenuate the signal to a level equal to -72 dB at half the sampling frequency.

Calculate the minimum filter order required if the sampling frequency is 6kHz.

Part 2:

We only have a 4th order Butterworth lowpass filter at our disposal.

Determine what sampling frequency F_s is required to obtain -72dB filter gain at $F_s/2$.

Part 3:

Conflicting design specifications defines a max sampling frequency of 4kHz.

What is the filter attenuation at $F_s/2$?

How many bits are toggled at this frequency?

Solutions

Problem 1 Determining the filter order

Part 1:

A signal has information bandwidth from 0 – 500Hz. It will be sampled with a 12-bit ADC.

The anti-aliasing filter is a Butterworth lowpass filter.

At 500Hz the signal must not be attenuated more than 3dB.

The anti-aliasing filter should attenuate the signal to a level equal to -72 dB at half the sampling frequency.

Calculate the minimum filter order required if the sampling frequency is 6kHz.

$$\text{Pass band: } G_p = -3\text{dB}$$

$$\omega_p = 3141.59\text{rad/s}$$

$$\text{Stop band: } \omega_s = \frac{\omega_{FS}}{2} = 18849.6\text{rad/s}$$

$$G_s = -72\text{dB} \approx 12 \text{ bit} \times (-6\text{dB})$$

Problem 1 Determining the filter order

Calculate the minimum filter order and the resulting cut-off frequency in radians:

$$\text{Pass band: } G_p = -3 \text{dB}$$

$$\omega_p = 3141.59 \text{ rad/s}$$

$$\text{Stop band: } \omega_s = \frac{\omega_{FS}}{2} = 18849.6 \text{ rad/s}$$

$$G_s = -72 \text{ dB} \approx 12 \text{ bit} \times (-6 \text{ dB})$$

We would need a 5th order filter if we would like to attenuate by 72 dB at 3kHz.

The frequency normalized filter must be frequency scaled by a factor:

$$n = \frac{\log\left(\frac{10^{-\frac{G_s}{10}} - 1}{10^{-\frac{G_p}{10}} - 1}\right)}{2 \log\left(\frac{\omega_s}{\omega_p}\right)}$$

$$n = \frac{\log\left(\frac{10^{-\frac{-72}{10}} - 1}{10^{-\frac{-3}{10}} - 1}\right)}{2 \log\left(\frac{18849.6}{3141.59}\right)} = 4.63 \approx 5$$

$$\omega_c = \frac{\omega_p}{[10^{-G_p/10} - 1]^{1/2n}}$$

$$\omega_c = \frac{942.5}{[10^{0.3} - 1]^{1/10}} = 942.9 \text{ rad/s}$$

$$K_F = \frac{\omega_c}{1 \text{ rad/s}} = 942.9$$

Problem 1 Determine the sampling frequency

Part 2:

We only have a 4th order Butterworth lowpass filter at our disposal.

Determine what sampling frequency F_s is required to obtain -72dB filter gain at $F_s/2$.

$$\frac{\omega_{FS}}{\omega_{3dB}} = 2^{\frac{N}{n}+1} = 2^{\frac{12}{4}+1} = 2^4 = 16$$

$$\omega_c = \frac{\omega_p}{[10^{-G_p/10} - 1]^{1/2n}}$$

$$\omega_c = \frac{3141.59}{[10^{-3/10} - 1]^{\frac{1}{2*4}}} = 3143.46 \text{rad/s}$$

$$\omega_{FS} = 16 \times \omega_c = 16 \times 3143.46 \frac{\text{rad}}{\text{s}} = 50295.36 \frac{\text{rad}}{\text{s}}$$

$$F_s = \frac{\omega_{FS}}{2\pi} = 8 \text{kHz}$$

Problem 1 Determine the attenuation obtained at the Nyquist frequency

Part 3:

Conflicting design specifications defines a max sampling frequency of 4kHz.

What is the filter attenuation at $F_s/2$?

A 1st order filter has a high frequency slope of -6dB/octave.

A 4th order filter has a high frequency slope of -24dB/octave.

How many bits are toggled at this frequency?

-48dB equals 8 bits. There are 12 bits in the ADC,
hence the 4 LSB are toggled.

f [Hz]	H _{dB}
500	0dB
1000	-24dB
2000	-48dB
4000	-72dB

Deciding on an acceptable sampling frequency

We see that we cannot avoid having some level of aliasing in the digitized data. The higher the sampling frequency the more insignificant the problem of aliasing will be.

We should always strive to reduce aliasing to an acceptable level. We do not want to sample an excessive amount of data, so we need to determine the smallest acceptable sampling frequency for our situation.

Trial and error: A typical approach is to perform a test sampling with a very high sampling frequency and then derive whatever quantitative features need to be measured on the over-sampled signal. Then a new trial is performed with the sampling frequency reduced by a factor of two. The same quantitative features are derived again. The procedure of reducing the sampling frequency is continued until one starts to observe unacceptable variance in the quantitative features measured on the digitized signals.

The minimum acceptable sampling frequency is then dependent on the nature of the quantitative features to be measured. Finding the time location of a peak or the time location of a maximum slope will require a higher sampling frequency than measuring the integral under the curve or the mean value.