

15 OCTOBER 2024

ASE 367K: FLIGHT DYNAMICS

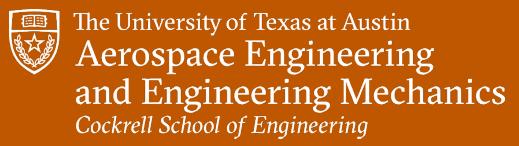
TTH 09:30-11:00
CMA 2.306

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Topics for Today

- Topic(s):
 - Linearized Longitudinal Equations of Motion



The University of Texas at Austin
Aerospace Engineering
and Engineering Mechanics
Cockrell School of Engineering

LINEARIZED LONGITUDINAL EOM

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Turning our attention to longitudinal motion...

- If we neglect small change in air density due to the small variations in altitude during cruise flight, then...
 - Position equations become uncoupled and can be integrated independently
 - We are left with the following 4 nonlinear equation of Longitudinal motion

$$X = m(\dot{u} + qw - rv + g \sin \theta)$$

$$Z = m(\dot{w} + pv - qu - g \cos \theta \cos \phi)$$

$$M = I_{yy}\dot{q} - I_{xz}(r^2 - p^2) - (I_{zz} - I_{xx})pr$$

$$q = \dot{\theta} \cos \phi + \dot{\psi} \cos \theta \sin \phi$$

Turning our attention to longitudinal motion...

- If we linearize the equations about some operating condition, which can be completely arbitrary, and ...
 - Denote the operating condition by the subscript “1”
 - Assume the perturbations about the operating condition are small
- We can represent the quantities of interest as the sum of the nominal operating condition and a perturbation from nominal

$$u = u_1 + \Delta u$$

$$p = p_1 + \Delta p$$

$$\phi = \phi_1 + \Delta \phi$$

$$v = v_1 + \Delta v$$

$$q = q_1 + \Delta q$$

$$\theta = \theta_1 + \Delta \theta$$

$$w = w_1 + \Delta w$$

$$r = r_1 + \Delta r$$

$$\psi = \psi_1 + \Delta \psi$$

$$\begin{aligned}
X &= m(\dot{u} + qw - rv + g \sin \theta) \\
&= m[(\dot{u}_1 + \Delta\dot{u}) + (q_1 + \Delta q)(w_1 + \Delta w) - (r_1 + \Delta r)(v_1 + \Delta v) \\
&\quad + g \sin(\theta_1 + \Delta\theta)] && \text{substitute in perturbations} \\
&= m[(\dot{u}_1 + \Delta\dot{u}) + (q_1 + \Delta q)(w_1 + \Delta w) - (r_1 + \Delta r)(v_1 + \Delta v) \\
&\quad + g(\sin \theta_1 \cos \Delta\theta + \cos \theta_1 \sin \Delta\theta)] && \text{angle addition formula} \\
&= m[(\dot{u}_1 + \Delta\dot{u}) + (q_1 + \Delta q)(w_1 + \Delta w) - (r_1 + \Delta r)(v_1 + \Delta v) \\
&\quad + g(\sin \theta_1 + \cos \theta_1 \Delta\theta)] && \text{small angle approximation} \\
&= m(\dot{u}_1 + q_1 w_1 - r_1 v_1 + g \sin \theta_1) \\
&\quad + m[(\Delta\dot{u} - r_1 \Delta v + q_1 \Delta w + w_1 \Delta q - v_1 \Delta r) + g \cos \theta_1 \Delta\theta] && \text{neglect higher-order terms} \\
&= X_1 + \Delta X
\end{aligned}$$

(8.2a)

(8.2b)

$$\begin{aligned}
Z &= m(\dot{w} + pv - qu - g \cos \theta \cos \phi) \\
&= m[(\dot{w}_1 + \Delta\dot{w}) + (p_1 + \Delta p)(v_1 + \Delta v) - (q_1 + \Delta q)(u_1 + \Delta u) \\
&\quad - g \cos(\theta_1 + \Delta\theta) \cos(\phi_1 + \Delta\phi)] && \text{substitute in perturbations} \\
&= m[(\dot{w}_1 + \Delta\dot{w}) + (p_1 + \Delta p)(v_1 + \Delta v) - (q_1 + \Delta q)(u_1 + \Delta u) \\
&\quad - g(\cos \theta_1 \cos \Delta\theta - \sin \theta_1 \sin \Delta\theta)(\cos \phi_1 \cos \Delta\phi - \sin \phi_1 \sin \Delta\phi)] && \text{angle addition formula} \\
&= m[(\dot{w}_1 + \Delta\dot{w}) + (p_1 + \Delta p)(v_1 + \Delta v) - (q_1 + \Delta q)(u_1 + \Delta u) \\
&\quad - g(\cos \theta_1 - \sin \theta_1 \Delta\theta)(\cos \phi_1 - \sin \phi_1 \Delta\phi)] && \text{small angle approximation} \\
&= m(\dot{w}_1 + p_1 v_1 - q_1 u_1 - g \cos \theta_1 \cos \phi_1) \\
&\quad + m[(\Delta\dot{w} - q_1 \Delta u + p_1 \Delta v + v_1 \Delta p - u_1 \Delta q) \\
&\quad + g \cos \theta_1 \sin \phi_1 \Delta\phi + g \sin \theta_1 \cos \phi_1 \Delta\theta] && (8.3a) \\
&= Z_1 + \Delta Z && (8.3b)
\end{aligned}$$

$$\begin{aligned}
M &= I_{yy}\dot{q} - I_{xz}(r^2 - p^2) - (I_{zz} - I_{xx})pr \\
&= I_{yy}(\dot{q}_1 + \Delta\dot{q}) - I_{xz}[(r_1 + \Delta r)(r_1 + \Delta r) - (p_1 + \Delta p)(p_1 + \Delta p)] \quad \text{substitute in perturbations} \\
&\quad - (I_{zz} - I_{xx})(p_1 + \Delta p)(r_1 + \Delta r)
\end{aligned}$$

$$\begin{aligned}
&= [I_{yy}\dot{q}_1 - I_{xz}(r_1^2 - p_1^2) - (I_{zz} - I_{xx})p_1r_1] \quad \text{neglect higher-order terms} \\
&\quad + I_{yy}\Delta\dot{q} + (2I_{xz}p_1 - (I_{zz} - I_{xx})r_1)\Delta p \\
&\quad - (2I_{xz}r_1 + (I_{zz} - I_{xx})p_1)\Delta r \tag{8.4a}
\end{aligned}$$

$$= M_1 + \Delta M \tag{8.4b}$$

$$\begin{aligned}
q &= \dot{\theta} \cos \phi + \dot{\psi} \cos \theta \sin \phi \\
&= (\dot{\theta}_1 + \Delta\dot{\theta}) \cos(\phi_1 + \Delta\phi) && \text{substitute in perturbations} \\
&\quad + (\dot{\psi}_1 + \Delta\dot{\psi}) \cos(\theta_1 + \Delta\theta) \sin(\phi_1 + \Delta\phi) \\
&= (\dot{\theta}_1 + \Delta\dot{\theta})(\cos \phi_1 \cos \Delta\phi - \sin \phi_1 \sin \Delta\phi) && \text{angle addition formula} \\
&\quad + (\dot{\psi}_1 + \Delta\dot{\psi})(\cos \theta_1 \cos \Delta\theta - \sin \theta_1 \sin \Delta\theta) \\
&\quad \times (\sin \phi_1 \cos \Delta\phi + \cos \phi_1 \sin \Delta\phi) \\
&= (\dot{\theta}_1 + \Delta\dot{\theta})(\cos \phi_1 - \sin \phi_1 \Delta\phi) && \text{small angle approximation} \\
&\quad + (\dot{\psi}_1 + \Delta\dot{\psi})(\cos \theta_1 - \sin \theta_1 \Delta\theta)(\sin \phi_1 + \cos \phi_1 \Delta\phi) \\
&= (\dot{\theta}_1 \cos \phi_1 + \dot{\psi}_1 \cos \theta_1 \sin \phi_1) && \text{neglect higher order terms} \\
&\quad + \cos \phi_1 \Delta\dot{\theta} + \cos \theta_1 \sin \phi_1 \Delta\dot{\psi} + \dot{\psi}_1 \cos \theta_1 \cos \phi_1 \Delta\phi \\
&\quad - \dot{\psi}_1 \sin \theta_1 \sin \phi_1 \Delta\theta && (8.5a) \\
&= q_1 + \Delta q && (8.5b)
\end{aligned}$$

Focusing in on the perturbations...

$$\Delta X = m(\Delta \dot{u} - r_1 \Delta v + q_1 \Delta w + w_1 \Delta q - v_1 \Delta r + g \cos \theta_1 \Delta \theta)$$

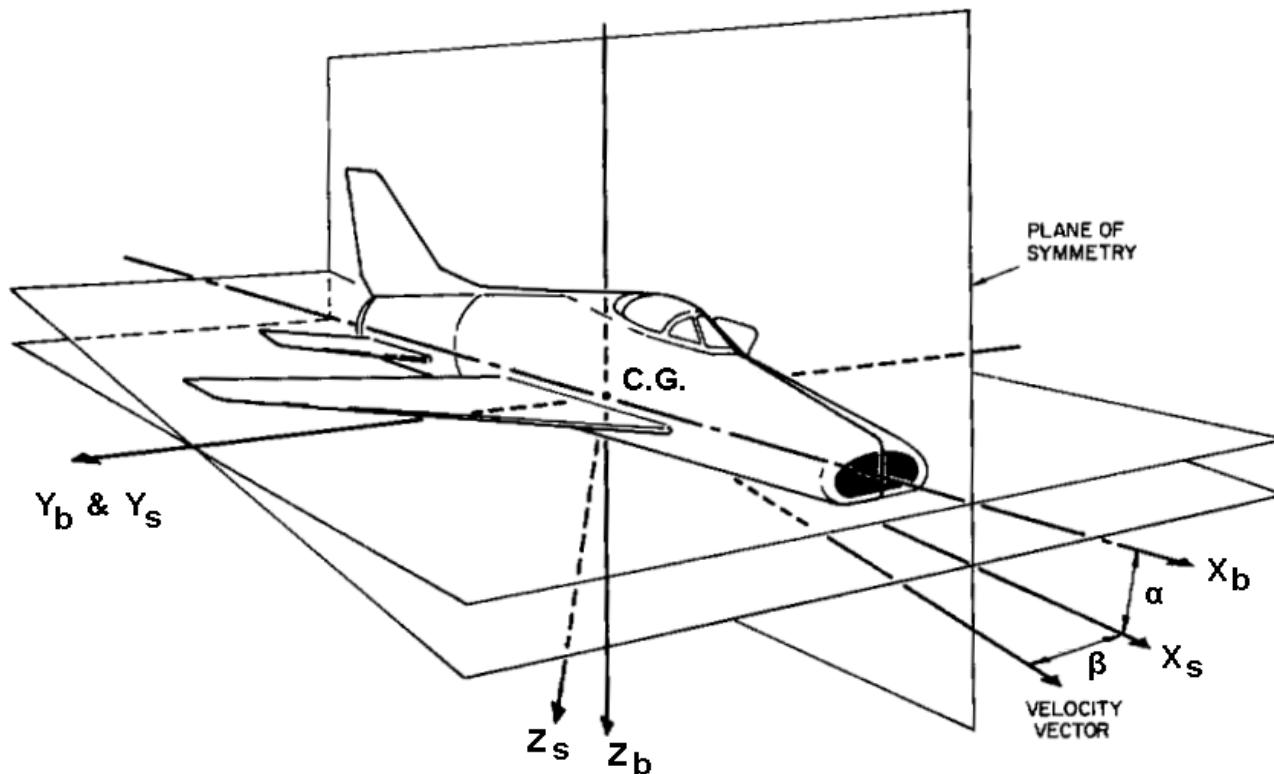
$$\Delta Z = m(\Delta \dot{w} - q_1 \Delta u + p_1 \Delta v + v_1 \Delta p - u_1 \Delta q + g \cos \theta_1 \sin \phi_1 \Delta \phi + g \sin \theta_1 \cos \phi_1 \Delta \theta)$$

$$\Delta M = I_{yy} \Delta \dot{q} + (2I_{xz}p_1 - (I_{zz} - I_{xx})r_1) \Delta p - (2I_{xz}r_1 + (I_{zz} - I_{xx})p_1) \Delta r$$

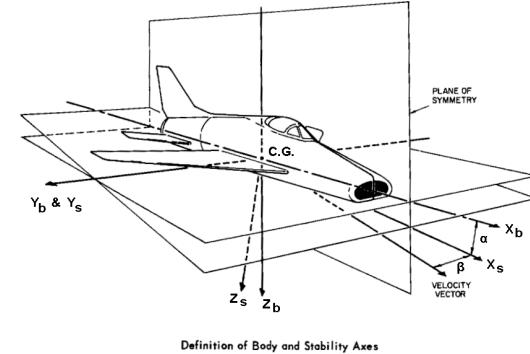
$$\Delta q = \cos \phi_1 \Delta \dot{\theta} + \cos \theta_1 \sin \phi_1 \Delta \dot{\psi} + \dot{\psi}_1 \cos \theta_1 \cos \phi_1 \Delta \phi - \dot{\psi}_1 \sin \theta_1 \sin \phi_1 \Delta \theta$$

Focusing in on the perturbations...

- If we linearize w.r.t. a constant velocity flight with zero beta and we choose a body-fixed frame that is aligned with the nominal velocity vector...



Definition of Body and Stability Axes



Focusing in on the perturbations...

- If we linearize w.r.t. a constant velocity flight with zero beta and we choose a body-fixed frame that is aligned with the nominal velocity vector, then ...

$$w_1 = 0 \quad v_1 = 0, \quad p_1 = 0, \quad q_1 = 0, \quad r_1 = 0, \quad \text{and} \quad \phi_1 = 0$$

$$\dot{\phi}_1 = 0, \quad \dot{\theta}_1 = 0, \quad \text{and} \quad \dot{\psi}_1 = 0$$

- and...

$$\Delta X = m(\Delta \dot{u} - \cancel{v}_1 \Delta v + \cancel{q}_1 \Delta w + \cancel{w}_1 \Delta q - \cancel{r}_1 \Delta r + g \cos \theta_1 \Delta \theta)$$

$$\Delta Z = m(\Delta \dot{w} - \cancel{q}_1 \Delta u + \cancel{p}_1 \Delta v + \cancel{v}_1 \Delta p - u_1 \Delta q + g \cos \theta_1 \sin \phi_1 \Delta \phi + g \sin \theta_1 \cos \phi_1 \Delta \theta)$$

$$\Delta M = I_{yy} \Delta \dot{q} + (2I_{xz} \cancel{p}_1 - (I_{zz} - I_{xx}) \cancel{r}_1) \Delta p - (2I_{xz} \cancel{r}_1 + (I_{zz} - I_{xx}) \cancel{p}_1) \Delta r$$

$$\Delta q = \cos \phi_1 \Delta \dot{\theta} + \cos \theta_1 \sin \phi_1 \Delta \dot{\psi} + \cancel{\dot{\psi}}_1 \cos \theta_1 \cos \phi_1 \Delta \phi - \cancel{\dot{\psi}}_1 \sin \theta_1 \sin \phi_1 \Delta \theta$$

Focusing in on the perturbations...

$$\Delta X = m(\Delta \dot{u} + g \cos \theta_1 \Delta \theta)$$

$$\Delta Z = m(\Delta \dot{w} - u_1 \Delta q + g \sin \theta_1 \Delta \theta)$$

$$\Delta M = I_{yy} \Delta \dot{q}$$

$$\Delta q = \Delta \dot{\theta}$$

Replacing Δw with $\Delta \alpha$...

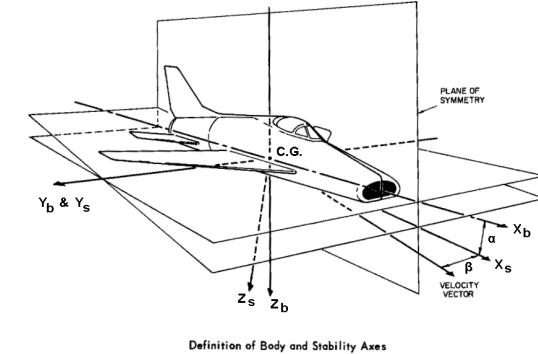
$$\begin{aligned}
\alpha &= \arctan \frac{w}{u} \\
&= \frac{w}{u} \\
&= \frac{w_1 + \Delta w}{u_1 + \Delta u} \\
&= (w_1 + \Delta w)(u_1 + \Delta u)^{-1} \\
&= \frac{w_1 + \Delta w}{u_1} \left(1 + \frac{\Delta u}{u_1}\right)^{-1} \\
&= \frac{w_1 + \Delta w}{u_1} \left(1 - \frac{\Delta u}{u_1} - \dots\right) \\
&= \frac{w_1}{u_1} + \frac{\Delta w}{u_1} - \frac{w_1 \Delta u}{u_1^2} \\
&= \alpha_1 + \Delta \alpha
\end{aligned}$$

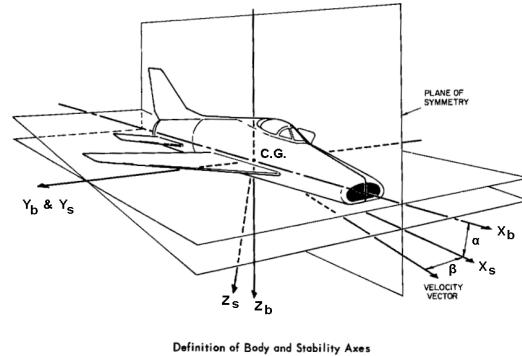
small angle approximation

substitute in perturbations

binomial expansion

neglect higher-order terms





Replacing Δw with $\Delta\alpha$...

$$w_1 = 0, \text{ therefore } \alpha_1 = 0, \text{ hence } \Delta\alpha = \frac{\Delta w}{u_1} \quad \text{or} \quad \Delta w = u_1 \Delta\alpha$$

Furthermore, since $\dot{u}_1 = 0$, it follows that

$$\Delta\dot{w} = u_1 \Delta\dot{\alpha}$$

$$\Delta X = m(\Delta\dot{u} + g \cos \theta_1 \Delta\theta)$$

$$\Delta Z = m(u_1 \Delta\dot{\alpha} - u_1 \Delta q + g \sin \theta_1 \Delta\theta)$$

$$\Delta M = I_{yy} \Delta \dot{q}$$

$$\Delta q = \Delta\dot{\theta}$$

Let's look at those forces...

$$X = T + L \sin \Delta\alpha - D \cos \Delta\alpha$$

$$Z = -L \cos \Delta\alpha - D \sin \Delta\alpha$$

$$T = T_1 + \Delta T, \quad L = L_1 + \Delta L, \quad \text{and} \quad D = D_1 + \Delta D$$

Let's look at those forces...

$$\begin{aligned} X &= T + L \sin \Delta\alpha - D \cos \Delta\alpha \\ &= (T_1 + \Delta T) + (L_1 + \Delta L) \sin \Delta\alpha - (D_1 + \Delta D) \cos \Delta\alpha && \text{substitute in perturbations} \\ &= (T_1 + \Delta T) + (L_1 + \Delta L)\Delta\alpha - (D_1 + \Delta D) && \text{small angle approximation} \\ &= (T_1 - D_1) + (\Delta T + L_1\Delta\alpha - \Delta D) && \text{neglect higher-order terms} \\ &= X_1 + \Delta X \end{aligned}$$

$$\begin{aligned} Z &= -L \cos \alpha - D \sin \alpha \\ &= -(L_1 + \Delta L) \cos \Delta\alpha - (D_1 + \Delta D) \sin \Delta\alpha && \text{substitute in perturbations} \\ &= -(L_1 + \Delta L) - (D_1 + \Delta D)\Delta\alpha && \text{small angle approximation} \\ &= (L_1 - D_1) + (-\Delta L - D_1\Delta\alpha) && \text{neglect higher-order terms} \\ &= Z_1 + \Delta Z \end{aligned}$$

Let's look at those forces...

$$\Delta X = \Delta T + L_1 \Delta \alpha - \Delta D$$

$$\Delta Z = -\Delta L - D_1 \Delta \alpha$$

$$\Delta \dot{u} = \Delta T/m + L_1/m \Delta \alpha - \Delta D/m - g \cos \theta_1 \Delta \theta$$

$$u_1 \Delta \dot{\alpha} = -\Delta L/m - D_1/m \Delta \alpha + u_1 \Delta q - g \sin \theta_1 \Delta \theta$$

$$\Delta \dot{q} = \Delta M/I_{yy}$$

$$\Delta \dot{\theta} = \Delta q$$

Forces are expressed as a product of the dynamic pressure, a reference area, and a non-dimensional coefficient, such that $F = \bar{q}SC_F$

$$\left(\frac{\partial F}{\partial u}\right)_1 = \left(\frac{\partial \bar{q}}{\partial u}\right)_1 SC_{F_1} + \bar{q}_1 S \left(\frac{\partial C_F}{\partial u}\right)_1$$

Since the dynamic pressure is given by

$$\bar{q} = \frac{1}{2}\rho(u^2 + v^2 + w^2)$$

we have

$$\left(\frac{\partial \bar{q}}{\partial u}\right)_1 = \left(\frac{1}{2}\rho \cdot 2u\right)_1 = \rho u_1 = \frac{2\bar{q}_1}{u_1}$$

such that

$$\left(\frac{\partial F}{\partial u}\right)_1 = \bar{q}_1 S \left(\frac{\partial C_F}{\partial u}\right)_1 + \frac{2\bar{q}_1 S}{u_1} C_{F_1}$$

Applying the chain rule to the first term

$$\left(\frac{\partial C_F}{\partial u} \right)_1 = \left(\frac{\partial C_F}{\partial(u/u_1)} \right)_1 \frac{\partial(u/u_1)}{\partial u} = \frac{1}{u_1} C_{F_u}$$

which gives

$$\left(\frac{\partial F}{\partial u} \right)_1 = \frac{\bar{q}_1 S}{u_1} (C_{F_u} + 2C_{F_1}) \quad \text{where} \quad C_{F_u} = \left(\frac{\partial C_F}{\partial(u/u_1)} \right)_1$$

$$\left(\frac{\partial F}{\partial \alpha} \right)_1 = \bar{q}_1 S C_{F_\alpha} \quad C_{F_\alpha} = \left(\frac{\partial C_F}{\partial \alpha} \right)_1$$

$$\left(\frac{\partial F}{\partial q} \right)_1 = \frac{\bar{q}_1 S \bar{c}}{2u_1} C_{F_q} \quad C_{F_q} = \left(\frac{\partial C_F}{\partial(q\bar{c}/2u_1)} \right)_1$$

$$\left(\frac{\partial F}{\partial \dot{\alpha}} \right)_1 = \frac{\bar{q}_1 S \bar{c}}{2u_1} C_{F_{\dot{\alpha}}} \quad C_{F_{\dot{\alpha}}} = \left(\frac{\partial C_F}{\partial(\dot{\alpha}\bar{c}/2u_1)} \right)_1$$

$$\left(\frac{\partial F}{\partial \delta_e} \right)_1 = \bar{q}_1 S C_{F_{\delta_e}} \quad C_{F_{\delta_e}} = \left(\frac{\partial C_F}{\partial \delta_e} \right)_1$$

Let's look at those forces again...

$$\Delta D = \frac{\bar{q}_1 S}{u_1} (C_{D_u} + 2C_{D_1}) \Delta u + \bar{q}_1 S C_{D_\alpha} \Delta \alpha + \frac{\bar{q}_1 S \bar{c}}{2u_1} C_{D_q} \Delta q + \frac{\bar{q}_1 S \bar{c}}{2u_1} C_{D_{\dot{\alpha}}} \Delta \dot{\alpha} + \bar{q}_1 S C_{D_{\delta_e}} \Delta \delta_e$$

$$\Delta T = \frac{\bar{q}_1 S}{u_1} (C_{T_u} + 2C_{T_1}) \Delta u + \bar{q}_1 S C_{T_\alpha} \Delta \alpha + \frac{\bar{q}_1 S \bar{c}}{2u_1} C_{T_q} \Delta q + \frac{\bar{q}_1 S \bar{c}}{2u_1} C_{T_{\dot{\alpha}}} \Delta \dot{\alpha} + \bar{q}_1 S C_{T_{\delta_e}} \Delta \delta_e$$

$$\Delta L = \frac{\bar{q}_1 S}{u_1} (C_{L_u} + 2C_{L_1}) \Delta u + \bar{q}_1 S C_{L_\alpha} \Delta \alpha + \frac{\bar{q}_1 S \bar{c}}{2u_1} C_{L_q} \Delta q + \frac{\bar{q}_1 S \bar{c}}{2u_1} C_{L_{\dot{\alpha}}} \Delta \dot{\alpha} + \bar{q}_1 S C_{L_{\delta_e}} \Delta \delta_e$$

Variations in the pitch rate and rate of change of the angle of attack mainly contribute to effects at the tail through the lift. This means that

$$C_{D_q} = 0, \quad C_{D_{\dot{\alpha}}} = 0, \quad C_{T_q} = 0, \quad \text{and} \quad C_{T_{\dot{\alpha}}} = 0$$

Furthermore, variations in the angle of attack and elevator deflection do not significantly contribute to variations in the thrust, giving

$$C_{T_\alpha} = 0 \quad \text{and} \quad C_{T_{\delta_e}} = 0$$

Making the above substitutions, ΔD , ΔT , and ΔL become

$$\Delta D = \frac{\bar{q}_1 S}{u_1} (C_{D_u} + 2C_{D_1}) \Delta u + \bar{q}_1 S C_{D_\alpha} \Delta \alpha + \bar{q}_1 S C_{D_{\delta_e}} \Delta \delta_e$$

$$\Delta T = \frac{\bar{q}_1 S}{u_1} (C_{T_u} + 2C_{T_1}) \Delta u$$

$$\Delta L = \frac{\bar{q}_1 S}{u_1} (C_{L_u} + 2C_{L_1}) \Delta u + \bar{q}_1 S C_{L_\alpha} \Delta \alpha + \frac{\bar{q}_1 S \bar{c}}{2u_1} C_{L_q} \Delta q + \frac{\bar{q}_1 S \bar{c}}{2u_1} C_{L_{\dot{\alpha}}} \Delta \dot{\alpha} + \bar{q}_1 S C_{L_{\delta_e}} \Delta \delta_e$$

Let's look at the moment...

$$M = \bar{q}S\bar{c}C_M$$

Following the same approach as taken for the force terms, it can be shown that

$$\left(\frac{\partial M}{\partial u} \right)_1 = \frac{\bar{q}_1 S \bar{c}}{u_1} (C_{M_u} + 2C_{M_1})$$

$$\left(\frac{\partial M}{\partial \alpha} \right)_1 = \bar{q}_1 S \bar{c} C_{M_\alpha}$$

$$\left(\frac{\partial M}{\partial q} \right)_1 = \frac{\bar{q}_1 S \bar{c}^2}{2u_1} C_{M_q}$$

$$\left(\frac{\partial M}{\partial \dot{\alpha}} \right)_1 = \frac{\bar{q}_1 S \bar{c}^2}{2u_1} C_{M_{\dot{\alpha}}}$$

$$\left(\frac{\partial M}{\partial \delta_e} \right)_1 = \bar{q}_1 S \bar{c} C_{M_{\delta_e}}$$

$$C_{M_u} = \left(\frac{\partial C_M}{\partial (u/u_1)} \right)_1$$

$$C_{M_\alpha} = \left(\frac{\partial C_M}{\partial \alpha} \right)_1$$

$$C_{M_q} = \left(\frac{\partial C_M}{\partial (q\bar{c}/2u_1)} \right)_1$$

$$C_{M_{\dot{\alpha}}} = \left(\frac{\partial C_M}{\partial (\dot{\alpha}\bar{c}/2u_1)} \right)_1$$

$$C_{M_{\delta_e}} = \left(\frac{\partial C_M}{\partial \delta_e} \right)_1$$

Let's look at the moment...

The aerodynamic and thrust pitch moment perturbations are then given by substituting the above relationships into Eq. (8.19), yielding

$$\Delta M_A = \frac{\bar{q}_1 S \bar{c}}{u_1} (C_{M_u} + 2C_{M_1}) \Delta u + \bar{q}_1 S \bar{c} C_{M_\alpha} \Delta \alpha + \frac{\bar{q}_1 S \bar{c}^2}{2u_1} C_{M_q} \Delta q + \frac{\bar{q}_1 S \bar{c}^2}{2u_1} C_{M_{\dot{\alpha}}} \Delta \dot{\alpha} + \bar{q}_1 S \bar{c} C_{M_{\delta_e}} \Delta \delta_e$$
$$\Delta M_T = \frac{\bar{q}_1 S \bar{c}}{u_1} (C_{M_{T_u}} + 2C_{M_{T_1}}) \Delta u + \bar{q}_1 S \bar{c} C_{M_{T_\alpha}} \Delta \alpha + \frac{\bar{q}_1 S \bar{c}^2}{2u_1} C_{M_{T_q}} \Delta q + \frac{\bar{q}_1 S \bar{c}^2}{2u_1} C_{M_{T_{\dot{\alpha}}}} \Delta \dot{\alpha} + \bar{q}_1 S \bar{c} C_{M_{T_{\delta_e}}} \Delta \delta_e$$

Let's look at the moment...

Variations in the pitch rate, rate of change of the angle of attack, and elevator deflection do not significantly contribute to the thrust pitch moment, i.e.

$$C_{M_{Tq}} = 0, \quad C_{M_{T\dot{\alpha}}} = 0, \quad \text{and} \quad C_{M_{T\delta_e}} = 0$$

which gives

$$\begin{aligned}\Delta M_A &= \frac{\bar{q}_1 S \bar{c}}{u_1} (C_{M_u} + 2C_{M_1}) \Delta u + \bar{q}_1 S \bar{c} C_{M_\alpha} \Delta \alpha + \frac{\bar{q}_1 S \bar{c}^2}{2u_1} C_{M_q} \Delta q \\ &\quad + \frac{\bar{q}_1 S \bar{c}^2}{2u_1} C_{M_{\dot{\alpha}}} \Delta \dot{\alpha} + \bar{q}_1 S \bar{c} C_{M_{\delta_e}} \Delta \delta_e\end{aligned}$$

$$\Delta M_T = \frac{\bar{q}_1 S \bar{c}}{u_1} (C_{M_{Tu}} + 2C_{M_{T1}}) \Delta u + \bar{q}_1 S \bar{c} C_{M_{T\alpha}} \Delta \alpha$$

Putting it all together...

$$\Delta D/m = \frac{\bar{q}_1 S}{m u_1} (C_{D_u} + 2C_{D_1}) \Delta u + \frac{\bar{q}_1 S}{m} C_{D_\alpha} \Delta \alpha + \frac{\bar{q}_1 S}{m} C_{D_{\delta_e}} \Delta \delta_e$$

$$\Delta T/m = \frac{\bar{q}_1 S}{m u_1} (C_{T_u} + 2C_{T_1}) \Delta u$$

$$\begin{aligned} \Delta L/m = & \frac{\bar{q}_1 S}{m u_1} (C_{L_u} + 2C_{L_1}) \Delta u + \frac{\bar{q}_1 S}{m} C_{L_\alpha} \Delta \alpha + \frac{\bar{q}_1 S \bar{c}}{2 m u_1} C_{L_q} \Delta q \\ & + \frac{\bar{q}_1 S \bar{c}}{2 m u_1} C_{L_{\dot{\alpha}}} \Delta \dot{\alpha} + \frac{\bar{q}_1 S}{m} C_{L_{\delta_e}} \Delta \delta_e \end{aligned}$$

$$\begin{aligned} \Delta M/I_{yy} = & \frac{\bar{q}_1 S \bar{c}}{I_{yy} u_1} (C_{M_u} + 2C_{M_1}) \Delta u + \frac{\bar{q}_1 S \bar{c}}{I_{yy} u_1} (C_{M_{T_u}} + 2C_{M_{T_1}}) \Delta u \\ & + \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_\alpha} \Delta \alpha + \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_{T_\alpha}} \Delta \alpha + \frac{\bar{q}_1 S \bar{c}^2}{2 I_{yy} u_1} C_{M_q} \Delta q \\ & + \frac{\bar{q}_1 S \bar{c}^2}{2 I_{yy} u_1} C_{M_{\dot{\alpha}}} \Delta \dot{\alpha} + \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_{\delta_e}} \Delta \delta_e \end{aligned}$$

Putting it all together...

$$\begin{aligned}
\Delta \dot{u} = & -\frac{\bar{q}_1 S}{m u_1} (C_{D_u} + 2C_{D_1}) \Delta u + \frac{\bar{q}_1 S}{m u_1} (C_{T_u} + 2C_{T_1}) \Delta u - \frac{\bar{q}_1 S}{m} (C_{D_\alpha} - C_{L_1}) \Delta \alpha \\
& - g \cos \theta_1 \Delta \theta - \frac{\bar{q}_1 S}{m} C_{D_{\delta_e}} \Delta \delta_e \\
\left(u_1 + \frac{\bar{q}_1 S \bar{c}}{2 m u_1} C_{L_{\dot{\alpha}}} \right) \Delta \dot{\alpha} = & -\frac{\bar{q}_1 S}{m u_1} (C_{L_u} + 2C_{L_1}) \Delta u - \frac{\bar{q}_1 S}{m} (C_{L_\alpha} + C_{D_1}) \Delta \alpha + \left(u_1 - \frac{\bar{q}_1 S \bar{c}}{2 m u_1} C_{L_q} \right) \Delta q \\
& - g \sin \theta_1 \Delta \theta - \frac{\bar{q}_1 S}{m} C_{L_{\delta_e}} \Delta \delta_e \\
\Delta \dot{q} - \frac{\bar{q}_1 S \bar{c}^2}{2 I_{yy} u_1} C_{M_{\dot{\alpha}}} \Delta \dot{\alpha} = & \frac{\bar{q}_1 S \bar{c}}{I_{yy} u_1} (C_{M_u} + 2C_{M_1}) \Delta u + \frac{\bar{q}_1 S \bar{c}}{I_{yy} u_1} (C_{M_{T_u}} + 2C_{M_{T_1}}) \Delta u + \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_\alpha} \Delta \alpha \\
& + \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_{T_\alpha}} \Delta \alpha + \frac{\bar{q}_1 S \bar{c}^2}{2 I_{yy} u_1} C_{M_q} \Delta q + \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_{\delta_e}} \Delta \delta_e \\
\Delta \dot{\theta} = \Delta q
\end{aligned}$$

Putting it all together...

To make the equations more manageable, let the dimensional stability derivatives be defined as

$$X_u = -\frac{\bar{q}_1 S}{m u_1} (C_{D_u} + 2C_{D_1})$$

$$X_{T_u} = \frac{\bar{q}_1 S}{m u_1} (C_{T_u} + 2C_{T_1})$$

$$X_\alpha = -\frac{\bar{q}_1 S}{m} (C_{D_\alpha} - C_{L_1})$$

$$X_{\delta_e} = -\frac{\bar{q}_1 S}{m} C_{D_{\delta_e}}$$

$$Z_u = -\frac{\bar{q}_1 S}{m u_1} (C_{L_u} + 2C_{L_1})$$

$$Z_\alpha = -\frac{\bar{q}_1 S}{m} (C_{L_\alpha} + C_{D_1})$$

$$Z_q = -\frac{\bar{q}_1 S \bar{c}}{2 m u_1} C_{L_q}$$

$$Z_{\dot{\alpha}} = -\frac{\bar{q}_1 S \bar{c}}{2 m u_1} C_{L_{\dot{\alpha}}}$$

$$Z_{\delta_e} = -\frac{\bar{q}_1 S}{m} C_{L_{\delta_e}}$$

$$M_u = \frac{\bar{q}_1 S \bar{c}}{I_{yy} u_1} (C_{M_u} + 2C_{M_1})$$

$$M_{T_u} = \frac{\bar{q}_1 S \bar{c}}{I_{yy} u_1} (C_{M_{T_u}} + 2C_{M_{T_1}})$$

$$M_\alpha = \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_\alpha}$$

$$M_{T_\alpha} = \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_{T_\alpha}}$$

$$M_q = \frac{\bar{q}_1 S \bar{c}^2}{2 I_{yy} u_1} C_{M_q}$$

$$M_{\dot{\alpha}} = \frac{\bar{q}_1 S \bar{c}^2}{2 I_{yy} u_1} C_{M_{\dot{\alpha}}}$$

$$M_{\delta_e} = \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_{\delta_e}}$$

Putting it all together...

Then, the linear longitudinal dynamics can be found in their final form by substituting the dimensional stability derivatives into the dynamics equations to get

$$\Delta\dot{u} = (X_u + X_{T_u})\Delta u + X_\alpha\Delta\alpha - g \cos \theta_1 \Delta\theta + X_{\delta_e}\Delta\delta_e \quad (8.22a)$$

$$(u_1 - Z_{\dot{\alpha}})\Delta\dot{\alpha} = Z_u\Delta u + Z_\alpha\Delta\alpha + (u_1 + Z_q)\Delta q - g \sin \theta_1 \Delta\theta + Z_{\delta_e}\Delta\delta_e \quad (8.22b)$$

$$\Delta\dot{q} - M_{\dot{\alpha}}\Delta\dot{\alpha} = (M_u + M_{T_u})\Delta u + (M_\alpha + M_{T_\alpha})\Delta\alpha + M_q\Delta q + M_{\delta_e}\Delta\delta_e \quad (8.22c)$$

$$\Delta\dot{\theta} = \Delta q \quad (8.22d)$$

The Linear Longitudinal Dynamics are...

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u_1 - Z_{\dot{\alpha}} & 0 & 0 \\ 0 & -M_{\dot{\alpha}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta \dot{u} \\ \Delta \dot{\alpha} \\ \Delta \dot{q} \\ \Delta \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u + X_{T_u} & X_\alpha & 0 & -g \cos \theta_1 \\ Z_u & Z_\alpha & u_1 + Z_q & -g \sin \theta_1 \\ M_u + M_{T_u} & M_\alpha + M_{T_\alpha} & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \alpha \\ \Delta q \\ \Delta \theta \end{bmatrix} + \begin{bmatrix} X_{\delta_e} \\ Z_{\delta_e} \\ M_{\delta_e} \\ 0 \end{bmatrix} \Delta \delta_e \quad (8.23)$$

Therefore, the longitudinal dynamics are given by the linear matrix equation

$$\mathbf{M}\dot{\mathbf{x}} = \mathbf{R}\mathbf{x} + \mathbf{F}\boldsymbol{\delta}$$

where

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u_1 - Z_{\dot{\alpha}} & 0 & 0 \\ 0 & -M_{\dot{\alpha}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} X_u + X_{T_u} & X_\alpha & 0 & -g \cos \theta_1 \\ Z_u & Z_\alpha & u_1 + Z_q & -g \sin \theta_1 \\ M_u + M_{T_u} & M_\alpha + M_{T_\alpha} & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} X_{\delta_e} \\ Z_{\delta_e} \\ M_{\delta_e} \\ 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \Delta u \\ \Delta \alpha \\ \Delta q \\ \Delta \theta \end{bmatrix}, \quad \dot{\mathbf{x}} = \begin{bmatrix} \Delta \dot{u} \\ \Delta \dot{\alpha} \\ \Delta \dot{q} \\ \Delta \dot{\theta} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{\delta} = \Delta \delta_e$$

In standard linear systems notation, this is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\delta}$$

$$\text{where } \mathbf{A} = \mathbf{M}^{-1}\mathbf{R} \quad \text{and} \quad \mathbf{B} = \mathbf{M}^{-1}\mathbf{F}$$

Example: Boeing 747 in High Cruise

A Boeing 747 airplane has the following physical characteristics:

$$W = 636,636 \text{ [lb]}, \quad I_{yy} = 3.31 \times 10^7 \text{ [slugs}\cdot\text{ft}^2\text]}, \quad S = 5,500 \text{ [ft}^2\text]}, \quad \text{and} \quad \bar{c} = 27.3 \text{ [ft]}$$

The aircraft is in trim at $u_1 = 516$ [knots] and $\theta_1 = 0$ [deg] when flying at an atmospheric density of $\rho = 5.8727 \times 10^{-4}$ [slugs/ft³], which corresponds to an altitude of $h = 40,000$ [ft]. The aerodynamic coefficients relevant to longitudinal dynamics of the Boeing 747 are given by

$$\begin{array}{llll} C_{L_1} = 0.52 & C_{L_{\delta_e}} = 0.3 & C_{T_1} = 0.045 & C_{M_{\dot{\alpha}}} = -9.0 \\ C_{L_u} = -0.23 & C_{D_1} = 0.045 & C_{T_u} = -0.95 & C_{M_q} = -25.0 \\ C_{L_\alpha} = 5.5 & C_{D_u} = 0.22 & C_{M_1} = 0.0 & C_{M_{\delta_e}} = -1.2 \\ C_{L_{\dot{\alpha}}} = 8.0 & C_{D_\alpha} = 0.50 & C_{M_u} = -0.09 & C_{M_{T_1}} = 0.0 \\ C_{L_q} = 7.8 & C_{D_{\delta_e}} = 0.0 & C_{M_\alpha} = -1.6 & C_{M_{T_u}} = 0.0 \end{array}$$

Using $g = 32.2$ [ft/s²], we wish to determine the matrices governing the linear longitudinal dynamics for the Boeing 747. First of all, we compute the dimensional stability derivatives:

$$\begin{array}{lll} X_u = -0.0221 & Z_q = -7.5742 & M_\alpha = -1.6165 \\ X_{T_u} = -0.0612 & Z_{\dot{\alpha}} = -7.7684 & M_{T_\alpha} = 0.0000 \\ X_\alpha = 1.2391 & Z_{\delta_e} = -18.5867 & M_q = -0.3959 \\ X_{\delta_e} = 0.0000 & M_u = -0.0001 & M_{\dot{\alpha}} = -0.1425 \\ Z_u = -0.0576 & M_{T_u} = 0.0000 & M_{\delta_e} = -1.2124 \\ Z_\alpha = -343.5450 & & \end{array}$$

These values can then be used to assemble the \mathbf{M} and \mathbf{R} matrices, yielding

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 878.6787 & 0 & 0 \\ 0 & 0.1425 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} -0.0832 & 1.2391 & 0 & -32.2000 \\ -0.0576 & -343.5450 & 863.3361 & 0 \\ -0.0001 & -1.6165 & -0.3959 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



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