PHYS 414 Final Project

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The following project consists of the calculations on the structures of various types of stars ranging from White Dwarfs (WDs) to Neutron Stars(NS) each of which required a different approach, namely Newtonian Gravity and General Relativity. The numerical calculations were conducted on Python and Mathematica.

NEWTON

In this part, some equations determining the structure of stars under Newtonian gravity were derived, then those equations solved numerically to further comprehend their structure.

A. Lane-Emden Equation

Starting with the hydro-static equilibrium of stars in Newtonian gravity, one can obtain the following ODEs.

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{1}$$

$$\frac{dp(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{2}$$

Using Eq. (2) we can separate the $m(r)$ and differentiate with respect to r.

$$\frac{d}{dr}\left(\frac{r^2}{G\rho(r)}\frac{dp(r)}{dr}\right) = -\frac{dm(r)}{dr} \tag{3}$$

Plugging the Eq. (1) to right hand side and rearranging.

$$\frac{1}{4\pi r^2}\frac{d}{dr}\left(\frac{r^2}{G\rho(r)}\frac{dp(r)}{dr}\right) + \rho(r) = 0 \tag{4}$$

Now the polytropic equation of states (EOS) can be utilized to find the $\frac{dp(r)}{dr}$.

$$\frac{dp(r)}{dr} = K\left(1+\frac{1}{n}\right)\rho^{1/n}\frac{d\rho}{dr} \tag{5}$$

Plugging the Eq. (5).

$$\frac{1}{4\pi r^2}\frac{d}{dr}\left(\frac{r^2}{G\rho(r)}K\left(1+\frac{1}{n}\right)\rho^{1/n}\frac{d\rho}{dr}\right) + \rho(r) = 0 \tag{6}$$

By following substitution $\rho = \rho_c\theta^n$, the equation becomes.

$$\frac{1}{4\pi r^2}\frac{d}{dr}\left(\frac{r^2}{G}K(1+n)\rho_c^{(1-n)/n}\frac{d\theta}{dr}\right) + \theta^n = 0 \tag{7}$$

Introducing the scaled mass $r/C = \xi$. The equation becomes.

$$\frac{K(1+n)\rho_c^{(1-n)/n}}{4\pi GC^2}\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{8}$$

This yields $C^2 = \frac{K(1+n)\rho_c^{(1-n)/n}}{4\pi G}$. Our equation finally takes the form of Lane-Emden Equation.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{9}$$

To solve it numerically using Mathematica, the Lane-Emden Equation was put into the following form with n=1.

$$\frac{d^2\theta}{d^2\xi} + \frac{2}{\xi}\frac{d\theta}{d\xi} + \theta^n = 0 \tag{10}$$

Using the AsymptoticDSolveValue the solution at the center was found up to the 5th order as following (lane_emden_analytical_part_a.nb).

$$\theta(\xi) = 1 - \frac{\xi^2}{6} + \frac{\xi^4}{120} \tag{11}$$

Secondly, we have to find the total mass. It is straigthforward integrating the Eq. (1) with our scaled values.

$$M = 4\pi C^3\rho_c\int_{\xi=0}^{\xi=R/C}\xi^2\theta^n(\xi)\,d\xi \tag{12}$$

We can use the Eq. (9) to express the integrand.

$$M = -4\pi C^3\rho_c\int_{\xi=0}^{\xi=R/C}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)d\xi \tag{13}$$

Introducing $\xi_n = R/C$.This integral results in the desired equation of.

$$M = 4\pi\rho_c R^3\left(-\theta'(\xi_n)/\xi_n\right) \tag{14}$$

The M and R values are already known; therefore, for the stars sharing the polytropic (EOS), M and R can be related. Using the definition of the C, ρ_c can be eliminated. Then the Eq. (14) yields:

$$M = 4R^3\pi\left(\frac{4\pi G}{(n+1)K}\right)^{\frac{n}{1-n}}C^{\frac{2n}{1-n}}\left(-\theta'(\xi_n)/\xi_n\right) \tag{15}$$

Using the relation $C = R/\xi_n$ yields the proportionality constant.

$$M = \left(-4\pi\left(\frac{4\pi G}{(n+1)K}\right)^{\frac{n}{1-n}}\theta'(\xi_n)\xi_n^{\frac{-n-1}{1-n}}\right)R^{\frac{3-n}{1-n}} \tag{16}$$

B. White Dwarf Data

The csv file containing the observational data consists of the rows that has star id, base ten logarithm of the surface gravity in CGS and the mass in solar mass. After the file was read using the csv library the radius of each star was calculated by the following formula.

$$R = \sqrt{\frac{GM}{g}} \tag{17}$$

In the provided code, the conversion to the mks units was done before the calculation of the R. The calculated R with the provided data yields the following Fig. 1.

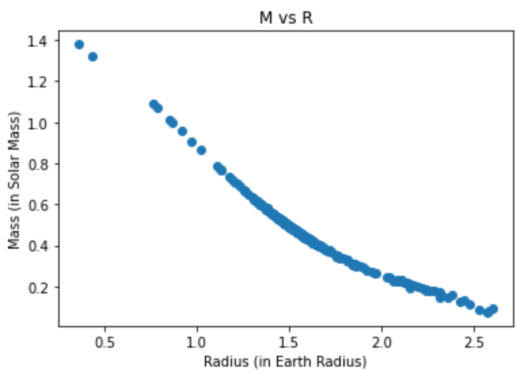

FIG. 1. Mass vs Radius graph from the observational white dwarf data, the scales are solar mass and earth radius respectively.

C. Low Mass WDs

For low mass WDs $x << 1$. This indicates that the surviving term for the solution of the EOS with electron degeneracy is the lowest degree in the series expansion. Solving the EOS for cold WDs in Mathematica with Series yields the following equation(newton_part_c.nb).

$$P = \frac{8}{5}Cx^5 - \frac{4}{7}Cx^7 + O(x^9) \tag{18}$$

Plugging $x = (\rho/D)^{\frac{1}{q}}$ and utilizing the $x << 1$ approximation.

$$P = \frac{8C}{5D^{\frac{5}{q}}}\rho^{1+\frac{5-q}{q}} \tag{19}$$

This is the form we desired. From here one can find n_* and K_* .

$$K_* = \frac{8C}{5D^{\frac{5}{q}}} \tag{20}$$

$$n_* = \frac{q}{5-q} \tag{21}$$

This indicates that with low mass WDs, we can use polytropic EOS and thus the relation found in Eq. (34) and Eq. (9). Now, it is required to choose a threshold such that below that mass the WDs are low mass. Based on the hint and the behaviour of the scatter plot, threshold was chosen to be 0.404 solar mass. Due to the power relation in Eq. (34), a log-log plot was utilized. Naming the proportionality constant as B for convenience log-log plot results in the following relation due to Eq. (34).

$$ln(M) = ln(B) + \frac{3-n_*}{1-n_*}ln(R) \tag{22}$$

The relation between ln(M) and ln(R) is linear; therefore, for the low mass data points a 1D line fit was applied.

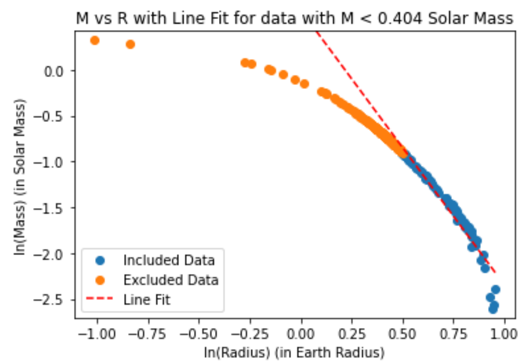

FIG. 2. Natural logarithm of mass vs natural logarithm of radius with a linear fit applied for the low mass WDs.

From the fit the slope was calculated.

$$\frac{3-n_*}{1-n_*} = -2.9995562983641317 \tag{23}$$

The slope was rounded to the closest integer and consequently $n_* = 1.5$ and $q = 3$. Using these values, the Lane-Emden equation was numerically solved. The results are as following:

$$\xi_n = 3.6537537362191217 \tag{24}$$

$$\theta(\xi_n) = 1.1893292290618252e - 17 \tag{25}$$

$$\theta'(\xi_n) = -0.20330128263855105 \tag{26}$$

Utilizing those values with the y-intercept(ln(B)), the K_* value was obtained from simply the proportionality constant

$$K^* = \frac{-4\pi G}{2.5}\left(\frac{4\pi\xi_n^5\theta'(\xi_n)}{B}\right)^{\frac{-1}{3}} \tag{27}$$

Scaled with respect to solar mass and earth radius the K_* is

$$K^* = 0.27065848274698195 \tag{28}$$

In mks,

$$K^* = 2819435.7864331105 \tag{29}$$

Finally, using the Lane-Emden solutions and n_*, ρ_c values were calculated using the total mass equation Eq. (14)

$$\rho_c = \frac{M}{4\pi R^3 \left(-\theta'(\xi_n)/\xi_n\right)} \tag{30}$$

Utilizing the Eq. (30) yields the following image.

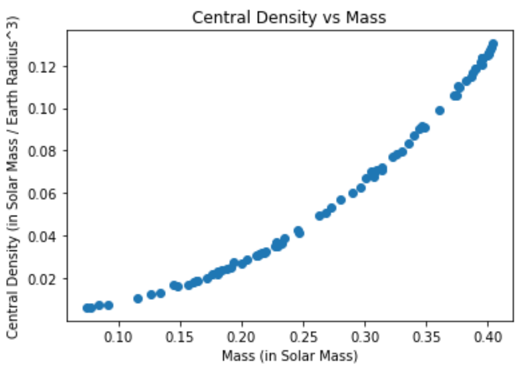

FIG. 3. Central Density vs Mass for Low Mass WDs.

E. High Mass WDs

Series expansion of EOS with electron degeneracy around ∞ in Mathematica results in the following expression.

$$P = 2Cx^4 - 2Cx^2 + (C/4)(-7 + 6log(4) + 12log(x)) + O(1/x)^2 \tag{31}$$

For $x >> 1$ the dominating term is the one with the highest power; therefore, we can approximately accept that

$$P = 2Cx^4 \tag{32}$$

Plugging the $x = (\rho/D)^{\frac{1}{q}}$

$$P = \frac{2C}{D^{\frac{4}{q}}} \rho^{1+\frac{4-q}{q}} \tag{33}$$

This is polytropic when n=3, q=3. Mass can be obtained from the Eq.(16):

$$M = \left(-4\pi \left(\frac{\pi G D^{\frac{4}{3}}}{2C}\right)^{\frac{-3}{2}} \theta'(\xi_n)\xi_n^2\right) \tag{34}$$

EINSTEIN

In this part, because of the general relativistic effects, our equations for hydrostatic equilibrium are modified. The new equations called the Tolman-Oppenheimer-Volkof(TOV) equations were utilized to calculate the structures of Neutran Stars.

A. M-R Curve

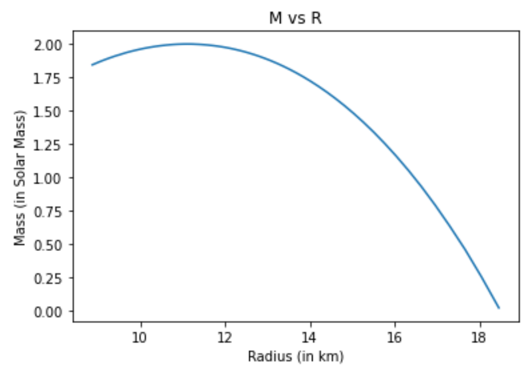

FIG. 4. Mass vs Radius for differernt values of ρ_c.

p_c values linearly spaced from $10^{(-5)}$ to the $10^{(-2)}$ was utilized to be used in the TOV equations.The resulting graph is FIG 4.

B. Baryonic Mass

For this part, the TOV solver was edited to additionally include the baryonic mass calculation as following.

$$m_p' = 4\pi \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} r^2 \rho \tag{35}$$

Fractional binding energy calculated from the given formula.

$$\Delta = \frac{M_p - M}{M} \tag{36}$$

The result is as following.

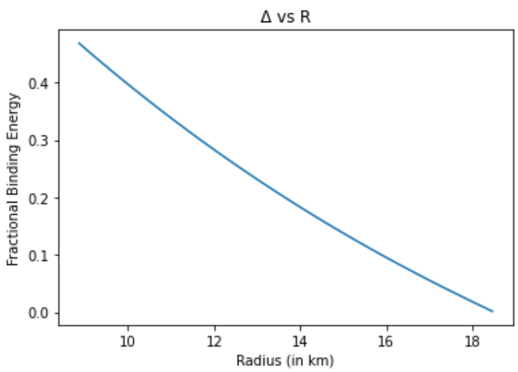

FIG. 5. Fractional binding energy vs radius for NS.

C. NS Stability

The stability of a NS depends on its reaction to an external pressure. If its mass increases with the pressure than it is stable ($\frac{dM}{d\rho_c} > 0$). In order to find the stable region of NS, a numerical differentiation with three point stencil was utilized. The result is as follows. The maximum possible mass value for a stable NS was found to be (in Solar Mass):

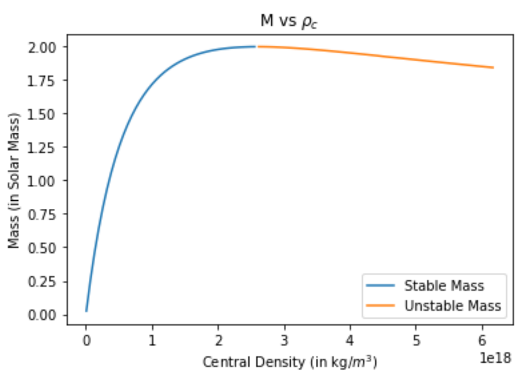

FIG. 6. The image showing the stable and unstable regions of NS

$$M_m = 1.994715369074918 \tag{37}$$

D. NS Max K Allowed

In this part our objective is to calculate the maximum K_{NS} value for the maximum mass observed for NS.To achieve it, a curve relating the mass and K_{NS} was required. It was numerically calculated by iterating over both K_{NS} and ρ_c values.At each K_{NS} value the maximum mass was found by applying first derivative. Since it was computationally costly, relatively smaller sample was measured and then a cubic spline was applied. This cubic spline was utilized in root finding to find the maximum possible K_{NS}

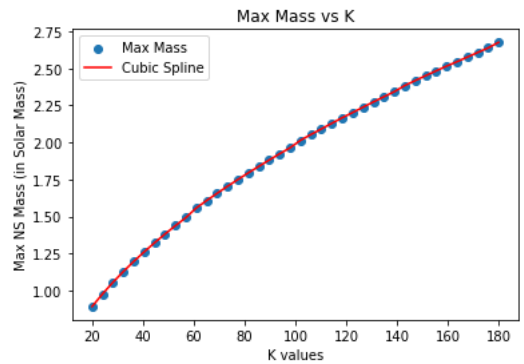

FIG. 7. The curve showing relation between K and maximum stable mass of NS

The max K_{NS} allowed was computed to be:

$$K_{NS} = 115.50967180507833 \tag{38}$$

E. Beyond the Star

Beyond the NS, the time dilation equation simplifies to

$$v' = \frac{2M}{r(r-2M)} \tag{39}$$

Solving this equation in Mathematica with DSolve yields the following equation(einstein_part_e.nb).

$$v(r) = c - 2M\left(\frac{log(r)}{2M} - \frac{log(-2M+r)}{2M}\right) \tag{40}$$

This equation simplifies to

$$v(r) = c + log(1 - 2M/r) \tag{41}$$

One can obtain the c by plugging r=R then $v(R) = v(R)$ and $c = v(R) - log(1 - 2M/R)$. The equation takes the desired form of

$$v(r) = v(R) - log(1 - 2M/R) + log(1 - 2M/r) \tag{42}$$