Three States PI discrimination

Step 1: adjust $|\psi_{1,2}\rangle$ on the opposite sides of Bloch sphere.

$$R_1 = \begin{pmatrix} \cos\frac{\theta_1}{2}; & \sin\frac{\theta_1}{2}e^{-i\phi_1}; \\ -\sin\frac{\theta_1}{2}e^{i\phi_1}; & \cos\frac{\theta_1}{2}; \end{pmatrix}; \quad R_2 = \begin{pmatrix} 1; & 0; \\ 0; & -ie^{-i\lambda-i\phi_2}; \end{pmatrix};$$

$$R_3 = \begin{pmatrix} \cos\frac{\pi-2\delta}{4}; & -i\sin\frac{\pi-2\delta}{4}; \\ -i\sin\frac{\pi-2\delta}{4}; & \cos\frac{\pi-2\delta}{4}; \end{pmatrix}$$

$$\begin{pmatrix} \cos\frac{\theta_1}{2}; & \sin\frac{\theta_1}{2}e^{-i\phi_1}; \\ -\sin\frac{\theta_1}{2}e^{i\phi_1}; & \cos\frac{\theta_1}{2}; \end{pmatrix}\begin{pmatrix} \cos\frac{\theta_1}{2} \\ e^{i\phi_1}\sin\frac{\theta_1}{2} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta_1}{2} + \sin^2\frac{\theta_1}{2} \\ \cancel{-\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}e^{i\phi_1}} + \cancel{\cos\frac{\theta_1}{2}\sin\frac{\theta_1}{2}e^{i\phi_1}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

$$\begin{pmatrix} \cos\frac{\theta_1}{2}; & \sin\frac{\theta_1}{2}e^{-i\phi_1}; \\ -\sin\frac{\theta_1}{2}e^{i\phi_1}; & \cos\frac{\theta_1}{2}; \end{pmatrix}\begin{pmatrix} \cos\frac{\theta_2}{2} \\ \sin\frac{\theta_2}{2}e^{i\phi_2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{i\phi_2-i\phi_1} \\ \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{i\phi_2} - \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{i\phi_1} \end{pmatrix}$$

$$\cos\delta = \sqrt{\left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos(\phi_2-\phi_1)\right)^2 + \left(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin(\phi_2-\phi_1)\right)^2}$$

$$= \sqrt{\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + \frac{1}{2}\sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1) + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\cos^2(\phi_2-\phi_1) + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\sin^2(\phi_2-\phi_1)}$$

$$= \sqrt{\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} + \frac{1}{2}\sin\theta_1\sin\theta_2 * \cos(\phi_2-\phi_1)}$$

$$\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} = \cos^2\frac{\theta_1}{2} - \cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} =$$

$$= \cos^2\frac{\theta_1}{2} - \cos\theta_1\sin^2\frac{\theta_2}{2} = \frac{1}{2}\left(2\cos^2\frac{\theta_1}{2} - 1 + 1\right) + \frac{\cos\theta_1}{2}\left(1 - 2\sin^2\frac{\theta_2}{2} - 1\right) =$$

$$= \frac{1}{2}\left(\cancel{\cos\theta_1} + 1\right) + \frac{\cos\theta_1}{2}\cos\theta_2 - \frac{\cancel{\cos\theta_1}}{\cancel{2}} = \frac{1 + \cos\theta_1\cos\theta_2}{2}$$

$$\boxed{\cos\sigma = \sqrt{\frac{1+\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1)}{2}}} \Rightarrow \boxed{|\psi_2\rangle = \begin{pmatrix}\cos\sigma\\ e^{i\varphi_2+i\lambda}\sin\sigma\end{pmatrix}}$$

$$R_1\begin{pmatrix}\cos\frac{\theta_2}{2}\\ e^{i\varphi_2}\sin\frac{\theta_2}{2}\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}+\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{i\varphi_2-i\varphi_1}\\ e^{i\varphi_2}\left(\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}-\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{i\varphi_1-i\varphi_2}\right)\end{pmatrix}$$

$$\boxed{\lambda = \arctan\left(\frac{\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin(\varphi_2-\phi_1)}{\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}-\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos(\phi_2-\phi_1)}\right) - \arctan\left(\frac{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin(\varphi_2-\phi_1)}{\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}+\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos(\varphi_2-\phi_1)}\right)}$$

$$\begin{pmatrix}\cos\left(\frac{\pi-2\sigma}{4}\right); & -i\sin\left(\frac{\pi-2\sigma}{4}\right)\\ -i\sin\left(\frac{\pi-2\sigma}{4}\right); & \cos\left(\frac{\pi-2\sigma}{4}\right)\end{pmatrix}\begin{pmatrix}\cos\sigma\\ -i\sin\sigma\end{pmatrix} = \begin{pmatrix}\cos\frac{\pi-2\sigma}{4}\cos\sigma-\sin\frac{\pi-2\sigma}{4}\sin\sigma\\ -i\sin\frac{\pi-2\sigma}{4}\cos\sigma - i\cos\frac{\pi-2\sigma}{4}\sin\sigma\end{pmatrix}$$

$$= \boxed{\begin{pmatrix}\cos\frac{\pi+2\sigma}{4}\\ -i\sin\frac{\pi+2\sigma}{4}\end{pmatrix}}$$

$$\begin{pmatrix}\cos\frac{\pi-2\sigma}{4}; & -i\sin\frac{\pi-2\sigma}{4}\\ -i\sin\frac{\pi-2\sigma}{4}; & \cos\frac{\pi-2\sigma}{4}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \boxed{\begin{pmatrix}\cos\frac{\pi-2\sigma}{4}\\ -i\sin\frac{\pi-2\sigma}{4}\end{pmatrix}}$$

$\Downarrow$

Third state?

$$\begin{pmatrix} \cos\frac{\pi-2\sigma}{4}; & -i\sin\frac{\pi-2\sigma}{4}; \\ -i\sin\frac{\pi-2\sigma}{4}; & \cos\frac{\pi-2\sigma}{4}; \end{pmatrix}\begin{pmatrix} \cos\frac{\theta_1}{2}\cos\frac{\theta_3}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_3}{2}e^{i\phi_3-i\phi_1} \\ -ie^{-i\lambda-i\phi_2}\Big(e^{i\phi_3}\Big(\cos\frac{\theta_1}{2}\sin\frac{\theta_3}{2} - \sin\frac{\theta_1}{2}\cos\frac{\theta_3}{2}e^{i\phi_1} \end{pmatrix}$$

$$= \begin{pmatrix} \beta \\ \gamma \end{pmatrix}; \quad \beta = \cos\frac{\pi-2\sigma}{4}\left(\cos\frac{\theta_1}{2}\cos\frac{\theta_3}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_3}{2}e^{i\phi_3-i\phi_1}\right) -$$

$$-\sin\frac{\pi-2\sigma}{4}\left(e^{-i\lambda-i\phi_2}\right) * \Big(e^{i\phi_3} * \Big(\cos\frac{\theta_1}{2}\sin\frac{\theta_3}{2} - \sin\frac{\theta_1}{2}\cos\frac{\theta_3}{2}e^{i\phi_1-i\phi_3}$$

$$= \sin\frac{\theta_1}{2}\sin\frac{\theta_3}{2}\cos\frac{\pi-2\sigma}{4}e^{i\phi_3-i\phi_1} + \cos\frac{\theta_1}{2}\cos\frac{\theta_3}{2}\cos\frac{\pi-2\sigma}{4} -$$

$$-\cos\frac{\theta_1}{2}\sin\frac{\theta_3}{2}\sin\frac{\pi-2\sigma}{4}e^{-i\lambda-i\phi_2+i\phi_3} +$$

$$+\sin\frac{\theta_1}{2}\cos\frac{\theta_3}{2}\sin\frac{\pi-2\sigma}{4}e^{-i\lambda-i\phi_2+\cancel{i\phi_3}+i\phi_1-\cancel{i\phi_3}} =$$

$$= \boxed{\begin{aligned}&\cos\frac{\theta_1}{2}\cos\frac{\theta_3}{2}\cos\frac{\pi-2\sigma}{4}\left(1+\tan\frac{\theta_1}{2}\tan\frac{\pi-2\sigma}{4}e^{i\phi_1-i\phi_2-i\lambda}\right) + \\ &+\sin\frac{\theta_1}{2}\sin\frac{\theta_3}{2}\cos\frac{\pi-2\sigma}{4}e^{i\phi_3-i\phi_1}\left(1-\cot\frac{\theta_1}{2}\cot\frac{\pi-2\sigma}{4}e^{i\phi_1-i\phi_2-i\lambda}\right)\end{aligned}}$$

$$\gamma = -i\cos\frac{\theta_1}{2}\cos\frac{\theta_3}{2}\sin\frac{\pi-2\sigma}{4} - i\sin\frac{\theta_1}{2}\sin\frac{\theta_3}{2}\sin\frac{\pi-2\sigma}{4}e^{i\phi_3-i\phi_1} -$$

$$-ie^{-i\lambda-i\phi_2+i\phi_3}\cos\frac{\theta_1}{2}\sin\frac{\theta_3}{2}\cos\frac{\pi-2\sigma}{4} +$$

$$+ie^{-i\lambda-i\phi_2+\cancel{i\phi_3}+i\phi_1-\cancel{i\phi_3}}\sin\frac{\theta_1}{2}\cos\frac{\theta_3}{2}\cos\frac{\pi-2\sigma}{4} =$$

$$= \boxed{\begin{aligned}&i\cos\frac{\theta_1}{2}\cos\frac{\theta_3}{2}\sin\frac{\pi-2\sigma}{4}\left(\tan\frac{\theta_1}{2}\cot\frac{\pi-2\sigma}{4}e^{i\phi_1-i\phi_2-i\lambda}-1\right) - \\ &-i\sin\frac{\theta_1}{2}\sin\frac{\theta_3}{2}\sin\frac{\pi-2\sigma}{4}e^{i\phi_3-i\phi_1}\left(1+\cot\frac{\theta_1}{2}\cot\frac{\pi-2\sigma}{4}e^{i\phi_1-i\phi_2-i\lambda}\right)\end{aligned}}$$

$$\Downarrow$$

$$|\psi_1\rangle \to \begin{pmatrix}\cos\frac{\pi-2\sigma}{4} \\ -i\sin\frac{\pi-2\sigma}{4}\end{pmatrix}; \quad |\psi_2\rangle \to \begin{pmatrix}\cos\frac{\pi+2\sigma}{4} \\ -i\sin\frac{\pi+2\sigma}{4}\end{pmatrix}$$

$$|\psi_3\rangle \to \begin{pmatrix}\cos(\mu/2) \\ e^{i\nu}\sin(\mu/2)\end{pmatrix}$$

Hamiltonian evolution:

$$e^{iH^+t}e^{-iHt}=\frac{1}{\cos^2\alpha}\begin{pmatrix}\cos(\omega t-\alpha); & i\sin(\omega t); \\ i\sin(\omega t); & \cos(\omega t+\alpha);\end{pmatrix}\begin{pmatrix}\cos(\omega t-\alpha); & -i\sin(\omega t) \\ -i\sin(\omega t); & \cos(\omega t+\alpha)\end{pmatrix}=$$

$$=\frac{1}{\cos^2\alpha}\begin{pmatrix}\cos^2(\omega t-\alpha)+\sin^2(\omega t); & -i\sin(\omega t)\cos(\omega t-\alpha)+i\sin(\omega t)\cos(\omega t+\alpha); \\ i\sin(\omega t)\cos(\omega t-\alpha)-i\sin(\omega t)\cos(\omega t+\alpha); & \cos^2(\omega t+\alpha)+\sin^2(\omega t);\end{pmatrix}$$

$$-i\sin(\omega t)\cos(\omega t)\cos(\alpha)-i\sin(\omega t)\sin(\omega t)\sin(\alpha)+i\sin(\omega t)\cos(\omega t)\cos\alpha-$$
$$-i\sin(\omega t)\sin(\omega t)\sin\alpha=-2i\sin^2(\omega t)\sin\alpha$$

$$\boxed{e^{iH^+t}e^{-iHt}=\frac{1}{\cos^2\alpha}\begin{pmatrix}\cos^2(\omega t-\alpha)+\sin^2(\omega t); & -2i\sin^2(\omega t)\sin\alpha; \\ 2i\sin^2(\omega t)\sin\alpha; & \cos^2(\omega t+\alpha)+\sin^2(\omega t);\end{pmatrix}}$$

First two states:

$$\frac{1}{\cos\alpha}\begin{pmatrix}\cos(\omega\tau-\alpha); & -i\sin(\omega\tau); \\ -i\sin(\omega\tau); & \cos(\omega\tau+\alpha);\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\pi-2\delta}{4}\right) \\ -i\sin\left(\frac{\pi-2\delta}{4}\right)\end{pmatrix}=$$

$$=\frac{1}{\cos\alpha}\begin{pmatrix}\cos(\omega\tau-\alpha)\cos\left(\frac{\pi-2\delta}{4}\right)-\sin(\omega\tau)\sin\left(\frac{\pi-2\delta}{4}\right) \\ -i\left(\sin(\omega\tau)\cos\left(\frac{\pi-2\delta}{4}\right)+\cos(\omega\tau+\alpha)\sin\left(\frac{\pi-2\delta}{4}\right)\right)\end{pmatrix}$$

$$N^2=\left(\cos(\omega\tau-\alpha)\cos\left(\frac{\pi-2\delta}{4}\right)-\sin(\omega\tau)\sin\left(\frac{\pi-2\delta}{4}\right)\right)^2+$$
$$+\left(\sin(\omega\tau)\cos\left(\frac{\pi-2\delta}{4}\right)+\cos(\omega\tau+\alpha)\sin\left(\frac{\pi-2\delta}{4}\right)\right)^2=$$

$$=\cos^2(\omega\tau-\alpha)\cos^2\left(\frac{\pi-2\delta}{4}\right)-2\cos(\omega\tau-\alpha)\sin(\omega\tau)\sin\left(\frac{\pi-2\delta}{4}\right)\cos\left(\frac{\pi-2\delta}{4}\right)+$$
$$+\sin^2(\omega\tau)\sin^2\left(\frac{\pi-2\delta}{4}\right)+\sin^2(\omega\tau)\cos^2\left(\frac{\pi-2\delta}{4}\right)+$$
$$+2\sin(\omega\tau)\cos(\omega\tau+\alpha)\sin\left(\frac{\pi-2\delta}{4}\right)\cos\left(\frac{\pi-2\delta}{4}\right)+$$
$$+\cos^2(\omega\tau+\alpha)\sin^2\left(\frac{\pi-2\delta}{4}\right)=$$

$$= \sin^2(\omega\tau) + \cos(\sigma) * \Big(\cancel{\cos(\omega\tau)\cos(\alpha)} - \sin(\omega\tau)\sin(\alpha) - \cancel{\cos(\omega\tau)\cos(\alpha)} - \sin(\omega\tau)\sin\alpha \Big) +$$

$$+ \cos^2(\omega\tau)\cos^2(\alpha)\cos^2\left(\frac{\pi-2\sigma}{4}\right) + 2\sin(\omega\tau)\cos(\omega\tau)\sin\alpha\cos\alpha\cos^2\left(\frac{\pi-2\sigma}{4}\right)$$

$$+ \sin^2(\omega\tau)\sin^2\alpha\cos^2\left(\frac{\pi-2\sigma}{4}\right) + \cos^2(\omega\tau)\cos^2\alpha\sin^2\left(\frac{\pi-2\sigma}{4}\right) -$$

$$- 2\sin\omega\tau\cos\omega\tau\sin\alpha\cos\alpha\sin^2\frac{\pi-2\sigma}{4} + \sin^2\omega\tau\sin^2\alpha\sin^2\frac{\pi-2\sigma}{4} =$$

$$= \sin^2(\omega\tau) - 2\cos(\sigma)\sin^2(\omega\tau)\sin(\alpha) + \cos^2(\omega\tau)\cos^2\alpha + \sin^2\omega\tau\sin^2\alpha +$$

$$+ 2\sin(\omega\tau)\cos(\omega\tau)\sin\alpha\cos\alpha\sin\sigma =$$

$$= \boxed{1 - \cos(2\omega\tau)\sin^2\alpha + 2\sin(\omega\tau)\sin(\alpha)\Big(\cos(\omega\tau)\cos(\alpha)\sin(\sigma) - \sin(\omega\tau)\cos(\sigma)\Big)}$$

$$\boxed{\cos(\delta/2) = \frac{\cos(\omega\tau-\alpha)\cos\left(\frac{\pi-2\sigma}{4}\right) - \sin(\omega\tau)\sin\left(\frac{\pi-2\sigma}{4}\right)}{\sqrt{1 - \cos(2\omega\tau)\sin^2\alpha + 2\sin(\omega\tau)\sin(\alpha)\Big(\cos(\omega\tau)\cos(\alpha)\sin(\sigma) - \sin(\omega\tau)\cos(\sigma)\Big)}}}$$

$$|\psi_1\rangle \longrightarrow \begin{pmatrix} \cos(\delta/2) \\ -i\sin(\delta/2) \end{pmatrix}; \quad |\psi_2\rangle \longrightarrow \begin{pmatrix} \sin(\delta/2) \\ i\cos(\delta/2) \end{pmatrix};$$

$$R_4 = \begin{pmatrix} \cos(\delta/2); & i\sin(\delta/2); \\ i\sin(\delta/2); & \cos(\delta/2); \end{pmatrix}$$

$$\begin{pmatrix} \cos(\delta/2); & i\sin(\delta/2); \\ i\sin(\delta/2); & \cos(\delta/2); \end{pmatrix} \begin{pmatrix} \cos(\delta/2) \\ -i\sin(\delta/2) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

$$\begin{pmatrix} \cos(\delta/2); & i\sin(\delta/2); \\ i\sin(\delta/2); & \cos(\delta/2); \end{pmatrix} \begin{pmatrix} \sin(\delta/2) \\ i\cos(\delta/2) \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i\begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

Third state after the first two are on North/South.

$$\begin{pmatrix} \cos(\omega\tau-\alpha) & -i\sin(\omega\tau) \\ -i\sin(\omega\tau) & \cos(\omega\tau+\alpha) \end{pmatrix}\begin{pmatrix} \cos\mu/2 \\ e^{i\nu}\sin(\mu/2) \end{pmatrix} =$$

$$= \begin{pmatrix} \cos(\omega\tau-\alpha)\cos(\mu/2) - i e^{i\nu}\sin(\omega\tau)\sin(\mu/2) \\ e^{i\nu}\cos(\omega\tau+\alpha)\sin(\mu/2) - i\sin(\omega\tau)\cos(\mu/2) \end{pmatrix} \longrightarrow$$

$$\boxed{R_4} = \begin{pmatrix} \cos(\delta/2) & i\sin(\delta/2) \\ i\sin(\delta/2) & \cos(\delta/2) \end{pmatrix}$$

$$\boxed{K} = \cos(\delta/2)\left(\cos(\omega\tau-\alpha)\cos(\mu/2) - i e^{i\nu}\sin(\omega\tau)\sin(\mu/2)\right) + $$
$$+ i\sin(\delta/2)\left(e^{i\nu}\cos(\omega\tau+\alpha)\sin(\mu/2) - i\sin(\omega\tau)\cos(\mu/2)\right) =$$

$$= \boxed{\cos(\mu/2)\left(\cos(\omega\tau-\alpha)\cos(\delta/2) + \sin(\omega\tau)\sin(\delta/2)\right) + i e^{i\nu}\sin(\mu/2)\left(\cos(\omega\tau+\alpha)\sin(\delta/2) - \sin(\omega\tau)\cos(\delta/2)\right)}$$

$$\boxed{S} = i\sin(\delta/2)\left(\cos(\omega\tau-\alpha)\cos(\mu/2) - i e^{i\nu}\sin(\omega\tau)\sin(\mu/2)\right) +$$
$$+ \cos(\delta/2)\left(e^{i\nu}\cos(\omega\tau+\alpha)\sin(\mu/2) - i\sin(\omega\tau)\cos(\mu/2)\right) =$$

$$= \boxed{i\cos(\mu/2)\left(\cos(\omega\tau-\alpha)\sin(\delta/2) - \sin(\omega\tau)\cos(\delta/2)\right) + e^{i\nu}\sin(\mu/2)\left(\cos(\omega\tau+\alpha)\cos(\delta/2) + \sin(\omega\tau)\sin(\delta/2)\right)}$$

$$|\psi_3\rangle \longrightarrow \begin{pmatrix} \cos\xi/2 \\ e^{i\chi}\sin\xi/2 \end{pmatrix}; \quad R_5 = \begin{pmatrix} 1 & 0 \\ 0 & ie^{-i\chi} \end{pmatrix}; \quad \Longrightarrow \quad \boxed{|\psi_3\rangle \longrightarrow \begin{pmatrix} \cos(\xi/2) \\ i\sin(\xi/2) \end{pmatrix}}$$

$$R_6 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}; \quad |4_1\rangle \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}}$$

$$|4_2\rangle \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}i\\ 1\end{pmatrix} = \boxed{\frac{i}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}}$$

$$|4_3\rangle \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}\cos \xi/2\\ i\sin \xi/2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos(\xi/2) - \sin(\xi/2)\\ i\left(\cos(\xi/2) + \sin(\xi/2)\right)\end{pmatrix}$$

$$\cos(\xi/2) - \sin(\xi/2) = \cos(\xi/2) - \cos\left(\frac{\pi}{2} - \frac{\xi}{2}\right) = -2\sin\left(\frac{\cancel{\xi/2} + \pi/2 - \cancel{\xi/2}}{2}\right) *$$

$$* \sin\left(\frac{\xi/2 - \pi/2 + \xi/2}{2}\right) = -\sqrt{2}\,\sin\left(\frac{\xi - \pi/2}{2}\right)$$

$$\cos(\xi/2) + \sin(\xi/2) = 2\cos\left(\frac{\cancel{\xi/2} + \pi/2 - \cancel{\xi/2}}{2}\right)\cos\left(\frac{\xi/2 - \pi/2 + \xi/2}{2}\right) =$$

$$= \sqrt{2}\,\cos\left(\frac{\xi - \pi/2}{2}\right); \quad \xi = \rho - \pi/2;$$

$$|4_3\rangle \longrightarrow \begin{pmatrix}-\sin\left(\frac{\xi - \pi/2}{2}\right)\\ i\cos\left(\frac{\xi - \pi/2}{2}\right)\end{pmatrix} = \begin{pmatrix}-\sin\left(\frac{\rho - \pi}{2}\right)\\ i\cos\left(\frac{\rho - \pi}{2}\right)\end{pmatrix} = \begin{pmatrix}\sin\left(\frac{\pi}{2} - \frac{\rho}{2}\right)\\ i\cos\left(\frac{\pi}{2} - \frac{\rho}{2}\right)\end{pmatrix} =$$

$$= \boxed{\begin{pmatrix}\cos(\rho/2)\\ i\sin(\rho/2)\end{pmatrix}}$$

III

CPT projections

$$C = \frac{1}{\cos\alpha}\begin{pmatrix}i\sin\alpha; & 1;\\ 1; & -i\sin\alpha;\end{pmatrix}; \quad P = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}; \quad CP = \frac{1}{\cos\alpha}\begin{pmatrix}1; & i\sin\alpha;\\ -i\sin\alpha; & 1;\end{pmatrix};$$

$$CPT\begin{pmatrix}\cos\frac{\pi - 2b}{4}\\ -i\sin\frac{\pi - 2b}{4}\end{pmatrix} = \frac{1}{\cos\alpha}\begin{pmatrix}1; & i\sin\alpha;\\ -i\sin\alpha; & 1;\end{pmatrix}\begin{pmatrix}\cos\frac{\pi - 2b}{4}\\ i\sin\frac{\pi - 2b}{4}\end{pmatrix} =$$

$$= \frac{1}{\cos\alpha}\left(\cos\frac{\pi-2\sigma}{4} - \sin\alpha \sin\frac{\pi-2\sigma}{4};\ -i\sin\alpha\cos\frac{\pi-2\sigma}{4} + i\sin\frac{\pi-2\sigma}{4}\right)$$

CPT on $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$

$$\langle\psi_1| = \frac{1}{\sqrt{2}\cos\alpha}\begin{pmatrix} 1; & i\sin\alpha; \\ -i\sin\alpha; & 1; \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}\cos\alpha}\begin{pmatrix} 1+\sin\alpha \\ -i(1+\sin\alpha) \end{pmatrix} =$$

$$= \boxed{\frac{1+\sin\alpha}{\sqrt{2}\cos\alpha}\begin{pmatrix} 1 \\ -i \end{pmatrix}^T}$$

$$\langle\psi_2| = \frac{1}{\sqrt{2}\cos\alpha}\begin{pmatrix} 1; & i\sin\alpha; \\ -i\sin\alpha; & 1; \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}\cos\alpha}\begin{pmatrix} 1-\sin\alpha \\ i(1-\sin\alpha) \end{pmatrix} =$$

$$= \frac{1}{\sqrt{2}\cos\alpha}\begin{pmatrix} 1 & i\sin\alpha \\ -i\sin\alpha & 1 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \boxed{\frac{1-\sin\alpha}{\sqrt{2}\cos\alpha}\begin{pmatrix} 1 \\ i \end{pmatrix}^T}$$

$$\langle\psi_3| = \frac{1}{\cos\alpha}\begin{pmatrix} 1; & i\sin\alpha \\ -i\sin\alpha; & 1; \end{pmatrix}\begin{pmatrix} \cos\rho/2 \\ -i\sin\rho/2 \end{pmatrix} =$$

$$= \boxed{\frac{1}{\cos\alpha}\begin{pmatrix} \cos(\rho/2) + \sin\alpha\sin(\rho/2) \\ -i\sin(\rho/2) - i\sin\alpha\cos(\rho/2) \end{pmatrix}^T}$$

$$\langle\psi_1|\psi_3\rangle = \boxed{\frac{(1+\sin\alpha)(\cos(\rho/2) + \sin(\rho/2))}{\sqrt{2}\cos\alpha}}$$

$$\langle\psi_2|\psi_3\rangle = \boxed{\frac{(1-\sin\alpha)(\cos(\rho/2) - \sin(\rho/2))}{\sqrt{2}\cos\alpha}}$$

$$\langle\psi_1|\psi_1\rangle = \frac{1+\sin\alpha}{\cancel{2}\cos\alpha} * \cancel{2} = \boxed{\frac{1+\sin\alpha}{\cos\alpha}}$$

$$\langle\psi_2|\psi_2\rangle = \frac{1-\sin\alpha}{\cancel{2}\cos\alpha} * \cancel{2} = \boxed{\frac{1-\sin\alpha}{\cos\alpha}}$$

$$\langle \psi_3 | \psi_3 \rangle = \frac{1}{\cos\alpha}\Big[\cos(\rho/2)\big(\cos(\rho/2) + \sin\alpha \sin(\rho/2)\big) + \sin(\rho/2)\big(\sin(\rho/2) + \sin\alpha \cos(\rho/2)\big)\Big] = \boxed{\frac{1}{\cos\alpha}\left[1 + \sin\alpha \sin\rho\right]}$$

$$\cos^2\theta_{13} = \frac{|\langle \psi_1 | \psi_3 \rangle|^2}{\langle \psi_1 | \psi_1 \rangle \langle \psi_3 | \psi_3 \rangle} = \frac{\dfrac{(1+\sin\alpha)^2 (\cos\rho/2 + \sin\rho/2)^2}{2\cancel{\cos^2\alpha}}}{\dfrac{\cancel{1+\sin\alpha}}{\cancel{\cos\alpha}} * \dfrac{1}{\cancel{\cos\alpha}}(1+\sin\alpha\sin\rho)} =$$

$$= \boxed{\frac{(1+\sin\alpha)(1+\sin\rho)}{2(1+\sin\alpha\sin\rho)}}$$

$$\cos^2\theta_{23} = \frac{\dfrac{(1-\sin\alpha)^2(1-\sin\rho)}{2\cancel{\cos^2\alpha}}}{\dfrac{\cancel{1-\sin\alpha}}{\cancel{\cos\alpha}} * \dfrac{1}{\cancel{\cos\alpha}}(1+\sin\alpha\sin\rho)} = \boxed{\frac{(1-\sin\alpha)(1-\sin\rho)}{2(1+\sin\alpha\sin\rho)}}$$

Projection operators

$$Pr1 = \frac{\dfrac{\cancel{1+\sin\alpha}}{\sqrt{2}\cancel{\cos\alpha}\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}(1\ -i)}{\dfrac{\cancel{1+\sin\alpha}}{\cancel{\cos\alpha}}} = \boxed{\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}}$$

$$Pr2 = \frac{\dfrac{\cancel{1-\sin\alpha}}{\sqrt{2}\cancel{\cos\alpha}}\dfrac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}(1\ i)}{\dfrac{\cancel{1-\sin\alpha}}{\cancel{\cos\alpha}}} = \boxed{\frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}}$$

Three States Hermitian

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}; \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}; \quad |\psi_3\rangle = \begin{pmatrix} \cos(\rho/2) \\ i\sin(\rho/2) \end{pmatrix};$$

$$e^{iH^+t}e^{-iHt} = \frac{1}{\cos^2\alpha}\begin{pmatrix} \cos^2(\omega t-\alpha)+\sin^2(\omega t); & -2i\sin^2(\omega t)\sin\alpha; \\ 2i\sin^2(\omega t)\sin\alpha; & \cos^2(\omega t+\alpha)+\sin^2(\omega t); \end{pmatrix}$$

$$(1; -i;)\begin{pmatrix} \cos^2(\omega t-\alpha)+\sin^2(\omega t); & -2i\sin^2(\omega t)\sin\alpha; \\ 2i\sin^2(\omega t)\sin\alpha; & \cos^2(\omega t+\alpha)+\sin^2(\omega t); \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} =$$

$$= (1; -i;)\begin{pmatrix} \cos^2(\omega t-\alpha)+\sin^2(\omega t) - 2\sin^2(\omega t)\sin\alpha; \\ i\left(2\sin^2(\omega t)\sin\alpha - \cos^2(\omega t+\alpha) - \sin^2(\omega t)\right); \end{pmatrix} =$$

$$= \cos^2(\omega t-\alpha) + \cancel{\sin^2(\omega t)} - \cancel{2\sin^2(\omega t)\sin\alpha} + \cancel{2\sin^2(\omega t)\sin\alpha} -$$
$$- \cos^2(\omega t+\alpha) - \cancel{\sin^2(\omega t)} = \cancel{\cos^2(\omega t)\cos^2\alpha} +$$
$$+ 2\sin\omega t\sin\alpha\cos\omega t\cos\alpha + \cancel{\sin^2\alpha\sin^2\omega t} - \left(\cancel{\cos^2(\omega t)\cos^2\alpha} -\right.$$
$$\left.- 2\sin\omega t\sin\alpha\cos\omega t\cos\alpha + \cancel{\sin^2\omega t\sin^2\alpha}\right) =$$
$$= 4\sin\omega t\cos\omega t\sin\alpha\cos\alpha = 2\sin(2\omega t)\sin\alpha\cos\alpha$$

$$\langle\psi_1|e^{iH^+t}e^{-iHt}|\psi_2\rangle = \frac{1}{\cancel{2}\cos^{\cancel{2}}\alpha}\cancel{2}\sin(2\omega t)\sin\alpha\cancel{\cos\alpha} =$$

$$= \sin(2\omega t)\frac{\sin\alpha}{\cos\alpha} = \sin(2\omega t)\tan(\alpha);$$

$$\boxed{\langle\psi_1|e^{iH^+t}e^{-iHt}|\psi_2\rangle = \sin(2\omega t)\tan(\alpha)}$$ ✓

$$(1; -i;)\begin{pmatrix} \cos^2(\omega t-\alpha)+\sin^2(\omega t); & -2i\sin^2(\omega t)\sin\alpha; \\ 2i\sin^2(\omega t)\sin\alpha; & \cos^2(\omega t-\alpha)+\sin^2(\omega t); \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} =$$

$$= (1; -i;)\begin{pmatrix} \cos^2(\omega t-\alpha)+\sin^2(\omega t)(1+2\sin\alpha) \\ i\left(\cos^2(\omega t+\alpha)+\sin^2(\omega t)(1+2\sin\alpha)\right) \end{pmatrix} =$$

$$= \cos^2(\omega t-\alpha)+\cos^2(\omega t+\alpha)+2\sin^2(\omega t)(1+2\sin\alpha) =$$

$$= 2\left(\cos^2\omega t\cos^2\alpha+\sin^2\omega t\sin^2\alpha+\sin^2(\omega t)(1+2\sin\alpha)\right) =$$

$$= 2\left(\cos^2\omega t\cos^2\alpha+\sin^2\omega t\,(1+\sin\alpha)^2\right)$$

$$\boxed{\langle\psi_1|\, e^{iH^+t}\, e^{-iHt}\,|\psi_1\rangle = \sec^2(\alpha)\left(\cos^2\omega t\cos^2\alpha+\sin^2(\omega t)(1+\sin\alpha)^2\right)}$$

$$(1; i;)\begin{pmatrix} \cos^2(\omega t-\alpha)+\sin^2(\omega t); & -2i\sin^2(\omega t)\sin\alpha; \\ 2i\sin^2(\omega t)\sin\alpha; & \cos^2(\omega t+\alpha)+\sin^2(\omega t); \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} =$$

$$= (1; i;)\begin{pmatrix} \cos^2(\omega t-\alpha)+\sin^2(\omega t)(1-2\sin\alpha) \\ i\left(-\cos^2(\omega t+\alpha)-\sin^2(\omega t)+2\sin^2(\omega t)\sin\alpha\right) \end{pmatrix} =$$

$$= \cos^2(\omega t-\alpha)+\sin^2(\omega t)(1-2\sin\alpha)+\cos^2(\omega t+\alpha)+$$

$$+\sin^2(\omega t)(1-2\sin\alpha) = 2\left(\cos^2\omega t\cos^2\alpha+\sin^2(\omega t)(1-\sin\alpha)^2\right)$$

$$\boxed{\langle\psi_2|\, e^{iH^+t}\, e^{-iHt}\,|\psi_2\rangle = \underset{\substack{\uparrow \\ \sec^2(\alpha)}}{}\left(\cos^2(\omega t)\cos^2\alpha+\sin^2(\omega t)(1-\sin\alpha)^2\right)} \checkmark$$

$$\langle\psi_1| e^{iH^+t} e^{-iHt} |\psi_1\rangle \langle\psi_2| e^{iH^+t} e^{-iHt} |\psi_2\rangle =$$

$$= \sec^4(\alpha)\left(\left(\cos^2\alpha \cos^2\omega t + \sin^2\omega t + \sin^2\alpha \sin^2\omega t\right)^2 - 4\sin^2\alpha \sin^4\omega t\right) = \sec^4\alpha\left(\left(\cos^2\omega t - \sin^2\alpha \cos^2\omega t + \sin^2\omega t + \sin^2\alpha \sin^2\omega t\right)^2 - 4\sin^2\alpha \sin^4\omega t\right) =$$

$$= \sec^4(\alpha)\left(\left(1 - \sin^2\alpha \cos(2\omega t)\right)^2 - 4\sin^2\alpha \sin^4\omega t\right) \ominus$$

$$\boxed{\begin{aligned} &\left(1 - \sin^2\alpha \cos 2\omega t\right)^2 = \left(1 - \cos(2\omega t) + \cos^2\alpha \cos(2\omega t)\right)^2 = \\ &= \left(1 - \cos^2(\omega t) + \sin^2(\omega t) + \cos^2\alpha \cos(2\omega t)\right)^2 = \\ &= \left(\cos(2\omega t)\cos^2\alpha + 2\sin^2(\omega t)\right)^2 \end{aligned}}$$

$$\ominus \frac{1}{\cos^4\alpha}\left(\cos^2(2\omega t)\cos^4\alpha + 4\sin^4\omega t + 4\cos(2\omega t)\sin^2(\omega t)\cos^2\alpha - 4\sin^4\omega t \sin^2\alpha\right) = \frac{1}{\cos^4\alpha}\left(\cos^2(2\omega t)\cos^4\alpha + 4\sin^4\omega t \cos^2\alpha + 4\cos(2\omega t)\sin^2(\omega t)\cos^2\alpha\right) =$$

$$= \frac{1}{\cos^4\alpha}\left(\cos^2(2\omega t)\cos^4(\alpha) + 4\cos^2\alpha \sin^2(\omega t)\left(\cancel{\sin^2(\omega t)} + \cos^2(\omega t) - \cancel{\sin^2(\omega t)}\right)\right) =$$

$$= \frac{1}{\cos^4(\alpha)}\left(\cos^2(2\omega t)\cos^4\alpha + \cos^2(\alpha)\sin^2(2\omega t)\right) =$$

$$= \frac{1}{\cos^4(\alpha)}\left(\left(\frac{1}{2}\cos(4\omega t) + \frac{1}{2}\right)\cos^4\alpha + \cos^2\alpha\left(\frac{1}{2} - \frac{1}{2}\cos(4\omega t)\right)\right) =$$

$$= \frac{1}{\cos^4(\alpha)}\left(\frac{1}{2}\cos(4\omega t)\cos^4(\alpha) + \frac{1}{2}\cos^4(\alpha) + \frac{1}{2}\cos^2(\alpha) - \frac{1}{2}\cos(4\omega t)\cos^2\alpha\right) = \frac{1}{2}\left(1 + \sec^2\alpha\right) + \frac{1}{2}\cos(4\omega t)\left(1 - \sec^2\alpha\right) =$$

$$= \frac{1}{2}\left(1 + \sec^2\alpha - \cos(4\omega t)\tan^2\alpha\right)$$

$$\boxed{\cos^2(\theta_{12}) = \frac{2\sin^2(2\omega t)\tan^2(\alpha)}{1+\sec^2(\alpha)-\tan^2\alpha\cos(4\omega t)}} \checkmark$$

$$(1;\ -i;)\begin{pmatrix}\cos^2(\omega t-\alpha)+\sin^2(\omega t); & -2i\sin^2(\omega t)\sin(\alpha); \\ 2i\sin^2(\omega t)\sin(\alpha); & \cos^2(\omega t+\alpha)+\sin^2(\omega t);\end{pmatrix}\begin{pmatrix}\cos(\rho/2)\\ i\sin(\rho/2)\end{pmatrix} =$$

$$= (1;\ -i;)\begin{pmatrix}\cos(\rho/2)\left(\cos^2(\omega t-\alpha)+\sin^2(\omega t)\right)+2\sin^2(\omega t)\sin(\alpha)\sin\left(\frac{\rho}{2}\right) \\ i\left(\sin\frac{\rho}{2}\right)\left(\cos^2(\omega t+\alpha)+\sin^2(\omega t)\right)+2\sin^2(\omega t)\sin(\alpha)\cos\left(\frac{\rho}{2}\right)\end{pmatrix}$$

$$\boxed{\begin{aligned}&\cos^2(\omega t-\alpha)=\cos^2(\omega t+\alpha)+\sin(2\omega t)\sin(2\alpha);\\ &\sin(\rho/2)+\cos(\rho/2)=\sqrt{2}\cos(\rho/2-\pi/4)=\sqrt{2}\sin\left(\frac{\pi+2\rho}{4}\right);\end{aligned}}$$

$$= \left(\cos(\rho/2)+\sin(\rho/2)\right)\left(\cos^2(\omega t+\alpha)+\sin^2(\omega t)\left(1+2\sin\alpha\right)\right)+$$

$$+\cos(\rho/2)\sin(2\omega t)\sin(2\alpha) =$$

$$= \sqrt{2}\sin\left(\frac{\pi+2\rho}{4}\right)\left(\cos^2(\omega t+\alpha)+\sin^2(\omega t)\left(1+2\sin\alpha\right)\right)+$$

$$+\sin(2\omega t)\sin(2\alpha)\cos(\rho/2);$$

$$\boxed{\langle\psi_1|\,e^{iH^+t}\,e^{-iHt}\,|\psi_3\rangle = \frac{\sec^2(\alpha)}{\sqrt{2}}\left[\sqrt{2}\sin\left(\frac{\pi+2\rho}{4}\right)\left(\cos^2(\omega t+\alpha)+\sin^2(\omega t)\left(1+2\sin\alpha\right)\right)+ \sin(2\omega t)\sin(2\alpha)\cos(\rho/2)\right]} \checkmark$$

Calculate $\langle \psi_2 | \psi_3 \rangle$, use

$$\cos(\rho/2) - \sin(\rho/2) = \sqrt{2}\,\sin\left(\frac{\pi}{4} - \frac{\rho}{2}\right)$$

$$(1;\ i;)\begin{pmatrix} \cos^2(\omega t - \alpha) + \sin^2(\omega t); & -2i\sin^2(\omega t)\sin\alpha; \\ 2i\sin^2(\omega t)\sin\alpha; & \cos^2(\omega t + \alpha) + \sin^2(\omega t); \end{pmatrix}\begin{pmatrix} \cos(\rho/2) \\ i\sin(\rho/2) \end{pmatrix} =$$

$$= (1;\ i;)\begin{pmatrix} \cos(\rho/2)\left(\cos^2(\omega t - \alpha) + \sin^2(\omega t)\right) + 2\sin^2(\omega t)\sin(\alpha)\sin\left(\frac{\rho}{2}\right) \\ i\left(2\sin^2(\omega t)\sin(\alpha)\cos(\rho/2) + \sin\left(\frac{\rho}{2}\right)\left(\cos^2(\omega t + \alpha) + \sin^2(\omega t)\right)\right) \end{pmatrix};$$

$$= \cos\left(\frac{\rho}{2}\right)\left(\cos^2(\omega t - \alpha) + \sin^2(\omega t)(1 - 2\sin\alpha)\right) -$$

$$- \sin(\rho/2)\left(\cos^2(\omega t + \alpha) + \sin^2(\omega t)(1 - 2\sin\alpha)\right) =$$

$$= \left(\cos\left(\frac{\rho}{2}\right) - \sin\left(\frac{\rho}{2}\right)\right)\left(\cos^2(\omega t + \alpha) + \sin^2(\omega t)(1 - 2\sin(\alpha))\right) +$$

$$+ \cos(\rho/2)\sin(2\omega t)\sin(2\alpha) =$$

$$= \sqrt{2}\,\sin\left(\frac{\pi}{4} - \frac{\rho}{2}\right)\left(\cos^2(\omega t + \alpha) + \sin^2(\omega t)(1 - 2\sin\alpha)\right) +$$

$$+ \sin(2\omega t)\sin(2\alpha)\cos(\rho/2)$$

$$\langle \psi_2 | \psi_3 \rangle = \frac{\sec^2(\alpha)}{\sqrt{2}}\left[\sqrt{2}\,\sin\left(\frac{\pi}{4} - \frac{\rho}{2}\right)\left(\cos^2(\omega t + \alpha) + \sin^2(\omega t)(1 - 2\sin\right.\right.$$

$$\left. + \sin(2\omega t)\sin(2\alpha)\cos(\rho/2)\right] \quad \checkmark$$

$$\left(\cos(\rho/2);\ -i\sin(\rho/2)\right)\begin{pmatrix}\cos^2(\omega t-\alpha)+\sin^2(\omega t); & -2i\sin^2(\omega t)\sin\alpha;\\ 2i\sin^2(\omega t)\sin\alpha; & \cos^2(\omega t+\alpha)+\sin^2(\omega t);\end{pmatrix}\begin{pmatrix}\cos(\rho/2)\\ i\sin(\rho/2)\end{pmatrix}=$$

$$=\left(\cos(\rho/2);\ -i\sin(\rho/2)\right)\begin{pmatrix}\cos(\rho/2)\left(\cos^2(\omega t-\alpha)+\sin^2(\omega t)\right)+2\sin^2(\omega t)\sin\alpha\sin(\rho/2)\\ i\sin(\rho/2)\left(\cos^2(\omega t+\alpha)+\sin^2(\omega t)+2\sin^2(\omega t)\sin\alpha\cos(\rho/2)\right)\end{pmatrix}=$$

$$=\cos^2(\rho/2)\left(\cos^2(\omega t-\alpha)+\sin^2(\omega t)\right)+\sin^2(\omega t)\sin\alpha\sin\rho+$$

$$+\sin^2(\rho/2)\left(\cos^2(\omega t+\alpha)+\sin^2(\omega t)\right)+\sin^2(\omega t)\sin\alpha\sin\rho=$$

$$=\cos^2(\omega t-\alpha)+\sin^2(\omega t)+2\sin^2(\omega t)\sin\alpha\sin\rho-\sin^2\left(\frac{\rho}{2}\right)\sin(2\omega t)\sin(2\alpha)=$$

$$=\cos^2(\omega t-\alpha)+\sin^2(\omega t)\left(1+2\sin\alpha\sin\rho\right)-\sin(2\omega t)\sin(2\alpha)\sin^2(\rho/2)$$

We used: $\cos^2(\omega t+\alpha)=\cos^2(\omega t-\alpha)-\sin(2\omega t)\sin(2\alpha)$;

$$\langle\psi_3|\psi_3\rangle=\sec^2(\alpha)\left[\cos^2(\omega t-\alpha)+\sin^2(\omega t)\left(1+2\sin\alpha\sin\rho\right)-\sin(2\omega t)\sin(2\alpha)\sin^2(\rho/2)\right]$$

V

$$\Downarrow$$

$$\cos^2(\theta_{13})=\frac{\left[\sqrt{2}\sin\left(\frac{\pi+2\rho}{4}\right)\left(\cos^2(\omega t+\alpha)+\sin^2(\omega t)\left(1+2\sin\alpha\right)\right)+\sin(2\omega t)\sin(2\alpha)\cos(\rho/2)\right]^2}{2\left(\cos^2(\omega t\quad)*\cos^2\alpha+\sin^2(\omega t)\left(1+\sin\alpha\right)^2\right)*\left(\cos^2(\omega t-\alpha)+\sin^2(\omega t)\left(1+2\sin\alpha\sin\rho\right)-\sin(2\omega t)\sin(2\alpha)\sin^2\left(\frac{\rho}{2}\right)\right)}$$

$$\cos^2(\theta_{23})=\frac{\left[\sqrt{2}\sin\left(\frac{\pi-2\rho}{4}\right)\left(\cos^2(\omega t+\alpha)+\sin^2(\omega t)\left(1-2\sin\alpha\right)\right)+\sin(2\omega t)\sin(2\alpha)\cos(\rho/2)\right]^2}{2\left(\cos^2(\omega t)\cos^2\alpha+\sin^2(\omega t)\left(1-\sin\alpha\right)^2\right)\left(\cos^2(\omega t-\alpha)+\sin^2(\omega t)\left(1+2\sin\alpha\sin\rho\right)-\sin(2\omega t)\sin(2\alpha)\sin^2(\rho/2)\right)}$$

$$\boxed{\text{Take } \omega t = \pi/2}$$

$$\underline{\cos^2(\theta_{13})} \rightarrow \frac{\left(\cancel{\sqrt{2}} \sin\left(\frac{\pi+2\rho}{4}\right)\left(\sin^2\alpha + 1 + 2\sin\alpha\right)\right)^2}{\cancel{2}\,(1+\sin\alpha)^2\left(\sin^2\alpha + 1 + 2\sin\alpha\sin\rho\right)} =$$

$$= \boxed{\frac{\boxed{(1+\sin\alpha)^2}\sin^2\left(\frac{\pi+2\rho}{4}\right)}{1+2\sin\alpha\sin\rho+\sin^2\alpha}} = \frac{(1+\sin\alpha)^2(1+\sin\rho)}{3-\cos(2\alpha)+4\sin\alpha\sin\rho}$$

$$\sin^2\left(\frac{\pi+2\rho}{4}\right) = \left(\sin\frac{\pi}{4}\cos\frac{\rho}{2} + \sin\frac{\rho}{2}\cos\frac{\pi}{4}\right)^2 = \frac{1}{2}\left(\cos^2\frac{\rho}{2} + \sin^2\frac{\rho}{2} + 2\sin\frac{\rho}{2}\cos\frac{\rho}{2}\right) =$$

$$= \frac{1+\sin\rho}{2}$$

$$\frac{3-\cos 2\alpha + 4\sin\alpha\sin\rho}{2} = \frac{3-1+2\sin^2\alpha+4\sin\alpha\sin\rho}{2} =$$

$$= \boxed{1+2\sin\alpha\sin\rho+\sin^2\alpha} = \boxed{1 - \frac{(1-\sin\rho)}{16(1+\sin\rho)}\left(\alpha-\frac{\pi}{2}\right)^4 + O\left(\alpha-\frac{\pi}{2}\right)^6}$$

$$\underline{\cos^2\theta_{23}} \rightarrow \frac{(1-\sin\alpha)^2(1-\sin\rho)}{3-\cos(2\alpha)+4\sin\alpha\sin\rho} =$$

$$= \frac{(1-\sin\alpha)^2\sin^2\left(\frac{\pi-2\rho}{4}\right)}{1+2\sin\alpha\sin\rho} = \boxed{\frac{(1-\sin\rho)}{16(1+\sin\rho)}\left(\alpha-\frac{\pi}{2}\right)^4 + O\left(\alpha-\frac{\pi}{2}\right)^6}$$

$\boxed{\mathcal{M} \text{ matrix}}$

$$e^{iH^+t}\, e^{-iHt} = \frac{1}{\cos^2\alpha}\begin{pmatrix} \cos^2(\omega t-\alpha)+\sin^2(\omega t); & -2i\sin^2(\omega t)\sin\alpha; \\ 2i\sin^2(\omega t)\sin\alpha; & \cos^2(\omega t+\alpha)+\sin^2(\omega t); \end{pmatrix}$$

$$e^{-iH^+t}\, e^{iHt} = \frac{1}{\cos^2\alpha}\begin{pmatrix} \cos^2(\omega t+\alpha)+\sin^2(\omega t); & -2i\sin^2(\omega t)\sin\alpha; \\ 2i\sin^2(\omega t)\sin\alpha; & \cos^2(\omega t-\alpha)+\sin^2(\omega t); \end{pmatrix}$$

$$e^{-iHt}\, e^{iH^+t} = \frac{1}{\cos^2\alpha}\begin{pmatrix} \cos^2(\omega t-\alpha)+\sin^2(\omega t); & 2i\sin^2(\omega t)\sin\alpha; \\ -2i\sin^2(\omega t)\sin\alpha; & \cos^2(\omega t+\alpha)+\sin^2(\omega t); \end{pmatrix}$$

$$\mathcal{M}(t) = \mathcal{M}_0\, e^{-iH^+t} e^{iHt} = \frac{\mathcal{M}_0}{\cos^2\alpha}\begin{pmatrix} \cos^2(\omega t+\alpha)+\sin^2(\omega t); & -2i\sin^2(\omega t)\sin\alpha \\ 2i\sin^2(\omega t)\sin\alpha; & \cos^2(\omega t-\alpha)+\sin^2(\omega t); \end{pmatrix}$$

$$\mathcal{M}(t) = \eta^2(t) + \mathbb{I}$$

$$\eta(t) = (\mathcal{M}(t) - \mathbb{I})^{1/2}$$

Apply on the ancilla:

$$\begin{cases} R_y(\theta) = \exp(-i\theta\sigma_y/2) = \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}); \\ \sin(\frac{\theta}{2}); & \cos(\frac{\theta}{2}); \end{pmatrix} \\ \theta = 2\tan^{-1}\eta_0 \end{cases}$$

Use: $\exp(i\theta\,\bar{n}\cdot\bar{\sigma}) = \cos\theta\cdot\mathbb{I} + i\sin(\theta)(\bar{n}\cdot\bar{\sigma})$;

$$\begin{pmatrix} \cos\frac{\theta}{2} & 0 \\ 0 & \cos\theta/2 \end{pmatrix} + i\sin\frac{\theta}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} * (-1) = \begin{pmatrix} \cos\frac{\theta}{2}; & -\sin\frac{\theta}{2}; \\ \sin\frac{\theta}{2}; & \cos\frac{\theta}{2}; \end{pmatrix}$$

Simulating the non-Hermitian Hamiltonian in the dilated space

$$i\frac{d}{dt}|\psi(t)\rangle_{a,q} = \mathcal{H}_{a,q}(t)\,|\psi(t)\rangle_{a,q}$$

$$|\psi(t)\rangle_{a,q} = |0\rangle_a |\psi(t)\rangle_q + |1\rangle_a |\tilde{\psi}(t)\rangle_q; \quad |\tilde{\psi}(t)\rangle_q = \eta(t)|\psi(t)\rangle_q$$

$$|\psi(t)\rangle_{a,q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes |\psi(t)\rangle_q + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \eta(t)|\psi(t)\rangle_q = \begin{pmatrix} |\psi(t)\rangle_q \\ \eta(t)|\psi(t)\rangle_q \end{pmatrix}$$

$$LHS = \begin{bmatrix} i\frac{d}{dt}|\psi(t)\rangle_q \\ \left(i\frac{d}{dt}\eta(t)\right)|\psi(t)\rangle_q + \eta(t)\left(i\frac{d}{dt}|\psi(t)\rangle_q\right) \end{bmatrix}$$

$$\mathcal{H}_{a,q}(t)\begin{pmatrix} |\psi(t)\rangle_q \\ \eta(t)|\psi(t)\rangle_q \end{pmatrix} = \left(\mathbb{I}\otimes\Lambda(t) + \sigma_y\otimes\Gamma(t)\right)\begin{pmatrix} |\psi(t)\rangle_q \\ \eta(t)|\psi(t)\rangle_q \end{pmatrix}$$

$$= \begin{pmatrix} \Lambda(t); & -i\Gamma(t); \\ i\Gamma(t); & \Lambda(t); \end{pmatrix}\begin{pmatrix} |\psi(t)\rangle_q \\ \eta(t)|\psi(t)\rangle_q \end{pmatrix} =$$

$$= \begin{pmatrix} \left(\Lambda(t) - i\Gamma(t)\eta(t)\right)|\psi(t)\rangle_q \\ \left(i\Gamma(t) + \Lambda(t)\eta(t)\right)|\psi(t)\rangle_q \end{pmatrix}$$

$$\boxed{\begin{aligned} &\Lambda(t) - i\Gamma(t)\eta(t) = H_q(t) \\ &i\Gamma(t) + \Lambda(t)\eta(t) = i\frac{d}{dt}\eta(t) + \eta(t)H_q(t) \end{aligned}}$$

$$\boxed{M(t) = \eta^2(t) + \mathbb{I};}$$

$$\begin{cases} \Lambda(t)\eta^2(t) + \cancel{i\Gamma(t)\eta(t)} = i\frac{d\eta(t)}{dt}\eta(t) + \eta(t)H_q(t)\eta(t); \\ \Lambda(t) - \cancel{i\Gamma(t)\eta(t)} = H_q(t) \end{cases} +$$

$$\Lambda(t)\underbrace{\left(\eta^2(t) + \mathbb{I}\right)}_{M(t)} = i\frac{d\eta(t)}{dt}\eta(t) + \eta(t)H_q(t)\eta(t) + H_q(t)$$

$$\boxed{\Lambda(t) = \left[H_q(t) + i\frac{d\eta(t)}{dt}\eta(t) + \eta(t)H_q(t)\eta(t)\right]M^{-1}(t)}$$

$$\begin{cases} \cancel{\Lambda(t)} - i\Gamma(t)\eta(t) = H_q(t) \\ -i\Gamma(t)\eta^{-1}(t) - \cancel{\Lambda(t)} = -i\left(\frac{d}{dt}\eta(t)\right)\eta^{-1}(t) - \eta(t)H_q(t)\eta^{-1}(t) \end{cases} +$$

$$-i\Gamma(t)\left(\eta(t) + \eta^{-1}(t)\right) = H_q(t) - i\left(\frac{d}{dt}\eta(t)\right)\eta^{-1}(t) -$$

$$-\eta(t)H_q(t)\eta^{-1}(t);$$

$$-i\Gamma(t)\left(\eta^2(t) + \mathbb{I}\right)\eta^{-1}(t) = H_q(t) - i\left(\frac{d\eta(t)}{dt}\right)\eta^{-1}(t) -$$

$$-\eta(t)H_q(t)\eta^{-1}(t)$$

$$-i\Gamma(t)\left(\eta^2(t) + \mathbb{I}\right) = H_q(t)\eta(t) - \eta(t)H_q(t) - i\frac{d\eta(t)}{dt}$$

$$\boxed{\Gamma(t) = i\left[H_q(t)\eta(t) - \eta(t)H_q(t) - i\frac{d\eta(t)}{dt}\right]M^{-1}(t)}$$

$$\Downarrow$$

Calculating metrics

$$\underbrace{\langle\psi(t)|_{a,q}\,|\psi(t)\rangle_{a,q}}_{\text{const}} = \left(\langle 0|_a\langle\psi(t)|_q + \langle 1|_a\langle\tilde{\psi}(t)|_q\right) \times$$

$$\times\left(|0\rangle_a|\psi(t)\rangle_q + |1\rangle_a|\tilde{\psi}(t)\rangle_q\right) =$$

$$= \langle \psi(t)|_q \left(\eta^2(t) + \mathbb{I}\right) |\psi(t)\rangle_q = \langle \psi(t)|_q M(t) |\psi(t)\rangle_q$$

$$0 = \langle \psi(t)|_q M(t)\, i\frac{d}{dt} |\psi(t)\rangle_q + \langle \psi(t)|_q\, i\frac{dM}{dt} |\psi(t)\rangle_q +$$

$$+ i\frac{d}{dt} \langle \psi(t)|_q M(t) |\psi(t)\rangle_q$$

$$i\frac{d}{dt} |\psi(t)\rangle_q = H_q(t) |\psi(t)\rangle_q;$$

$$-i\frac{d}{dt} \langle \psi(t)|_q = \langle \psi(t)|_q H_q^+(t)$$

$$i\frac{d}{dt} \langle \psi(t)|_q = - \langle \psi(t)|_q H_q^+(t)$$

$$M(t) H_q(t) + i\frac{dM}{dt} - H^+(t) M(t) = 0$$

$$\boxed{i\frac{dM(t)}{dt} = H_q^+(t) M(t) - M(t) H_q(t)}$$

$$\Downarrow$$

$$\boxed{M(t) = T\exp\left[-i \int_0^t d\tau\, H_q^+(\tau)\right] M(0)\, \tilde{T} \exp\left[i \int_0^t d\tau\, H_q(\tau)\right]}$$

$$\Downarrow$$

$\boxed{\text{Initial conditions}}$

Need to ensure that $M(t) - \mathbb{I} > 0$ for all time interval.

$$\eta(t=0) = \left(\frac{m_0}{\mu_{min}} * f - 1\right)^{1/2} * \mathbb{I} = \eta_0 * \mathbb{I}.$$

μ_{min} $\downarrow$ min eigenvalue

$$\eta_0 = \left(\frac{m_0}{\mu_{min}} * f - 1\right)^{1/2}$$

$$i\frac{d}{dt}|\psi(t)\rangle_{a,q} = \mathcal{H}_{a,q}|\psi(t)\rangle_{a,q};$$

$$|\psi(t)\rangle_{a,q} = |0\rangle_a|\psi(t)\rangle_q + |1\rangle|\tilde{\psi}(t)\rangle_q$$

$$i\frac{d}{dt}|\psi(t)\rangle_q = H_q|\psi(t)\rangle_q : |\tilde{\psi}(t)\rangle_q = \eta(t)|\psi(t)\rangle_q$$

$$|\psi(0)\rangle_{a,q} = |0\rangle_a|\psi(0)\rangle_q + |1\rangle_a|\tilde{\psi}(0)\rangle_q =$$

$$= |0\rangle_a|\psi(0)\rangle_q + |1\rangle_a\eta(0)|\psi(0)\rangle_q =$$

$$= (|0\rangle_a + \eta_0|1\rangle_a) \otimes |\psi(0)\rangle_q = |\psi(0)\rangle_a \otimes |\psi(0)\rangle_q$$

Subjet: $|0\rangle_a$; $R_y(\theta) = \exp(-i\theta\sigma_y/2)$

$$\boxed{\theta = 2\tan^{-1}\eta_0}$$

$\Downarrow$

Evolution and measurement in the dilated space

$$i\frac{d}{dt}|\psi(t)\rangle_{a,q} = \mathcal{H}_{a,q}(t)|\psi(0)\rangle_{a,q}$$

$$|\psi(t)\rangle_{a,q} = \mathrm{Texp}\left[i\int_0^t \mathcal{H}_{a,q}(\tau)d\tau\right]|\psi(0)\rangle_{a,q} = U_{a,q}(t)|\psi(0)\rangle_{a,q}$$

Solve numerically:

$$i\frac{d}{dt}|\psi_{ke}(t)\rangle_{a,q} = \mathcal{H}_{a,q}(t)|\psi_{ke}(t)\rangle_{a,q}$$

$$|\psi_{ke}(0)\rangle_{k,e} = |ke\rangle_{a,q}$$

$\Longrightarrow$

Need to plug numerica

$$U_{a,q}(t) = |\psi_{00}(t)\rangle\langle 00| +$$

$$+ |\psi_{01}(t)\rangle\langle 01| +$$

$$+ |\psi_{10}(t)\rangle\langle 10| +$$

$$+ |\psi_{11}(t)\rangle\langle 11|$$