Ising spin model simulation on Barabási-Albert network topologies

Bálint Hantos

Theoretical underpinnings

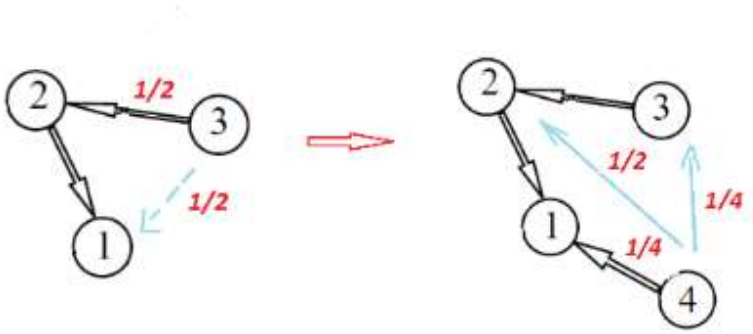
Preferential networks

In academic life it is a common occurrence, that solutions for problems arising in one discipline serves well other fields of study also. Network science, as well as physics, uses mathematical constructions from a practical point of view, adjusting it to the task at hand.

One example of a network is the Barabási-Albert network, which was first discussed first in [1]. This model can be defined by the following algorithm. A network is instantiated with n_0 randomly connected nodes. New nodes connect to the network in a probabilistic way: the node connects to the preexisting nodes with a chance proportional to the function of their degree. This is called preferential attachment [2]:

$$p_i = \frac{f(k_i)}{\sum_j k_j},$$

where p_i is the probability of the new node connecting to the i-th preexisting node and k_i is the degree of the i-th node.



1. Figure: preferential attachment example (linear case)[3]

The Ising-model

Consider N magnetic dipoles fixed in place. The details of the Ising-model can be found in [4], most of this summary comes from this book. The dipoles interact only with their neighbours, however global patterns can be observed stemming from this property. Dipoles can be linked together in a variety of ways. For this project's sake, I'm going to use the topology of a preferential attachment model. Every dipole has a value of either $s_i = \pm 1$ at a time. The system behaves in a way which can be described Boltzmann distribution, meaning that the system is more likely to be found in a lower energy state. The energy of the system is defined by the following formula, when no magnetic field is present:

$$E_{\alpha_k} = -J \sum_{i=1}^{N-1} s_i s_{i+1},$$

where α_k is the configuration denoted by k and J is a constant setting the energy scale.

I'm going to conduct the simulation with the Metropolis Monte Carlo algorithm, which is discussed in great details in [4]. I intend to examine such network topologies, with different preferential attachment dynamics (eg. p_i~k^2, k^3, e^k).

$$< m >= \frac{1}{N} \sum_{i=1}^{N} m_i$$

Also, I'm going to compare critical exponents, <m> and their relaxation time on different βJ (different temperatures) values. For the network generations I'm going to try to use `networkx` under `python`.

Methods

Monte-Carlo algorithm

Firstly, the steps of the simulations are established in the following way:

0. Initialize the grid of spins
1. Choose another spin configuration.
2. Calculate the ΔE difference in energy between the original and the new configuration.
3. Based on ΔE decide:

- if $\Delta E \leq 0$, pick the new spin configuration with certainty
- if $\Delta E > 0$, pick the new spin configuration with $exp(-\beta J)$ probability, otherwise leave it as it was

4. Loop through 1-3.

Secondly, I'm going to elaborate on my implementation. I've initialized the grid of spins on a 2D plane. Then I've chosen a spin, and calculated the change in energy in case it was flippeda based on its 8 neighbours' spin -on the sides, up-down and diagonally.

Ising spins on a network

Another thing that can be done, that instead of a grid, we use a network, where one spin's neighbours are trivially defined by the network edges. The use cases of such models can be find in the area sociophysics, for example the Sznajd-model[5].

Sznajd-model[6]

The Sznajd-model is a social network model for voter behaviour. In one paper it was used to model the general election of Brasil in 2002[7]. It's basically the same (-1,1/up-down) notion alike spins, but now it represents binary opinions (eg: political affiliation, stance on social issues). The algorithm uses the steps as follows:

0. Take a preexisting/build a network, initialize a spin for each node.
1. Choose an individual (node).
2. One changes their mind (flip spin) with probability p or *yield to social pressure* with $1-p$ probability.
   Yield to social pressure: if the ratio of the neighbouring nodes having a different opinion, is greater than r, then the chosen individual changes their mind.
3. Loop through 1-2.

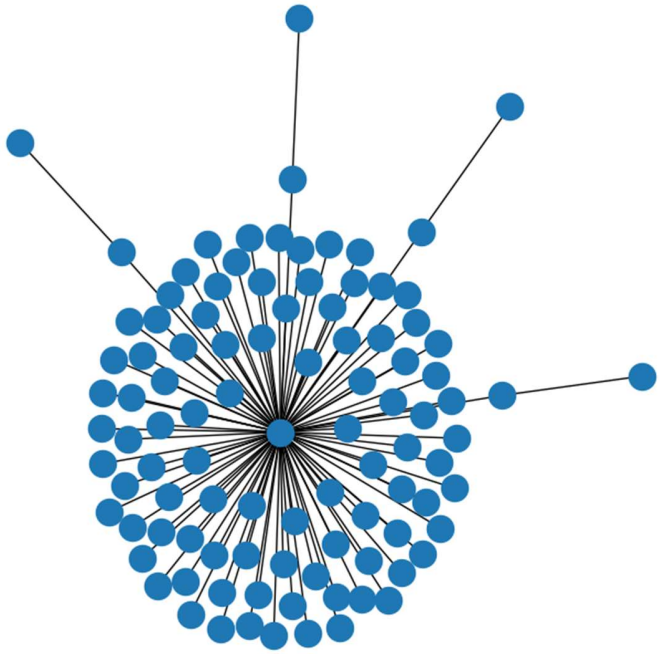
I infer that r, p parameter values are somewhat limited for the model to make sense. For example, a person changes their mind not so often, so p has to be less than ~10%. Also, social pressure is only present, when more than 50% of the neighbours have different opinion, than the chosen individual.

Results

Building the networks

For building a linear preferential attachment network, I begin the network with two nodes and an edge between, then followed with the original regime.

In the case when the preferential attachment was following a power law, I had to use a different approach. The reason for that when starting with only one edge between two nodes, it was like an egocentric network, in the sense that most of the nodes were connected to one node, which is



2. Figure: nearly every node connects to the same initial node,
this is not appropriate for this task, because it's just an egocentric network,
not a social network between multiple people

described visually on Figure 2.

Instead I used an Erdős-Rényi network consisting of 10 and p = 0.2. On that base network I built upon using the preferential attachment regime.

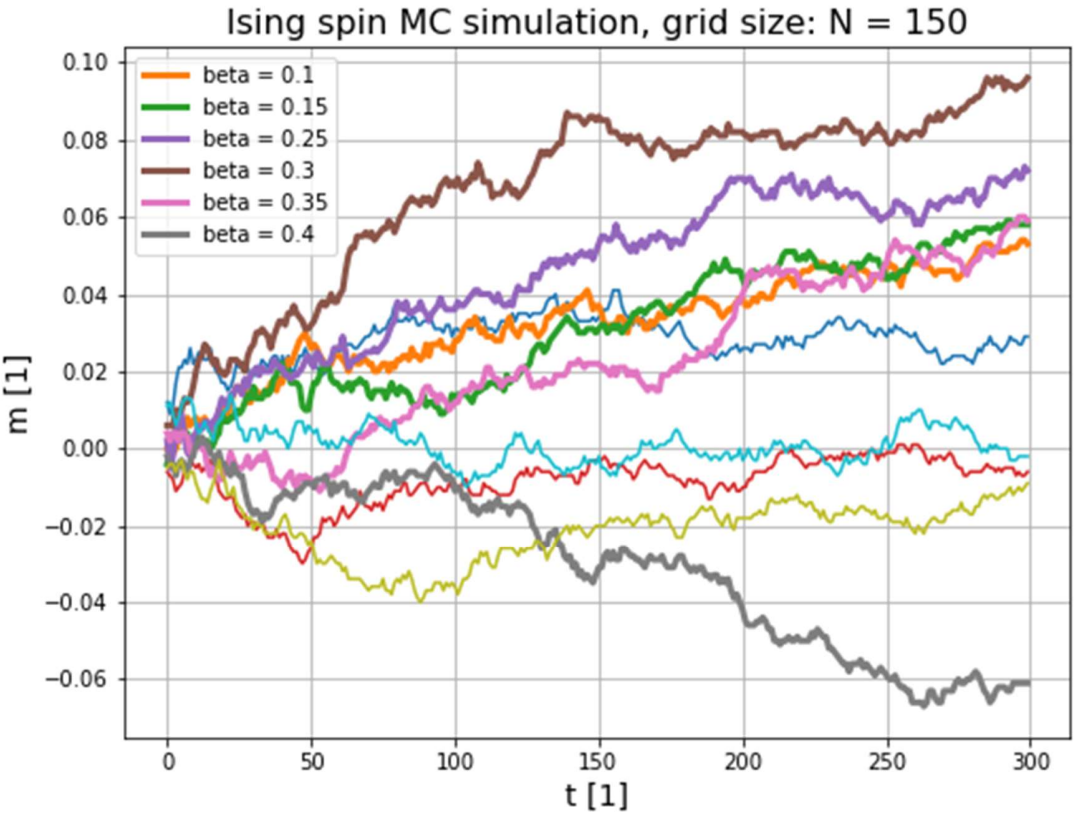
Ising-spins on a two dimensional grid

In ferromagnetic materials (for example iron, cobalt and nickel and their alloys) the spins points in random directions at high temperatures. At some T_C critical temperature all atoms orient their

spins in the same direction. Thus, the metal turns into a magnet. Such phenomena can be observed with Ising-spins. The temperature is represented in the model through the $\beta = \frac{1}{k_B T}$ parameter, where k_B is the Boltzmann-constant.

The main task was to find the critical exponents of with the simulation as a benchmark of how good it is working. I feel somewhat ambivalent about my results. According to [8], the critical exponents are $\alpha = 0, \beta = 0.125, \gamma = 1.75$. For investigating my implementation, I used two methods to find the critical exponents.

Firstly, I did many simulations with different βJ, and examined how magnetization changes with time. I found two trends: one was that magnetization oscillates around zero, another, where magnetization converges to some (positive or negative) value. When the magnetization converges, that means, that there is a phase transition.



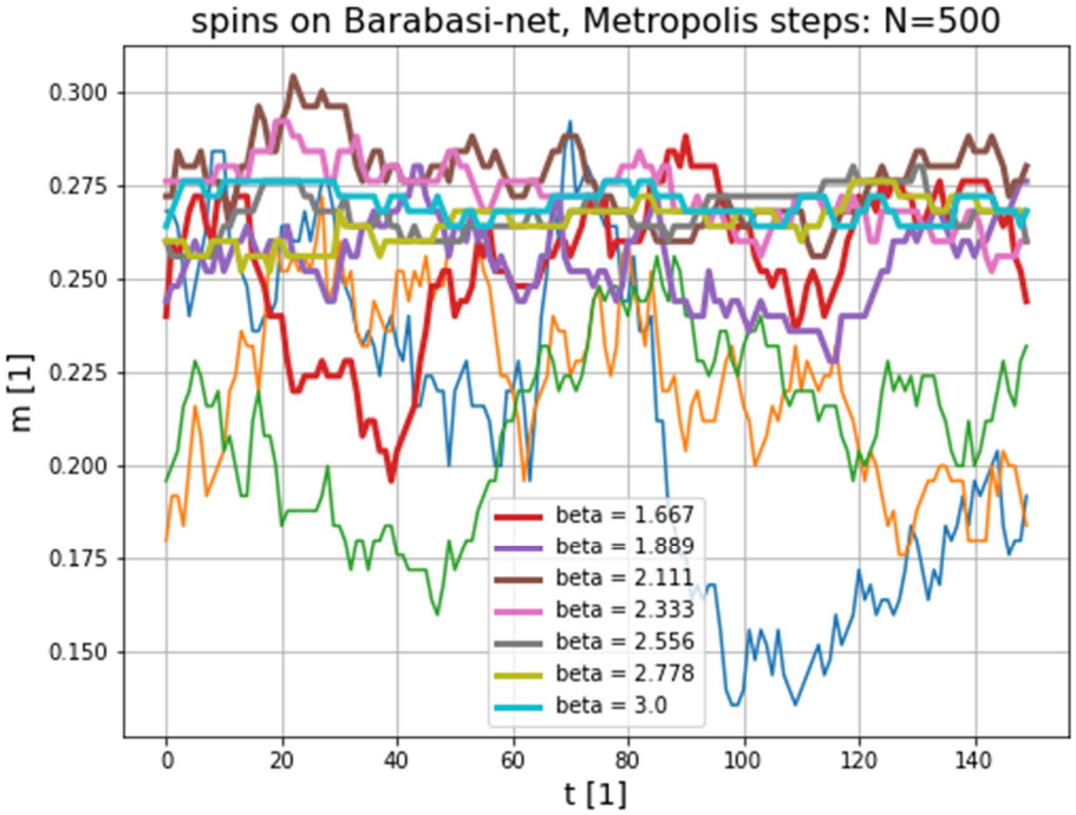
3. Figure: m(t) of an Ising-spin simulation.
The series, which I suspected that converge
are thicker lines.

On the graph above, we can see the magnetization by time of numerous Monte Carlo simulations for a 50x50 grid of Ising spins. We can see on the graph, that βJs, from 0.1 are converging, the other values are in [0, 0.1]. In each realization, a different beta was chosen, the grid was randomly initialized and then evolved in time. This method was not very efficient, because each step took about ~50ms, and one plot took about 3000. Also, it is somewhat inaccurate, because one Monte Carlo run with the given parameter is hardly enough to extract some generality from the model. However, to run multiple MCs would've been much more time consuming.

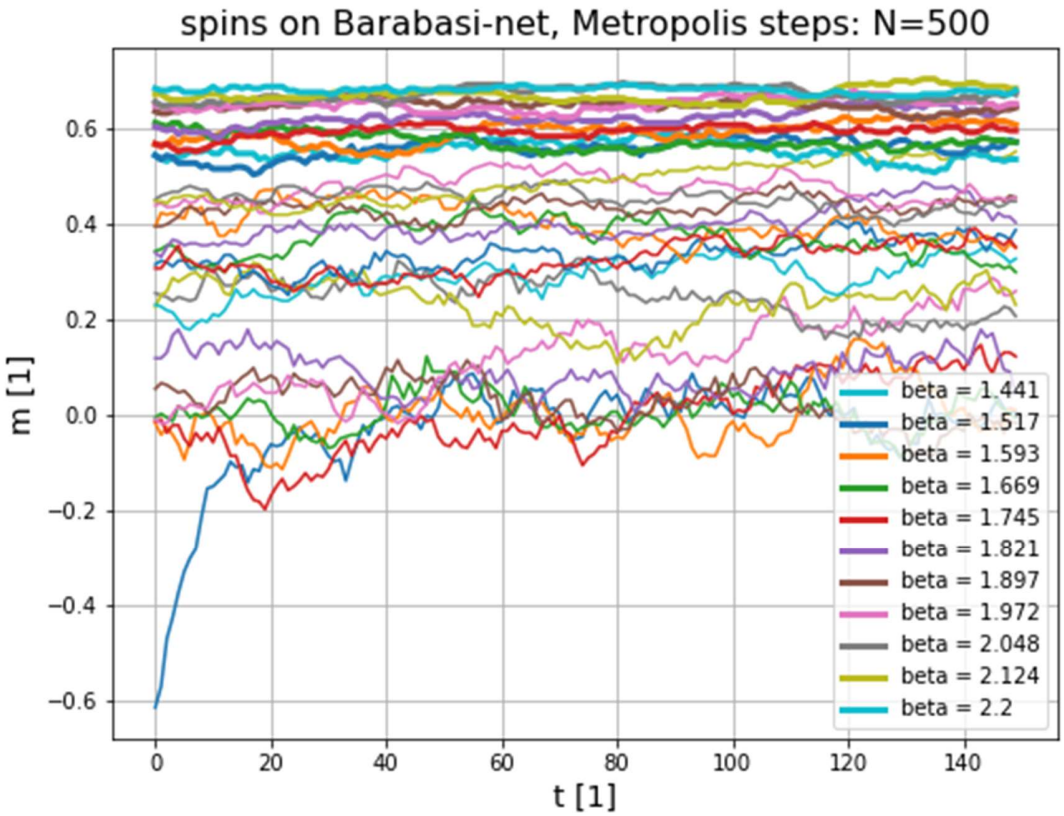
Secondly, I made animations, where between each step βJ was increased a little. This method's performance was stellar comparing to the previous one. The results are in the `pics/2d_grid_beta.gif` and `pics/2d_grid_beta.gif` path. The phase transition is very clearly visible at around $\beta J \approx 0.13$, which is in accordance with the 0.125 critical exponent mentioned above.

Ising-spins on Barabási-Albert network with Metropolis steps

I used the same two approach as in case of 2D grid. Firstly, I animated the spin flips on the network at different βJs on a larger interval to have a guess on a smaller interval, where the critical exponent might be. Then I plotted some MC runs on this smaller interval.



4. Figure: Monte Carlo runs on $\beta J \in [1,3]$. We can observe two behaviour:
one, where m oscillates with times with a greater amplitude,
and the other, which has only minor oscillations and is very angular.



5. Figure: Another observation can be made: the higher the absolute value of the magnetization, the smaller the smaller the oscillations. $\beta J \in [0,2.2]$

So, we might find a critical magnetization value, which can signify a phase transition. After looking at `pics/Ising_metro.gif` I guess the critical exponent to be somewhere around $\beta J \approx 1.2$. This is very rough method, we could use some more Monte Carlo runs, to estimate it better.

Technical remarks

Both implementations, the two dimensional grid and the spins on a network, can be read in `simulate_2dGrid.py` and `simulate_barabasi.py`.

At first, I tried to be consistent about only using `numpy` functions when writing the simulation for the 2D grid case, thinking this consistency would result in faster runtimes. Surprisingly, this had a few drawbacks, namely, the code run so slowly, that it was inefficient to explore to model with. Each run took minutes, and only after that, I could make the animations. Instead, I tried to use

standard python library functions, with which I could reach shorter and more practical runtimes. In a post[8], I found that the reason for that can be, that `numpy` has a much bigger overhead, because it makes a lot of effort for optimal memory management.

Conculsion

I investigated the critical exponents of the Ising-model for different topologies: a 2D grid and linear preferential networks. One of the critical exponents was confirmed with visual methods, so the implementation proved to be correct.

Sources

[1] Albert, Réka, and Albert-László Barabási. "Statistical mechanics of complex networks." Reviews of modern physics 74.1 (2002): 47.

[2] Barabási–Albert model on wiki: https://en.wikipedia.org/wiki/Barabási–Albert_model

[3] Zoltán Rácz – Random physical processes course notes

[4] Rubin H. Landau: Computational physics

[5] Sznajd-model on wiki: https://en.wikipedia.org/wiki/Sznajd_model

[6] Nowak, Bartłomiej, and Katarzyna Sznajd-Weron. "In search for the social hysteresis--the symmetrical threshold model with independence on Watts-Strogatz graphs." *arXiv preprint arXiv:2002.10395* (2020).

[7] Bernardes, Americo T., Dietrich Stauffer, and Janos Kertész. "Election results and the Sznajd model on Barabasi network." *The European Physical Journal B-Condensed Matter and Complex Systems* 25.1 (2002): 123-127.

[8] SklogWiki, encyclopedia dedicated to thermodynamics and statistical mechanics: http://www.sklogwiki.org/SklogWiki/index.php/Ising_model

[9] https://stackoverflow.com/questions/52603487/speed-comparison-numpy-vs-python-standard