Optimisation (COMP331/557) 2022-23 Class Test
Time Allowed: 1 hr
Exam Policy: Closed Book and Calculators NOT Allowed

1. **LP-Basics / Geometry**

   Among the vectors $(2,-2,2)$, $(1,-1,1)$, $(-1,0,1)$, $(3,1,0)$ find a maximum size subset of linearly independent vectors and provide an argument that they indeed are linearly independent.

   [5 marks]

   Model Solution:
   Because we consider vectors in dimension 3, the maximum number of linearly independent vectors in such space is 3. An example of a set of independent vectors is, for instance, the set $\{(-1,0,1),(3,1,0),(1,-1,1)\}$, and other such sets exist as well (thus such a set is not unique). An argument that the vectors in the set $\{(-1,0,1),(3,1,0),(1,-1,1)\}$ are linearly independent vectors is that

   $$\det\begin{pmatrix}-1 & 0 & 1\\ 3 & 1 & 0\\ 1 & -1 & 1\end{pmatrix} = (-1)\cdot(1-0)+0+1\cdot(-3-1) = -5 \neq 0,$$

   so it is non-singular, meaning that these vectors indeed are linearly independent.

2. **Simplex**

   Consider the following linear program:

   $$\begin{array}{lrcrcrcl} \min & -3x_1 & - & 2x_2 & - & x_3 & & \\ s.t. & x_1 & + & 2x_2 & + & 3x_3 & \leq & 4 \\ & 2x_1 & + & 2x_2 & + & x_3 & \leq & 5 \\ & 2x_1 & + & x_2 & & & \leq & 4 \\ & & & x_1, & x_2, & x_3 & \geq & 0 \end{array}$$

   (a) Convert the problem into standard form and construct a basic feasible solution at which $(x_1,x_2,x_3) = (0,0,0)$.
   [4 marks]
   Model Solution:

$$
\begin{array}{llllllllllllll}
\min & -3x_1 & - & 2x_2 & - & x_3 & & & & & & & & \\
s.t. & x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & & & & & = & 4 \\
 & 2x_1 & + & 2x_2 & + & x_3 & & & + & x_5 & & & = & 5 \\
 & 2x_1 & + & x_2 & & & & & & & + & x_6 & = & 4 \\
 & & & & & & & x_1, & x_2, & x_3, & x_4, & x_5, \ x_6 & \geq & 0
\end{array}
$$

BFS: $x = (0,0,0,4,5,4)$

(b) Carry out the tableau implementation for the first two iterations of the simplex method, starting at the basic feasible solution of part (a). Use Bland's rule to determine the pivot element. In every step mark the pivot element and provide the current values of the objective function and the variables. **[6 marks]**

Model Solution:

Pivot elements are in bold face with an asterix.

Initial tableau:

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-3	-2	-1	0	0	0
x_4	4	1	2	3	1	0	0
x_5	5	2	2	1	0	1	0
x_6	4	$\mathbf{2}^*$	1	0	0	0	1

$x = (0,0,0,4,5,4)$ $\quad z = 0$

After 1st pivot:

		x_1	x_2	x_3	x_4	x_5	x_6
	6	0	$-1/2$	-1	0	0	$3/2$
x_4	2	0	$3/2$	3	1	0	$-1/2$
x_5	1	0	$\mathbf{1}^*$	1	0	1	-1
x_1	2	1	$1/2$	0	0	0	$1/2$

$x = (2,0,0,2,1,0)$ $\quad z = -6$

After 2nd pivot:

		x_1	x_2	x_3	x_4	x_5	x_6
	$13/2$	0	0	$-1/2$	0	$1/2$	1
x_4	$1/2$	0	0	$\mathbf{3/2}^*$	1	$-3/2$	1
x_2	1	0	1	1	0	1	-1
x_1	$3/2$	1	0	$-1/2$	0	$-1/2$	1

$x = (\frac{3}{2}, 1, 0, \frac{1}{2}, 0, 0)$ $\quad z = -\frac{13}{2}$

After 3rd pivot:

		x_1	x_2	x_3	x_4	x_5	x_6
	$20/3$	0	0	0	$1/3$	0	$4/3$
x_3	$1/3$	0	0	1	$2/3$	-1	$2/3$
x_2	$2/3$	0	1	0	$-2/3$	2	$-5/3$
x_1	$5/3$	1	0	0	$1/3$	-1	$4/3$

$x = (\frac{5}{3}, \frac{2}{3}, \frac{1}{3}, 0, 0, 0)$ $\quad z = -\frac{20}{3}$