$$\underbrace{\mathcal{F}_0 \xrightarrow[\theta]{\vec{u}_3^{(0)}} \mathcal{F}_a \xrightarrow[\phi]{\vec{u}_2^{(a)}} \mathcal{F}_b \xrightarrow[\psi]{\vec{u}_1^{(b)}} \mathcal{F}_7}$$

Euler angle sequence (321)

$\hat{C}$: rotation matrix

$\hat{C}^{(0,7)}$: rotation of $\mathcal{F}_0$ into $\mathcal{F}_7$

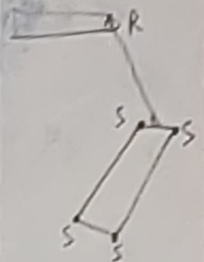
$$\hat{C}^{(0,7)} = \hat{C}^{(0,a)} \hat{C}^{(a,b)} \hat{C}^{(b,7)}$$

$$= e^{\tilde{u}_3 \theta} \cdot e^{\tilde{u}_2 \phi} \cdot e^{\tilde{u}_1 \psi}$$

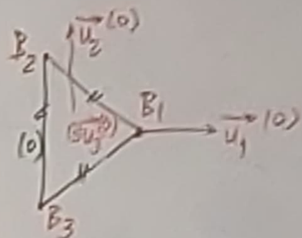
$\hat{C}$ is also used as a transformation matrix

$$\left[\hat{C}^{(0,7)}\right]^{-1} = \hat{C}^{(7,0)} = \left[\hat{C}^{(0,7)}\right]^{t}$$

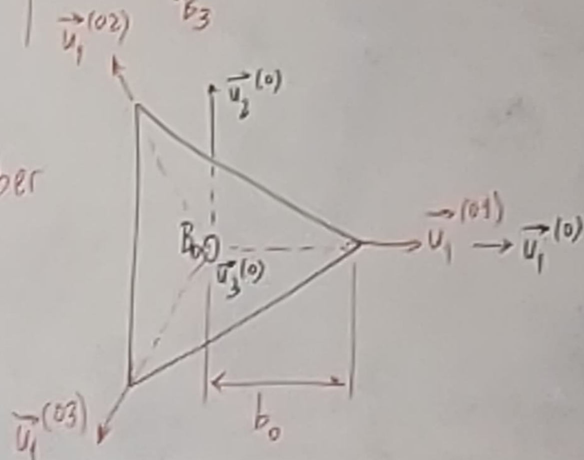
$[\hat{C}]^{-1} = \hat{C}^{t}$, $\hat{C}$: orthogonal matrix

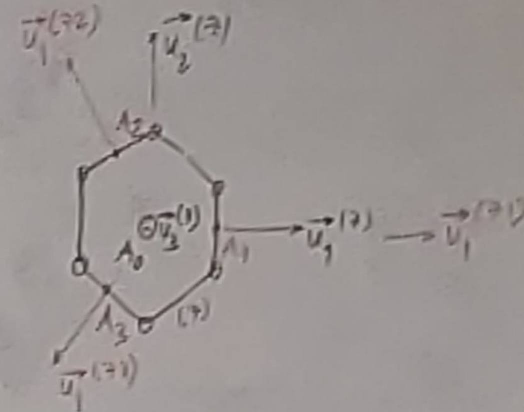


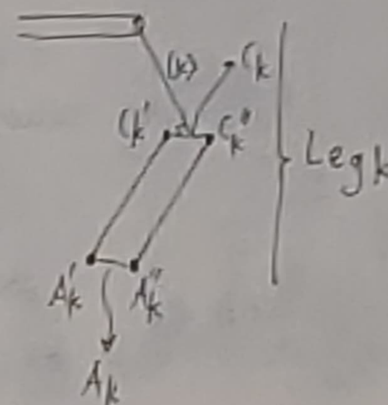
Top View



k: leg number

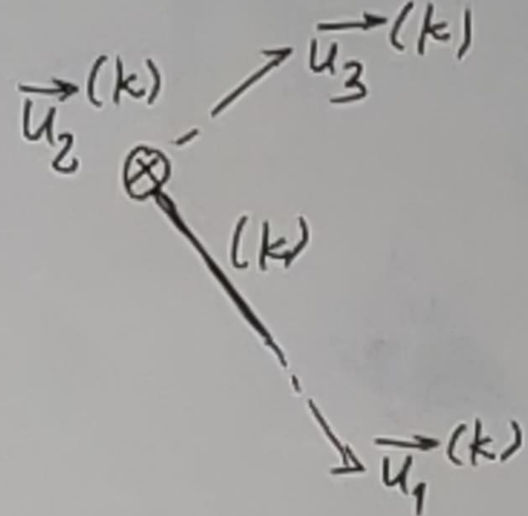




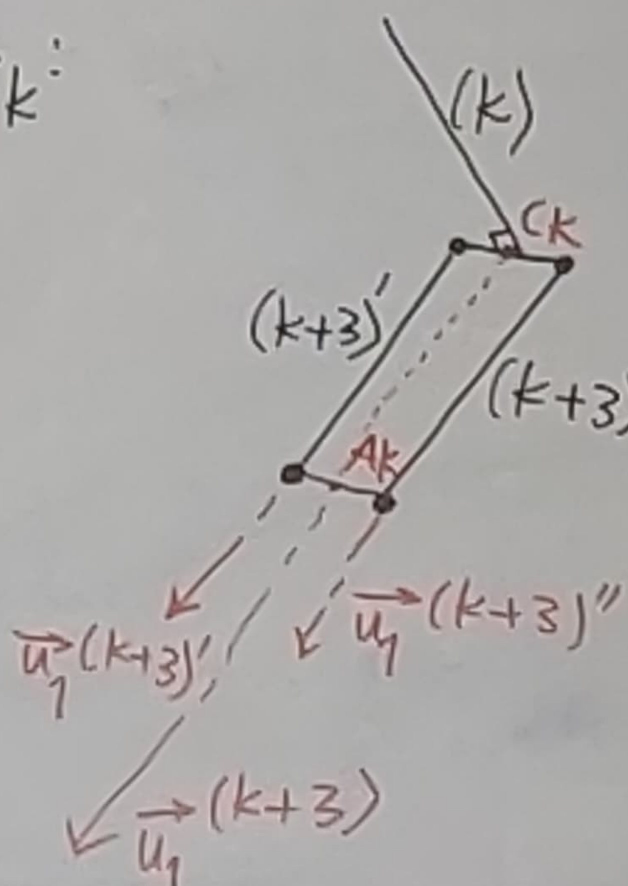


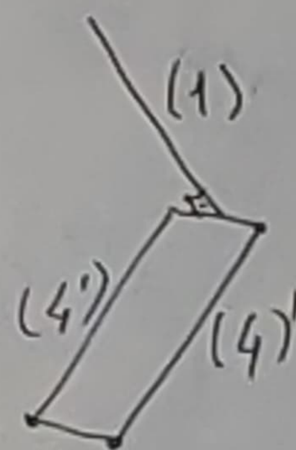
$$\hat{C} = e^{\tilde{n}\theta} \Rightarrow (\hat{C})^{-1} = e^{-\tilde{n}\theta} = (\hat{C})^t$$

Upper leg_k:



Lower legs_k:





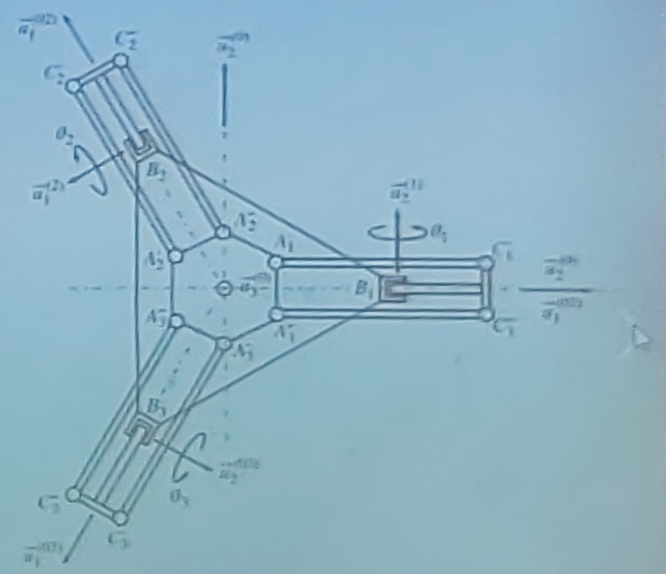
($\mathcal{L}_0$) can be modeled geometrically as an equilateral triangle. The centers of the revolute joints (B_1, B_2, B_3) are located at the corners of this triangle. Considering one of the three legs, say L_k, its upper link is $\mathcal{L}_k$ and its lower parallel links are $\mathcal{L}'_{k+3}$ and $\mathcal{L}''_{k+3}$. On these parallel links, the centers of the upper spherical joints are C'_k and C''_k with the mid point C_k between them. Similarly, the centers of the lower spherical joints are A'_k and A''_k with the mid point A_k between them. The lower spherical joints are shared by the moving platform $\mathcal{L}_7$.

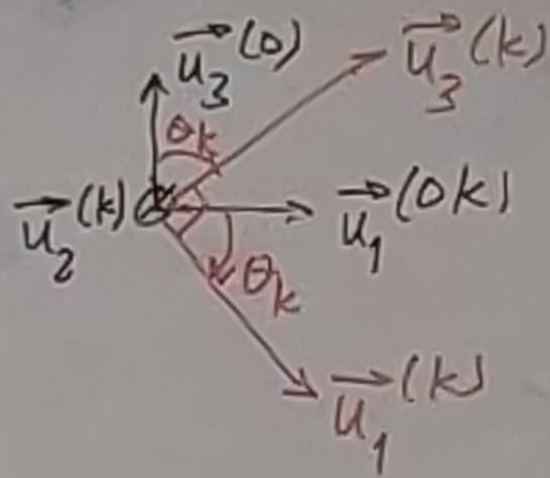
The geometric parameters associated with Figure 10.17 are indicated below.

$$\beta_1 = \sphericalangle[\overrightarrow{B_0B_1}] = 0,\ \beta_2 = \sphericalangle[\overrightarrow{B_0B_2}] = \beta_0,\ \beta_3 = \sphericalangle[\overrightarrow{B_0B_3}] = -\beta_0;\ \beta_0 = 2\pi/3$$

$$C'_kC''_k = A'_kA''_k = 2d_0\ \forall\ k = 1, 2, 3$$

Figure 10.18 illustrates the side view of the leg L_k of the manipulator together with the relevant geometric parameters and the active joint variable θ_k. The side views of the other two legs are of course similar due to the symmetric configuration of the manipulator.

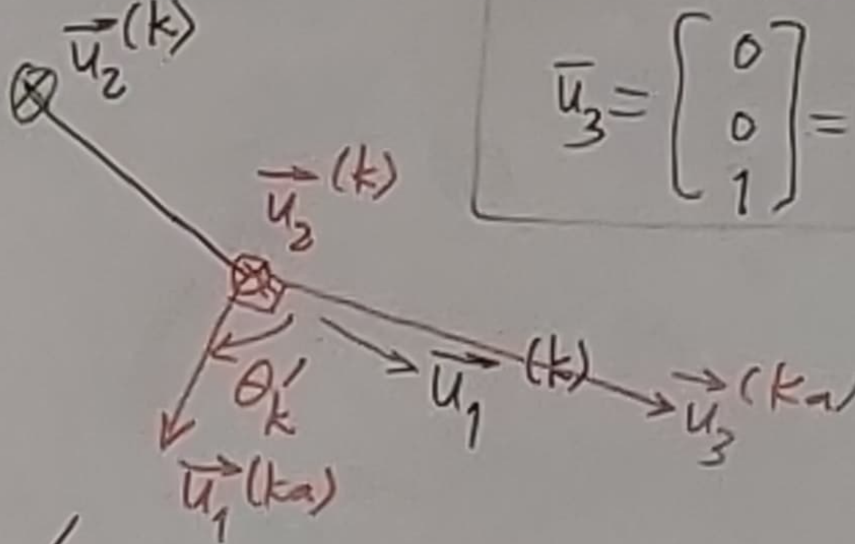


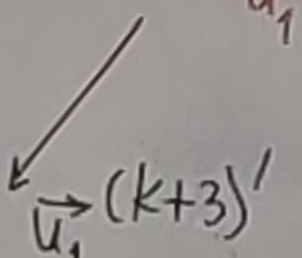


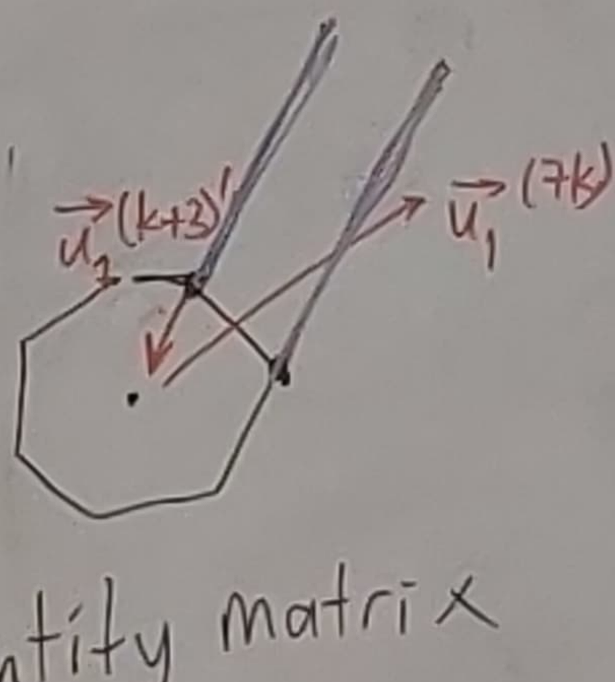
$$\bar{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} = \bar{u}_1^{(n/n)}$$

$$\bar{u}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} = \bar{u}_2^{(n/n)}$$

$$\bar{u}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} = \bar{u}_3^{(n/n)}$$







$$\hat{C}^{(0,7)} = \underline{\hat{I}}$$

↓ (3×3) identity matrix

(a) End-Effector Orientation Equations Through the Legs

For $k = 1, 2, 3, 7$ and the "lefty" (left-hand side) legs,

$$\hat{C} = \hat{C}^{(0,7)} = \hat{C}^{(0,0k)}\hat{C}^{(0k,k)}\hat{C}^{(k,k+3)'}\hat{C}^{(k+3,7k)'}\hat{C}^{(7k,7)'} \Rightarrow$$

$$\hat{C} = [e^{\tilde{u}_3\beta_k}][e^{\tilde{u}_2\theta_k}][e^{\tilde{u}_2\theta'_k}e^{\tilde{u}_3\phi'_k}e^{\tilde{u}_1\psi'_k}][e^{\tilde{u}_1\psi'_{k+3}}e^{\tilde{u}_3\phi'_{k+3}}e^{\tilde{u}_2\theta'_{k+3}}][e^{\tilde{u}_3\beta_k}]^t \Rightarrow \tag{10.441}$$

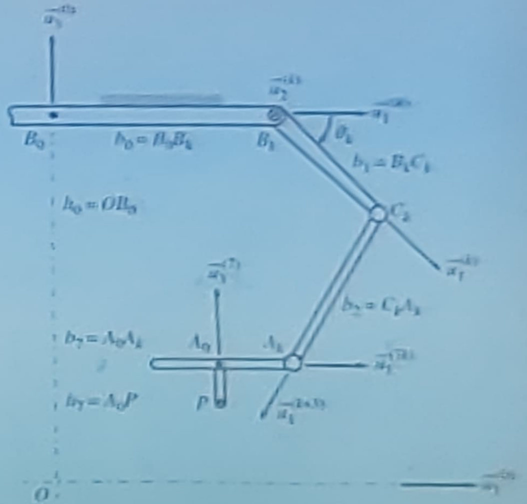
$$\hat{C} = e^{\tilde{u}_3\beta_k}e^{\tilde{u}_2(\theta_k+\theta'_k)}e^{\tilde{u}_3\phi'_k}e^{\tilde{u}_1(\psi'_k+\psi'_{k+3})}e^{\tilde{u}_3\phi'_{k+3}}e^{\tilde{u}_2\theta'_{k+3}}e^{-\tilde{u}_3\beta_k}$$

For $k = 1, 2, 3, 7$ and the "righty" (right-hand side) legs,

$$\hat{C} = \hat{C}^{(0,7)} = \hat{C}^{(0,0k)}\hat{C}^{(0k,k)}\hat{C}^{(k,k+3)''}\hat{C}^{(k+3,7k)''}\hat{C}^{(7k,7)''} \Rightarrow$$

$$\hat{C} = [e^{\tilde{u}_3\beta_k}][e^{\tilde{u}_2\theta_k}][e^{\tilde{u}_2\theta''_k}e^{\tilde{u}_3\phi''_k}e^{\tilde{u}_1\psi''_k}][e^{\tilde{u}_1\psi''_{k+3}}e^{\tilde{u}_3\phi''_{k+3}}e^{\tilde{u}_2\theta''_{k+3}}][e^{\tilde{u}_3\beta_k}]^t \Rightarrow \tag{10.442}$$

$$\hat{C} = e^{\tilde{u}_3\beta_k}e^{\tilde{u}_2(\theta_k+\theta''_k)}e^{\tilde{u}_3\phi''_k}e^{\tilde{u}_1(\psi''_k+\psi''_{k+3})}e^{\tilde{u}_3\phi''_{k+3}}e^{\tilde{u}_2\theta''_{k+3}}e^{-\tilde{u}_3\beta_k}$$



Note that the Euler angle sequences associated with the upper and lower spherical joints are selected differently (i.e. respectively, as 2-3-1 and 1-3-2) for the sake of kinematic convenience as it will be evidenced in the sequel. Here, it can be said that the major convenience of these sequences is that the indefinite angle pairs (ψ'_k, ψ'_{k+3}) and (ψ''_k, ψ''_{k+3}) that describe the insignificant spinning motions of the lower legs get combined into single equivalent spin angles.

(b) **Tip Point Location Equations Through the Legs**

- Tip point location equations through the *lefty legs* for $k = 1, 2, 3$:

$$\bar{p} = \overline{OP} = \overline{OB_0} + \overline{B_0B_k} + \overline{B_kC_k} + \overline{C_kC'_k} + \overline{C'_kA'_k} + \overline{A'_kA_k} + \overline{A_kA_0} + \overline{A_0P} \Rightarrow$$

$$\begin{aligned} \bar{p} = {} & h_0\vec{u}_3^{(0)} + b_0\vec{u}_1^{(0k)} + b_1\vec{u}_1^{(k)} + d_0\vec{u}_2^{(k)} + b_2\vec{u}_1^{(k+3)'} \\ & - d_0\vec{u}_2^{(7k)'} - b_7\vec{u}_1^{(7k)'} - h_7\vec{u}_3^{(7)} \end{aligned} \tag{10.443}$$

Equation (10.443) can be written as the following matrix equation in the base frame.

$$\begin{aligned} \bar{p} = \vec{p}^{(0)} = {} & h_0\vec{u}_3^{(0/0)} + b_0\vec{u}_1^{(0k/0)} + b_1\vec{u}_1^{(k/0)} + d_0\vec{u}_2^{(k/0)} + b_2\vec{u}_1^{(k+3/0)'} \\ & - d_0\vec{u}_2^{(7k/0)'} - b_7\vec{u}_1^{(7k/0)'} - h_7\vec{u}_3^{(7/0)} \Rightarrow \end{aligned}$$

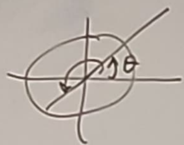
$$\begin{aligned} \bar{p} = {} & h_0\bar{u}_3 + b_0\hat{C}^{(0,0k)}\bar{u}_1 + b_1\hat{C}^{(0,k)}\bar{u}_1 + d_0\hat{C}^{(0,k)}\bar{u}_2 \\ & + b_2\hat{C}^{(0,k+3)'}\bar{u}_1 - d_0\hat{C}^{(0,7k)'}\bar{u}_2 - b_7\hat{C}^{(0,7k)'}\bar{u}_1 - h_7\hat{C}^{(0,7)}\bar{u}_3 \Rightarrow \end{aligned}$$

$$\begin{aligned} \bar{p} = {} & h_0\bar{u}_3 + b_0 e^{\tilde{u}_3\beta_k}\bar{u}_1 + b_1 e^{\tilde{u}_3\beta_k} e^{\tilde{u}_2\theta_k}\bar{u}_1 + d_0 e^{\tilde{u}_3\beta_k} e^{\tilde{u}_2\theta_k}\bar{u}_2 \\ & + b_2 e^{\tilde{u}_3\beta_k} e^{\tilde{u}_2(\theta_k+\theta'_k)} e^{\tilde{u}_3\phi'_k} e^{\tilde{u}_1\psi'_k}\bar{u}_1 \\ & - d_0 e^{\tilde{u}_3\beta_k} e^{\tilde{u}_2(\theta_k+\theta'_k)} e^{\tilde{u}_3\phi'_k} e^{\tilde{u}_1(\psi'_k+\psi'_{k+3})} e^{\tilde{u}_3\phi'_{k+3}} e^{\tilde{u}_2\theta'_{k+3}}\bar{u}_2 \\ & - b_7 e^{\tilde{u}_3\beta_k} e^{\tilde{u}_2(\theta_k+\theta'_k)} e^{\tilde{u}_3\phi'_k} e^{\tilde{u}_1(\psi'_k+\psi'_{k+3})} e^{\tilde{u}_3\phi'_{k+3}} e^{\tilde{u}_2\theta'_{k+3}}\bar{u}_1 \\ & - h_7 e^{\tilde{u}_3\beta_k} e^{\tilde{u}_2(\theta_k+\theta'_k)} e^{\tilde{u}_3\phi'_k} e^{\tilde{u}_1(\psi'_k+\psi'_{k+3})} e^{\tilde{u}_3\phi'_{k+3}} e^{\tilde{u}_2\theta'_{k+3}} e^{-\tilde{u}_3\beta_k}\bar{u}_3 \Rightarrow \end{aligned}$$

$$\begin{aligned} \bar{p} = {} & h_0\bar{u}_3 + e^{\tilde{u}_3\beta_k}[\bar{u}_1(b_0 + b_1 c\theta_k) + \bar{u}_2 d_0 - \bar{u}_3 b_1 s\theta_k] \\ & + b_2 e^{\tilde{u}_3\beta_k} e^{\tilde{u}_2(\theta_k+\theta'_k)} e^{\tilde{u}_3\phi'_k}\bar{u}_1 \\ & - e^{\tilde{u}_3\beta_k} e^{\tilde{u}_2(\theta_k+\theta'_k)} e^{\tilde{u}_3\phi'_k} e^{\tilde{u}_1(\psi'_k+\psi'_{k+3})} e^{\tilde{u}_3\phi'_{k+3}} e^{\tilde{u}_2\theta'_{k+3}}(\bar{u}_1 b_7 + \bar{u}_2 d_0 + \bar{u}_3 h_7) \end{aligned} \tag{10.444}$$

Equation (10.444) can also be written more compactly as follows upon inserting the matrix $\hat{C}$ given by Eq. (10.441).

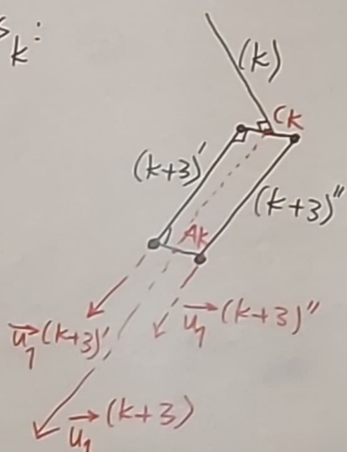
$$\begin{aligned} \bar{p} = {} & h_0\bar{u}_3 + e^{\tilde{u}_3\beta_k}[\bar{u}_1(b_0 + b_1 c\theta_k) + \bar{u}_2 d_0 - \bar{u}_3 b_1 s\theta_k] \\ & + b_2 e^{\tilde{u}_3\beta_k} e^{\tilde{u}_2(\theta_k+\theta'_k)} e^{\tilde{u}_3\phi'_k}\bar{u}_1 - \hat{C} e^{\tilde{u}_3\beta_k}(\bar{u}_1 b_7 + \bar{u}_2 d_0 + \bar{u}_3 h_7) \end{aligned} \tag{10.445}$$

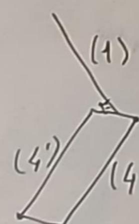


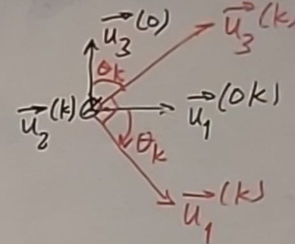
$$\left.\begin{matrix}\sin\theta\\ \cos\theta\end{matrix}\right\}\ \theta = \mathrm{atan}_2(\sin\theta, \cos\theta)$$

atan2

Lower legs$_k$:



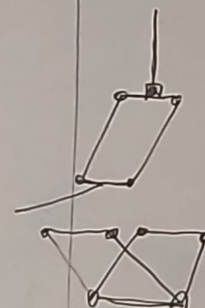


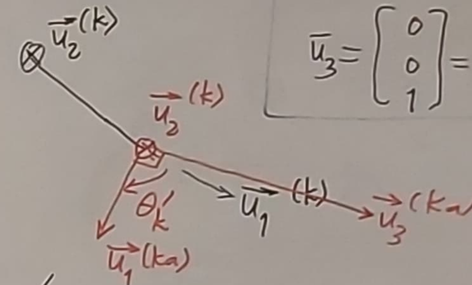


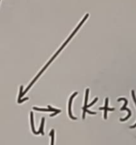
$$\bar{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} = \bar{u}_1^{(n/n)}$$

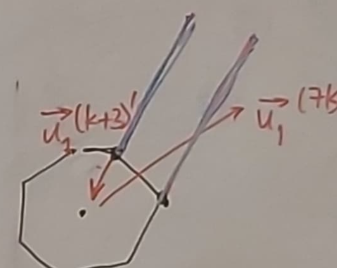
$$\bar{u}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} = \bar{u}_2^{(n/n)}$$

$$\bar{u}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} = \bar{u}_3^{(n/n)}$$









$$\hat{C}^{(0,7)} = \hat{I}$$

$\downarrow$ (3×3) identity matrix