ASE 389P-7 Problem Set 5

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You need not hand in anything. Instead, be prepared to answer any of these problems—or similar problems—on an upcoming take-home exam. You may discuss your solutions with classmates up until the time that the exam becomes available, but do not swap work (including code).

Readings

All background reading material is found on Canvas. You'll find notes on FFT-based code search for GNSS acquisition in `fftAcqTheory.pdf`. This draws from original work in [1].

Problems

1. In lecture we showed a *recipe* for computing the complex correlation product S_k, and we introduced a *model* for S_k that relates it to code and carrier phase errors. We can use such a model to design acquisition and tracking algorithms. In lecture we considered the case of a random spreading code $C(\tau)$ and we claimed that under this assumption S_k can be modeled as

$$S_k = \frac{N_k \bar{A}_k d_i}{2} \bar{R}(\Delta t_k) \left[\frac{1}{N_k} \sum_{j \in \mathcal{J}k} \exp\left[i \Delta\theta(\tau_j) \right] \right] + n_k$$

   Derive this model from the recipe for S_k given in the notes under the assumption of a random spreading code. Derive the mean and variance of n_k in terms of σ_n and show that $E[n_k n_l^*] = 0$ for $k \neq l$.

   Hint: The signal component of S_k will boil down to a summation involving three functions of time: $A(\tau_j)$, $R_c[t_s(\tau_j) - \hat{t}_{s,k}] = C[\tau_j - t_s(\tau_j)]C[\tau_j - \hat{t}_{s,k}]$, and $\exp[i\Delta\theta(\tau_j)]$. Assume that $A(\tau_j)$ and $R_c[t_s(\tau_j) - \hat{t}_{s,k}]$ are statistically independent from each other and vary slowly enough over the accumulation that they can be modeled as constant in τ_j. Justify the choice of $\bar{A}_k$ and $\bar{R}(\Delta t_k)$ as the appropriate constants. Assume that for the modeled S_k we take the expectation of R_c with respect to the underlying code, which we have assumed is perfectly random.

2. Derive the expression for carrier-to-noise ratio

$$C/N_0 = \frac{A^2}{4\sigma_n^2 T}$$

   introduced above under the assumption that the analog-to-digital converter that produces the samples $x(j)$ is preceded by a bandpass anti-aliasing filter with (one-sided) noise-equivalent bandwidth $B_{ne} = 1/2T$. In other words, assume that the bandpass analog signal is sampled at exactly the Nyquist frequency according to the bandpass sampling theorem. Also assume that the noise component of the analog signal is spectrally flat (white) with two-sided power density $N_0/2$.

Hint: You'll find the derivation in the lecture notes.

3. The "High-Sensitivity GNSS Acquisition" notes pointed out that the summation in the model for S_k is closely related to the discrete-time coherence function

$$C_{\text{coh}}(N) = \left| \frac{1}{N} \sum_{j=0}^{N-1} \exp\left[i\Delta\theta(\tau_j)\right] \right|$$

where N is the number of samples that participate in a coherent accumulation.

To prevent signal power from being lost due to variations in $\Delta\theta(\tau_j)$, the value of the coherence function must be maintained near unity. Explain in your own words why this is so. To accompany your explanation, draw a picture that illustrates the concept of signal amplitude degradation due to variations in $\Delta\theta(\tau_j)$.

The coherence time is defined as $\tau_{\text{coh}} \triangleq TN_{\text{coh}}$, where T is the sampling interval and N_{coh} is the value of N for which the quantity

$$E\left[C_{\text{coh}}^2(N)\right] = \frac{1}{N^2} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} E\left[\exp\left\{i\left[\Delta\theta(\tau_j) - \Delta\theta(\tau_k)\right]\right\}\right]$$

drops to some specified value. For our purposes, we will define this value to be 0.5. (Here, $E[\xi]$ denotes the expected value of the random variable ξ). Derive this expression for $E\left[C_{\text{coh}}^2(N)\right]$ from the foregoing definition of $C_{\text{coh}}(N)$.

4. Write a function in Matlab that computes the coherence. Your function should adhere to the following interface

```
function [Ccoh] = computeCoherence(DeltaThetaVec,N)
% computeCoherence : Compute the value of the discrete-time coherence function
%                    Ccoh(N).
%
%
% INPUTS
%
% DeltaThetaVec ----- Ns-by-1 vector representing a sampled carrier phase
%                     error time history, in rad.
%
% N ----------------- The number of samples that will be used to evaluate
%                     the coherence Ccoh(N).
%
%
```

```
% OUTPUTS
%
% Ccoh -------------- The value of the discrete-time coherence function for
%                     the first N samples of DeltaThetaVec.
%
%+------------------------------------------------------------------------------+
% References:
%
%
%+==============================================================================+
```

Develop a simulation that employs your function `computeCoherence` to examine the coherence of three common processes that drive variations in the time history $\Delta\theta(\tau_j)$:

White phase noise: $\Delta\theta(\tau_j)$ is a sample from a discrete-time white Gaussian sequence with $E[\Delta\theta(\tau_j)\Delta\theta(\tau_m)] = \sigma_\theta^2 \delta_{j,m}$. The units of σ_θ are radians.

White frequency noise (phase random walk): The instantaneous Doppler frequency error $\Delta\omega(\tau_j)$ is a sample from a discrete-time white Gaussian sequence with $E[\Delta\omega(\tau_j)\Delta\omega(\tau_m)] = \sigma_\omega^2 \delta_{j,m}$. $\Delta\theta(\tau_j)$ is the integral of the time history $\Delta\omega(\tau_j)$. This process is sometimes called phase random walk because the integrated white frequency noise "walks" around randomly. The units of σ_ω are radians per sampling interval.

White frequency rate noise (frequency random walk): The instantaneous Doppler frequency rate error $\Delta\alpha(\tau_j)$ is a sample from a discrete-time white Gaussian sequence with $E[\Delta\alpha(\tau_j)\Delta\alpha(\tau_m)] = \sigma_\alpha^2 \delta_{j,m}$. $\Delta\omega(\tau_j)$ is the integral of the time history $\Delta\alpha(\tau_j)$, and $\Delta\theta(\tau_j)$ is the integral of the time history $\Delta\omega(\tau_j)$. This process is sometimes called frequency random walk because the integrated white frequency rate noise "walks" around randomly. The units of σ_α are radians per sampling interval squared.

Using Monte-Carlo-type simulations, estimate the coherence time $\tau_{\text{coh}} = TN_{\text{coh}}$ for the white frequency noise and white frequency rate noise processes with $\sigma_\omega = 0.01$ radians per sampling interval and $\sigma_\alpha = 0.0001$ radians per sampling interval squared. In other words, generate a large number (many thousands) of realizations of the random process $\Delta\theta(\tau_j)$ using Matlab's random number generator and then look at the mean of $C_{\text{coh}}^2(N)$ to estimate the coherence time. Assume that the sampling interval T is equal to 1 ms. Use simple Euler-type integration to derive $\Delta\theta(\tau_j)$ from $\Delta\omega(\tau_j)$ and $\Delta\alpha(\tau_j)$.

Experiment with the value of $C_{\text{coh}}(N)$ for white phase noise with, say, $\sigma_\theta = 0.8$ radians, and several different values of N. Why doesn't it make sense to estimate the coherence time for the white phase noise process?

5. Let

$$s(j) \triangleq D[\tau_j - t_d(\tau_j)]C[\tau_j - t_s(\tau_j)] \cos[2\pi f_{\text{IF}}\tau_j + \theta(\tau_j)]$$

so that we can abbreviate our model for $x(j)$ over $j \in \mathcal{J}_k$ as

$$x(j) = \theta s(j) + n(j)$$

where $\theta \in \{0, A\}$, with $A > 0$, and where the noise samples $\{n(j) \mid j \in \mathcal{J}_k\}$ are taken to be independent, identically-distributed random variables with the distribution $\mathcal{N}(0, \sigma_n^2)$. Assume that both A and $s(j)$ are perfectly known to us for $j \in \mathcal{J}_k$. Suppose we take N_k observations of the samples $x(j)$ and stack these in an observation vector $\boldsymbol{z}$ as

$$\boldsymbol{z} = [x(j_k), x(j_k + 1), \ldots, x(j_k + N_k - 1)]$$

A hypothesis test for the presence of the signal $s(j)$ can be written as

$$\begin{aligned} H_0 &: \ \theta = \theta_0 = 0 \\ H_1 &: \ \theta = \theta_1 = A \end{aligned}$$

Since θ can assume only a single point under each of H_0 and H_1, this is a *simple* hypothesis test. Show that the optimal detection statistic implies an architecture wherein the incoming signal is correlated with $s(j), j \in \mathcal{J}_k$. Explain why it is not necessary to know the value of A to form the detection statistic.

Hint: This problem can be cast as a special case of the General Gaussian Problem discussed in the lecture notes.

6. Repeat Problem 5 for the case where $\theta(\tau_j) = \tilde{\theta}_2$ is constant over $j \in \mathcal{J}_k$ and modeled as a random variable uniformly distributed on $[0, 2\pi]$ and independent of the noise samples $\{n(j)\}$. Assume that all else about Problem 5 remains unchanged. Show that the optimal detection statistic implies a quadrature correlation architecture wherein the incoming signal is correlated against both

$$s_c(j) \triangleq D[\tau_j - t_d(\tau_j)]C[\tau_j - t_s(\tau_j)] \cos[2\pi f_{\mathrm{IF}} \tau_j]$$

and

$$s_s(j) \triangleq D[\tau_j - t_d(\tau_j)]C[\tau_j - t_s(\tau_j)] \sin[2\pi f_{\mathrm{IF}} \tau_j]$$

Is the value of A needed for the detection statistic in this case?

Hints: This is a case where the parameter vector can be modeled as $\boldsymbol{\theta} = [\tilde{\theta}_1, \tilde{\theta}_2]^\mathsf{T}$, with $\tilde{\theta}_1 \in \{0, A\}$, and $\tilde{\theta}_2 \sim \mathcal{U}(0, 2\pi)$. The hypothesis test can be written

$$\begin{aligned} H_0 &: \boldsymbol{\theta} \in \mathcal{X}_0 \triangleq \{\boldsymbol{\theta} \in \mathcal{X} | \tilde{\theta}_1 = 0\} \\ H_1 &: \boldsymbol{\theta} \in \mathcal{X}_1 \triangleq \{\boldsymbol{\theta} \in \mathcal{X} | \tilde{\theta}_1 = A\} \end{aligned}$$

The likelihood ratio test then becomes

$$\Lambda(\boldsymbol{z}) = \frac{p(\boldsymbol{z} | \boldsymbol{\theta} \in \mathcal{X}_1)}{p(\boldsymbol{z} | \boldsymbol{\theta} \in \mathcal{X}_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \nu$$

Finding $p(\boldsymbol{z} | \boldsymbol{\theta} \in \mathcal{X}_0)$ is straightforward, since $\tilde{\theta}_1 = 0$ in this case. To find $p(\boldsymbol{z} | \boldsymbol{\theta} \in \mathcal{X}_1)$, average over $\tilde{\theta}_2$:

$$p(\boldsymbol{z} | \boldsymbol{\theta} \in \mathcal{X}_1) = \int_0^{2\pi} p(\boldsymbol{z} | \tilde{\theta}_1 = A, \tilde{\theta}_2 = \xi) \underbrace{p_2(\xi)}_{\mathcal{U}[\xi; 0, 2\pi]} \, \mathrm{d}\xi$$

The full likelihood ratio then becomes an integral having a term that looks like

$$\sum_j x(j) A s(j, \xi)$$

where

$$s(j, \xi) \triangleq D[\tau_j - t_d(\tau_j)] C[\tau_j - t_s(\tau_j)] \cos\left[2\pi f_{\mathrm{IF}} \tau_j + \xi\right]$$

Use the identity $\cos(a + b) = \cos a \cos b - \sin a \sin b$ to break this term into two components. You will then find similarity between your likelihood function and the one in the example under Composite Hypothesis Testing in the lecture notes.

7. In lecture, we considered the following general acquisition statistic:

$$Z = \sum_{k=1}^{N} |S_k|^2$$

Important parameters here are the number of samples in each coherent accumulation N_k and the number of accumulations S_k summed noncoherently to get Z, N. Assuming $\Delta f_k = 0$ and $\Delta t_k = 0$, the normalized accumulation S_k can be modeled as

$$S_k = \rho_k \exp\left[i \Delta\theta(\tau_{j_k})\right] + n_k$$

where $\rho_k = \frac{N_k \bar{A}_k}{2\sigma_{IQ}}$, and $n_k = n_{Ik} + j n_{Qk}$, with

$$n_{Ik} \sim N(0, 1), \quad n_{Qk} \sim N(0, 1), \quad E[n_{Ik} n_{Ii}] = E[n_{Qk} n_{Qi}] = \delta_{k,i}, \quad E[n_{Ik} n_{Qi}] = 0 \ \ \forall k, i$$

Let $\boldsymbol{\theta} = [\rho_k, \theta(\tau_{j_k})]^\mathsf{T}$ and assume $\rho_k = \rho$ is constant over the acquisition interval. We can then state the acquisition hypotheses as follows:

$$H_0 : \boldsymbol{\theta} \in \mathcal{X}_0 \triangleq \{\boldsymbol{\theta} \in \mathcal{X} \mid \rho = 0\}$$
$$H_1 : \boldsymbol{\theta} \in \mathcal{X}_0 \triangleq \{\boldsymbol{\theta} \in \mathcal{X} \mid \rho > 0\}$$

To formulate a Neyman-Pearson hypothesis test, we will need to derive the likelihood functions $p(Z|\boldsymbol{\theta} \in \mathcal{X}_0)$ and $p(Z|\boldsymbol{\theta} \in \mathcal{X}_1)$.

Show that $p(Z|\boldsymbol{\theta} \in \mathcal{X}_0) = \chi^2_{2N}$. In other words, show that $p(Z|\boldsymbol{\theta} \in \mathcal{X}_0)$ is a chi-square distribution with $2N$ degrees of freedom. Thus, under H_0, Z has mean $\mathbb{E}\left[Z|\boldsymbol{\theta} \in \mathcal{X}_0\right] = 2N$ and variance $\mathrm{Var}\left[Z|\boldsymbol{\theta} \in \mathcal{X}_0\right] = 4N$.

Show that $p(Z|\boldsymbol{\theta} \in \mathcal{X}_1) = \chi^2_{2N}(\lambda)$. In other words, show that $p(Z|\boldsymbol{\theta} \in \mathcal{X}_1)$ is a noncentral chi-square distribution with $2N$ degrees of freedom and noncentrality parameter $\lambda = N\rho^2$. Thus, under H_1, Z has mean $\mathbb{E}\left[Z|\boldsymbol{\theta} \in \mathcal{X}_1\right] = 2N + \lambda$ and variance $\mathrm{Var}\left[Z|\boldsymbol{\theta} \in \mathcal{X}_1\right] = 4(N + \lambda)$.

Hints: Let X_i be N independent, normally distributed random variables with means μ_i and variances σ_i. Then the random variable

$$Z = \sum_{i=1}^{N} \left(\frac{X_i}{\sigma_i}\right)^2$$

has a noncentral chi-square distribution with N degrees of freedom and noncentrality parameter

$$\lambda = \sum_{i=1}^{N} \left(\frac{\mu_i}{\sigma_i} \right)^2$$

In the special case that $\mu_i = 0 \quad \forall i$, then Z has a chi-square distribution with N degrees of freedom.

8. Write a Matlab function to calculate the null hypothesis and alternative hypothesis probability density functions and the decision threshold corresponding to a GNSS signal acquisition problem. Your function should adhere to the following interface:

```
function [pZ_H0,pZ_H1,lambda0,Pd,ZVec] = performAcqHypothesisCalcs(s)
% performAcqHypothesisCalcs : Calculate the null-hypothesis and alternative
%                             hypothesis probability density functions and the
%                             decision threshold corresponding to GNSS signal
%                             acquisition with the given inputs.
%
% Z is the acquisition statistic:
%
%              N  |      |^2
%  Z =        sum |  Sk  |
%             k=1 |      |
%
%
%                  _           _
%              N  |             |
%    =        sum | Ik^2 + Qk^2 |
%             k=1 |_           _|
%
%
%
% where Sk = rhok + nk
%          = Ik + j*Qk
%
% and   nk = nIk + j*nQk
%
% with nIk ~ N(0,1), nQk ~ N(0,1), E[nIk nIi] = E[nQk nQi] = 1 for k = i and 0
% for k != i, and E[nIk nQi] = 0 for all k,i.  The amplitude rhok is related
% to familiar parameters Nk, Abark, and sigma_IQ by rhok =
% (Nk*Abark)/(2*sigma_IQ), i.e., it is the magnitude of the usual complex
% baseband phasor normalized by sigma_IQ.
%
% Under H0, the statistic Z is distributed as a chi-square distribution with
% 2*N degrees of freedom; under H1, it is distributed as a noncentral
% chi-square distribution with lambda = N*rhok^2 and 2*N degrees of freedom.
%
% The total number of cells in the search grid is assumed to be nCells =
% nCodeOffsets*nFreqOffsets, where nFreqOffsets = 2*fMax*Ta and Ta = Na*T is
% the total coherent accumulation time.  Here, Na is the average value of the
% number of samples in each accumulation, Nk.
%
% INPUTS
%
% s -------- A structure containing the following fields:
%
%         C_N0dBHz ------- Carrier to noise ratio in dB-Hz.
%
%         Ta ------------- Coherent accumulation interval, in seconds.
%
%
%         N -------------- The number of accumulations summed noncoherently to
%                          get Z.
%
```

```
%         fMax ----------- Frequency search range delimiter.  The total
%                          frequency search range is +/- fMax.
%
%         nCodeOffsets --- Number of statistically independent code offsets in
%                          the search range.
%
%         PfaAcq --------- The total acquisition false alarm probability.
%                          This is the probability that the statistic Z
%                          exceeds the threshold lambda in any one of the
%                          search cells under the hypothesis H0.  One can
%                          derive the false alarm probability for *each*
%                          search cell from PfaAcq.  This procedure is
%                          straightforward if we assume that the detection
%                          statistics from the search cells are independent
%                          of one another.
%
%         ZMax ----------- The maximum value of Z that will be considered.
%
%         delZ ----------- The discretization interval used for the
%                          independent variable Z.  The full vector of Z
%                          values considered is thus ZVec = [0:delZ:ZMax].
%
%
% OUTPUTS
%
% pZ_H0 ---------- The probability density of Z under hypothesis H0.
%
% pZ_H1 ---------- The probability density of Z under hypothesis H1.
%
% lambda0 -------- The detection threshold.
%
% Pd ------------- The probability of detection.
%
% Zvec ----------- The vector of Z values considered.
%
%+------------------------------------------------------------------------------+
% References:
%
%
%+==============================================================================+
```

Note that `pZ_H0` and `pZ_H1` are vectors of length `length(ZVec)` that represent the corresponding continuous densities sampled at the points in `ZVec` and scaled such that the vectors each sum to 1.

You may use the script `topPerformAcqHypothesisCalcs.m` found on Canvas to test your function. Use your function and the following values for some of the sub-fields of `s`:

```
s.PfaAcq = 0.0001;
s.Ta = 0.001;
s.fMax = 7000
s.nCodeOffsets = 1023*5;
s.ZMax = 1000;
s.delZ = 0.1;
```

to plot the following two curves:

(a) Plot the N required to achieve a probability of detection $P_D = 0.95$ or more for each of the C/N_0 values in the set $\{27, 30, 33, 36, 39, 42, 45, 48, 51\}$.

(b) Set $N = 1$ and vary T_a. Plot the T_a required to achieve a probability of detection $P_D = 0.95$ or more for each of the C/N_0 values in the set $\{5, 10, 15, ..., 45, 50\}$.

What conclusions can you draw about the efficiency of coherent vs. noncoherent integration as regards the total length of data required to achieve a given P_D? For example, to achieve $P_D = 0.95$ at $C/N_0 = 30$, do you require more total data for noncoherent integration or for coherent integration?

Hints: You may find the function `chi2pdf`, `ncx2pdf`, `chi2inv`, and `ncx2cdf` from Matlab's statistics toolbox useful in implementing `performAcqHypothesisCalcs`.

9. Repeat Problem 8 of Problem Set 4 except for two differences:

   (a) Use FFT-based correlation to expedite acquisition by exploring all possible code offsets simultaneously, as described in lecture.

   (b) Use data from the following file:

   `http://radionavlab.ae.utexas.edu/datastore/gnssSigProcCourse/dfDataHead.bin`

   which includes 70 seconds of digitized IF data from the GP2015 front end.

Hints: Load samples from `dfDataHead.bin` into your worspace as shown in `gridPSD.m` from Problem Set 2. The first 14 samples should have values

```
1   3   3   -3   -3   1   1   -1   3   3   -1   -3   1   1
```

Use the document `fftAcqTheory.pdf` found on Canvas as your guide. Recall that you studied the GP2015 front end in Problem Set 4. The GP2015 produces digitized data with an intermediate frequency $f_{\text{IF}} = 1.405396825396879$ MHz and a sampling rate $N_s = 40e6/7$ samples per second. In the absence of Doppler, there would be $N_s/1000 = 40000/7 \approx 5714$ samples per GPS L1 C/A code.

The FFT implementation in Matlab, `fft`, is actually a discrete Fourier transform (DFT) and works for any number of input samples, not just a power-of-two number of samples. Thus, you may use the technique described in `fftAcqTheory.pdf` by assuming N_s samples per C/A code. However, if you wish to make the FFTs in your FFT-based acquisition run faster, you can resample the incoming data so that there are $8192 = 2^{13}$ samples per nominal C/A code.

Check to see that your results for two strong signals in the data, corresponding to PRNs 31 and 14, approximately match those given below.

```
PRN 31:
C/No (dB-Hz): 52.5912
maxidoffset: 5022
Code start time from first sample (us): 878.85
Measured Doppler (Hz): 175

PNR 14:
C/No (dB-Hz): 48.6108
maxidoffset: 2943
Code start time from first sample (us): 515.025
Measured Doppler (Hz): 2275
```

References

[1] D. Van Nee and A. Coenen, "New fast GPS code-acquisition technique using FFT," *Electronics Letters*, vol. 27, no. 2, pp. 158–160, 1991.