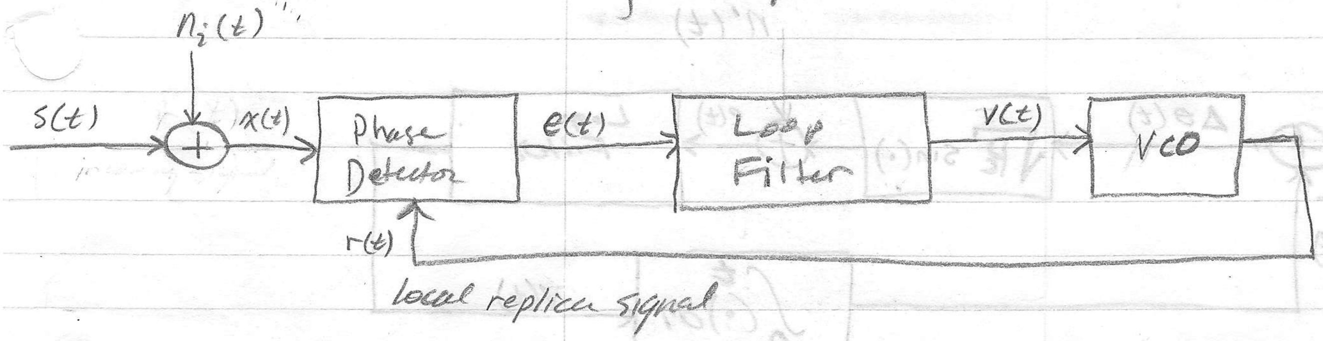
Phase Tracking Loops (Notes from Lindsey)



$$x(t) = \sqrt{2P_c}\, \sin[2\pi f_0 t + \theta(t)] + n_i(t)$$

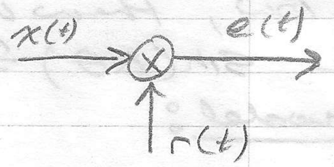
$$n_i(t) = \sqrt{2}\,[n_c(t)\cos 2\pi f_0 t - n_s(t) \sin 2\pi f_0 t] \quad \left(\begin{array}{c}\text{bandpass}\\ \text{noise}\end{array}\right)$$

$n_c(t)$ and $n_s(t)$ are independent, stationary white noise processes each with (two-sided) density $\frac{N_0}{2}$ W/Hz.

The voltage controlled oscillator generates a signal

$$r(t) = \sqrt{2}\cos\left[2\pi f_0 t + \int_0^t v(\tau)\,d\tau\right] = \sqrt{2}\cos[2\pi f_0 t + \hat{\theta}(t)]$$

The simplest phase detector is a mixer:



$$\sin a \cos b = \tfrac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

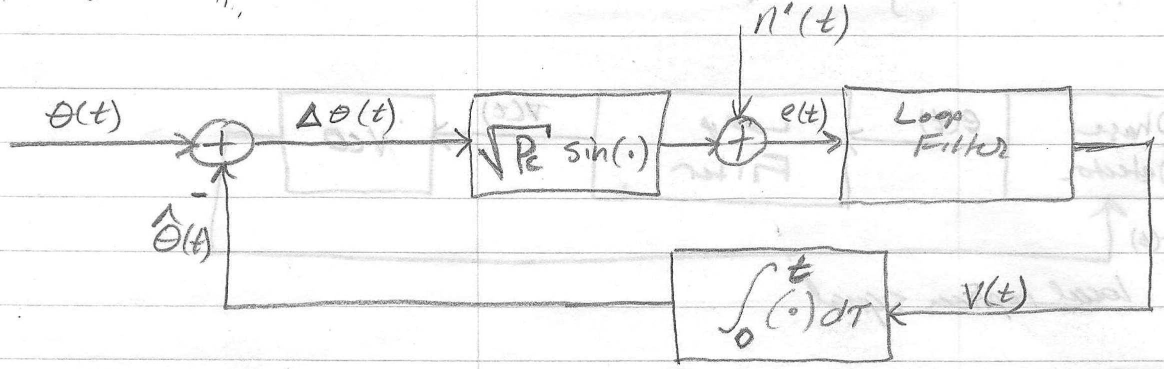
Then
$$\begin{aligned} e(t) = x(t)\,r(t) = {} & \sqrt{P_c}\,\sin(4\pi f_0 t + \theta + \hat{\theta}) \\ & + \sqrt{P_c}\,\sin(\theta - \hat{\theta}) \\ & + n_c(t)\cos(4\pi f_0 t + \hat{\theta}) + n_c(t)\cos(\hat{\theta}) \\ & - n_s(t)\sin(4\pi f_0 t + \hat{\theta}) + n_s(t)\sin(\hat{\theta}) \end{aligned}$$

Double-freq terms get filtered out by loop filter. Anticipating this, we can write

$$e(t) = \sqrt{P_c}\,\sin(\theta - \hat{\theta}) + n'(t), \quad \text{where}$$

$$n'(t) = n_c(t)\cos\hat{\theta} + n_s(t)\sin\hat{\theta}$$

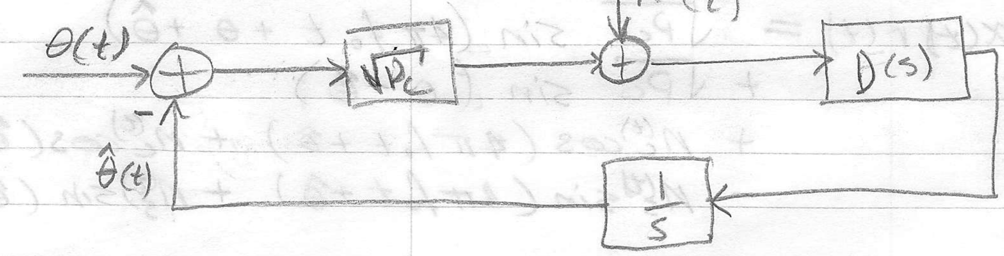
Equivalent Nonlinear Baseband Model:



$$\Delta\theta(t) = \theta(t) - \hat{\theta}(t)$$

This is a nonlinear, stochastic system. Very interesting behavior: ① Finite "pull in" **range**: If $\theta(t) = 2\pi f_D t + \theta_0$ (phase ramp), then there exists an $f_{D\max}$ beyond which a 1st-order loop can't lock.
② Cycle slipping phenomena
- phase unlock
- frequency unlock

If noise and $\theta(t)$ dynamics are low, then we can assume $\Delta\theta(t) \ll 1$ and hence $\sin(\Delta\theta) \cong \Delta\theta$. This gives rise to the <u>linear</u> <u>model</u>:



Costas Loop

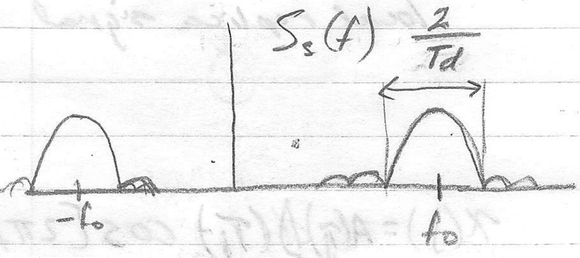
Suppose the received signal is modulated with BPSK modulation:

$$s(t) = \sqrt{2P_c}\, D(t) \sin[2\pi f_0 t + \theta(t)]$$

$$D(t) = \sum_{i=-\infty}^{\infty} d_i \, \Pi_{T_d}(t - iT_d)$$

$$d_i \in \{-1, 1\}$$

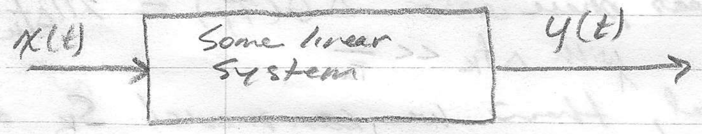
The power spectrum of $s(t)$ looks like



Notice that there remains no pure carrier component: there is no impulse in $S_s(f)$ at f_0. $s(t)$ is called a suppressed carrier signal.

Q: How can we extract the carrier signal component from $x(t) = s(t) + n_i(t)$?

Note:



$y(t)$ can never contain an impulse at f_0. Lin. sys. can only change phase & amp. of components of $x(t)$.

A: (1) Get the data bits from a side channel and "wipe off!"

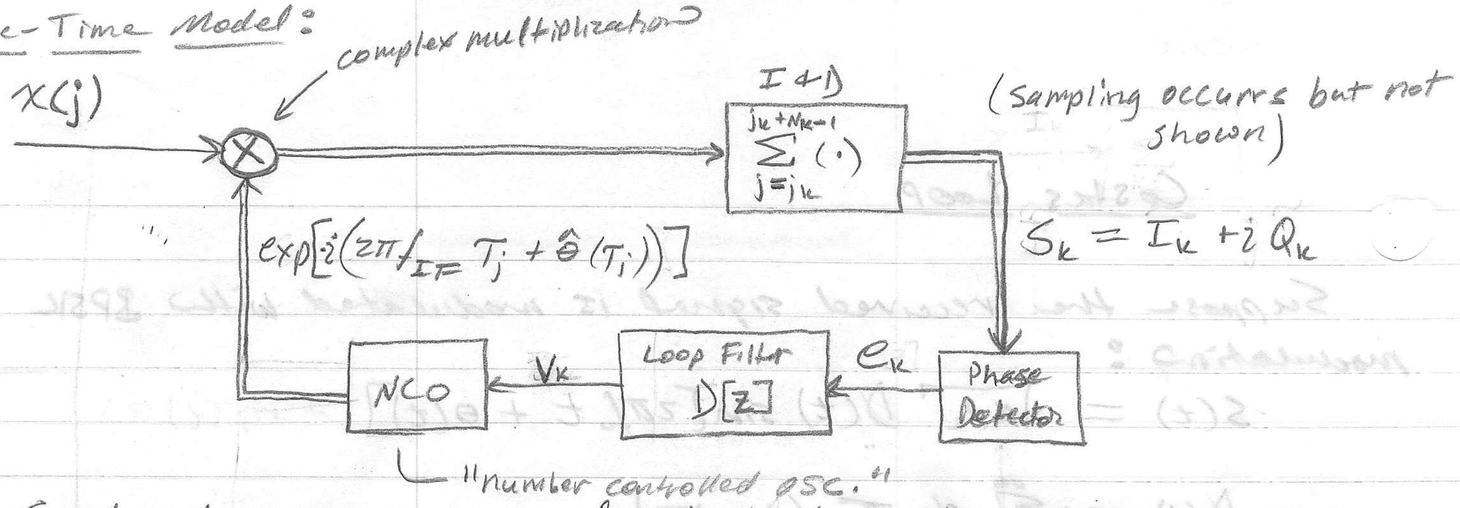
(2) Square the signal $x(t)$ and track the carrier at $2f_0$

(3) Employ Costas loop (same result as squaring)

~~In the following, we will introduce~~

With the introduction of the Costas loop, we will also transition from continuous to discrete time.

Discrete-Time Model:



Costas loop is simply a feedback loop with a complex-valued local replica signal and a data-insensitive phase detector.

$$x(j) = A(T_j)D(T_j)\cos[2\pi f_{IF} T_j + \theta(T_j)] + n(j)$$

$$S_k = \frac{N_k \bar{A}_k d_i}{2}\left[\frac{1}{N_k}\sum_{j=j_k}^{j_k+N_k-1} \exp[i\Delta\theta(T_j)]\right] + n_k$$

If we assume a linear phase error $\Delta\theta(T_j) = 2\pi\Delta f_k\cdot(T_j - T_{j_k}) + \Delta\theta(T_{j_k})$ with $\Delta f_k \ll \frac{1}{T_a}$ over the k^{th} interval, then we can model S_k as

$$S_k \cong \frac{N_k \bar{A}_k d_i}{2}\exp[i\Delta\theta_k] + n_k \quad , \quad \Delta\theta_k = \pi\Delta f_k(T_{j_{k+1}} - T_{j_k}) + \Delta\theta(T_{j_k})$$

$$\cong \pi\Delta f_k T_a + \Delta\theta(T_{j_k})$$

Phase Detectors (AKA discriminators):

1. Arctangent: $e_k = \mathrm{atan}(Q_k / I_k)$ (AT)

2. Conventional Costas: $e_k = Q_k \cdot I_k$ $\quad \bar{S}_k \in \{-1, 1\}$ = Best guess of symbol value at end of k^{th} accumulation

3. Decision-Directed: $e_k = \bar{S}_k \cdot Q_k$,

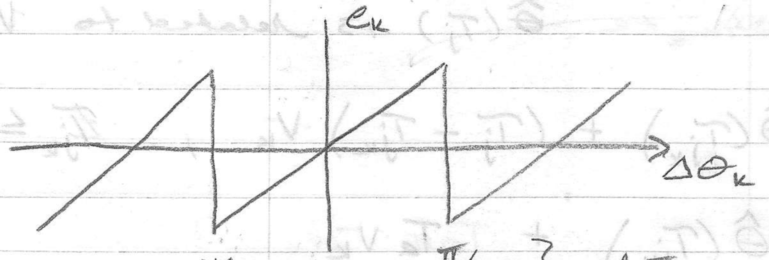
   [Can be based on FEC, known patterns, measured value]

4. Decision-Directed arctangent: (DDAT)

~~Note: If you happen to know the correct data bit value d_i substitute this for $\bar{S}_k$.~~

$$e_k = \mathrm{atan2}(\bar{S}_k \cdot Q_k, \bar{S}_k \cdot I_k)$$

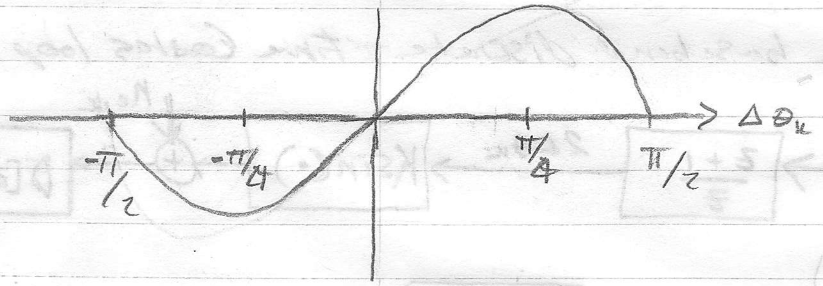
Arctangent phase detectors:



$-\pi/2 \quad \pi/2$ } AT

$-\pi \quad \pi$ } DDAT with correct symbols

NB: The phase detector has a linear response for small $\Delta\theta_k$! As we showed earlier, the AT disc. is ML estimate when freq known. Conventional Costas Detector characteristic looks similar for small $\Delta\theta_k$:



To arrive at a baseband discrete-time model, it's easiest to work with the CC detector:

$$e_k = Q_k \cdot I_k = \frac{N_k^2 \bar{A}_k^2}{8} \sin(2\Delta\theta_k) + n_{e,k}$$

(note that d_i vanished — Costas at work!)

Statistics of $n_{e,k}$ are

HW
$$E[n_{e,k}] = 0$$
$$E[n_{e,k} n_{e,i}] = \left[\frac{N_k^2 \bar{A}_k^2}{4}\sigma_{IQ}^2 + \sigma_{IQ}^4\right]\delta_{k,i}$$

$D[z]$ filters e_k to produce V_k.

$\hat{\theta}(T_j)$ is related to V_k by

$$\hat{\theta}(T_j) = \hat{\theta}(T_{j_k}) + (T_j - T_{j_k})\, V_k \,, \qquad T_{j_k} \le T_j < T_{j_{k+1}}$$

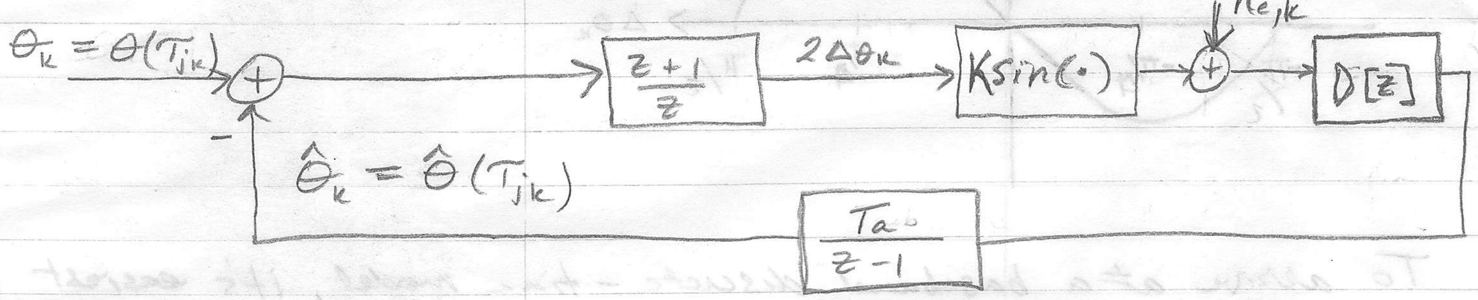
$$\hat{\theta}(T_{j_{k+1}}) \cong \hat{\theta}(T_{j_k}) + T_a V_k$$

The average phase error $\Delta\theta_k$ is related to θ and $\hat{\theta}$ by

$$\Delta\theta_k = \frac{1}{2}\left\{\left[\theta(T_{j_{k+1}}) - \hat{\theta}(T_{j_{k+1}})\right] + \left[\theta(T_{j_k}) - \hat{\theta}(T_{j_k})\right]\right\}$$

$$= \frac{1}{2}\left[\Delta\theta(T_{j_{k+1}}) + \Delta\theta(T_{j_k})\right]$$

We can now diagram the baseband discrete-time Costas loop model:



where $K = \dfrac{N_k^2 \bar{A}_k^2}{8}$

Now assume: ① Loop in lock ($\sin 2\Delta\theta_k \cong 2\Delta\theta_k$)

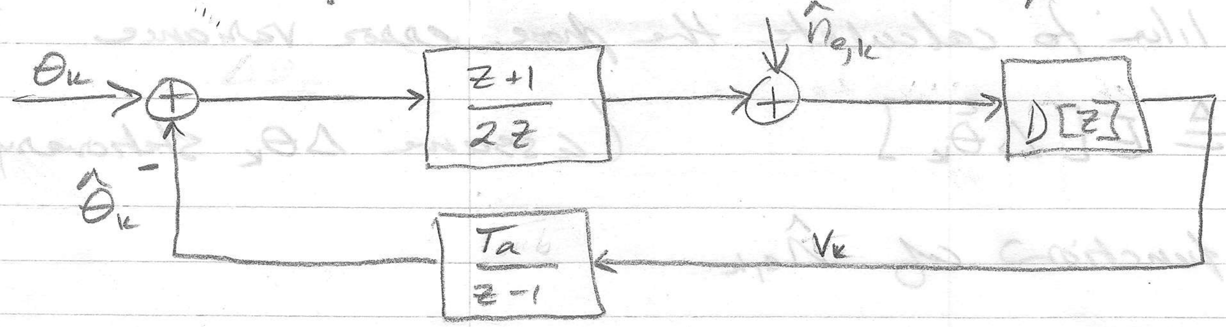
② A gain factor $\dfrac{4}{N_k^2 \bar{A}_k^2}$ is introduced to loop just before $D[z]$.

— or —

The AT or DDAT detector is used instead of CC.

$\Longrightarrow$ then $K = 1/2$

Linearized Discrete-time Costas Loop:



$\hat{n}_{e,k} = \dfrac{4 n_{e,k}}{N_k^2 \bar{A}_k^2}$ is normalized noise

$E[\hat{n}_{e,k}] = 0$

$$E[\hat{n}_{e,k}\hat{n}_{e,i}] = \underbrace{\left(\frac{4}{N_k^2 \bar{A}_k^2}\right)^2 \left[\frac{N_k^2 \bar{A}_k^2}{4}\sigma_{IQ}^2 + \sigma_{IQ}^4\right]}_{\triangleq\, \sigma_{\hat{n}_e}^2} \delta_{k,i}$$

Loop transfer fcn:

$$H[z] = \frac{\left(\frac{z+1}{2z}\right)\left(\frac{T_a}{z-1}\right) D[z]}{1 + \left(\frac{z+1}{2z}\right)\left(\frac{T_a}{z-1}\right) D[z]}$$

Loop noise transfer fcn: (from noise to $\hat{\theta}_k$)

$$H_n[z] = \frac{\left(\frac{T_a}{z-1}\right) D[z]}{1 + \left(\frac{z+1}{2z}\right)\left(\frac{T_a}{z-1}\right) D[z]}$$

We'd like to calculate the phase error variance

$\sigma_{\Delta\theta}^2 \triangleq E[\Delta\theta_k^2]$ (assume $\Delta\theta_k$ stationary)

as a function of $\hat{n}_{e,k}$

Let $H(s)$ and $H_n(s)$ be the continuous-time equivalents of $H[z]$ and $H_n[z]$ (assume $B_n \ll \frac{1}{T_a}$)

HW [then we can show that when loop is in linear regime

$$S_{\hat{n}_e} = \frac{N_0}{2C} \underbrace{\left(1 + \frac{N_0}{2CT_a}\right)}_{S_L^{-1}} = \frac{N_0}{2C S_L}$$

S_L is called the "squaring loss" factor: penalty paid relative to standard (nonsquaring) PLL, for multiplying I_k & Q_k in the CC phase detector (also applies approximately to AT detector).

$$\boxed{\sigma_{\Delta\theta}^2 \triangleq E[\Delta\theta_k^2] = \int_{-\infty}^{\infty} |H_n(f)|^2 S_{\hat{n}_e}(f)\, df = \frac{N_0 B_n}{C S_L}} \quad (\text{rad}^2)$$

$$B_n = \frac{1}{2}\int_{-\infty}^{\infty} |H_n(f)|^2 df$$

$$\cong \frac{1}{2}\int_{-\infty}^{\infty} |H(f)|^2 df$$

Loop Signal-to-Noise Ratio, ρ_L

ρ_L is defined for standard (non-squaring) phase tracking loops as

$$\rho_L \triangleq \frac{1}{\sigma^2_{\Delta\theta}}$$

When the loop is in its linear regime, $\rho_L = \frac{C}{N_0 B_n}$.

For a squaring-type loop (e.g., Costas) we track a signal at $2f_0$ so phase error is $2\Delta\theta$, with a corresponding variance $\sigma^2_{2\Delta\theta} = 4\sigma^2_{\Delta\theta}$. [A phase error of 0.2 rad at $2f_0$ would lead to an error of 0.1 rad at f_0]

Thus $\sigma^2_{\Delta\theta} = \frac{\sigma^2_{2\Delta\theta}}{4} \cong \frac{1}{\rho_L S_L}$

is a good approximation to $\sigma^2_{\Delta\theta}$ in the linear regime.

For analysis of the squaring loop, an equivalent loop SNR is defined as:

$$\boxed{\rho_{eq} \triangleq \frac{\rho_L S_L}{4}}$$

which leads to $\rho_{eq} \cong \frac{1}{\sigma^2_{2\Delta\theta}}$ for small $2\Delta\theta$.

Rule of thumb: the linear model is reasonably accurate for

$$\begin{cases} \rho_L > 4 & \text{(standard loop)} \quad (\sigma_{\Delta\theta} < 28^\circ) \\ \rho_{eq} > 4 & \text{(squaring loop)} \quad (\sigma_{\Delta\theta} < 14^\circ) \end{cases}$$

We define the mean time to first cycle slip T_s as the average time required for phase error to reach

$$\begin{cases} \pm 2\pi & \text{(standard loop)} \\ \pm \pi & \text{(squaring loop)} \end{cases}$$

T_s can be calculated analytically for a 1st-order unstressed PLL driven by white noise:

(Tikhanov)

$$T_s = \frac{\pi^2 \rho\, I_0^2(\rho)}{2 B_n}, \quad \rho \in \{\rho_L, \rho_{eq}\}$$

mod. Bessel fcn.

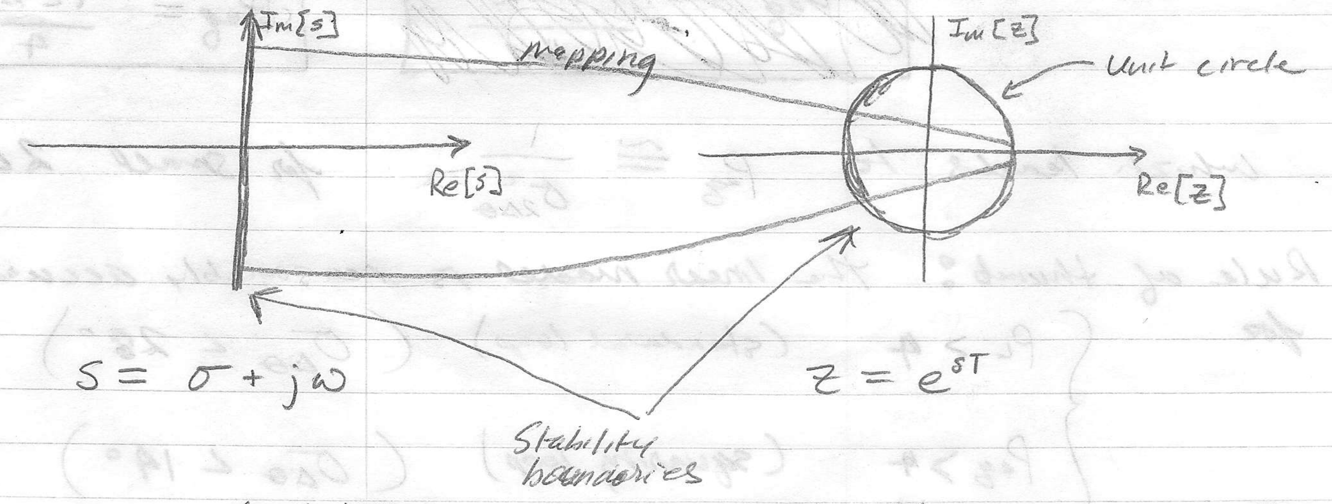
This is a useful approx even for higher-order loops.

Discretizing $D(s)$ to get $D[z]$

The right way to design $D[z]$ is to do the design in the z-domain. (See paper by Stephens (1995) on Canvas).

But the lazy/ignorant/expedient way is to design $D(s)$ and then convert $D(s)$ to $D[z]$.

Fundamental relationship: $z = e^{sT}$
where T is sampling interval. Defines a conformal mapping from s-plane to z-plane:



$$z = e^{sT} = e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T} = |z| \left[\cos\omega T + j \sin\omega T \right]$$

When T is small then a large range of ω along the $Im[s]$ axis maps into a small region of the unit circle. Thus, the unit circle looks "flat" to nearby poles → the discrete-time system behaves much like its continuous-time counterpart.

See c2d and d2c in Matlab controls toolbox.

HW [Look at freq response and step response of $H(s)$ and $H[z]$ for all three loops presented earlier.

HW [Design a Costas loop in Matlab. Make sure to get the output of $F[z]$ right. Take in estimate of data bits, if available. Inject the right amount of noise. (Maybe give them some code here

Note: We won't discuss FLLs, but these are generally more robust than PLLs and are useful for doing the initial transition from acquisition to phase tracking.