

Some applications of Deep Learning in Finance

AI for Global Goals
OxML 2023

Blanka Horvath

University of Oxford, Oxford Man Institute &
The Alan Turing Institute

Part II: Market Generation

10th July 2023, Oxford

The talk is based on joint work with

Hans Bühler, Terry Lyons, Imanol Perez Arribas, Ben Wood



Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

- ▶ Changes made possible by the increase in computing power availability.

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

- ▶ Changes made possible by the increase in computing power availability.
- ▶ Changes also necessary? (Some use cases, later.)

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

- ▶ Changes made possible by the increase in computing power availability.
- ▶ Changes also necessary? (Some use cases, later.)
- ▶ Classical models historically a simplified but tractable model of “reality”:

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

- ▶ Changes made possible by the increase in computing power availability.
- ▶ Changes also necessary? (Some use cases, later.)
- ▶ Classical models historically a simplified but tractable model of “reality”: often: few parameters, but role of the parameters clear,

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

- ▶ Changes made possible by the increase in computing power availability.
- ▶ Changes also necessary? (Some use cases, later.)
- ▶ Classical models historically a simplified but tractable model of “reality”:
often: few parameters, but role of the parameters clear, properties and limitations of the model well-understood, numerical or asymptotic approximations with error analysis available,

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

- ▶ Changes made possible by the increase in computing power availability.
- ▶ Changes also necessary? (Some use cases, later.)
- ▶ Classical models historically a simplified but tractable model of “reality”:
often: few parameters, but role of the parameters clear, properties and limitations of the model well-understood, numerical or asymptotic approximations with error analysis available, well-established risk-management and model governance.

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

- ▶ Changes made possible by the increase in computing power availability.
- ▶ Changes also necessary? (Some use cases, later.)
- ▶ Classical models historically a simplified but tractable model of “reality”:
often: few parameters, but role of the parameters clear, properties and limitations of the model well-understood, numerical or asymptotic approximations with error analysis available, well-established risk-management and model governance.
- ▶ Trend towards more accurate but more complex models:
often: more accurate reflection of stylised facts of markets, more accurate fits to option prices,

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

- ▶ Changes made possible by the increase in computing power availability.
- ▶ Changes also necessary? (Some use cases, later.)
- ▶ Classical models historically a simplified but tractable model of “reality”:
often: few parameters, but role of the parameters clear, properties and limitations of the model well-understood, numerical or asymptotic approximations with error analysis available, well-established risk-management and model governance.
- ▶ Trend towards more accurate but more complex models:
often: more accurate reflection of stylised facts of markets, more accurate fits to option prices, path dependent considerations,

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

- ▶ Changes made possible by the increase in computing power availability.
- ▶ Changes also necessary? (Some use cases, later.)
- ▶ Classical models historically a simplified but tractable model of “reality”:
often: few parameters, but role of the parameters clear, properties and limitations of the model well-understood, numerical or asymptotic approximations with error analysis available, well-established risk-management and model governance.
- ▶ Trend towards more accurate but more complex models:
often: more accurate reflection of stylised facts of markets, more accurate fits to option prices, path dependent considerations, consolidate modelling of different markets.

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

- ▶ Changes made possible by the increase in computing power availability.
- ▶ Changes also necessary? (Some use cases, later.)
- ▶ Classical models historically a simplified but tractable model of “reality”:
often: few parameters, but role of the parameters clear, properties and limitations of the model well-understood, numerical or asymptotic approximations with error analysis available, well-established risk-management and model governance.
- ▶ Trend towards more accurate but more complex models:
often: more accurate reflection of stylised facts of markets, more accurate fits to option prices, path dependent considerations, consolidate modelling of different markets.
Sometimes at a trade-off with tractability, control over errors, or interpretability, ...

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

(1) Classical and Neo-classical stochastic market models:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = s.$$

Black-Scholes, diffusions, models with jumps, stochastic volatility, path-dependent models, LSV, multifactor models, Rough Volatility, ...

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

(1) Classical and Neo-classical stochastic market models:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = s.$$

Black-Scholes, diffusions, models with jumps, stochastic volatility, path-dependent models, LSV, multifactor models, Rough Volatility, ...

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

(1) Classical and Neo-classical stochastic market models:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = s.$$

Black-Scholes, diffusions, models with jumps, stochastic volatility, path-dependent models, LSV, multifactor models, Rough Volatility, ...

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?
Data driven

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

(1) Classical and Neo-classical stochastic market models:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = s.$$

Black-Scholes, diffusions, models with jumps, stochastic volatility, path-dependent models, LSV, multifactor models, Rough Volatility, ...

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?
Data driven (no a-priori assumption on distribution of stochastic process)

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

(1) Classical and Neo-classical stochastic market models:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = s.$$

Black-Scholes, diffusions, models with jumps, stochastic volatility, path-dependent models, LSV, multifactor models, Rough Volatility, ...

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?
Data driven (no a-priori assumption on distribution of stochastic process) non-parametric?

$$f \in \mathcal{N}_r(I, d_1, \dots, d_{r-1}, O; \sigma_1, \dots, \sigma_r)$$

⇒ very flexible.

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

(1) Classical and Neo-classical stochastic market models:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = s.$$

Black-Scholes, diffusions, models with jumps, stochastic volatility, path-dependent models, LSV, multifactor models, Rough Volatility, ...

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?
Data driven (no a-priori assumption on distribution of stochastic process) non-parametric?

$$f \in \mathcal{N}_r(I, d_1, \dots, d_{r-1}, O; \sigma_1, \dots, \sigma_r)$$

⇒ very flexible. Originally developed for static problems, adaptation to financial time-series modelling not straightforward (see later).

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

(1) Classical and Neo-classical stochastic market models:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = s.$$

Black-Scholes, diffusions, models with jumps, stochastic volatility, path-dependent models, LSV, multifactor models, Rough Volatility, ...

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?
Data driven (no a-priori assumption on distribution of stochastic process) non-parametric?

$$f \in \mathcal{N}_r(I, d_1, \dots, d_{r-1}, O; \sigma_1, \dots, \sigma_r)$$

⇒ very flexible. Originally developed for static problems, adaptation to financial time-series modelling not straightforward (see later). Interpretability, risk-management...

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

(1) Classical and Neo-classical stochastic market models:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = s.$$

Black-Scholes, diffusions, models with jumps, stochastic volatility, path-dependent models, LSV, multifactor models, Rough Volatility, ...

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance? Data driven (no a-priori assumption on distribution of stochastic process) non-parametric?

$$f \in \mathcal{N}_r(I, d_1, \dots, d_{r-1}, O; \sigma_1, \dots, \sigma_r)$$

⇒ very flexible. Originally developed for static problems, adaptation to financial time-series modelling not straightforward (see later). Interpretability, risk-management...

(1.5) Extending (or augmenting) currently prevalent stochastic models:

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

(1) Classical and Neo-classical stochastic market models:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = s.$$

Black-Scholes, diffusions, models with jumps, stochastic volatility, path-dependent models, LSV, multifactor models, Rough Volatility, ...

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance? Data driven (no a-priori assumption on distribution of stochastic process) non-parametric?

$$f \in \mathcal{N}_r(I, d_1, \dots, d_{r-1}, O; \sigma_1, \dots, \sigma_r)$$

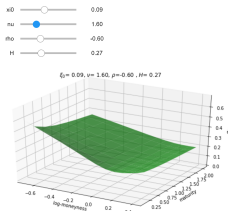
⇒ very flexible. Originally developed for static problems, adaptation to financial time-series modelling not straightforward (see later). Interpretability, risk-management...

(1.5) Extending (or augmenting) currently prevalent stochastic models: Models that are more adaptive to market environments by creating mixtures of (neo-)classical models

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

- (1.5) Augmenting and extending currently prevalent stochastic models
models that are adaptive to market environments by mixing of (neo-)classical models
- **Mixture of expert models:** A first step towards this was demonstrated in the Deep Learning Volatility framework: Take a mixture of two (or more) stochastic volatility models and calibrate it to data including the mixture parameter a .
 $a \times \text{Heston} + (1 - a) \times \text{rBergomi}$



Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

(1.5) Augmenting and extending currently prevalent stochastic models
models that are adaptive to market environments

- ▶ **Mixture of expert models:** First step as demonstrated in the Deep Learning Volatility (2019) framework: Eg. $a \times \text{Heston} + (1 - a) \times \text{rBergomi}$.

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

(1.5) Augmenting and extending currently prevalent stochastic models
models that are adaptive to market environments

- ▶ **Mixture of expert models:** First step as demonstrated in the Deep Learning Volatility (2019) framework: Eg. $a \times \text{Heston} + (1 - a) \times \text{rBergomi}$.
- ▶ **More flexible mixtures:** Neural SDEs parametrising diffusion parameters by Neural Networks (see L. Szpruch et al., 2020)

Market Simulation in Mathematical Finance

Transforming horizons of mathematical modelling towards models that can more and more accurately fit market data:

- (1.5) Augmenting and extending currently prevalent stochastic models
models that are adaptive to market environments
- ▶ **Mixture of expert models:** First step as demonstrated in the Deep Learning Volatility (2019) framework: Eg. $a \times \text{Heston} + (1 - a) \times \text{rBergomi}$.
 - ▶ **More flexible mixtures:** Neural SDEs parametrising diffusion parameters by Neural Networks (see L. Szpruch et al., 2020)
 - ▶ **OT approach:** From Optimal Transport to Robust Pricing and Hedging via Neural Networks (see J. Obłój et al., 2018), a framework that interpolates between model-specific and model independent settings.
 - ▶ **Non-parametric mixtures:** Signature-based interpolation and approx. of models (I. Perez-Arribas and C. Salvi, 2020) or directly mixing classical with data driven models (C.A. Lehalle).

Generative Models in Machine Learning

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?

Generative Models in Machine Learning

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?

Generative models can be trained to transport **some source distribution** μ

Generative Models in Machine Learning

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?

Generative models can be trained to transport **some source distribution** μ (say $\mu \sim \mathcal{U}[0, 1]$ or $\mu \sim \mathcal{N}(0, 1)$) to some **target distribution** observed in the data

Generative Models in Machine Learning

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?

Generative models can be trained to transport **some source distribution** μ (say $\mu \sim \mathcal{U}[0, 1]$ or $\mu \sim \mathcal{N}(0, 1)$) to some **target distribution** observed in the data and generate more samples that are similar/indistinguishable from the ones observed (Data-driven as there are no assumptions made on the latter distribution).

Generative Models in Machine Learning

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?

Generative models can be trained to transport **some source distribution** μ (say $\mu \sim \mathcal{U}[0, 1]$ or $\mu \sim \mathcal{N}(0, 1)$) to some **target distribution** observed in the data and generate more samples that are similar/indistinguishable from the ones observed (Data-driven as there are no assumptions made on the latter distribution).

Currently the most popular DNN-based generative models

- ▶ Restricted Boltzman Machine (RBM)

Generative Models in Machine Learning

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?

Generative models can be trained to transport **some source distribution** μ (say $\mu \sim \mathcal{U}[0, 1]$ or $\mu \sim \mathcal{N}(0, 1)$) to some **target distribution** observed in the data and generate more samples that are similar/indistinguishable from the ones observed (Data-driven as there are no assumptions made on the latter distribution).

Currently the most popular DNN-based generative models

- ▶ Restricted Boltzman Machine (RBM) Kondratyev & Schwarz (2019)
- ▶ Generative Adversarial Networks (GAN)

Generative Models in Machine Learning

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?

Generative models can be trained to transport **some source distribution** μ (say $\mu \sim \mathcal{U}[0, 1]$ or $\mu \sim \mathcal{N}(0, 1)$) to some **target distribution** observed in the data and generate more samples that are similar/indistinguishable from the ones observed (Data-driven as there are no assumptions made on the latter distribution).

Currently the most popular DNN-based generative models

- ▶ Restricted Boltzman Machine (RBM) Kondratyev & Schwarz (2019)
- ▶ Generative Adversarial Networks (GAN) Wiese, Knobloch, Korn & Kretschmer (2019)
- ▶ Variational Autoencoders (VAE)

Generative Models in Machine Learning

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?

Generative models can be trained to transport **some source distribution** μ (say $\mu \sim \mathcal{U}[0, 1]$ or $\mu \sim \mathcal{N}(0, 1)$) to some **target distribution** observed in the data and generate more samples that are similar/indistinguishable from the ones observed (Data-driven as there are no assumptions made on the latter distribution).

Currently the most popular DNN-based generative models

- ▶ Restricted Boltzman Machine (RBM) Kondratyev & Schwarz (2019)
- ▶ Generative Adversarial Networks (GAN) Wiese, Knobloch, Korn & Kretschmer (2019)
- ▶ Variational Autoencoders (VAE) **Bühler, H., Lyons, Perez-Arribas, Wood (2020)**

Generative Models in Machine Learning

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?

Generative models can be trained to transport **some source distribution** μ (say $\mu \sim \mathcal{U}[0, 1]$ or $\mu \sim \mathcal{N}(0, 1)$) to some **target distribution** observed in the data and generate more samples that are similar/indistinguishable from the ones observed (Data-driven as there are no assumptions made on the latter distribution).

Currently the most popular DNN-based generative models

- ▶ Restricted Boltzman Machine (RBM) Kondratyev & Schwarz (2019)
- ▶ Generative Adversarial Networks (GAN) Wiese, Knobloch, Korn & Kretschmer (2019)
- ▶ Variational Autoencoders (VAE) **Bühler, H., Lyons, Perez-Arribas, Wood (2020)**

Data driven, flexible, originally developed for static problems, adaptation to financial time-series modelling not straightforward.

Generative Models in Machine Learning

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?

Generative models can be trained to transport **some source distribution** μ (say $\mu \sim \mathcal{U}[0, 1]$ or $\mu \sim \mathcal{N}(0, 1)$) to some **target distribution** observed in the data and generate more samples that are similar/indistinguishable from the ones observed (Data-driven as there are no assumptions made on the latter distribution).

Currently the most popular DNN-based generative models

- ▶ Restricted Boltzman Machine (RBM) Kondratyev & Schwarz (2019)
- ▶ Generative Adversarial Networks (GAN) Wiese, Knobloch, Korn & Kretschmer (2019)
- ▶ Variational Autoencoders (VAE) **Bühler, H., Lyons, Perez-Arribas, Wood (2020)**

Data driven, flexible, originally developed for static problems, adaptation to financial time-series modelling not straightforward.

- ▶ One approach to the incorporation of time-series aspect is by Causal Optimal Transport (see ongoing work by B. Acciaio, T. Xu and collaborators)

Generative Models in Machine Learning

(2) DNN-based Generative Modelling in other AI applications: Adapt to Finance?

Generative models can be trained to transport **some source distribution** μ (say $\mu \sim \mathcal{U}[0, 1]$ or $\mu \sim \mathcal{N}(0, 1)$) to some **target distribution** observed in the data and generate more samples that are similar/indistinguishable from the ones observed (Data-driven as there are no assumptions made on the latter distribution).

Currently the most popular DNN-based generative models

- ▶ Restricted Boltzman Machine (RBM) Kondratyev & Schwarz (2019)
- ▶ Generative Adversarial Networks (GAN) Wiese, Knobloch, Korn & Kretschmer (2019)
- ▶ Variational Autoencoders (VAE) **Bühler, H., Lyons, Perez-Arribas, Wood (2020)**

Data driven, flexible, originally developed for static problems, adaptation to financial time-series modelling not straightforward.

- ▶ One approach to the incorporation of time-series aspect is by Causal Optimal Transport (see ongoing work by B. Acciaio, T. Xu and collaborators)
- ▶ In this work we take an approach via **Rough Paths** using **Signatures**.

Motivation for our Market Generators

The concept of “Model” (here, in form of a numerical program):

- ▶ Classical: (Program; Data) \Rightarrow Output
- ▶ **Now:** (Architecture, ObjF; TrainData) \Rightarrow Program
(Program, TestData) \Rightarrow Output

⇒ This may redefine the concept of model governance too:

Model=($\underbrace{\text{Architecture, ObjF}}_{\text{Network}}$; Dataset) “Quality of” training data shapes the DNN!

See: Deep Hedging, **Bühler, Gonon, Teichmann, Wood (2019)**

Motivation for our Market Generators

The concept of “Model” (here, in form of a numerical program):

- ▶ Classical: (Program; Data) \Rightarrow Output
- ▶ **Now:** (Architecture, ObjF; TrainData) \Rightarrow Program
(Program, TestData) \Rightarrow Output

⇒ This may redefine the concept of model governance too:

Model=($\underbrace{\text{Architecture, ObjF}}_{\text{Network}}$; Dataset) “Quality of” training data shapes the DNN!

See: Deep Hedging, **Bühler, Gonon, Teichmann, Wood (2019)**

General Strategy

(Step 1) **Data extraction from time series** we subdivide original (say daily S&P index) data into partitions of: (1) daily data, (3) weekly path segments, i.e. 5 days, and (3) monthly path segments, i.e. 20 days.

General Strategy

- (Step 1) **Data extraction from time series** we subdivide original (say daily S&P index) data into partitions of: (1) daily data, (3) weekly path segments, i.e. 5 days, and (3) monthly path segments, i.e. 20 days.
- (Step 2) **Preprocessing the data** transforming data into (i) returns for all (1), (2), (3) and into (ii) log-signatures for (2), (3) of (leadlag) paths.

General Strategy

- (Step 1) **Data extraction from time series** we subdivide original (say daily S&P index) data into partitions of: (1) daily data, (3) weekly path segments, i.e. 5 days, and (3) monthly path segments, i.e. 20 days.
- (Step 2) **Preprocessing the data** transforming data into (i) returns for all (1), (2), (3) and into (ii) log-signatures for (2), (3) of (leadlag) paths.
- (Step 3) **Creating and training the VAE and the CVAE network:** VAE, a parsimonious generator model with “bottleneck structure”. Conditional Variational Autoencoder is learned to condition VAE on current market conditions (a) current level of the index (b) instantaneous volatility (c) signature of the previous path segment.

General Strategy

- (Step 1) **Data extraction from time series** we subdivide original (say daily S&P index) data into partitions of: (1) daily data, (3) weekly path segments, i.e. 5 days, and (3) monthly path segments, i.e. 20 days.
- (Step 2) **Preprocessing the data** transforming data into (i) returns for all (1), (2), (3) and into (ii) log-signatures for (2), (3) of (leadlag) paths.
- (Step 3) **Creating and training the VAE and the CVAE network:** VAE, a parsimonious generator model with “bottleneck structure”. Conditional Variational Autoencoder is learned to condition VAE on current market conditions (a) current level of the index (b) instantaneous volatility (c) signature of the previous path segment.
- (Step 4) **Postprocessing of the outputs of the VAEs** transforming (i), (ii) back to paths; such as building paths of arbitrary length

General Strategy

- (Step 1) **Data extraction from time series** we subdivide original (say daily S&P index) data into partitions of: (1) daily data, (3) weekly path segments, i.e. 5 days, and (3) monthly path segments, i.e. 20 days.
- (Step 2) **Preprocessing the data** transforming data into (i) returns for all (1), (2), (3) and into (ii) log-signatures for (2), (3) of (leadlag) paths.
- (Step 3) **Creating and training the VAE and the CVAE network:** VAE, a parsimonious generator model with “bottleneck structure”. Conditional Variational Autoencoder is learned to condition VAE on current market conditions (a) current level of the index (b) instantaneous volatility (c) signature of the previous path segment.
- (Step 4) **Postprocessing of the outputs of the VAEs** transforming (i), (ii) back to paths; such as building paths of arbitrary length
- (Step 5) **Performance evaluation** similar to the role of the discriminator in GANs, but here without feeding back to the generator.

General Strategy

- (Step 1) **Data extraction from time series** we subdivide original (say daily S&P index) data into partitions of: (1) daily data, (3) weekly path segments, i.e. 5 days, and (3) monthly path segments, i.e. 20 days.
- (Step 2) **Preprocessing the data** transforming data into (i) returns for all (1), (2), (3) and into (ii) log-signatures for (2), (3) of (leadlag) paths.
- (Step 3) **Creating and training the VAE and the CVAE network:** VAE, a parsimonious generator model with “bottleneck structure”. Conditional Variational Autoencoder is learned to condition VAE on current market conditions (a) current level of the index (b) instantaneous volatility (c) signature of the previous path segment.
- (Step 4) **Postprocessing of the outputs of the VAEs** transforming (i), (ii) back to paths; such as building paths of arbitrary length
- (Step 5) **Performance evaluation** similar to the role of the discriminator in GANs, but here without feeding back to the generator. **Good performance evaluation metrics?**

Rough Paths Approach to Generative Modelling of Markets

Levin, Lyons, & Ni. (2013) firstly proposed the signature of a path as a basis of functions for a functional on path space.

Definition (Signature of a path)

Let $X : [0, T] \rightarrow \mathbb{R}^d$ be a continuous path of bounded variation. The signature of X is then defined by the sequence of iterated integrals given by

$$\mathbb{X}_T^{\leq \infty} := (1, \mathbb{X}_T^1, \dots, \mathbb{X}_T^n, \dots), \quad \text{where}$$

$$\mathbb{X}_T^n := \int_{0 < u_1 < \dots < u_n < T} dX_{u_1} \otimes \dots \otimes dX_{u_n} \in (\mathbb{R}^d)^{\otimes n}$$

with \otimes the tensor product. Similarly, given $N \in \mathbb{N}$, the truncated signature of order N is defined by

$$\mathbb{X}_T^{\leq N} := (1, \mathbb{X}_T^1, \dots, \mathbb{X}_T^N).$$

If the path X has bounded variation – which is the case of discrete data – the integrals above can be defined using Riemann-Stieltjes integrals.

Rough Paths Approach to Generative Modelling

Why signatures?

Rough Paths Approach to Generative Modelling

Why signatures? Signatures provide a basis of functions for a functional on path space.

- ▶ While Fourier transforms and wavelets have a similar role approximating curves as a linear combination of basis functions, signatures do so in an un-parametrised way

Rough Paths Approach to Generative Modelling

Why signatures? Signatures provide a basis of functions for a functional on path space.

- ▶ While Fourier transforms and wavelets have a similar role approximating curves as a linear combination of basis functions, signatures do so in an un-parametrised way (model free, path by path characterisation possible).
- ▶ Robustness to missing data and irregular sampling and

Rough Paths Approach to Generative Modelling

Why signatures? Signatures provide a basis of functions for a functional on path space.

- ▶ While Fourier transforms and wavelets have a similar role approximating curves as a linear combination of basis functions, signatures do so in an un-parametrised way (model free, path by path characterisation possible).
- ▶ Robustness to missing data and irregular sampling and towards highly oscillatory data, and

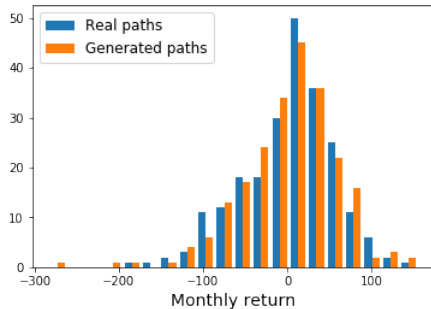
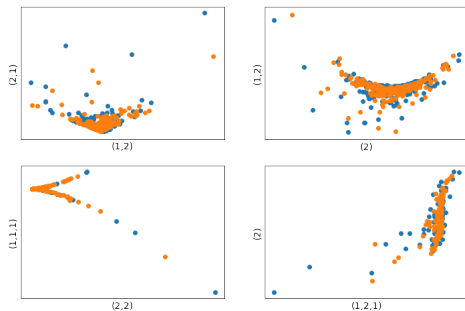
Rough Paths Approach to Generative Modelling

Why signatures? Signatures provide a basis of functions for a functional on path space.

- ▶ While Fourier transforms and wavelets have a similar role approximating curves as a linear combination of basis functions, signatures do so in an un-parametrised way (model free, path by path characterisation possible).
- ▶ Robustness to missing data and irregular sampling and towards highly oscillatory data, and invariance under time re-parametrisation. **Liao, Lyons, Ni, Yang (2019)**
- ▶ Signatures provide the right framework for performance evaluation metrics on pathspace, one of the difficulties being that the pathspace $C([0, 1], \mathbb{R}^d)$ is infinite-dimensional and not locally compact. **Chevyrev, Oberhauser (2018)**
- ▶ In fact, **Liao, Lyons, Ni, Yang (2019)** demonstrated the advantages of log-signatures (with all positive properties listed above, but lower dimensionality)

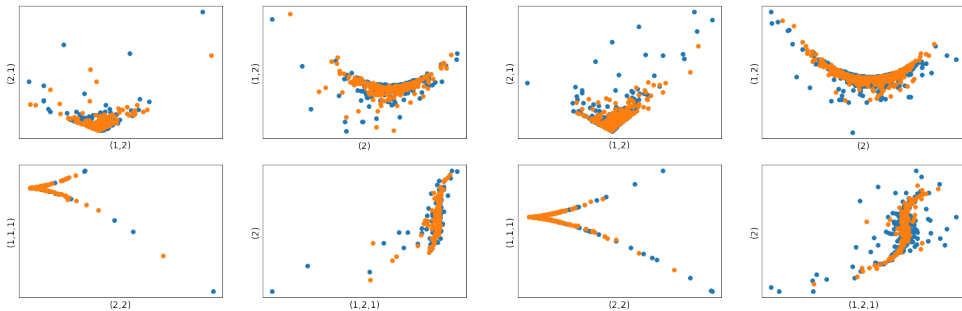
Numerical Results

- Signature based training (ii) at least as accurate as returns-based training (i).



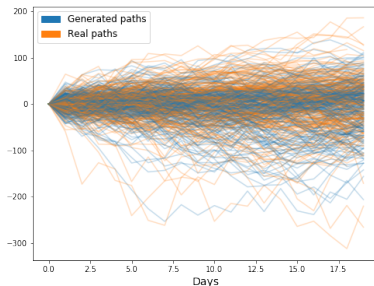
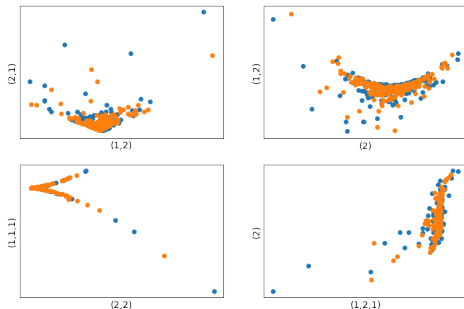
Numerical Results

- ▶ Signature-based generative model works well already with the low number of training samples available in data (1000 samples weekly-unconditioned **left image**, 250 samples monthly **right image** unconditioned).
- ▶ Increasing the number of samples in numerically generated data does not lead to better performance.



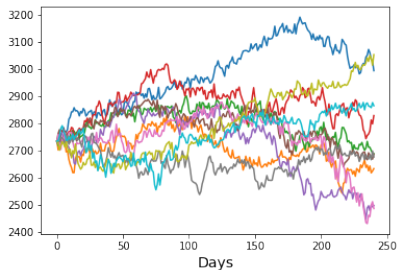
Numerical Results

- ▶ Signature-based generative model works well already with the low number of training samples available in data (1000 samples weekly-unconditioned, 250 samples monthly unconditioned).
- ▶ Increasing the number of samples in numerically generated data does not lead to better performance.



Conditional VAE and Postprocessing

- ▶ Conditional Variational Autoencoder is learned to condition VAE on current market conditions (a) current level of the index (b) instantaneous volatility (c) signature of the previous path segment.
- ▶ By conditioning, the number of available training samples is even smaller \Rightarrow powerful parsimonious generative model crucial.
- ▶ By conditioning on the signature of the previous path segment, one can generate + build paths far longer than the direct output of the generative model. Numerical results consistent between weekly vs. monthly paths. **Below: Yearly paths**



Performance evaluation metrics

- For classical models and continuous time stochastic processes popular performance evaluation metrics were whether the process reflects **stylised facts** of financial markets (many of these concern the returns)

Performance evaluation metrics

- ▶ For classical models and continuous time stochastic processes popular performance evaluation metrics were whether the process reflects **stylised facts** of financial markets (many of these concern the returns) or provides **close fits to option prices** etc.

Performance evaluation metrics

- ▶ For classical models and continuous time stochastic processes popular performance evaluation metrics were whether the process reflects **stylised facts** of financial markets (many of these concern the returns) or provides **close fits to option prices** etc.
- ▶ If returns of the target distribution are only observable on a time-grid with pre-defined mesh size and a generative model is fitted to observed distributions of returns marginal by marginal this can become problematic between \mathbb{P} and \mathbb{Q} measures. See **Brigo (2019)** and **Armstrong, Bellani, Brigo, Cass (2019)** underline the necessity of pathwise evaluation metrics.
- ▶ While in the finite dimensional case it is well-understood how moments describe the law of a random variable this question is more subtle in the infinite-dimensional case of path-valued random variables. **Chevryrev, Oberhauser (2018)** prove an analogous result for path-valued random variables by using the normalized sequence of signature moments.

Issues and Solutions offered by of Market Generators

Important concerns regarding training data:

- Issue 1** Data availability (some data is naturally scarce & not sufficient for training of neural networks)
- Issue 2** Data privacy (data often proprietary and not accessible, difficult to compare/regulate models' performance consistently across the industry)
- Issue 3** Data quality (noise, biases, outliers, expressiveness)

⇒ We need standardised and widely accessible testing datasets for innovation and benchmarking of DNN models.

Issues and Solutions offered by Market Generators

Important concerns regarding training data:

- Issue 1** Data availability (some data is naturally scarce & not sufficient for training of neural networks)
- Issue 2** Data privacy (data often proprietary and not accessible, difficult to compare/regulate models' performance consistently across the industry)
- Issue 3** Data quality (noise, biases, outliers, expressiveness)

⇒ We need standardised and widely accessible testing datasets for innovation and benchmarking of DNN models.

With this in mind we demonstrate in a joint work with A. Kondratyev and C. Schwarz the potential of generative models towards **data anonymisation, outlier detection & combating overfitting**, in several experiments **Kondratyev, Schwarz, H. (2020)**.

Thank you for your attention.

Rough Paths Approach to Generative Modelling

In what follows we work instead with the log-signatures

Definition (Log-signature)

Let $X : [0, T] \rightarrow \mathbb{R}^d$ be a path such that its signature $\mathbb{X}_{0,T}^{<\infty}$ is well-defined. The log-signature is then defined by

$$\log \mathbb{X}_T^{<\infty} := -\mathbb{X}_T^{<\infty} + \frac{1}{2}(\mathbb{X}_T^{<\infty})^{\otimes 2} - \frac{1}{3}(\mathbb{X}_T^{<\infty})^{\otimes 3} + \dots + (-1)^n \frac{1}{n}(\mathbb{X}_T^{<\infty})^{\otimes n} + \dots,$$

which can be shown to be well-defined.

- ▶ There is a one-to-one map between signatures and log-signatures.
- ▶ Log-signatures have all positive properties listed above.
- ▶ They allow for lower dimensional representation and are better suited to VAE.