In[13]:=
```
(*Our matrix which we need to decompose*)
UThreeStateQKD = {{-6.347620784914656`*^-7 - 0.0973860882661547` ⅈ,
    -0.8721441186779699` - 6.97989071263283`*^-8 ⅈ, -4.6128673457032643`*^-7 +
     0.43865680004380286` ⅈ, -0.19369856564145735` - 2.4076816894369468`*^-8 ⅈ},
   {-0.8721534195528541` - 1.9078761579957582`*^-7 ⅈ, -3.449747074208906`*^-7 +
     0.09739506377346524` ⅈ, -0.19372073482014773` - 5.80156005357095`*^-7 ⅈ,
    3.298500406388738`*^-7 - 0.43863536362097916` ⅈ},
   {-8.75371528933494`*^-7 - 0.43859533121604716` ⅈ,
    0.19366914843267485` - 5.601793151392698`*^-8 ⅈ, 4.118582682278299`*^-7 -
     0.09743869406996503` ⅈ, -0.872118885951191` - 1.2799996305934425`*^-9 ⅈ},
   {0.19365824662777398` - 1.9745487718578897`*^-7 ⅈ, -5.275845184945366`*^-7 +
     0.43860584469231045` ⅈ, -0.8721000179264191` + 4.7419741025858664`*^-7 ⅈ,
    -2.2566669351806526`*^-7 + 0.09742036629032869` ⅈ}}
```

Out[13]=
$$\{\{-6.34762\times10^{-7}-0.0973861\,i,\ -0.872144-6.97989\times10^{-8}\,i,\ -4.61287\times10^{-7}+0.438657\,i,\ -0.193699-2.40768\times10^{-8}\,i\},$$
$$\{-0.872153-1.90788\times10^{-7}\,i,\ -3.44975\times10^{-7}+0.0973951\,i,\ -0.193721-5.80156\times10^{-7}\,i,\ 3.2985\times10^{-7}-0.438635\,i\},$$
$$\{-8.75372\times10^{-7}-0.438595\,i,\ 0.193669-5.60179\times10^{-8}\,i,\ 4.11858\times10^{-7}-0.0974387\,i,\ -0.872119-1.28\times10^{-9}\,i\},$$
$$\{0.193658-1.97455\times10^{-7}\,i,\ -5.27585\times10^{-7}+0.438606\,i,\ -0.8721+4.74197\times10^{-7}\,i,\ -2.25667\times10^{-7}+0.0974204\,i\}\}$$

In[14]:=
```
(*It is unitary, vectors and eigenvalues are found with high accuracy*)
Transpose[Eigenvectors[UThreeStateQKD]].{{Eigenvalues[UThreeStateQKD][[1]], 0, 0, 0},
    {0, Eigenvalues[UThreeStateQKD][[2]], 0, 0}, {0, 0,
     Eigenvalues[UThreeStateQKD][[3]], 0}, {0, 0, 0, Eigenvalues[UThreeStateQKD][[4]]}}.
  Conjugate[Eigenvectors[UThreeStateQKD]] - UThreeStateQKD
ConjugateTranspose[UThreeStateQKD].UThreeStateQKD
Max[Abs[ConjugateTranspose[UThreeStateQKD].UThreeStateQKD -
    {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}]]
```

Out[14]=
$$\{\{-1.51038\times10^{-6}-0.0000301335\,i,\ -0.0000867372-5.82632\times10^{-7}\,i,\ 1.21046\times10^{-6}-0.0000164347\,i,\ 0.000103662-8.77156\times10^{-7}\,i\},$$
$$\{-0.0000811274+4.70158\times10^{-7}\,i,\ -1.46074\times10^{-6}+0.0000212787\,i,\ 0.000120522+8.522\times10^{-7}\,i,\ 9.86732\times10^{-7}+0.0000209397\,i\},$$
$$\{9.86494\times10^{-7}-0.0000158936\,i,\ 0.000101724-6.17091\times10^{-7}\,i,\ 1.46073\times10^{-6}+0.000027779\,i,\ 0.0000898391+5.22317\times10^{-7}\,i\},$$
$$\{0.000120073+6.42077\times10^{-7}\,i,\ 1.21018\times10^{-6}+0.0000204174\,i,\ 0.0000774953-6.34803\times10^{-7}\,i,\ 1.51039\times10^{-6}-0.0000189242\,i\}\}$$

Out[15]=
$$\{\{1.00001+0.\,i,\ 5.08951\times10^{-7}+3.88111\times10^{-6}\,i,\ 0.0000819274+3.31226\times10^{-7}\,i,\ 6.22355\times10^{-7}+0.0000527724\,i\},$$
$$\{5.08951\times10^{-7}-3.88111\times10^{-6}\,i,\ 1.+0.\,i,\ 1.13533\times10^{-6}-0.0000672088\,i,\ 0.0000387657+1.2515\times10^{-7}\,i\},$$
$$\{0.0000819274-3.31226\times10^{-7}\,i,\ 1.13533\times10^{-6}+0.0000672088\,i,\ 1.+0.\,i,\ 1.53303\times10^{-7}+1.52943\times10^{-6}\,i\},$$
$$\{6.22355\times10^{-7}-0.0000527724\,i,\ 0.0000387657-1.2515\times10^{-7}\,i,\ 1.53303\times10^{-7}-1.52943\times10^{-6}\,i,\ 1.+0.\,i\}\}$$

Out[16]= 0.0000819281

In[29]:=
```
(*Do SVD on the UR and UI*)
M = 1/Sqrt[2] {{1, 0, 0, I}, {0, I, 1, 0}, {0, I, -1, 0}, {1, 0, 0, -I}}
MatrixForm[M]
Λ = {{1, 1, -1, 1}, {1, 1, 1, -1}, {1, -1, -1, -1}, {1, -1, 1, 1}}
MatrixForm[Λ]
UP = ConjugateTranspose[M].UThreeStateQKD.M
UR = (UP + Conjugate[UP]) / 2
UI = (UP - Conjugate[UP]) / (2 * I)
{a, b, c} = SingularValueDecomposition[UR]
Max[Abs[a.b.ConjugateTranspose[c] - UR]]
{d, e, f} = SingularValueDecomposition[UI]
Max[Abs[d.e.ConjugateTranspose[f] - UI]]
MatrixForm[M]
MatrixForm[M.ConjugateTranspose[M]]
```

Out[29]= $\left\{\left\{\frac{1}{\sqrt{2}}, 0, 0, \frac{i}{\sqrt{2}}\right\}, \left\{0, \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\}, \left\{0, \frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\}, \left\{\frac{1}{\sqrt{2}}, 0, 0, -\frac{i}{\sqrt{2}}\right\}\right\}$

Out[30]//MatrixForm=
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

Out[31]= {{1, 1, -1, 1}, {1, 1, 1, -1}, {1, -1, -1, -1}, {1, -1, 1, 1}}

Out[32]//MatrixForm=
$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

Out[33]= {{-0.0000205897 + 0.0000170282 i, -0.438632 - 0.872123 i,
-0.0000220835 - 0.0000257497 i, 0.0974033 + 0.193678 i},
{-0.438615 + 0.872136 i, -0.0000257598 - 0.0000221332 i, 0.0974171 - 0.193695 i,
-0.0000178694 + 0.0000199214 i}, {-0.0000166642 - 0.000020111 i, -0.0974166 - 0.193695 i,
0.0000258266 - 0.0000214971 i, -0.438615 - 0.872136 i}, {-0.0974031 + 0.193679 i,
-0.0000220172 + 0.0000252057 i, -0.438631 + 0.872122 i, 0.0000197293 + 0.0000172498 i}}

Out[34]= {{-0.0000205897 + 0. i, -0.438632 + 0. i, -0.0000220835 + 0. i, 0.0974033 + 0. i},
{-0.438615 + 0. i, -0.0000257598 + 0. i, 0.0974171 + 0. i, -0.0000178694 + 0. i},
{-0.0000166642 + 0. i, -0.0974166 + 0. i, 0.0000258266 + 0. i, -0.438615 + 0. i},
{-0.0974031 + 0. i, -0.0000220172 + 0. i, -0.438631 + 0. i, 0.0000197293 + 0. i}}

Out[35]= {{0.0000170282 + 0. i, -0.872123 + 0. i, -0.0000257497 + 0. i, 0.193678 + 0. i},
{0.872136 + 0. i, -0.0000221332 + 0. i, -0.193695 + 0. i, 0.0000199214 + 0. i},
{-0.000020111 + 0. i, -0.193695 + 0. i, -0.0000214971 + 0. i, -0.872136 + 0. i},
{0.193679 + 0. i, 0.0000252057 + 0. i, 0.872122 + 0. i, 0.0000172498 + 0. i}}

Out[36]= {{{0.712476 + 0. i, -0.305828 + 0. i, 0.0872313 + 0. i, -0.62549 + 0. i},
{0.131312 + 0. i, -0.622386 + 0. i, -0.683227 + 0. i, 0.3586 + 0. i},
{-0.0871949 + 0. i, -0.624967 + 0. i, 0.713 + 0. i, 0.305687 + 0. i},
{0.683763 + 0. i, 0.358498 + 0. i, 0.131238 + 0. i, 0.621871 + 0. i}}, {{0.449343, 0., 0., 0.},
{0., 0.44934, 0., 0.}, {0., 0., 0.449279, 0.}, {0., 0., 0., 0.449276}},
{{-0.276425 + 0. i, 0.529857 + 0. i, 0.638527 + 0. i, -0.484896 + 0. i},
{-0.676629 + 0. i, 0.43405 + 0. i, -0.23973 + 0. i, 0.544338 + 0. i},
{-0.639034 + 0. i, -0.484909 + 0. i, -0.276235 + 0. i, -0.529333 + 0. i},
{0.23958 + 0. i, 0.543797 + 0. i, -0.677131 + 0. i, -0.434027 + 0. i}}}

Out[37]= 3.88578×10^{-16}

Out[38]= {{{-0.577536 + 0. i, 0.19051 + 0. i, -0.417282 + 0. i, -0.675303 + 0. i},
{-0.429137 + 0. i, 0.67443 + 0. i, 0.563202 + 0. i, 0.20926 + 0. i},
{-0.412127 + 0. i, -0.680929 + 0. i, 0.57346 + 0. i, -0.193987 + 0. i},
{0.558968 + 0. i, 0.212571 + 0. i, 0.424056 + 0. i, -0.680107 + 0. i}},
{{0.893413, 0., 0., 0.}, {0., 0.893406, 0., 0.},
{0., 0., 0.893349, 0.}, {0., 0., 0., 0.893343}},
{{-0.297743 + 0. i, 0.704475 + 0. i, 0.641743 + 0. i, 0.0568359 + 0. i},
{0.65315 + 0. i, -0.0383536 + 0. i, 0.283028 + 0. i, 0.701298 + 0. i},
{0.638712 + 0. i, 0.0612976 + 0. i, 0.291866 + 0. i, -0.709299 + 0. i},
{0.277113 + 0. i, 0.706036 + 0. i, -0.650289 + 0. i, 0.0429669 + 0. i}}}

Out[39]= 6.52256×10^{-16}

Out[40]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

Out[41]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In[42]:=
```
(*Real and symmetric matrices, with high accuracy*)
MatrixForm[UI.ConjugateTranspose[UR]]
```

Out[42]//MatrixForm=

$$\begin{pmatrix} 0.401405 + 0.\,i & 9.02743\times10^{-6} + 0.\,i & 9.01552\times10^{-6} + 0.\,i & 0.0000326589 + 0.\,i \\ -2.03084\times10^{-6} + 0.\,i & -0.401402 + 0.\,i & -0.0000261176 + 0.\,i & 0.0000116372 + 0.\,i \\ 0.0000119495 + 0.\,i & 0.0000273009 + 0.\,i & 0.401401 + 0.\,i & -1.55384\times10^{-6} + 0.\,i \\ -0.0000326231 + 0.\,i & 9.15965\times10^{-6} + 0.\,i & 9.275\times10^{-6} + 0.\,i & -0.401405 + 0.\,i \end{pmatrix}$$

In[43]:=
```
Max[Abs[UI.ConjugateTranspose[UR] - Transpose[UI.ConjugateTranspose[UR]]]]
```

Out[43]= 0.000065282

In[44]:=
```
MatrixForm[UR.ConjugateTranspose[UI]]
```

Out[44]//MatrixForm=

$$\begin{pmatrix} 0.401405 + 0.\,i & -2.03084\times 10^{-6} + 0.\,i & 0.0000119495 + 0.\,i & -0.0000326231 + 0.\,i \\ 9.02743\times 10^{-6} + 0.\,i & -0.401402 + 0.\,i & 0.0000273009 + 0.\,i & 9.15965\times 10^{-6} + 0.\,i \\ 9.01552\times 10^{-6} + 0.\,i & -0.0000261176 + 0.\,i & 0.401401 + 0.\,i & 9.275\times 10^{-6} + 0.\,i \\ 0.0000326589 + 0.\,i & 0.0000116372 + 0.\,i & -1.55384\times 10^{-6} + 0.\,i & -0.401405 + 0.\,i \end{pmatrix}$$

In[45]:=
```
Max[Abs[UR.ConjugateTranspose[UI] - Transpose[UR.ConjugateTranspose[UI]]]]
```

Out[45]= 0.000065282

In[46]:=
```
A = {{a11, a12, a13, a14}, {a21, a22, a23, a24}, {a31, a32, a33, a34}, {a41, a42, a43, a44}}
```

Out[46]= {{a11, a12, a13, a14}, {a21, a22, a23, a24}, {a31, a32, a33, a34}, {a41, a42, a43, a44}}

In[47]:=
```
Solve[c.A == f, {a11, a12, a13, a14, a21, a22, a23, a24, a31, a32, a33, a34, a41, a42, a43, a44}]
```

Out[47]= {{a11 → -0.701405 + 0. i, a12 → -0.0388021 + 0. i, a13 → -0.711207 + 0. i,
a14 → -0.0266692 + 0. i, a21 → -0.0332851 + 0. i, a22 → 0.71084 + 0. i,
a23 → -0.0322731 + 0. i, a24 → 0.701824 + 0. i, a31 → -0.710773 + 0. i,
a32 → -0.0359911 + 0. i, a33 → 0.701628 + 0. i, a34 → 0.0350081 + 0. i,
a41 → 0.0415437 + 0. i, a42 → -0.70136 + 0. i, a43 → -0.0293673 + 0. i, a44 → 0.71099 + 0. i}}

In[ ]:=

In[48]:=
```
a11 = -0.7014045825013059` + 0.` i
a12 = -0.03880213525614083` + 0.` i
a13 = -0.7112065533058932` + 0.` i
a14 = -0.026669167225180708` + 0.` i
a21 = -0.03328509257843116` + 0.` i
a22 = 0.7108401221356002` + 0.` i
a23 = -0.032273125867894555` + 0.` i
a24 = 0.7018239584974315` + 0.` i
a31 = -0.710772700749082` + 0.` i
a32 = -0.03599109098631475` + 0.` i
a33 = 0.7016275688551669` + 0.` i
a34 = 0.03500805424269116` + 0.` i
a41 = 0.041543737531303145` + 0.` i
a42 = -0.7013596484197129` + 0.` i
a43 = -0.029367303312622058` + 0.` i
a44 = 0.7109896785021927` + 0.` i
```

Out[48]= -0.701405 + 0. i

Out[49]= -0.0388021 + 0. i

Out[50]= -0.711207 + 0. i

Out[51]= -0.0266692 + 0. i

Out[52]= -0.0332851 + 0. i

Out[53]= 0.71084 + 0. i

Out[54]= -0.0322731 + 0. i

Out[55]= 0.701824 + 0. i

Out[56]= -0.710773 + 0. i

Out[57]= -0.0359911 + 0. i

Out[58]= 0.701628 + 0. i

Out[59]= 0.0350081 + 0. i

Out[60]= 0.0415437 + 0. i

Out[61]= -0.70136 + 0. i

Out[62]= -0.0293673 + 0. i

Out[63]= 0.71099 + 0. i

In[64]:=
```
d.e.ConjugateTranspose[A].ConjugateTranspose[c]
```

Out[64]=
```
{{0.0000170282 + 0. i, -0.872123 + 0. i, -0.0000257497 + 0. i, 0.193678 + 0. i},
 {0.872136 + 0. i, -0.0000221332 + 0. i, -0.193695 + 0. i, 0.0000199214 + 0. i},
 {-0.000020111 + 0. i, -0.193695 + 0. i, -0.0000214971 + 0. i, -0.872136 + 0. i},
 {0.193679 + 0. i, 0.0000252057 + 0. i, 0.872122 + 0. i, 0.0000172498 + 0. i}}
```

In[65]:= **UI**

Out[65]= {{0.0000170282 + 0. i, -0.872123 + 0. i, -0.0000257497 + 0. i, 0.193678 + 0. i},
{0.872136 + 0. i, -0.0000221332 + 0. i, -0.193695 + 0. i, 0.0000199214 + 0. i},
{-0.000020111 + 0. i, -0.193695 + 0. i, -0.0000214971 + 0. i, -0.872136 + 0. i},
{0.193679 + 0. i, 0.0000252057 + 0. i, 0.872122 + 0. i, 0.0000172498 + 0. i}}

In[66]:= **Max[Abs[d.e.ConjugateTranspose[A].ConjugateTranspose[c] - UI]]**

Out[66]= 7.21645×10^{-16}

In[67]:= **F = {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, -1}}**

Out[67]= {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, -1}}

In[68]:= **MatrixForm[a]**

Out[68]//MatrixForm=

$$\begin{pmatrix} 0.712476+0.\,i & -0.305828+0.\,i & 0.0872313+0.\,i & -0.62549+0.\,i \\ 0.131312+0.\,i & -0.622386+0.\,i & -0.683227+0.\,i & 0.3586+0.\,i \\ -0.0871949+0.\,i & -0.624967+0.\,i & 0.713+0.\,i & 0.305687+0.\,i \\ 0.683763+0.\,i & 0.358498+0.\,i & 0.131238+0.\,i & 0.621871+0.\,i \end{pmatrix}$$

In[69]:= **MatrixForm[F.d.ConjugateTranspose[A]]**

Out[69]//MatrixForm=

$$\begin{pmatrix} 0.712477+0.\,i & -0.305831+0.\,i & 0.0872227+0.\,i & -0.625488+0.\,i \\ 0.131304+0.\,i & -0.622383+0.\,i & -0.683229+0.\,i & 0.358604+0.\,i \\ -0.0871856+0.\,i & -0.624966+0.\,i & 0.713+0.\,i & 0.305691+0.\,i \\ 0.683764+0.\,i & 0.358502+0.\,i & 0.131229+0.\,i & 0.621869+0.\,i \end{pmatrix}$$

In[70]:= **Max[Abs[F.d.ConjugateTranspose[A] - a]]**

Out[70]= 9.37172×10^{-6}

In[71]:= **a.Inverse[d.ConjugateTranspose[A]]**

Out[71]= $\{\{1.+0.\,i,\ 8.71514\times10^{-6}+0.\,i,\ 3.65198\times10^{-6}+0.\,i,\ 4.45102\times10^{-8}+0.\,i\},$
$\{8.71513\times10^{-6}+0.\,i,\ -1.+0.\,i,\ 1.47394\times10^{-6}+0.\,i,\ -3.08359\times10^{-6}+0.\,i\},$
$\{-3.65199\times10^{-6}+0.\,i,\ 1.47388\times10^{-6}+0.\,i,\ 1.+0.\,i,\ 9.6177\times10^{-6}+0.\,i\},$
$\{4.44482\times10^{-8}+0.\,i,\ 3.0836\times10^{-6}+0.\,i,\ 9.61769\times10^{-6}+0.\,i,\ -1.+0.\,i\}\}$

In[72]:= **a.b.ConjugateTranspose[c] + I * d.ConjugateTranspose[A].e.ConjugateTranspose[c] - UP**

Out[72]= $\{\{-9.71445\times10^{-17}-0.0000426721\,i,\ -1.11022\times10^{-16}-0.0000147215\,i,$
$-2.77556\times10^{-17}+3.2578\times10^{-6}\,i,\ 2.77556\times10^{-17}+1.94267\times10^{-6}\,i\},$
$\{-5.55112\times10^{-17}-0.0000151915\,i,\ -3.88578\times10^{-16}+0.0000426746\,i,$
$2.77556\times10^{-17}+1.01572\times10^{-6}\,i,\ 1.38778\times10^{-16}-3.2465\times10^{-6}\,i\},$
$\{-8.32667\times10^{-17}+3.09761\times10^{-6}\,i,\ -1.66533\times10^{-16}+1.96389\times10^{-6}\,i,$
$1.38778\times10^{-17}+0.0000420042\,i,\ 0.+0.0000148172\,i\},$
$\{0.+1.03695\times10^{-6}\,i,\ -3.05311\times10^{-16}-3.10893\times10^{-6}\,i,$
$5.55112\times10^{-17}+0.0000152872\,i,\ 4.16334\times10^{-17}-0.0000420017\,i\}\}$

In[73]:= **Max[**
Abs[a.b.ConjugateTranspose[c] + I * d.ConjugateTranspose[A].e.ConjugateTranspose[c] - UP]]

Out[73]= 0.0000426746

```
In[74]:= {b, e}

Out[74]= {{{0.449343, 0., 0., 0.}, {0., 0.44934, 0., 0.},
    {0., 0., 0.449279, 0.}, {0., 0., 0., 0.449276}}, {{0.893413, 0., 0., 0.},
    {0., 0.893406, 0., 0.}, {0., 0., 0.893349, 0.}, {0., 0., 0., 0.893343}}}

In[75]:= Max[Abs[b.a.ConjugateTranspose[c] + I * F.a.e.ConjugateTranspose[c] - UP]]

Out[75]= 0.0000554328

In[76]:= Max[Abs[(b + I * F.e).a.ConjugateTranspose[c] - UP]]

Out[76]= 0.0000483441

In[77]:= DiagonalPart = b + I * F.e

Out[77]= {{0.449343 + 0.893413 ⅈ, 0. + 0. ⅈ, 0. + 0. ⅈ, 0. + 0. ⅈ},
   {0. + 0. ⅈ, 0.44934 - 0.893406 ⅈ, 0. + 0. ⅈ, 0. + 0. ⅈ},
   {0. + 0. ⅈ, 0. + 0. ⅈ, 0.449279 + 0.893349 ⅈ, 0. + 0. ⅈ},
   {0. + 0. ⅈ, 0. + 0. ⅈ, 0. + 0. ⅈ, 0.449276 - 0.893343 ⅈ}}

In[78]:= 𝓔 = a.ConjugateTranspose[c]
```

Out[78]=

$$\{\{6.41477\times 10^{-6}+0.\,i,\ -0.976216+0.\,i,\ -1.71104\times 10^{-6}+0.\,i,\ 0.216799+0.\,i\},$$
$$\{-0.976216+0.\,i,\ -6.75575\times 10^{-6}+0.\,i,\ 0.2168+0.\,i,\ 1.75639\times 10^{-7}+0.\,i\},$$
$$\{2.97846\times 10^{-6}+0.\,i,\ -0.216799+0.\,i,\ 7.44671\times 10^{-6}+0.\,i,\ -0.976216+0.\,i\},$$
$$\{-0.2168+0.\,i,\ -1.44307\times 10^{-6}+0.\,i,\ -0.976216+0.\,i,\ -7.78769\times 10^{-6}+0.\,i\}\}$$

```
In[79]:= Max[Abs[DiagonalPart.𝓔 - UP]]

Out[79]= 0.0000483441

In[80]:= Max[Abs[UThreeStateQKD - M.UP.ConjugateTranspose[M]]]
```

Out[80]= 3.33085×10^{-16}

```
In[81]:= Max[Abs[UThreeStateQKD - M.DiagonalPart.ConjugateTranspose[M].M.𝓔.ConjugateTranspose[M]]]

Out[81]= 0.0000465227

In[82]:= (*Now we decompose this happiness*)
         Search = KroneckerProduct[{{UA11, UA12}, {UA21, UA22}}, {{UB11, UB12}, {UB21, UB22}}]

Out[82]= {{UA11 UB11, UA11 UB12, UA12 UB11, UA12 UB12}, {UA11 UB21, UA11 UB22, UA12 UB21, UA12 UB22},
   {UA21 UB11, UA21 UB12, UA22 UB11, UA22 UB12}, {UA21 UB21, UA21 UB22, UA22 UB21, UA22 UB22}}

In[ ]:=
```

In[83]:=
```
WeHave = M.ξ.ConjugateTranspose[M]
Search = KroneckerProduct[{{UA11, UA12}, {UA21, UA22}}, {{UB11, UB12}, {UB21, UB22}}]
UAMatr = {{UA11, UA12}, {UA21, UA22}}
UBMatr = {{UB11, UB12}, {UB21, UB22}}
Reverse[{WeHave[[1]][[1]] / WeHave[[1]][[2]], Search[[1]][[1]] / Search[[1]][[2]]}]
Reverse[{WeHave[[1]][[3]] / WeHave[[1]][[4]], Search[[1]][[3]] / Search[[1]][[4]]}]
Reverse[{WeHave[[1]][[1]] / WeHave[[1]][[3]], Search[[1]][[1]] / Search[[1]][[3]]}]
Reverse[{WeHave[[1]][[2]] / WeHave[[1]][[4]], Search[[1]][[2]] / Search[[1]][[4]]}]
Reverse[{WeHave[[2]][[1]] / WeHave[[2]][[2]], Search[[2]][[1]] / Search[[2]][[2]]}]
Reverse[{WeHave[[2]][[3]] / WeHave[[2]][[4]], Search[[2]][[3]] / Search[[2]][[4]]}]
Reverse[{WeHave[[2]][[1]] / WeHave[[2]][[3]], Search[[2]][[1]] / Search[[2]][[3]]}]
Reverse[{WeHave[[2]][[2]] / WeHave[[2]][[4]], Search[[2]][[2]] / Search[[2]][[4]]}]
Reverse[{WeHave[[3]][[1]] / WeHave[[3]][[2]], Search[[3]][[1]] / Search[[3]][[2]]}]
Reverse[{WeHave[[3]][[3]] / WeHave[[3]][[4]], Search[[3]][[3]] / Search[[3]][[4]]}]
Reverse[{WeHave[[4]][[1]] / WeHave[[4]][[2]], Search[[4]][[1]] / Search[[4]][[2]]}]
Reverse[{WeHave[[4]][[3]] / WeHave[[4]][[4]], Search[[4]][[3]] / Search[[4]][[4]]}]
Reverse[{WeHave[[3]][[1]] / WeHave[[1]][[4]], Search[[3]][[1]] / Search[[1]][[4]]}]
Reverse[{WeHave[[4]][[1]] / WeHave[[1]][[4]], Search[[4]][[1]] / Search[[1]][[4]]}]
Reverse[{WeHave[[4]][[2]] / WeHave[[3]][[3]], Search[[4]][[2]] / Search[[3]][[3]]}]
Reverse[{WeHave[[1]][[2]] / WeHave[[3]][[1]], Search[[1]][[2]] / Search[[3]][[1]]}]
```

Out[83]= $\{\{-6.86461\times10^{-7}-0.216799\,i,\ -1.57705\times10^{-6}+4.68503\times10^{-8}\,i,\ 1.33985\times10^{-7}+0.976216\,i,\ 7.10123\times10^{-6}-2.10939\times10^{-7}\,i\},\ \{1.57705\times10^{-6}+4.68428\times10^{-8}\,i,\ 3.45478\times10^{-7}+0.216799\,i,\ -7.10123\times10^{-6}-2.10948\times10^{-7}\,i,\ 1.40141\times10^{-6}-0.976216\,i\},\ \{-1.40141\times10^{-6}-0.976216\,i,\ -7.10123\times10^{-6}+2.10948\times10^{-7}\,i,\ 3.45478\times10^{-7}-0.216799\,i,\ -1.57705\times10^{-6}+4.68428\times10^{-8}\,i\},\ \{7.10123\times10^{-6}+2.10939\times10^{-7}\,i,\ -1.33985\times10^{-7}+0.976216\,i,\ 1.57705\times10^{-6}+4.68503\times10^{-8}\,i,\ -6.86461\times10^{-7}+0.216799\,i\}\}$

Out[84]= {{UA11 UB11, UA11 UB12, UA12 UB11, UA12 UB12}, {UA11 UB21, UA11 UB22, UA12 UB21, UA12 UB22}, {UA21 UB11, UA21 UB12, UA22 UB11, UA22 UB12}, {UA21 UB21, UA21 UB22, UA22 UB21, UA22 UB22}}

Out[85]= {{UA11, UA12}, {UA21, UA22}}

Out[86]= {{UB11, UB12}, {UB21, UB22}}

Out[87]= $\left\{\frac{UB11}{UB12},\ -4079.9+137\,350.\,i\right\}$

Out[88]= $\left\{\frac{UB11}{UB12},\ -4079.9+137\,350.\,i\right\}$

Out[89]= $\left\{\frac{UA11}{UA12},\ -0.222081+6.72705\times10^{-7}\,i\right\}$

Out[90]= $\left\{\frac{UA11}{UA12},\ -0.222081+6.72737\times10^{-7}\,i\right\}$

Out[91]= $\left\{\frac{UB21}{UB22},\ 2.16077\times10^{-7}-7.27424\times10^{-6}\,i\right\}$

Out[92]= $\left\{\frac{\text{UB21}}{\text{UB22}},\ 2.16077\times10^{-7}-7.27424\times10^{-6}\ i\right\}$

Out[93]= $\left\{\frac{\text{UA11}}{\text{UA12}},\ -0.222081+6.72725\times10^{-7}\ i\right\}$

Out[94]= $\left\{\frac{\text{UA11}}{\text{UA12}},\ -0.222081+6.72705\times10^{-7}\ i\right\}$

Out[95]= $\left\{\frac{\text{UB11}}{\text{UB12}},\ -4079.9+137\,350.\ i\right\}$

Out[96]= $\left\{\frac{\text{UB11}}{\text{UB12}},\ -4079.9+137\,350.\ i\right\}$

Out[97]= $\left\{\frac{\text{UB21}}{\text{UB22}},\ 2.16077\times10^{-7}-7.27424\times10^{-6}\ i\right\}$

Out[98]= $\left\{\frac{\text{UB21}}{\text{UB22}},\ 2.16077\times10^{-7}-7.27424\times10^{-6}\ i\right\}$

Out[99]= $\left\{\frac{\text{UA21 UB11}}{\text{UA12 UB12}},\ 4079.73-137\,350.\ i\right\}$

Out[100]= $\left\{\frac{\text{UA21 UB21}}{\text{UA12 UB12}},\ 0.998237+0.0593567\ i\right\}$

Out[101]= $\left\{\frac{\text{UA21 UB22}}{\text{UA22 UB11}},\ -4.50285+6.55747\times10^{-6}\ i\right\}$

Out[102]= $\left\{\frac{\text{UA11 UB12}}{\text{UA21 UB11}},\ -4.79894\times10^{-8}-1.61547\times10^{-6}\ i\right\}$

In[103]:=
```
UB11 = (-4079.9039963680075` + 137350.20046536557` I) * UB12
UA11 =
 UA12 * (-0.22208132892984822` + 6.727054068557072`*^-7 I) / (0.9999999999999999` + 0.` I)
UB21 = UB22 * (2.1607688690084087`*^-7 - 7.274240761984574`*^-6 I) /
   ((1.` + 4.235164736271502`*^-22 I))
UA21 = -UA12 * (4079.725654981561` - 137350.20578687568` I) /
   (4079.9039963680075` - 137350.20046536557` I)
UB22 = -UB12 * (0.9982368378013047` + 0.059356681649596815` I) /
   ((2.1606744171489688`*^-7 - 7.2742410438184935`*^-6 I))
UA12 = UA22 * (-4.502854899207825` + 6.557465169755572`*^-6 I) /
   (1.0000000001536578` + 2.74437222613777`*^-7 I)
```

Out[103]= $(-4079.9 + 137\,350.\ i)\ \mathrm{UB12}$

Out[104]= $(-0.222081 + 6.72705\times10^{-7}\ i)\ \mathrm{UA12}$

Out[105]= $(2.16077\times10^{-7} - 7.27424\times10^{-6}\ i)\ \mathrm{UB22}$

Out[106]= $(-1. + 1.29845\times10^{-6}\ i)\ \mathrm{UA12}$

Out[107]= $(4080.12 - 137\,350.\ i)\ \mathrm{UB12}$

Out[108]= $(-4.50285 + 7.79322\times10^{-6}\ i)\ \mathrm{UA22}$

In[109]:=
```
ConjugateTranspose[{{UA11, UA12}, {UA21, UA22}}].{{UA11, UA12}, {UA21, UA22}}
```

Out[109]=
```
{{(21.2757 + 0. i) UA22 Conjugate[UA22],
   (1.48158×10^-9 + 3.64749×10^-10 i) UA22 Conjugate[UA22]},
 {(1.48158×10^-9 - 3.64749×10^-10 i) UA22 Conjugate[UA22],
   (21.2757 + 0. i) UA22 Conjugate[UA22]}}
```

In[110]:=
```
UA22 = (1 / ((21.27570224393311` + 0.` I)^(1/2))) * Exp[I * ψ]
```

Out[110]= $(0.216799 + 0.\ i)\ e^{i\psi}$

In[111]:=
```
ConjugateTranspose[{{UB11, UB12}, {UB21, UB22}}].{{UB11, UB12}, {UB21, UB22}}
```

Out[111]=
```
{{(1.88817×10^10 + 0. i) UB12 Conjugate[UB12],
   (-0.0000315153 - 0.0000261427 i) UB12 Conjugate[UB12]},
 {(-0.0000315153 + 0.0000261427 i) UB12 Conjugate[UB12],
   (1.88817×10^10 + 0. i) UB12 Conjugate[UB12]}}
```

In[112]:=
```
UB12 = (1 / ((1.888172318549569`*^10 + 0.` I)^(1/2))) * Exp[I * ϕ]
```

Out[112]= $(7.27745\times10^{-6} + 0.\ i)\ e^{i\phi}$

In[113]:=
```
ϕ = 0
```

Out[113]= 0

In[114]:= **Search**

Out[114]= $\{\{(-0.00643602+0.216704\,i)\,e^{i\psi},\ (1.57775\times10^{-6}-7.50979\times10^{-12}\,i)\,e^{i\psi},$
$(0.0289834-0.975786\,i)\,e^{i\psi},\ (-7.10436\times10^{-6}+1.22957\times10^{-11}\,i)\,e^{i\psi}\},$
$\{(-1.57497\times10^{-6}-9.36443\times10^{-8}\,i)\,e^{i\psi},\ (0.00643636-0.216704\,i)\,e^{i\psi},$
$(7.09184\times10^{-6}+4.21688\times10^{-7}\,i)\,e^{i\psi},\ (-0.028985+0.975786\,i)\,e^{i\psi}\},$
$\{(-0.0289822+0.975786\,i)\,e^{i\psi},\ (7.10436\times10^{-6}-2.15204\times10^{-11}\,i)\,e^{i\psi},$
$(-0.00643705+0.216704\,i)\,e^{i\psi},\ (1.57775\times10^{-6}+0.\,i)\,e^{i\psi}\},$
$\{(-7.09184\times10^{-6}-4.21679\times10^{-7}\,i)\,e^{i\psi},\ (0.0289837-0.975786\,i)\,e^{i\psi},$
$(-1.57496\times10^{-6}-9.36518\times10^{-8}\,i)\,e^{i\psi},\ (0.0064374-0.216704\,i)\,e^{i\psi}\}\}$

In[115]:= **ψ = Log[WeHave[[1]][[1]] / (-0.006436022919153112` + 0.21670383341807445` i)] / I**

Out[115]= $3.1119-1.67095\times10^{-10}\,i$

In[116]:= **Max[Abs[Search - WeHave]]**

Out[116]= 2.20626×10^{-10}

In[117]:= **UAMatr**

Out[117]= $\{\{-0.216704+0.00643774\,i,\ 0.975786-0.0289853\,i\},$
$\{-0.975786+0.0289865\,i,\ -0.216704+0.00643671\,i\}\}$

In[118]:= **UBMatr**

Out[118]= $\{\{-0.0296913+0.999559\,i,\ 7.27745\times10^{-6}+0.\,i\},$
$\{-7.26462\times10^{-6}-4.31975\times10^{-7}\,i,\ 0.0296929-0.999559\,i\}\}$

In[119]:= **Max[Abs[Search - KroneckerProduct[UAMatr, UBMatr]]]**

Out[119]= 0.

In[120]:= **Max[Abs[Search - M.ℰ.ConjugateTranspose[M]]]**

Out[120]= 2.20626×10^{-10}

In[122]:= **Max[Abs[UThreeStateQKD - M.DiagonalPart.ConjugateTranspose[M].M.ℰ.ConjugateTranspose[M]]]**

Out[122]= 0.0000465227

In[123]:= **Max[Abs[
UThreeStateQKD - M.DiagonalPart.ConjugateTranspose[M].KroneckerProduct[UAMatr, UBMatr]]]**

Out[123]= 0.0000465227

In[138]:=
```
Λ = {{1, 1, -1, 1}, {1, 1, 1, -1}, {1, -1, -1, -1}, {1, -1, 1, 1}}
MatrixForm[Λ]
M
MatrixForm[M]
θ0 = (Inverse[Λ].{Φ0, Φ1, Φ2, Φ3})[[1]]
θ1 = (Inverse[Λ].{Φ0, Φ1, Φ2, Φ3})[[2]]
θ2 = (Inverse[Λ].{Φ0, Φ1, Φ2, Φ3})[[3]]
θ3 = (Inverse[Λ].{Φ0, Φ1, Φ2, Φ3})[[4]]
σx = {{0, 1}, {1, 0}}
σy = {{0, -I}, {I, 0}}
σz = {{1, 0}, {0, -1}}
ϕ1 = 1/Sqrt[2] * (KroneckerProduct[{{1, 0}}, {{1, 0}}] + KroneckerProduct[{{0, 1}}, {{0, 1}}])
ϕ2 = -I/Sqrt[2] * (KroneckerProduct[{{1, 0}}, {{1, 0}}] - KroneckerProduct[{{0, 1}}, {{0, 1}}])
ϕ3 = 1/Sqrt[2] * (KroneckerProduct[{{1, 0}}, {{0, 1}}] - KroneckerProduct[{{0, 1}}, {{1, 0}}])
ϕ4 = -I/Sqrt[2] * (KroneckerProduct[{{1, 0}}, {{0, 1}}] + KroneckerProduct[{{0, 1}}, {{1, 0}}])
DMatrix =
  {{Exp[I * Φ0], 0, 0, 0}, {0, Exp[I * Φ1], 0, 0}, {0, 0, Exp[I * Φ2], 0}, {0, 0, 0, Exp[I * Φ3]}}
FullSimplify[M.DMatrix.ConjugateTranspose[M] -
   Exp[I * θ0] * MatrixExp[I * (θ1 * KroneckerProduct[σx, σx] +
        θ2 * KroneckerProduct[σy, σy] + θ3 * KroneckerProduct[σz, σz])]]
```

Out[138]= {{1, 1, -1, 1}, {1, 1, 1, -1}, {1, -1, -1, -1}, {1, -1, 1, 1}}

Out[139]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

Out[140]= $\left\{\left\{\frac{1}{\sqrt{2}}, 0, 0, \frac{i}{\sqrt{2}}\right\}, \left\{0, \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\}, \left\{0, \frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\}, \left\{\frac{1}{\sqrt{2}}, 0, 0, -\frac{i}{\sqrt{2}}\right\}\right\}$

Out[141]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

Out[142]= $\frac{\Phi0}{4} + \frac{\Phi1}{4} + \frac{\Phi2}{4} + \frac{\Phi3}{4}$

Out[143]= $\frac{\Phi0}{4} + \frac{\Phi1}{4} - \frac{\Phi2}{4} - \frac{\Phi3}{4}$

Out[144]= $-\frac{\Phi0}{4} + \frac{\Phi1}{4} - \frac{\Phi2}{4} + \frac{\Phi3}{4}$

Out[145]= $\frac{\Phi 0}{4}-\frac{\Phi 1}{4}-\frac{\Phi 2}{4}+\frac{\Phi 3}{4}$

Out[146]= $\{\{0, 1\}, \{1, 0\}\}$

Out[147]= $\{\{0, -i\}, \{i, 0\}\}$

Out[148]= $\{\{1, 0\}, \{0, -1\}\}$

Out[149]= $\left\{\left\{\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right\}\right\}$

Out[150]= $\left\{\left\{-\frac{i}{\sqrt{2}}, 0, 0, \frac{i}{\sqrt{2}}\right\}\right\}$

Out[151]= $\left\{\left\{0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\}\right\}$

Out[152]= $\left\{\left\{0, -\frac{i}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, 0\right\}\right\}$

Out[153]= $\left\{\left\{e^{i\,\Phi 0}, 0, 0, 0\right\}, \left\{0, e^{i\,\Phi 1}, 0, 0\right\}, \left\{0, 0, e^{i\,\Phi 2}, 0\right\}, \left\{0, 0, 0, e^{i\,\Phi 3}\right\}\right\}$

Out[154]= $\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}$

In[155]:=
```
σx = {{0, 1}, {1, 0}}
σy = {{0, -I}, {I, 0}}
σz = {{1, 0}, {0, -1}}
CNOT1 = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}}
CNOT2 = {{1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}}
MatrixForm[CNOT1]
MatrixForm[CNOT2]
Ry[θ_] := {{Cos[θ/2], Sin[θ/2]}, {-Sin[θ/2], Cos[θ/2]}}
Rz[α_] := {{ⅇ^(ⅈ α/2), 0}, {0, ⅇ^(-ⅈ α/2)}}
Unit2 = {{1, 0}, {0, 1}}
σx = {{0, 1}, {1, 0}}
σy = {{0, -I}, {I, 0}}
σz = {{1, 0}, {0, -1}}
ϕ1 = 1/Sqrt[2] * (KroneckerProduct[{{1, 0}}, {{1, 0}}] + KroneckerProduct[{{0, 1}}, {{0, 1}}])
ϕ2 = -I/Sqrt[2] * (KroneckerProduct[{{1, 0}}, {{1, 0}}] - KroneckerProduct[{{0, 1}}, {{0, 1}}])
ϕ3 = 1/Sqrt[2] * (KroneckerProduct[{{1, 0}}, {{0, 1}}] - KroneckerProduct[{{0, 1}}, {{1, 0}}])
ϕ4 = -I/Sqrt[2] * (KroneckerProduct[{{1, 0}}, {{0, 1}}] + KroneckerProduct[{{0, 1}}, {{1, 0}}])
Λ = {{1, 1, -1, 1}, {1, 1, 1, -1}, {1, -1, -1, -1}, {1, -1, 1, 1}}
θ0 = (Inverse[Λ].{Φ0, Φ1, Φ2, Φ3})[[1]]
θ1 = (Inverse[Λ].{Φ0, Φ1, Φ2, Φ3})[[2]]
θ2 = (Inverse[Λ].{Φ0, Φ1, Φ2, Φ3})[[3]]
θ3 = (Inverse[Λ].{Φ0, Φ1, Φ2, Φ3})[[4]]
CNOT1 = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}}
CNOT2 = {{1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}}
H = 1/Sqrt[2] * {{1, 1}, {1, -1}}
CNOT2Trial = KroneckerProduct[H, H].CNOT1.KroneckerProduct[H, H]
CNOT2Trial - CNOT2
U3[θ_, ϕ_, λ_] :=
  {{Cos[θ/2], -Exp[I*λ] * Sin[θ/2]}, {Exp[I*ϕ] * Sin[θ/2], Exp[I*(ϕ+λ)] * Cos[θ/2]}}
FullSimplify[Exp[I*π/4] * KroneckerProduct[Rz[-π/2], Unit2].CNOT2.
    KroneckerProduct[Unit2, Ry[2*θ2 - π/2]].CNOT1.KroneckerProduct[
      Rz[2*θ3 - π/2], Ry[π/2 - 2*θ1]].CNOT2.KroneckerProduct[Unit2, Rz[π/2]] -
  MatrixExp[I * (θ1 * KroneckerProduct[σx, σx] + θ2 * KroneckerProduct[σy, σy] +
       θ3 * KroneckerProduct[σz, σz])]]
```

Out[155]= `{{0, 1}, {1, 0}}`

Out[156]= `{{0, -ⅈ}, {ⅈ, 0}}`

Out[157]= `{{1, 0}, {0, -1}}`

Out[158]= `{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}}`

Out[159]= `{{1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}}`

Out[160]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Out[161]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Out[164]= $\{\{1, 0\}, \{0, 1\}\}$

Out[165]= $\{\{0, 1\}, \{1, 0\}\}$

Out[166]= $\{\{0, -i\}, \{i, 0\}\}$

Out[167]= $\{\{1, 0\}, \{0, -1\}\}$

Out[168]= $\left\{\left\{\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right\}\right\}$

Out[169]= $\left\{\left\{-\frac{i}{\sqrt{2}}, 0, 0, \frac{i}{\sqrt{2}}\right\}\right\}$

Out[170]= $\left\{\left\{0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\}\right\}$

Out[171]= $\left\{\left\{0, -\frac{i}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, 0\right\}\right\}$

Out[172]= $\{\{1, 1, -1, 1\}, \{1, 1, 1, -1\}, \{1, -1, -1, -1\}, \{1, -1, 1, 1\}\}$

Out[173]= $\frac{\Phi0}{4} + \frac{\Phi1}{4} + \frac{\Phi2}{4} + \frac{\Phi3}{4}$

Out[174]= $\frac{\Phi0}{4} + \frac{\Phi1}{4} - \frac{\Phi2}{4} - \frac{\Phi3}{4}$

Out[175]= $-\frac{\Phi0}{4} + \frac{\Phi1}{4} - \frac{\Phi2}{4} + \frac{\Phi3}{4}$

Out[176]= $\frac{\Phi0}{4} - \frac{\Phi1}{4} - \frac{\Phi2}{4} + \frac{\Phi3}{4}$

Out[177]= $\{\{1, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, 0, 1\}, \{0, 0, 1, 0\}\}$

Out[178]= $\{\{1, 0, 0, 0\}, \{0, 0, 0, 1\}, \{0, 0, 1, 0\}, \{0, 1, 0, 0\}\}$

Out[179]= $\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}\right\}$

Out[180]= $\{\{1, 0, 0, 0\}, \{0, 0, 0, 1\}, \{0, 0, 1, 0\}, \{0, 1, 0, 0\}\}$

Out[181]= $\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}$

Out[183]= $\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}$

```
In[184]:= Φ0 = Log[DiagonalPart[[1]][[1]]] / I
          Φ1 = Log[DiagonalPart[[2]][[2]]] / I
          Φ2 = Log[DiagonalPart[[3]][[3]]] / I
          Φ3 = Log[DiagonalPart[[4]][[4]]] / I

Out[184]= 1.10479 - 0.0000479157 i

Out[185]= -1.10479 - 0.0000406347 i

Out[186]= 1.10482 + 0.0000378283 i

Out[187]= -1.10482 + 0.0000450026 i

In[188]:= Max[Abs[UThreeStateQKD - Exp[I * θ0] * Exp[I * π / 4] *
             KroneckerProduct[Rz[-π / 2], Unit2].CNOT2.KroneckerProduct[Unit2, Ry[2 * θ2 - π / 2]].
              CNOT1.KroneckerProduct[Rz[2 * θ3 - π / 2], Ry[π / 2 - 2 * θ1]].CNOT2.
              KroneckerProduct[Unit2, Rz[π / 2]].KroneckerProduct[UAMatr, UBMatr]]]

Out[188]= 0.0000465227

In[189]:= Max[Abs[UThreeStateQKD -
            Exp[I * θ0] * Exp[I * π / 4] * KroneckerProduct[Rz[-π / 2], Unit2].KroneckerProduct[H, H].
              CNOT1.KroneckerProduct[H, H].KroneckerProduct[Unit2, Ry[2 * θ2 - π / 2]].
              CNOT1.KroneckerProduct[Rz[2 * θ3 - π / 2], Ry[π / 2 - 2 * θ1]].
              KroneckerProduct[H, H].CNOT1.KroneckerProduct[H, H].
              KroneckerProduct[Unit2, Rz[π / 2]].KroneckerProduct[UAMatr, UBMatr]]]

Out[189]= 0.0000465227

In[190]:= Max[Abs[UThreeStateQKD - Exp[I * (θ0 + π / 4)] * KroneckerProduct[Rz[-π / 2].H, Unit2.H].
              CNOT1.KroneckerProduct[H.Unit2, H.Ry[2 * θ2 - π / 2]].CNOT1.
              KroneckerProduct[Rz[2 * θ3 - π / 2].H, Ry[π / 2 - 2 * θ1].H].CNOT1.
              KroneckerProduct[H.Unit2.UAMatr, H.Rz[π / 2].UBMatr]]]

Out[190]= 0.0000465227
```

In[191]:=
```
U11 = Rz[-π/2].H
U12 = Unit2.H
U21 = H.Unit2
U22 = H.Ry[2*θ2-π/2]
U31 = Rz[2*θ3-π/2].H
U32 = Ry[π/2-2*θ1].H
U41 = H.Unit2.UAMatr
U42 = H.Rz[π/2].UBMatr
```

Out[191]= $\left\{\left\{\frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}, \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}\right\}, \left\{\frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}, -\frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}\right\}\right\}$

Out[192]= $\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}\right\}$

Out[193]= $\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}\right\}$

Out[194]= $\{\{0.449309 + 3.2285\times10^{-6}\, i, -0.893376 + 1.62372\times10^{-6}\, i\}, \{-0.893376 + 1.62372\times10^{-6}\, i, -0.449309 - 3.2285\times10^{-6}\, i\}\}$

Out[195]= $\{\{0.499993 - 0.500007\, i, 0.499993 - 0.500007\, i\}, \{0.499993 + 0.500007\, i, -0.499993 - 0.500007\, i\}\}$

Out[196]= $\{\{1. + 2.96833\times10^{-13}\, i, 6.92801\times10^{-9} - 0.0000428453\, i\}, \{6.92801\times10^{-9} - 0.0000428453\, i, -1. - 2.96833\times10^{-13}\, i\}\}$

Out[197]= $\{\{-0.843217 + 0.0250487\, i, 0.536752 - 0.0159442\, i\}, \{0.536752 - 0.0159444\, i, 0.843217 - 0.0250471\, i\}\}$

Out[198]= $\{\{-0.514629 + 0.484937\, i, -0.484929 - 0.514622\, i\}, \{-0.514621 + 0.48493\, i, 0.484937 + 0.51463\, i\}\}$

In[199]:=
```
Max[Abs[UThreeStateQKD -
    Exp[I*(θ0+π/4)] * KroneckerProduct[U11, U12].CNOT1.KroneckerProduct[U21, U22].
      CNOT1.KroneckerProduct[U31, U32].CNOT1.KroneckerProduct[U41, U42]]]
```

Out[199]= 0.0000465227

In[200]:=
```
N0 = 1.733
```

Out[200]= 1.733

In[ ]:=

In[201]:=
```
z1 = 2*ArcTan[(N0-1)^(1/2)]
Rrot = MatrixExp[-I*z1*σy/2]
Ancilla = N[Rrot.Transpose[{{1, 0}}]]
```

Out[201]= 1.41611

Out[202]= $\{\{0.759628 + 0.\, i, -0.650358 + 0.\, i\}, \{0.650358 + 0.\, i, 0.759628 + 0.\, i\}\}$

Out[203]= $\{\{0.759628 + 0.\, i\}, \{0.650358 + 0.\, i\}\}$

In[204]:=
```
α = ArcSin[Sec[2 * π / 3] - Tan[2 * π / 3]]
```
Out[204]= $-\text{ArcSin}\left[2-\sqrt{3}\right]$

In[205]:=
```
τ = π / 2
```
Out[205]= $\frac{\pi}{2}$

In[206]:=
```
v1Ref = {{Cos[1/4 (π - 2 * 2 * π / 3)], -ⅈ Sin[1/4 (π - 2 * 2 * π / 3)]}}
```
Out[206]= $\left\{\left\{\frac{1+\sqrt{3}}{2\sqrt{2}}, \frac{i\left(-1+\sqrt{3}\right)}{2\sqrt{2}}\right\}\right\}$

In[207]:=
```
v2Ref = {{Cos[1/4 (π + 2 * 2 * π / 3)], -ⅈ Sin[1/4 (π + 2 * 2 * π / 3)]}}
```
Out[207]= $\left\{\left\{-\frac{-1+\sqrt{3}}{2\sqrt{2}}, -\frac{i\left(1+\sqrt{3}\right)}{2\sqrt{2}}\right\}\right\}$

In[208]:=
```
Evolution = {{Cos[τ - α], -ⅈ * Sin[τ]}, {-ⅈ * Sin[τ], Cos[α + τ]}} * Sec[α]
```
Out[208]= $\left\{\left\{\frac{-2+\sqrt{3}}{\sqrt{1-\left(2-\sqrt{3}\right)^2}}, -\frac{i}{\sqrt{1-\left(2-\sqrt{3}\right)^2}}\right\}, \left\{-\frac{i}{\sqrt{1-\left(2-\sqrt{3}\right)^2}}, \frac{2-\sqrt{3}}{\sqrt{1-\left(2-\sqrt{3}\right)^2}}\right\}\right\}$

In[ ]:=

In[209]:=
```
Simplify[Evolution.Transpose[v1Ref]]
```
Out[209]= $\left\{\{0\}, \left\{\frac{i\left(-3+\sqrt{3}\right)}{2\sqrt{-3+2\sqrt{3}}}\right\}\right\}$

In[210]:=
```
Simplify[Evolution.Transpose[v2Ref]]
```
Out[210]= $\left\{\left\{\frac{-3+\sqrt{3}}{2\sqrt{-3+2\sqrt{3}}}\right\}, \{0\}\right\}$

In[ ]:=

In[ ]:=

```
(*No need to perform final rotation*)
```

In[ ]:=

In[ ]:=

In[ ]:=

In[211]:=
```
U12P = U12
```

Out[211]= $\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}\right\}$

In[212]:=
```
(*U11*)
U11OverallPhase = Arg[U11[[1]][[1]]]
U11Dephased = U11 * Exp[-I * U11OverallPhase]
θU11 = 2 * ArcTan[Abs[U11Dephased[[1]][[2]]] / Abs[U11Dephased[[1]][[1]]]]
ϕU11 = Arg[U11Dephased[[2]][[1]]]
λU11 = Arg[U11Dephased[[2]][[2]]] - ϕU11
Max[N[Abs[Exp[I * U11OverallPhase] * U3[θU11, ϕU11, λU11] - U11]]]
(*U12P*)
U12POverallPhase = Arg[U12P[[1]][[1]]]
U12PDephased = U12P * Exp[-I * U12POverallPhase]
θU12P = 2 * ArcTan[Abs[U12PDephased[[1]][[2]]] / Abs[U12PDephased[[1]][[1]]]]
ϕU12P = Arg[U12PDephased[[2]][[1]]]
λU12P = Arg[U12PDephased[[2]][[2]]] - ϕU12P
Max[N[Abs[Exp[I * U12POverallPhase] * U3[θU12P, ϕU12P, λU12P] - U12P]]]
(*U21*)
U21OverallPhase = Arg[U21[[1]][[1]]]
U21Dephased = U21 * Exp[-I * U21OverallPhase]
θU21 = 2 * ArcTan[Abs[U21Dephased[[1]][[2]]] / Abs[U21Dephased[[1]][[1]]]]
ϕU21 = Arg[U21Dephased[[2]][[1]]]
λU21 = Arg[U21Dephased[[2]][[2]]] - ϕU21
Max[N[Abs[Exp[I * U21OverallPhase] * U3[θU21, ϕU21, λU21] - U21]]]
(*U22*)
U22OverallPhase = Arg[U22[[1]][[1]]]
U22Dephased = U22 * Exp[-I * U22OverallPhase]
θU22 = 2 * ArcTan[Abs[U22Dephased[[1]][[2]]] / Abs[U22Dephased[[1]][[1]]]]
ϕU22 = Arg[U22Dephased[[2]][[1]]]
λU22 = Arg[U22Dephased[[2]][[2]]] - ϕU22
Max[N[Abs[Exp[I * U22OverallPhase] * U3[θU22, ϕU22, λU22] - U22]]]
(*U31*)
U31OverallPhase = Arg[U31[[1]][[1]]]
U31Dephased = U31 * Exp[-I * U31OverallPhase]
θU31 = 2 * ArcTan[Abs[U31Dephased[[1]][[2]]] / Abs[U31Dephased[[1]][[1]]]]
ϕU31 = Arg[U31Dephased[[2]][[1]]]
λU31 = Arg[U31Dephased[[2]][[2]]] - ϕU31
Max[N[Abs[Exp[I * U31OverallPhase] * U3[θU31, ϕU31, λU31] - U31]]]
(*U32*)
U32OverallPhase = Arg[U32[[1]][[1]]]
U32Dephased = U32 * Exp[-I * U32OverallPhase]
θU32 = 2 * ArcTan[Abs[U32Dephased[[1]][[2]]] / Abs[U32Dephased[[1]][[1]]]]
ϕU32 = Arg[U32Dephased[[2]][[1]]]
λU32 = Arg[U32Dephased[[2]][[2]]] - ϕU32
Max[N[Abs[Exp[I * U32OverallPhase] * U3[θU32, ϕU32, λU32] - U32]]]
(*U41R*)
U41OverallPhaseR = Arg[(U41.Rrot)[[1]][[1]]]
U41DephasedR = (U41.Rrot) * Exp[-I * U41OverallPhaseR]
θU41R = 2 * ArcTan[Abs[U41DephasedR[[1]][[2]]] / Abs[U41DephasedR[[1]][[1]]]]
ϕU41R = Arg[U41DephasedR[[2]][[1]]]
λU41R = Arg[U41DephasedR[[2]][[2]]] - ϕU41R
Max[N[Abs[Exp[I * U41OverallPhaseR] * U3[θU41R, ϕU41R, λU41R] - U41.Rrot]]]
```

Out[212]= $-\frac{\pi}{4}$

Out[213]= $\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{i}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right\}\right\}$

Out[214]= $\frac{\pi}{2}$

Out[215]= $\frac{\pi}{2}$

Out[216]= $-\pi$

Out[217]= 0.

Out[218]= 0

Out[219]= $\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}\right\}$

Out[220]= $\frac{\pi}{2}$

Out[221]= 0

Out[222]= π

Out[223]= 0.

Out[224]= 0

Out[225]= $\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}\right\}$

Out[226]= $\frac{\pi}{2}$

Out[227]= 0

Out[228]= π

Out[229]= 0.

Out[230]= 7.18548×10^{-6}

Out[231]= $\{\{0.449309 + 0.\, i, -0.893376 + 8.04305 \times 10^{-6}\, i\}, \{-0.893376 + 8.04305 \times 10^{-6}\, i, -0.449309 + 0.\, i\}\}$

Out[232]= 2.20961

Out[233]= 3.14158

Out[234]= 9.00299×10^{-6}

Out[235]= 0.0000160861

Out[236]= -0.785412

Out[237]= $\{\{0.707107 + 0.\, i, 0.707107 + 0.\, i\}, \{-0.0000202021 + 0.707107\, i, 0.0000202021 - 0.707107\, i\}\}$

Out[238]= 1.5708

Out[239]= 1.57082

Out[240]= -3.14159

Out[241]= 1.88622×10^{-8}

Out[242]= 2.96833×10^{-13}

Out[243]= $\{\{1.+0.\,i,\ 6.92801\times10^{-9}-0.0000428453\,i\},\ \{6.92801\times10^{-9}-0.0000428453\,i,\ -1.+0.\,i\}\}$

Out[244]= 0.0000856906

Out[245]= -1.57063

Out[246]= 4.71223

Out[247]= 0.0000856906

Out[248]= 3.11189

Out[249]= $\{\{0.291579+1.73472\times10^{-17}\,i,\ -0.956547-1.73826\times10^{-6}\,i\},$
$\{-0.956547-2.66917\times10^{-6}\,i,\ -0.291579-1.34351\times10^{-6}\,i\}\}$

Out[250]= 2.54984

Out[251]= -3.14159

Out[252]= 1.81729×10^{-6}

Out[253]= 1.41999×10^{-10}

In[254]:=
```
(*Constant input*)
{θU11, ϕU11, λU11}
```

Out[254]= $\{\frac{\pi}{2},\ \frac{\pi}{2},\ -\pi\}$

In[ ]:= **{θU12P, ϕU12P, λU12P}**

Out[ ]= $\{\frac{\pi}{2},\ 0,\ \pi\}$

In[ ]:= **{θU21, ϕU21, λU21}**

Out[ ]= $\{\frac{\pi}{2},\ 0,\ \pi\}$

In[ ]:= **{θU22, ϕU22, λU22}**

Out[ ]= $\{2.20961,\ 3.14158,\ 9.00299\times10^{-6}\}$

In[ ]:= **{θU31, ϕU31, λU31}**

Out[ ]= {1.5708, 1.57082, -3.14159}

In[ ]:= **{θU32, ϕU32, λU32}**

Out[ ]= {0.0000856906, -1.57063, 4.71223}

In[ ]:= {θU41R, ϕU41R, λU41R}

Out[ ]= $\{2.54984, -3.14159, 1.81729\times10^{-6}\}$

In[ ]:=

In[264]:= σ = 2 * π / 3

Out[264]= $\frac{2\pi}{3}$

In[265]:=
```
(*Varying input*)
(*U42*)
δ = -σ * 5.0
Initializer =
  {{Cos[1/4 (π - 2 δ)], -ⅈ Sin[1/4 (π - 2 δ)]}, {-ⅈ Sin[1/4 (π - 2 δ)], Cos[1/4 (π - 2 δ)]}}
U42POverallPhase = Arg[(U42.Initializer)[[1]][[1]]]
U42PDephased = (U42.Initializer) * Exp[-I * U42POverallPhase]
θU42P = 2 * ArcTan[Abs[U42PDephased[[1]][[2]]] / Abs[U42PDephased[[1]][[1]]]]
ϕU42P = Arg[U42PDephased[[2]][[1]]]
λU42P = Arg[U42PDephased[[2]][[2]]] - ϕU42P
Max[N[Abs[Exp[I * U42POverallPhase] * U3[θU42P, ϕU42P, λU42P] - (U42.Initializer)]]]
{θU42P, ϕU42P, λU42P}
```

Out[265]= -10.472

Out[266]= {{0.965926, 0. + 0.258819 ⅈ}, {0. + 0.258819 ⅈ, 0.965926}}

Out[267]= 2.38589

Out[268]= $\{\{0.500006 + 0.\,i, -1.57287\times10^{-6} + 0.866022\,i\},$
$\{0.866022 + 4.1216\times10^{-7}\,i, 1.14608\times10^{-6} - 0.500006\,i\}\}$

Out[269]= 2.09438

Out[270]= 4.75923×10^{-7}

Out[271]= -1.57079

Out[272]= 2.41517×10^{-11}

Out[273]= $\{2.09438, 4.75923\times10^{-7}, -1.57079\}$

In[ ]:=

```
(* δ = σ * 1.0 *)
{2.094380553889995`, 4.7592325951135496`*^-7, -1.5707945105950663`}

(* δ = σ * 0.8 *)
{1.6755015334119399`, -2.6685967690114113`*^-7, -1.570794745245003`}

(* δ = σ * 0.6 *)
{1.2566225129338329`, -9.432414952159787`*^-7, -1.5707946729543403`}

(* δ = σ * 0.4 *)
{0.8377434924559058`, -1.8484302891241022`*^-6, -1.570794210237056`}
```

```
(* δ = σ * 0.2 *)
{0.4188644719789314`, -3.965057294769772`*^-6, -1.5707924595755849`}

(* δ = σ * 0.0 *)
{0.000014633280648220358`, -3.033897643545013`, 1.4631008845848992`}

(* δ = - σ * 0.2 *)
{0.41889356898390323`, -3.1415895531166917`, 1.5707924598283103`}

(* δ = - σ * 0.4 *)
{0.837772589460878`, -3.141591669549925`, 1.5707942102925065`}

(* δ = - σ * 0.6 *)
{1.2566516099388054`, -3.1415925747064466`, 1.570794672969976`}

(* δ = - σ * 0.8 *)
{1.6755306304169124`, 3.1415920560956474`, -4.71239056193942`}

(* δ = - σ * 1.0 *)
{2.094409650894967`, 3.141591313297961`, -4.712390796615031`}

(* δ = - σ * 1.2 *)
{2.513288671372618`, 3.141590056474617`, -4.712391656395857`}

(* δ = - σ * 1.4 *)
{2.9321676918471393`, 3.1415848210484585`, -4.712396546077896`}

(* δ = - σ * 1.6 *)
{2.932138594842169`, 6.9671750459055595`*^-6, -1.570788762137289`}

(* δ = - σ * 1.8 *)
{2.513259574367646`, 1.7326751576918632`*^-6, -1.5707936508908968`}

(* δ = - σ * 2.0 *)
{2.0943805538899953`, 4.7592325944725595`*^-7, -1.5707945105950663`}

(* δ = - σ * 2.2 *)
{1.6755015334119399`, -2.668596769758392`*^-7, -1.5707947452450028`}

(* δ = - σ * 2.4 *)
{1.256622512933833`, -9.432414954048626`*^-7, -1.5707946729543403`}

(* δ = - σ * 2.6 *)
{0.8377434924559056`, -1.8484302893970656`*^-6, -1.5707942102370558`}

(* δ = - σ * 2.8 *)
{0.41886447197893195`, -3.965057294436012`*^-6, -1.5707924595755849`}

(* δ = - σ * 3.0 *)
{0.000014633280647907295`, -3.033897643542074`, 1.4631008845819602`}

(* δ = - σ * 3.2*)
{0.41889356898390384`, -3.1415895531166917`, 1.57079245982831`}

(* δ = - σ * 3.4*)
{0.837772589460877`, -3.141591669549925`, 1.5707942102925068`}

(* δ = - σ * 3.6*)
{1.256651609938805`, -3.1415925747064466`, 1.5707946729699755`}
```

```
(* δ = - σ * 3.8*)
{1.6755306304169129`, 3.1415920560956474`, -4.71239056193942`}

(* δ = - σ * 4.0 *)
{2.0944096508949666`, 3.1415913132979614`, -4.712390796615031`}



(* δ = - σ * 4.2 *)
{2.5132886713726177`, 3.1415900564746173`, -4.712391656395858`}
(* δ = - σ * 4.4 *)
{2.93216769184714`, 3.1415848210484585`, -4.712396546077897`}
(* δ = - σ * 4.6 *)
{2.932138594842169`, 6.967175046017194`*^-6, -1.5707887621372887`}
(* δ = - σ * 4.8 *)
{2.5132595743676465`, 1.7326751577502308`*^-6, -1.5707936508908966`}
(* δ = - σ * 5.0 *)
{2.0943805538899967`, 4.7592325951135475`*^-7, -1.5707945105950663`}
```

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