

**22 OCTOBER 2024**

# **ASE 367K: FLIGHT DYNAMICS**

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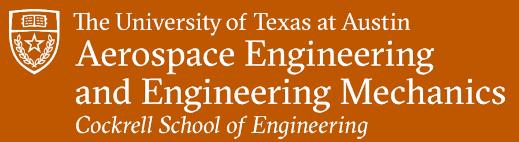
TTH 09:30-11:00  
CMA 2.306

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# Topics for Today

- Topic(s):
  - Videos of Longitudinal Dynamics
  - Thoughts on HW6
  - Linearized Lateral Equations of Motion



# USEFUL LONGITUDINAL DYNAMICS VIDEOS

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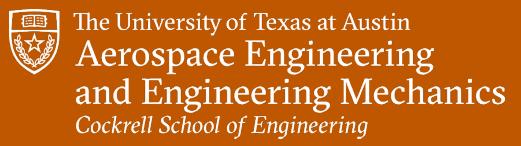
# Useful Videos

## ■ Phugoid

- <https://www.youtube.com/watch?v=9DnSG4H2UjE>
- [https://www.youtube.com/watch?v=vbKaCqF\\_E1M](https://www.youtube.com/watch?v=vbKaCqF_E1M)

## ■ Short Period

- [https://www.youtube.com/watch?v=1O7ZqBS0\\_B8](https://www.youtube.com/watch?v=1O7ZqBS0_B8)



# THOUGHTS ON HW6

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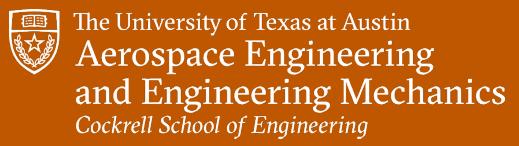
# Problem 1

Given the linear system model provided on Slide 8 of Lecture 15 and the dimensional stability derivatives for the B747 at “low cruise” provided on Slide 13 of Lecture 15:

- a. Determine the damped frequency, natural frequency, damping ratio, the time to damp to half the initial amplitude, and the number of cycles to damp to half the initial amplitude for BOTH the short period and the Phugoid (long period) modes.
- b. Develop a simulation (using either MATLAB or Python) for this model and use it to determine and plot the values of the state variables  $\Delta u$ ,  $\Delta \alpha$ ,  $\Delta q$ , and  $\Delta \theta$  (as functions of time) in response to the step and impulse elevator deflections of maximum amplitude.

## Problem 1b

- How do you determine  $u_1$  and  $\theta_1$  for your simulation?
  - Assume level flight
  - Select altitude and weight
  - Set  $u_1$  equal to the velocity of your choice
  - Compute  $C_L$  then  $\alpha$
  - Set  $\theta_1$  equal to  $\alpha$



# LINEARIZED LATERAL EOM

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## Starting with the nonlinear lateral equations of motion...

- Because the lateral position equation is uncoupled and can be integrated independently... We get the following 5 nonlinear lateral equations of motion

$$Y = m(\dot{v} + ru - pw - g \cos \theta \sin \phi)$$

$$L = I_{xx}\dot{p} - I_{xz}(\dot{r} + pq) - (I_{yy} - I_{zz})qr$$

$$N = I_{zz}\dot{r} - I_{xz}(\dot{p} - qr) - (I_{xx} - I_{yy})pq$$

$$p = \dot{\phi} - \dot{\psi} \sin \theta$$

$$r = -\dot{\theta} \sin \phi + \dot{\psi} \cos \theta \cos \phi$$

## Substituting speed and angular rate perturbations...

- We can represent the quantities of interest as the sum of the nominal operating condition and a perturbation from nominal

$$u = u_1 + \Delta u$$

$$v = v_1 + \Delta v$$

$$w = w_1 + \Delta w$$

$$p = p_1 + \Delta p$$

$$q = q_1 + \Delta q$$

$$r = r_1 + \Delta r$$

$$\phi = \phi_1 + \Delta \phi$$

$$\theta = \theta_1 + \Delta \theta$$

$$\psi = \psi_1 + \Delta \psi$$

## Focusing in on the effects of these perturbations...

$$\Delta Y = m(\Delta \dot{v} + r_1 \Delta u - p_1 \Delta w - w_1 \Delta p + u_1 \Delta r - g \cos \theta_1 \cos \phi_1 \Delta \phi + g \sin \theta_1 \sin \phi_1 \Delta \theta)$$

$$\Delta L = I_{xx} \Delta \dot{p} - I_{xz} \Delta \dot{r} - I_{xz} p_1 \Delta q - I_{xz} q_1 \Delta p - (I_{yy} - I_{zz}) q_1 \Delta r - (I_{yy} - I_{zz}) r_1 \Delta q$$

$$\Delta N = I_{zz} \Delta \dot{r} - I_{xz} \Delta \dot{p} + I_{xz} r_1 \Delta q + I_{xz} q_1 \Delta r - (I_{xx} - I_{yy}) q_1 \Delta p - (I_{xx} - I_{yy}) p_1 \Delta q$$

$$\Delta p = \Delta \dot{\phi} - \dot{\psi}_1 \cos \theta_1 \Delta \theta - \sin \theta_1 \Delta \dot{\psi}$$

$$\Delta r = -\sin \phi_1 \Delta \dot{\theta} + \cos \theta_1 \cos \phi_1 \Delta \dot{\psi} - \dot{\theta}_1 \cos \phi_1 \Delta \phi - \dot{\psi}_1 \cos \theta_1 \sin \phi_1 \Delta \phi - \dot{\psi}_1 \sin \theta_1 \cos \phi_1 \Delta \theta$$

## Eliminating steady-state values that are zero...

- If we linearize w.r.t. a constant velocity flight with zero beta and we choose a body-fixed frame that is aligned with the nominal velocity vector, then ...

$$w_1 = 0 \quad v_1 = 0, \quad p_1 = 0, \quad q_1 = 0, \quad r_1 = 0, \quad \text{and} \quad \phi_1 = 0$$

$$\dot{\phi}_1 = 0, \quad \dot{\theta}_1 = 0, \quad \text{and} \quad \dot{\psi}_1 = 0$$

- and...

$$\Delta Y = m(\Delta \dot{v} + r_1 \Delta u - p_1 \Delta w - q_1 \Delta p + u_1 \Delta r - g \cos \theta_1 \cos \phi_1 \Delta \phi + g \sin \theta_1 \sin \phi_1 \Delta \theta)$$

$$\Delta L = I_{xx} \Delta \dot{p} - I_{xz} \Delta \dot{r} - I_{xz} p_1 \Delta q - I_{xz} q_1 \Delta p - (I_{yy} - I_{zz}) q_1 \Delta r - (I_{yy} - I_{zz}) r_1 \Delta q$$

$$\Delta N = I_{zz} \Delta \dot{r} - I_{xz} \Delta \dot{p} + I_{xz} r_1 \Delta q + I_{xz} q_1 \Delta r - (I_{xx} - I_{yy}) q_1 \Delta p - (I_{xx} - I_{yy}) p_1 \Delta q$$

$$\Delta p = \Delta \dot{\phi} - \dot{\psi}_1 \cos \theta_1 \Delta \theta - \sin \theta_1 \Delta \dot{\psi}$$

$$\Delta r = -\sin \phi_1 \Delta \dot{\theta} + \cos \theta_1 \cos \phi_1 \Delta \dot{\psi} - \dot{\theta}_1 \cos \phi_1 \Delta \phi - \dot{\psi}_1 \cos \theta_1 \sin \phi_1 \Delta \phi - \dot{\phi}_1 \sin \theta_1 \cos \phi_1 \Delta \theta$$

**Gives us the linearized lateral equations of motion**

$$\Delta Y = m(\Delta \dot{v} + u_1 \Delta r - g \cos \theta_1 \Delta \phi)$$

$$\Delta L = I_{xx} \Delta \dot{p} - I_{xz} \Delta \dot{r}$$

$$\Delta N = I_{zz} \Delta \dot{r} - I_{xz} \Delta \dot{p}$$

$$\Delta p = \Delta \dot{\phi} - \sin \theta_1 \Delta \dot{\psi}$$

$$\Delta r = \cos \theta_1 \Delta \dot{\psi}$$

**Then replacing  $\Delta w$  with  $\Delta\beta$  gives us...**

$$v_1 = 0, \text{ therefore } \beta_1 = 0, \text{ hence} \quad \Delta\beta = \frac{\Delta v}{u_1} \quad \text{or} \quad \Delta v = u_1 \Delta\beta$$

Furthermore, since  $\dot{u}_1 = 0$ , it follows that  $\Delta\dot{v} = u_1 \Delta\dot{\beta}$

$$\Delta Y = m(u_1 \Delta\dot{\beta} + u_1 \Delta r - g \cos \theta_1 \Delta\phi)$$

$$\Delta L = I_{xx} \Delta \dot{p} - I_{xz} \Delta \dot{r}$$

$$\Delta N = I_{zz} \Delta \dot{r} - I_{xz} \Delta \dot{p}$$

$$\Delta p = \Delta\dot{\phi} - \sin \theta_1 \Delta\dot{\psi}$$

$$\Delta r = \cos \theta_1 \Delta\dot{\psi}$$

**Which can be reorganized as...**

$$u_1 \Delta \dot{\beta} = \Delta Y/m - u_1 \Delta r + g \cos \theta_1 \Delta \phi$$

$$\Delta \dot{p} - (I_{xz}/I_{xx}) \Delta \dot{r} = \Delta L/I_{xx}$$

$$\Delta \dot{r} - (I_{xz}/I_{zz}) \Delta \dot{p} = \Delta N/I_{zz}$$

$$\Delta \dot{\phi} = \Delta p + \tan \theta_1 \Delta r$$

$$\Delta \dot{\psi} = \sec \theta_1 \Delta r$$

## Applying the chain rule to the force and moments...

$$\Delta Y = \left( \frac{\partial Y}{\partial \beta} \right)_1 \Delta \beta + \left( \frac{\partial Y}{\partial p} \right)_1 \Delta p + \left( \frac{\partial Y}{\partial r} \right)_1 \Delta r + \left( \frac{\partial Y}{\partial \delta_a} \right)_1 \Delta \delta_a + \left( \frac{\partial Y}{\partial \delta_r} \right)_1 \Delta \delta_r$$

$$\Delta L = \left( \frac{\partial L}{\partial \beta} \right)_1 \Delta \beta + \left( \frac{\partial L}{\partial p} \right)_1 \Delta p + \left( \frac{\partial L}{\partial r} \right)_1 \Delta r + \left( \frac{\partial L}{\partial \delta_a} \right)_1 \Delta \delta_a + \left( \frac{\partial L}{\partial \delta_r} \right)_1 \Delta \delta_r$$

$$\begin{aligned} \Delta N = & \left( \frac{\partial N_A}{\partial \beta} + \frac{\partial N_T}{\partial \beta} \right)_1 \Delta \beta + \left( \frac{\partial N_A}{\partial p} + \frac{\partial N_T}{\partial p} \right)_1 \Delta p + \left( \frac{\partial N_A}{\partial r} + \frac{\partial N_T}{\partial r} \right)_1 \Delta r \\ & + \left( \frac{\partial N_A}{\partial \delta_a} + \frac{\partial N_T}{\partial \delta_a} \right)_1 \Delta \delta_a + \left( \frac{\partial N_A}{\partial \delta_r} + \frac{\partial N_T}{\partial \delta_r} \right)_1 \Delta \delta_r \end{aligned}$$

# Neglecting effects that are insignificant...

Typically, the contribution to the thrust yaw moment due to rolling angular velocity and the lateral controls is zero, such that

$$\Delta Y = \left( \frac{\partial Y}{\partial \beta} \right)_1 \Delta \beta + \left( \frac{\partial Y}{\partial p} \right)_1 \Delta p + \left( \frac{\partial Y}{\partial r} \right)_1 \Delta r + \left( \frac{\partial Y}{\partial \delta_a} \right)_1 \Delta \delta_a + \left( \frac{\partial Y}{\partial \delta_r} \right)_1 \Delta \delta_r$$

$$\Delta L = \left( \frac{\partial L}{\partial \beta} \right)_1 \Delta \beta + \left( \frac{\partial L}{\partial p} \right)_1 \Delta p + \left( \frac{\partial L}{\partial r} \right)_1 \Delta r + \left( \frac{\partial L}{\partial \delta_a} \right)_1 \Delta \delta_a + \left( \frac{\partial L}{\partial \delta_r} \right)_1 \Delta \delta_r$$

$$\Delta N = \left( \frac{\partial N_A}{\partial \beta} + \frac{\partial N_T}{\partial \beta} \right)_1 \Delta \beta + \left( \frac{\partial N_A}{\partial p} \right)_1 \Delta p + \left( \frac{\partial N_A}{\partial r} + \frac{\partial N_T}{\partial r} \right)_1 \Delta r + \left( \frac{\partial N_A}{\partial \delta_a} \right)_1 \Delta \delta_a + \left( \frac{\partial N_A}{\partial \delta_r} \right)_1 \Delta \delta_r$$

# Substituting the stability derivatives...

Let the dimensional stability derivatives be defined as

$$Y_\beta = \frac{\bar{q}_1 S}{m} C_{y_\beta}$$

$$Y_p = \frac{\bar{q}_1 Sb}{2mu_1} C_{y_p}$$

$$Y_r = \frac{\bar{q}_1 Sb}{2mu_1} C_{y_r}$$

$$Y_{\delta_a} = \frac{\bar{q}_1 S}{m} C_{y_{\delta_a}}$$

$$Y_{\delta_r} = \frac{\bar{q}_1 S}{m} C_{y_{\delta_r}}$$

$$L_\beta = \frac{\bar{q}_1 Sb}{I_{xx}} C_{\ell_\beta}$$

$$L_p = \frac{\bar{q}_1 Sb^2}{2I_{xx}u_1} C_{\ell_p}$$

$$L_r = \frac{\bar{q}_1 Sb^2}{2I_{xx}u_1} C_{\ell_r}$$

$$L_{\delta_a} = \frac{\bar{q}_1 Sb}{I_{xx}} C_{\ell_{\delta_a}}$$

$$L_{\delta_r} = \frac{\bar{q}_1 Sb}{I_{xx}} C_{\ell_{\delta_r}}$$

$$N_\beta = \frac{\bar{q}_1 Sb}{I_{zz}} C_{n_\beta}$$

$$N_{T_\beta} = \frac{\bar{q}_1 Sb}{I_{zz}} C_{n_{T_\beta}}$$

$$N_p = \frac{\bar{q}_1 Sb^2}{2I_{zz}u_1} C_{n_p}$$

$$N_r = \frac{\bar{q}_1 Sb^2}{2I_{zz}u_1} C_{n_r}$$

$$N_{T_r} = \frac{\bar{q}_1 Sb^2}{2I_{zz}u_1} C_{n_{T_r}}$$

$$N_{\delta_a} = \frac{\bar{q}_1 Sb}{I_{zz}} C_{n_{\delta_a}}$$

$$N_{\delta_r} = \frac{\bar{q}_1 Sb}{I_{zz}} C_{n_{\delta_r}}$$

## Gives the following accelerations...

$$\Delta Y/m = Y_\beta \Delta \beta + Y_p \Delta p + Y_r \Delta r + Y_{\delta_a} \Delta \delta_a + Y_{\delta_r} \Delta \delta_r$$

$$\Delta L/I_{xx} = L_\beta \Delta \beta + L_p \Delta p + L_r \Delta r + L_{\delta_a} \Delta \delta_a + L_{\delta_r} \Delta \delta_r$$

$$\Delta N/I_{zz} = (N_\beta + N_{T_\beta}) \Delta \beta + N_p \Delta p + (N_r + N_{T_r}) \Delta r + N_{\delta_a} \Delta \delta_a + N_{\delta_r} \Delta \delta_r$$

**Which after putting it all together becomes...**

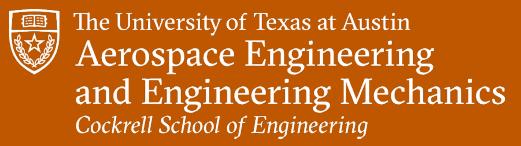
$$u_1 \Delta \dot{\beta} = Y_\beta \Delta \beta + Y_p \Delta p + Y_r \Delta r + Y_{\delta_a} \Delta \delta_a + Y_{\delta_r} \Delta \delta_r - u_1 \Delta r + g \cos \theta_1 \Delta \phi$$

$$\Delta \dot{p} - (I_{xz}/I_{xx}) \Delta \dot{r} = L_\beta \Delta \beta + L_p \Delta p + L_r \Delta r + L_{\delta_a} \Delta \delta_a + L_{\delta_r} \Delta \delta_r$$

$$\Delta \dot{r} - (I_{xz}/I_{zz}) \Delta \dot{p} = (N_\beta + N_{T_\beta}) \Delta \beta + N_p \Delta p + (N_r + N_{T_r}) \Delta r + N_{\delta_a} \Delta \delta_a + N_{\delta_r} \Delta \delta_r$$

$$\Delta \dot{\phi} = \Delta p + \tan \theta_1 \Delta r$$

$$\Delta \dot{\psi} = \sec \theta_1 \Delta r$$



# LATERAL MODES

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# The linearized lateral EOM in Matrix Form is...

$$\begin{bmatrix} u_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -(I_{xz}/I_{xx}) & 0 & 0 \\ 0 & -(I_{xz}/I_{zz}) & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\phi} \\ \Delta\dot{\psi} \end{bmatrix} = \begin{bmatrix} Y_\beta & Y_p & (Y_r - u_1) & g \cos \theta_1 & 0 \\ L_\beta & L_p & L_r & 0 & 0 \\ (N_\beta + N_{T_\beta}) & N_p & (N_r + N_{T_r}) & 0 & 0 \\ 0 & 1 & \tan \theta_1 & 0 & 0 \\ 0 & 0 & 0 & \sec \theta_1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \end{bmatrix} + \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix}$$

Therefore, the lateral dynamics are given by the linear matrix equation

$$\mathbf{M}\dot{\mathbf{x}} = \mathbf{R}\mathbf{x} + \mathbf{F}\boldsymbol{\delta}$$

where

$$\mathbf{M} = \begin{bmatrix} u_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -(I_{xz}/I_{xx}) & 0 & 0 \\ 0 & -(I_{xz}/I_{zz}) & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} Y_\beta & Y_p & (Y_r - u_1) & g \cos \theta_1 & 0 \\ L_\beta & L_p & L_r & 0 & 0 \\ (N_\beta + N_{T_\beta}) & N_p & (N_r + N_{T_r}) & 0 & 0 \\ 0 & 1 & \tan \theta_1 & 0 & 0 \\ 0 & 0 & 0 & \sec \theta_1 & 0 \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \end{bmatrix}, \quad \dot{\mathbf{x}} = \begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\phi} \\ \Delta\dot{\psi} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{\delta} = \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix}$$

In standard linear systems notation, this is  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$  where  $\mathbf{A} = \mathbf{M}^{-1}\mathbf{R}$  and  $\mathbf{B} = \mathbf{M}^{-1}\mathbf{F}$

A Boeing 747 airplane has the following characteristics

$$W = 636,636 \text{ [lb]}, \quad I_{xx}^b = 1.82 \times 10^7 \text{ [slugs}\cdot\text{ft}^2], \quad I_{zz}^b = 4.97 \times 10^7 \text{ [slugs}\cdot\text{ft}^2], \\ I_{xz}^b = 9.70 \times 10^5 \text{ [slugs}\cdot\text{ft}^2], \quad S = 5,500 \text{ [ft}^2], \quad \text{and} \quad b = 195.7 \text{ [ft]}$$

The aircraft is in trim at  $u_1 = 399$  [knots],  $\theta_1 = 2.4$  [deg], and  $\alpha_1 = 2.4$  [deg] when flying at an atmospheric density of  $\rho = 1.2673 \times 10^{-3}$  [slugs/ft<sup>3</sup>]. The aerodynamic coefficients relevant to lateral dynamic stability of the 747 are given by

$$\begin{array}{lll} C_{y_\beta} = -0.9000 & C_{\ell_p} = -0.3400 & C_{n_p} = -0.0260 \\ C_{y_p} = 0.0000 & C_{\ell_r} = 0.1300 & C_{n_r} = -0.2800 \\ C_{y_r} = 0.0000 & C_{\ell_{\delta_a}} = 0.0130 & C_{n_{T_r}} = 0.0000 \\ C_{y_{\delta_a}} = 0.0000 & C_{\ell_{\delta_r}} = 0.0080 & C_{n_{\delta_a}} = 0.0018 \\ C_{y_{\delta_r}} = 0.1200 & C_{n_\beta} = 0.1600 & C_{n_{\delta_r}} = -0.1000 \\ C_{\ell_\beta} = -0.1600 & C_{n_{T_\beta}} = 0.0000 & \end{array}$$

The A and B matrices are:

$$\mathbf{A} = \begin{bmatrix} -0.1067 & 0 & -1.0000 & 0.0477 & 0 \\ -2.7427 & -0.8404 & 0.3264 & 0 & 0 \\ 1.0146 & -0.0176 & -0.2554 & 0 & 0 \\ 0 & 1.0000 & 0.0419 & 0 & 0 \\ 0 & 0 & 1.0009 & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 0.0142 \\ 0.2211 & 0.1482 \\ 0.0096 & -0.6231 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

With the matrix  $\mathbf{A}$  built, we can compute the eigenvalues and eigenvectors of the matrix in order to determine stability and some modal characteristics of the system. For this case, we find that the eigenvalues are given by

$$\lambda_1 = 0, \quad \lambda_2 = -0.9388, \quad \lambda_3 = -0.0171, \quad \lambda_{4,5} = -0.1234 \pm 1.0416i$$

and that the normalized non-dimensional eigenvector magnitudes are

$$\|\tilde{\mathbf{v}}\|_1 = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 1.0000 \end{bmatrix}, \quad \|\tilde{\mathbf{v}}\|_2 = \begin{bmatrix} 0.0311 \\ 0.1364 \\ 0.0032 \\ 1.0000 \\ 0.0234 \end{bmatrix}, \quad \|\tilde{\mathbf{v}}\|_3 = \begin{bmatrix} 0.0039 \\ 0.0009 \\ 0.0025 \\ 0.3648 \\ 1.0000 \end{bmatrix}, \quad \|\tilde{\mathbf{v}}\|_{4,5} = \begin{bmatrix} 0.4859 \\ 0.1524 \\ 0.0699 \\ 1.0000 \\ 0.4589 \end{bmatrix}$$

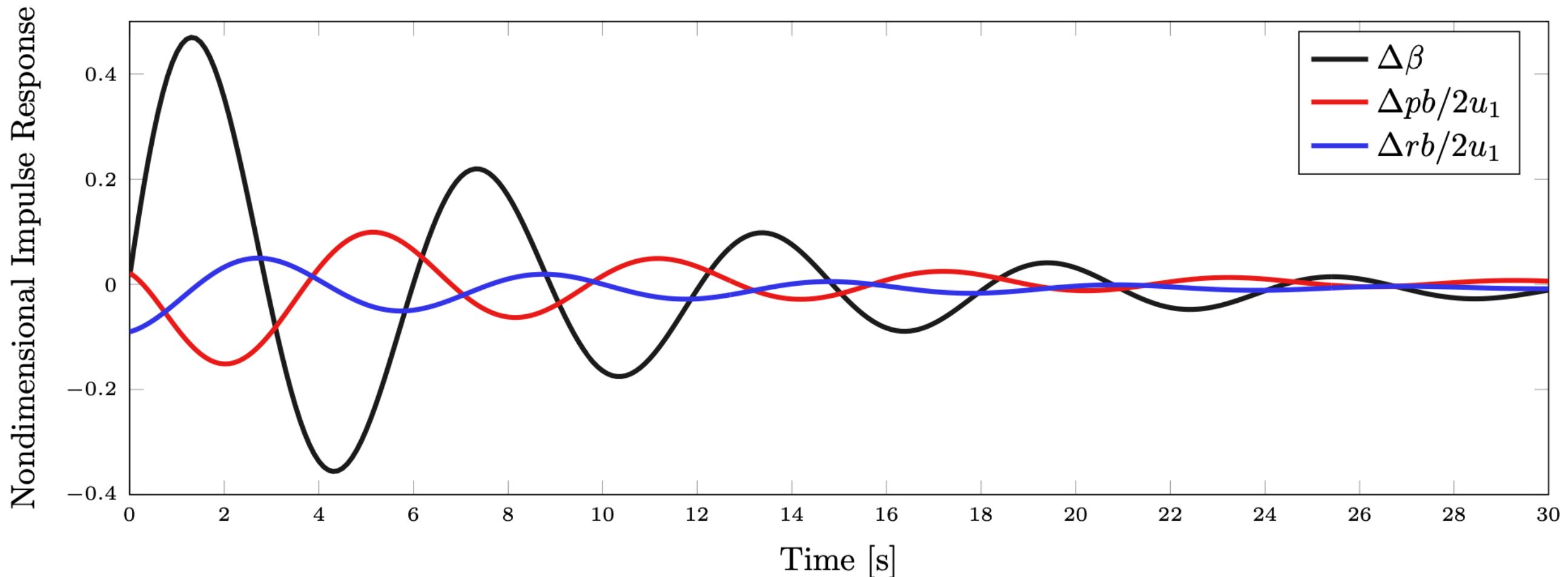
**Remember the  
5 components  
of the state vector:**

$$\mathbf{x} = \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \end{bmatrix}$$

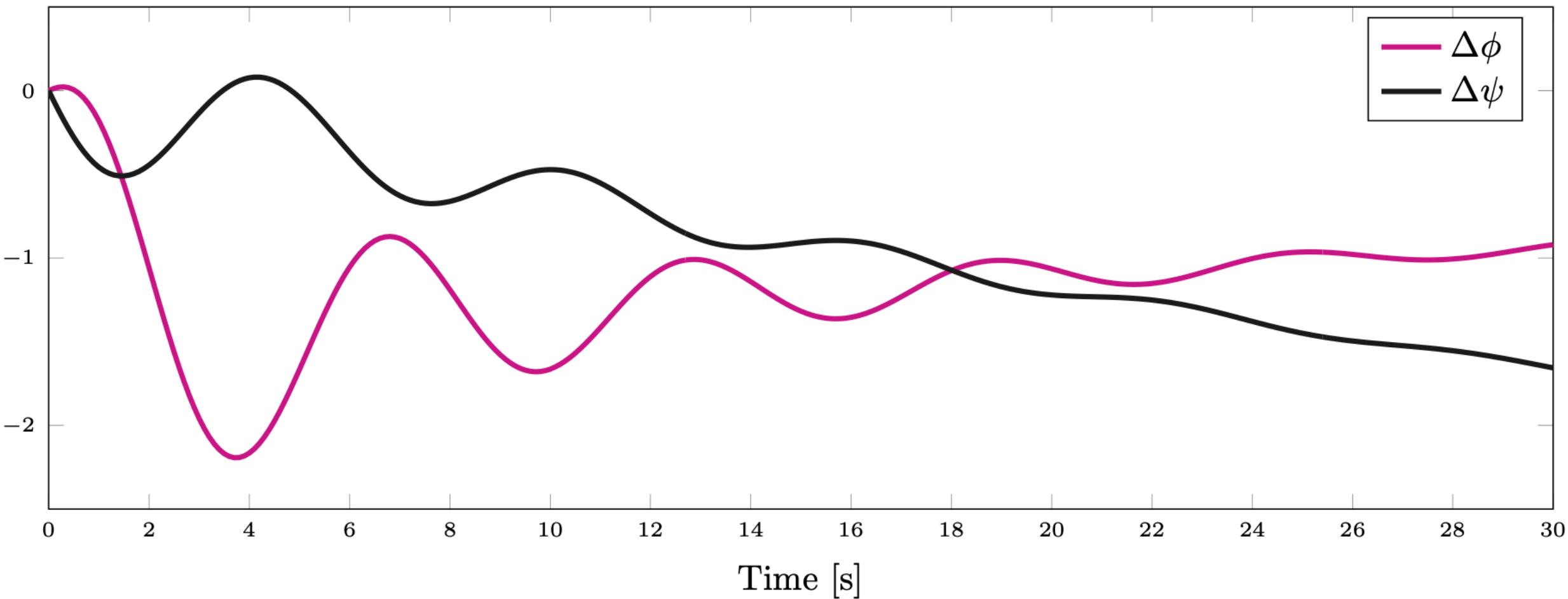
where normalization is done such that the largest element is one. Based on the above data, complete the following table:

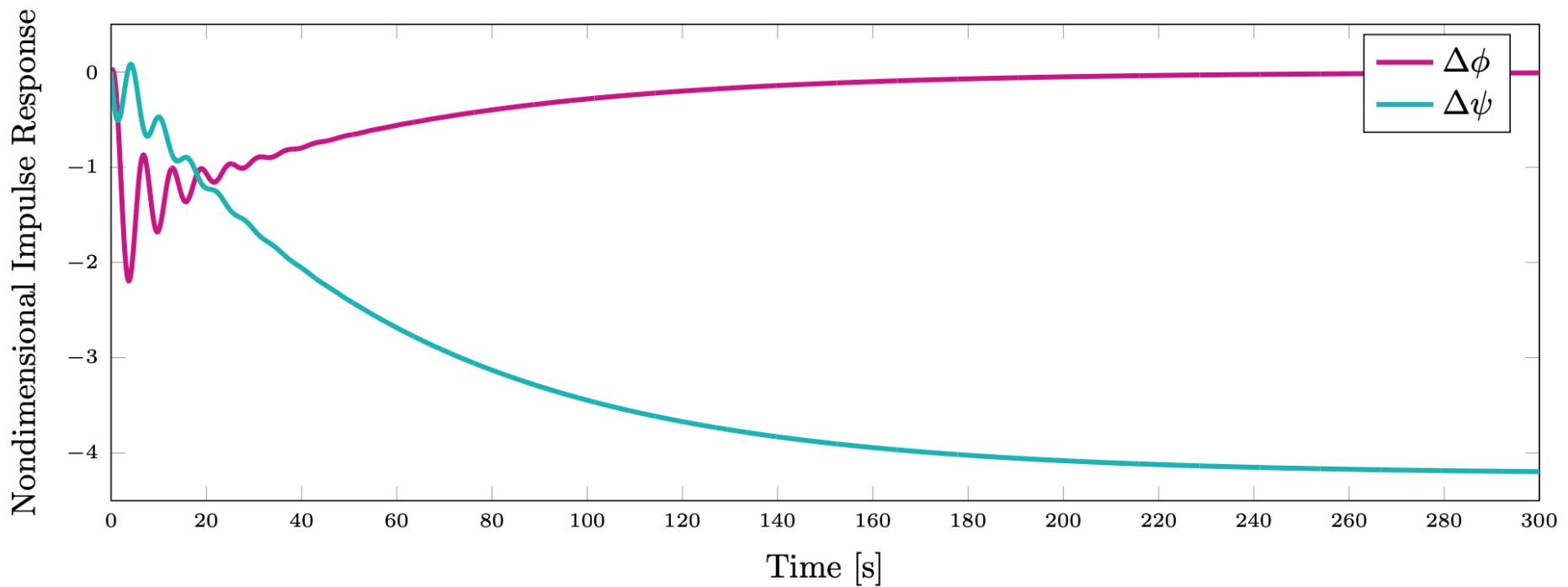
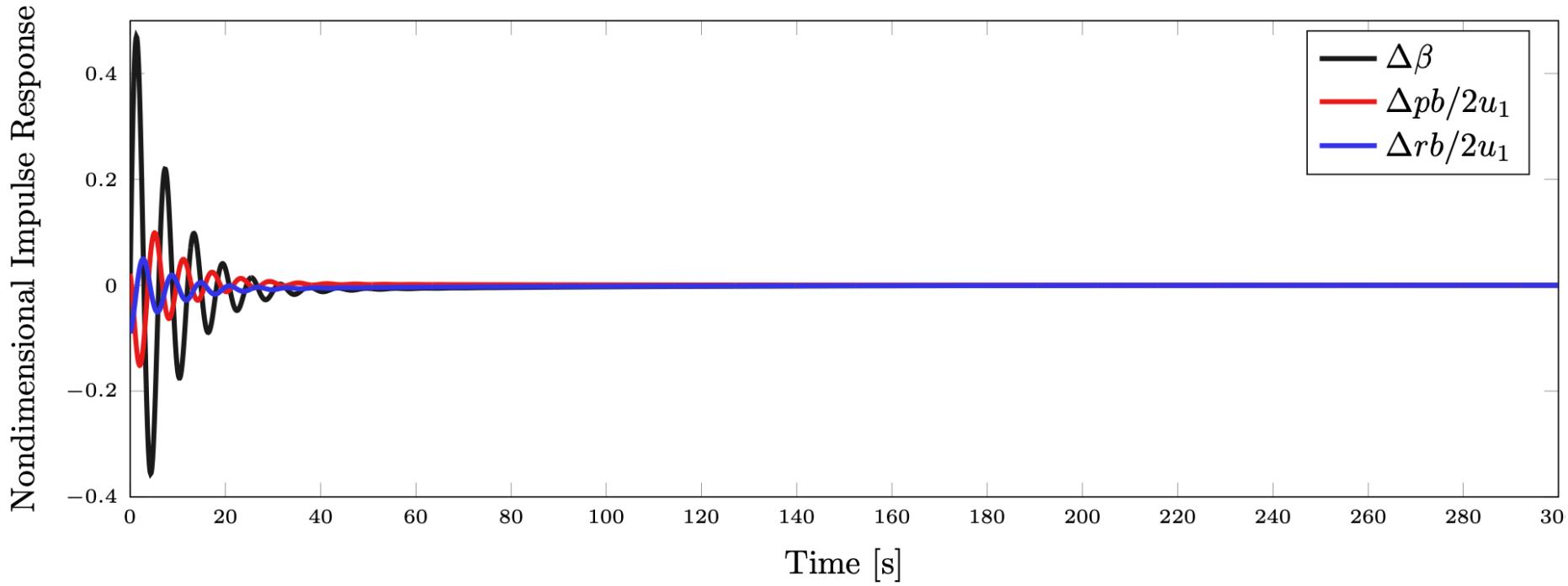
| Eigenvalue      | Stable? | Oscillatory? | Mode? | Dominant Motion? |
|-----------------|---------|--------------|-------|------------------|
| $\lambda_1$     |         |              |       |                  |
| $\lambda_2$     |         |              |       |                  |
| $\lambda_3$     |         |              |       |                  |
| $\lambda_{4,5}$ |         |              |       |                  |

Rudder Impulse Response of the Linear System:

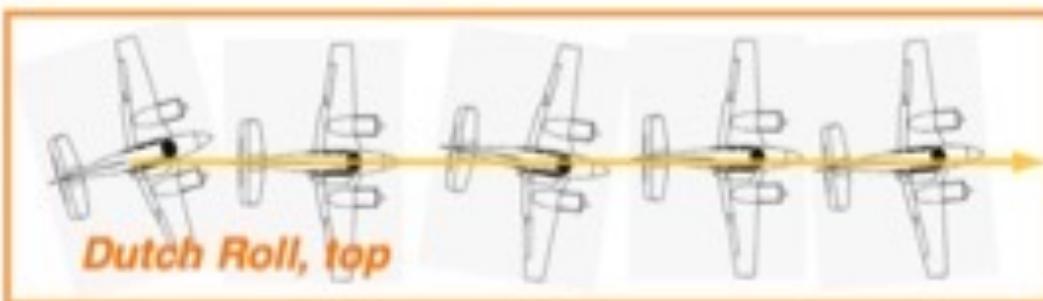


Nondimensional Impulse Response





**Dutch-roll motion is primarily described by stability-axis yaw rate and sideslip angle**



**Roll and spiral motions are primarily described by stability-axis roll rate and roll angle**



## Roll Approximation

We start from the equation that contains the Roll torque

$$\Delta\dot{p} - (I_{xz}/I_{xx})\Delta\dot{r} = L_\beta\Delta\beta + L_p\Delta p + L_r\Delta r + L_{\delta_a}\Delta\delta_a + L_{\delta_r}\Delta\delta_r$$

The roll mode occurs at approximately constant (and zero) sideslip and yaw, such that  $\Delta r = \Delta\beta = 0$  and  $\Delta\dot{r} = \Delta\dot{\beta} = 0$ , which gives

$$\Delta\dot{p} = L_p\Delta p + L_{\delta_a}\Delta\delta_a + L_{\delta_r}\Delta\delta_r$$

The primary control action is through the aileron deflection,

$$\Delta\dot{p} = L_p\Delta p + L_{\delta_a}\Delta\delta_a$$

let the control term be set to zero, it can be shown that

$$\Delta p(t) = \Delta p_0 e^{L_p t}, \quad t \geq 0$$

## Spiral Approximation

The spiral mode is characterized by changes in the bank angle,  $\phi$ , and the heading angle,  $\psi$ . The sideslip angle is usually quite small, but cannot be fully neglected. Typically, the spiral mode is very slow to develop following a disturbance, so it is usually assumed that  $\Delta\beta$ ,  $\Delta p$ , and  $\Delta r$  are quasi-steady relative to the time scale of the mode, hence  $\Delta\dot{\beta} = \Delta\dot{p} = \Delta\dot{r} = 0$ . Making these substitutions

$$0 = -u_1\Delta r + g\Delta\phi + Y_{\delta_a}\Delta\delta_a \quad (11.24a)$$

$$0 = L_\beta\Delta\beta + L_p\Delta p + L_r\Delta r + L_{\delta_a}\Delta\delta_a \quad (11.24b)$$

$$0 = N_\beta\Delta\beta + N_p\Delta p + N_r\Delta r + N_{\delta_a}\Delta\delta_a \quad (11.24c)$$

$$\Delta\dot{\phi} = \Delta p \quad (11.24d)$$

Note that the yaw angle does not influence the above equations, so the last equation of Eqs. (11.23) has been omitted in Eqs. (11.24). Multiplying Eq. (11.24b) by  $N_\beta$  and Eq. (11.24c) by  $L_\beta$ , subtracting the resulting equations, and solving for  $\Delta r$  gives

$$\Delta r = \frac{L_\beta N_p - N_\beta L_p}{N_\beta L_r - L_\beta N_r} \Delta p + \frac{L_\beta N_{\delta_a} - N_\beta L_{\delta_a}}{N_\beta L_r - L_\beta N_r} \Delta \delta_a \quad (11.25)$$

Substituting Eq. (11.25) into Eq. (11.24a) and clearing the fractions yields

$$0 = -u_1(L_\beta N_p - N_\beta L_p)\Delta p + g(N_\beta L_r - L_\beta N_r)\Delta \dot{\phi} + (Y_{\delta_a} - u_1(L_\beta N_{\delta_a} - N_\beta L_{\delta_a}))\Delta \delta_a$$

Finally, from Eq. (11.24d),  $\Delta \dot{\phi} = \Delta p$ , such that

$$0 = -u_1(L_\beta N_p - N_\beta L_p)\Delta \dot{\phi} + g(N_\beta L_r - L_\beta N_r)\Delta \phi + (Y_{\delta_a} - u_1(L_\beta N_{\delta_a} - N_\beta L_{\delta_a}))\Delta \delta_a$$

or

$$u_1(L_\beta N_p - N_\beta L_p)\Delta \dot{\phi} + g(L_\beta N_r - N_\beta L_r)\Delta \phi = (Y_{\delta_a} - u_1(L_\beta N_{\delta_a} - N_\beta L_{\delta_a}))\Delta \delta_a$$

In the presence of no control action

$$\Delta\dot{\phi} + \frac{g(L_\beta N_r - N_\beta L_r)}{u_1(L_\beta N_p - N_\beta L_p)} \Delta\phi = 0$$

Applying this solution to the zero-input governing equation for the spiral approximation yields

$$\Delta\phi(t) = \Delta\phi_0 e^{-(g/u_1)[(L_\beta N_r - N_\beta L_r)/(L_\beta N_p - N_\beta L_p)]t}, \quad t \geq 0$$

It is therefore clear that  $\Delta\phi(t) \rightarrow \infty$  as  $t \rightarrow \infty$  if

$$\frac{g(L_\beta N_r - N_\beta L_r)}{u_1(L_\beta N_p - N_\beta L_p)} < 0$$

In the case that this term is identically zero, the roll angle remains fixed at its initial value, but the response does not decay to zero. Thus, to obtain a stable spiral model, we require

$$\frac{g(L_\beta N_r - N_\beta L_r)}{u_1(L_\beta N_p - N_\beta L_p)} > 0 \tag{11.27}$$

Typically,  $(L_\beta N_p - N_\beta L_p) > 0$ , which means that we get the classical requirement that

$$L_\beta N_r > N_\beta L_r$$

# Dutch Roll Approximation

In the prior two modes we focused on the roll angle and roll rate. We study the Dutch Roll approximation by focusing the other two: sideslip angle and yaw rate. Neglecting roll in Dutch roll is clearly contradictory, but it is based on the fact that the mode is first a yawing oscillation and aerodynamic coupling causes rolling motion as a secondary effect. It is generally true that, for most aircraft, the roll to yaw ratio in Dutch rolling motion is less than one; in some cases, it may be much less than one, lending the assumption some small credibility. Given that the Dutch roll mode consists of primarily sideslipping and yawing motions, the approximation is:

$$\begin{aligned} u_1 \Delta \dot{\beta} &= Y_\beta \Delta \beta + Y_p \Delta p + (Y_r - u_1) \Delta r + g \cos \theta_1 \Delta \phi + Y_{\delta_a} \Delta \delta_a + Y_{\delta_r} \Delta \delta_r \\ \Delta \dot{r} - (I_{xz}/I_{zz}) \Delta \dot{p} &= (N_\beta + N_{T_\beta}) \Delta \beta + N_p \Delta p + (N_r + N_{T_r}) \Delta r + N_{\delta_a} \Delta \delta_a + N_{\delta_r} \Delta \delta_r \end{aligned}$$

From the assumption that no rolling motion is involved, setting  $\Delta p$ ,  $\Delta \dot{p}$ ,  $\Delta \phi$ , and  $\Delta \dot{\phi}$  to zero leads to

$$\begin{aligned} u_1 \Delta \dot{\beta} &= Y_\beta \Delta \beta + (Y_r - u_1) \Delta r + Y_{\delta_a} \Delta \delta_a + Y_{\delta_r} \Delta \delta_r \\ \Delta \dot{r} &= (N_\beta + N_{T_\beta}) \Delta \beta + (N_r + N_{T_r}) \Delta r + N_{\delta_a} \Delta \delta_a + N_{\delta_r} \Delta \delta_r \end{aligned}$$

Finally, to arrive at the Dutch roll approximation, neglect the propulsive effects and note that the primary control action occurs through the rudder deflection, such that

$$\begin{aligned} u_1 \Delta \dot{\beta} &= Y_\beta \Delta \beta + (Y_r - u_1) \Delta r + Y_{\delta_r} \Delta \delta_r \\ \Delta \dot{r} &= N_\beta \Delta \beta + N_r \Delta r + N_{\delta_r} \Delta \delta_r \end{aligned}$$

or, in standard matrix form

$$\begin{bmatrix} \dot{\Delta\beta} \\ \dot{\Delta r} \end{bmatrix} = \begin{bmatrix} Y_\beta/u_1 & Y_r/u_1 - 1 \\ N_\beta & N_r \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta r \end{bmatrix} + \begin{bmatrix} Y_{\delta_r}/u_1 \\ N_{\delta_r} \end{bmatrix} \Delta\delta_r$$

$$\begin{vmatrix} Y_\beta/u_1 - \lambda & Y_r/u_1 - 1 \\ N_\beta & N_r - \lambda \end{vmatrix} = 0$$

or

$$(Y_\beta/u_1 - \lambda)(N_r - \lambda) - (Y_r/u_1 - 1)N_\beta = 0$$

Reducing this to a standard degree two polynomial in  $\lambda$  gives the characteristic equation

$$\lambda^2 - \frac{Y_\beta + N_r u_1}{u_1} \lambda + \frac{N_r Y_\beta - N_\beta Y_r + N_\beta u_1}{u_1} = 0$$

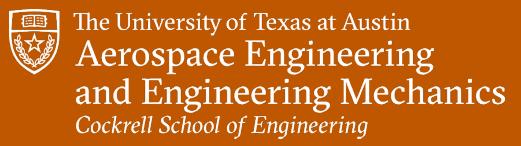
$$\omega_n = \sqrt{\frac{N_r Y_\beta - N_\beta Y_r + N_\beta u_1}{u_1}} \quad \zeta = -\frac{1}{2\omega_n} \frac{Y_\beta + N_r u_1}{u_1}$$

**Dutch-roll motion is primarily described by stability-axis yaw rate and sideslip angle**



**Roll and spiral motions are primarily described by stability-axis roll rate and roll angle**





# BACKUP SLIDES

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**JOHN-PAUL CLARKE**

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$$\begin{aligned}
Y &= m(\dot{v} + ru - pw - g \cos \theta \sin \phi) \\
&= m[(\dot{v}_1 + \Delta\dot{v}) + (r_1 + \Delta r)(u_1 + \Delta u) - (p_1 + \Delta p)(w_1 + \Delta w) \\
&\quad - g \cos(\theta_1 + \Delta\theta) \sin(\phi_1 + \Delta\phi)] && \text{substitute in perturbations} \\
&= m[(\dot{v}_1 + \Delta\dot{v}) + (r_1 + \Delta r)(u_1 + \Delta u) - (p_1 + \Delta p)(w_1 + \Delta w) \\
&\quad - g(\cos \theta_1 \cos \Delta\theta - \sin \theta_1 \sin \Delta\theta)(\sin \phi_1 \cos \Delta\phi + \cos \phi_1 \sin \Delta\phi)] && \text{angle addition formula} \\
&= m[(\dot{v}_1 + \Delta\dot{v}) + (r_1 + \Delta r)(u_1 + \Delta u) - (p_1 + \Delta p)(w_1 + \Delta w) \\
&\quad - g(\cos \theta_1 - \sin \theta_1 \Delta\theta)(\sin \phi_1 + \cos \phi_1 \Delta\phi)] && \text{small angle approximation} \\
&= m(\dot{v}_1 + r_1 u_1 - p_1 w_1 - g \cos \theta_1 \sin \phi_1) \\
&\quad + m[(\Delta\dot{v} + r_1 \Delta u - p_1 \Delta w - w_1 \Delta p + u_1 \Delta r) \\
&\quad - g \cos \theta_1 \cos \phi_1 \Delta\phi + g \sin \theta_1 \sin \phi_1 \Delta\theta] && \text{neglect higher-order terms} \\
&= Y_1 + \Delta Y
\end{aligned}$$

$$\begin{aligned}
L &= I_{xx}\dot{p} - I_{xz}(\dot{r} + pq) - (I_{yy} - I_{zz})qr \\
&= I_{xx}(\dot{p}_1 + \Delta\dot{p}) - I_{xz}[(\dot{r}_1 + \Delta\dot{r}) + (p_1 + \Delta p)(q_1 + \Delta q)] \quad \text{substitute in perturbations} \\
&\quad - (I_{yy} - I_{zz})(q_1 + \Delta q)(r_1 + \Delta r) \\
&= [I_{xx}\dot{p}_1 - I_{xz}(\dot{r}_1 + p_1q_1) - (I_{yy} - I_{zz})q_1r_1] \quad \text{neglect higher-order terms} \\
&\quad + I_{xx}\Delta\dot{p} - I_{xz}\Delta\dot{r} - I_{xz}p_1\Delta q - I_{xz}q_1\Delta p \\
&\quad - (I_{yy} - I_{zz})q_1\Delta r - (I_{yy} - I_{zz})r_1\Delta q \\
&= L_1 + \Delta L
\end{aligned}$$

$$\begin{aligned}
N &= I_{zz}\dot{r} - I_{xz}(\dot{p} - qr) - (I_{xx} - I_{yy})pq \\
&= I_{zz}(\dot{r}_1 + \Delta\dot{r}) - I_{xz}[(\dot{p}_1 + \Delta\dot{p}) - (q_1 + \Delta q)(r_1 + \Delta r)] \quad \text{substitute in perturbations} \\
&\quad - (I_{xx} - I_{yy})(p_1 + \Delta p)(q_1 + \Delta q) \\
&= [I_{zz}\dot{r}_1 - I_{xz}(\dot{p}_1 - q_1 r_1) - (I_{xx} - I_{yy})p_1 q_1] \quad \text{neglect higher-order terms} \\
&\quad + I_{zz}\Delta\dot{r} - I_{xz}\Delta\dot{p} + I_{xz}r_1\Delta q + I_{xz}q_1\Delta r \\
&\quad - (I_{xx} - I_{yy})q_1\Delta p - (I_{xx} - I_{yy})p_1\Delta q \\
&= N_1 + \Delta N
\end{aligned}$$

$$\begin{aligned} p &= \dot{\phi} - \dot{\psi} \sin \theta \\ &= (\dot{\phi}_1 + \Delta\dot{\phi}) - (\dot{\psi}_1 + \Delta\dot{\psi}) \sin(\theta_1 + \Delta\theta_1) && \text{substitute in perturbations} \\ &= (\dot{\phi}_1 + \Delta\dot{\phi}) - (\dot{\psi}_1 + \Delta\dot{\psi})(\sin \theta_1 \cos \Delta\theta + \cos \theta_1 \sin \Delta\theta) && \text{angle addition formula} \\ &= (\dot{\phi}_1 + \Delta\dot{\phi}) - (\dot{\psi}_1 + \Delta\dot{\psi})(\sin \theta_1 + \cos \theta_1 \Delta\theta) && \text{small angle approximation} \\ &= (\dot{\phi}_1 - \dot{\psi}_1 \sin \theta_1) + (\Delta\dot{\phi} - \dot{\psi}_1 \cos \theta_1 \Delta\theta - \sin \theta_1 \Delta\dot{\psi}) && \text{neglect higher-order terms} \\ &= p_1 + \Delta p \end{aligned}$$

$$\begin{aligned}
r &= -\dot{\theta} \sin \phi + \dot{\psi} \cos \theta \cos \phi \\
&= -(\dot{\theta}_1 + \Delta\dot{\theta}) \sin(\phi_1 + \Delta\phi) && \text{substitute in perturbations} \\
&\quad + (\dot{\psi}_1 + \Delta\dot{\psi}) \cos(\theta_1 + \Delta\theta) \cos(\phi_1 + \Delta\phi) \\
&= -(\dot{\theta}_1 + \Delta\dot{\theta})(\sin \phi_1 \cos \Delta\phi + \cos \phi_1 \sin \Delta\phi) && \text{angle addition formula} \\
&\quad + (\dot{\psi}_1 + \Delta\dot{\psi})(\cos \theta_1 \cos \Delta\theta - \sin \theta_1 \sin \Delta\theta) \\
&\quad \times (\cos \phi_1 \cos \Delta\phi - \sin \phi_1 \sin \Delta\phi) \\
&= -(\dot{\theta}_1 + \Delta\dot{\theta})(\sin \phi_1 + \cos \phi_1 \Delta\phi) && \text{small angle approximation} \\
&\quad + (\dot{\psi}_1 + \Delta\dot{\psi})(\cos \theta_1 - \sin \theta_1 \Delta\theta)(\cos \phi_1 - \sin \phi_1 \Delta\phi) \\
&= (-\dot{\theta}_1 \sin \phi_1 + \dot{\psi}_1 \cos \theta_1 \cos \phi_1) && \text{neglect higher order terms} \\
&\quad + (-\sin \phi_1 \Delta\dot{\theta} + \cos \theta_1 \cos \phi_1 \Delta\dot{\psi} - \dot{\theta}_1 \cos \phi_1 \Delta\phi \\
&\quad - \dot{\psi}_1 \cos \theta_1 \sin \phi_1 \Delta\phi - \dot{\psi}_1 \sin \theta_1 \cos \phi_1 \Delta\theta) \\
&= r_1 + \Delta r
\end{aligned}$$

$$\begin{aligned}
\beta &= \arctan \frac{v}{u} && \\
&= \frac{v}{u} && \text{small angle approximation} \\
&= \frac{v_1 + \Delta v}{u_1 + \Delta u} && \\
&= (v_1 + \Delta v)(u_1 + \Delta u)^{-1} && \\
&= \frac{v_1 + \Delta v}{u_1} \left(1 + \frac{\Delta u}{u_1}\right)^{-1} && \\
&= \frac{v_1 + \Delta v}{u_1} \left(1 - \frac{\Delta u}{u_1} - \dots\right) && \text{binomial expansion} \\
&= \frac{v_1}{u_1} + \frac{\Delta v}{u_1} - \frac{v_1 \Delta u}{u_1^2} && \text{neglect higher-order terms} \\
&= \frac{v_1}{u_1} + \frac{\Delta v}{u_1} && \text{trim conditions} \\
&= \beta_1 + \Delta \beta
\end{aligned}$$

$$\Delta Y = m(u_1 \Delta \dot{\beta} + u_1 \Delta r - g \cos \theta_1 \Delta \phi) \quad (11.11a)$$

$$\Delta L = I_{xx} \Delta \dot{p} - I_{xz} \Delta \dot{r} \quad (11.11b)$$

$$\Delta N = I_{zz} \Delta \dot{r} - I_{xz} \Delta \dot{p} \quad (11.11c)$$

$$\Delta p = \Delta \dot{\phi} - \sin \theta_1 \Delta \dot{\psi} \quad (11.11d)$$

$$\Delta r = \cos \theta_1 \Delta \dot{\psi} \quad (11.11e)$$

Eq. (11.11e) can be solved for  $\Delta \dot{\psi}$  as

$$\Delta \dot{\psi} = \sec \theta_1 \Delta r \quad (11.12)$$

Eq. (11.12) can then be substituted into Eq. (11.11d) to give

$$\begin{aligned}\Delta p &= \Delta \dot{\phi} - \sin \theta_1 \Delta \dot{\psi} \\ &= \Delta \dot{\phi} - \sin \theta_1 \sec \theta_1 \Delta r \\ &= \Delta \dot{\phi} - \tan \theta_1 \Delta r\end{aligned}$$

or, solving for  $\Delta \dot{\phi}$ ,

$$\Delta \dot{\phi} = \Delta p + \tan \theta_1 \Delta r$$

## Let's look at the force and moments...

$$Y = \bar{q}SC_y$$

$$\left( \frac{\partial Y}{\partial \beta} \right)_1 = \bar{q}_1 SC_{y_\beta}$$

$$\left( \frac{\partial Y}{\partial p} \right)_1 = \frac{\bar{q}_1 Sb}{2u_1} C_{y_p}$$

$$\left( \frac{\partial Y}{\partial r} \right)_1 = \frac{\bar{q}_1 Sb}{2u_1} C_{y_r}$$

$$\left( \frac{\partial Y}{\partial \delta_a} \right)_1 = \bar{q}_1 SC_{y_{\delta_a}}$$

$$\left( \frac{\partial Y}{\partial \delta_r} \right)_1 = \bar{q}_1 SC_{y_{\delta_r}}$$

$$C_{y_\beta} = \left( \frac{\partial C_y}{\partial \beta} \right)_1$$

$$C_{y_p} = \left( \frac{\partial C_y}{\partial (pb/2u_1)} \right)_1$$

$$C_{y_r} = \left( \frac{\partial C_y}{\partial (rb/2u_1)} \right)_1$$

$$C_{y_{\delta_a}} = \left( \frac{\partial C_y}{\partial \delta_a} \right)_1$$

$$C_{y_{\delta_r}} = \left( \frac{\partial C_y}{\partial \delta_r} \right)_1$$

## Let's look at the force and moments...

$$L = \bar{q}SbC_\ell$$

$$\left( \frac{\partial L}{\partial \beta} \right)_1 = \bar{q}_1 Sb C_{\ell_\beta}$$

$$\left( \frac{\partial L}{\partial p} \right)_1 = \frac{\bar{q}_1 Sb^2}{2u_1} C_{\ell_p}$$

$$\left( \frac{\partial L}{\partial r} \right)_1 = \frac{\bar{q}_1 Sb^2}{2u_1} C_{\ell_r}$$

$$\left( \frac{\partial L}{\partial \delta_a} \right)_1 = \bar{q}_1 Sb C_{\ell_{\delta_a}}$$

$$\left( \frac{\partial L}{\partial \delta_r} \right)_1 = \bar{q}_1 Sb C_{\ell_{\delta_r}}$$

$$C_{\ell_\beta} = \left( \frac{\partial C_\ell}{\partial \beta} \right)_1$$

$$C_{\ell_p} = \left( \frac{\partial C_\ell}{\partial (pb/2u_1)} \right)_1$$

$$C_{\ell_r} = \left( \frac{\partial C_\ell}{\partial (rb/2u_1)} \right)_1$$

$$C_{\ell_{\delta_a}} = \left( \frac{\partial C_\ell}{\partial \delta_a} \right)_1$$

$$C_{\ell_{\delta_r}} = \left( \frac{\partial C_\ell}{\partial \delta_r} \right)_1$$

## Let's look at the force and moments...

$$N_A = \bar{q}SbC_n$$

$$\left( \frac{\partial N_A}{\partial \beta} \right)_1 = \bar{q}_1 Sb C_{n_\beta}$$

$$\left( \frac{\partial N_A}{\partial p} \right)_1 = \frac{\bar{q}_1 Sb^2}{2u_1} C_{n_p}$$

$$\left( \frac{\partial N_A}{\partial r} \right)_1 = \frac{\bar{q}_1 Sb^2}{2u_1} C_{n_r}$$

$$\left( \frac{\partial N_A}{\partial \delta_a} \right)_1 = \bar{q}_1 Sb C_{n_{\delta_a}}$$

$$\left( \frac{\partial N_A}{\partial \delta_r} \right)_1 = \bar{q}_1 Sb C_{n_{\delta_r}}$$

$$C_{n_\beta} = \left( \frac{\partial C_n}{\partial \beta} \right)_1$$

$$C_{n_p} = \left( \frac{\partial C_n}{\partial (pb/2u_1)} \right)_1$$

$$C_{n_r} = \left( \frac{\partial C_n}{\partial (rb/2u_1)} \right)_1$$

$$C_{n_{\delta_a}} = \left( \frac{\partial C_n}{\partial \delta_a} \right)_1$$

$$C_{n_{\delta_r}} = \left( \frac{\partial C_n}{\partial \delta_r} \right)_1$$

## Let's look at the force and moments...

$$N_T = \bar{q}SbC_{n_T}$$

$$\left( \frac{\partial N_T}{\partial \beta} \right)_1 = \bar{q}_1 Sb C_{n_{T_\beta}}$$

$$\left( \frac{\partial N_T}{\partial r} \right)_1 = \frac{\bar{q}_1 Sb^2}{2u_1} C_{n_{T_r}}$$

$$C_{n_{T_\beta}} = \left( \frac{\partial C_{n_T}}{\partial \beta} \right)_1$$

$$C_{n_{T_r}} = \left( \frac{\partial C_{n_T}}{\partial (rb/2u_1)} \right)_1$$

## Let's look at the force and moments...

$$\Delta Y = \bar{q}_1 S C_{y_\beta} \Delta \beta + \frac{\bar{q}_1 S b}{2u_1} C_{y_p} \Delta p + \frac{\bar{q}_1 S b}{2u_1} C_{y_r} \Delta r + \bar{q}_1 S C_{y_{\delta_a}} \Delta \delta_a + \bar{q}_1 S C_{y_{\delta_r}} \Delta \delta_r$$

$$\Delta L = \bar{q}_1 S b C_{\ell_\beta} \Delta \beta + \frac{\bar{q}_1 S b^2}{2u_1} C_{\ell_p} \Delta p + \frac{\bar{q}_1 S b^2}{2u_1} C_{\ell_r} \Delta r + \bar{q}_1 S b C_{\ell_{\delta_a}} \Delta \delta_a + \bar{q}_1 S b C_{\ell_{\delta_r}} \Delta \delta_r$$

$$\begin{aligned} \Delta N = & \left( \bar{q}_1 S b C_{n_\beta} + \bar{q}_1 S b C_{n_{T_\beta}} \right) \Delta \beta + \frac{\bar{q}_1 S b^2}{2u_1} C_{n_p} \Delta p + \left( \frac{\bar{q}_1 S b^2}{2u_1} C_{n_r} + \frac{b^2}{\bar{q}_1 S 2u_1} C_{n_{T_r}} \right) \Delta r \\ & + \bar{q}_1 S b C_{n_{\delta_a}} \Delta \delta_a + \bar{q}_1 S b C_{n_{\delta_r}} \Delta \delta_r \end{aligned}$$

## Which gives the accelerations...

$$\Delta Y/m = \frac{\bar{q}_1 S}{m} C_{y_\beta} \Delta \beta + \frac{\bar{q}_1 S b}{2mu_1} C_{y_p} \Delta p + \frac{\bar{q}_1 S b}{2mu_1} C_{y_r} \Delta r + \frac{\bar{q}_1 S}{m} C_{y_{\delta_a}} \Delta \delta_a + \frac{\bar{q}_1 S}{m} C_{y_{\delta_r}} \Delta \delta_r$$

$$\Delta L/I_{xx} = \frac{\bar{q}_1 S b}{I_{xx}} C_{\ell_\beta} \Delta \beta + \frac{\bar{q}_1 S b^2}{2I_{xx}u_1} C_{\ell_p} \Delta p + \frac{\bar{q}_1 S b^2}{2I_{xx}u_1} C_{\ell_r} \Delta r + \frac{\bar{q}_1 S b}{I_{xx}} C_{\ell_{\delta_a}} \Delta \delta_a + \frac{\bar{q}_1 S b}{I_{xx}} C_{\ell_{\delta_r}} \Delta \delta_r$$

$$\begin{aligned} \Delta N/I_{zz} = & \left( \frac{\bar{q}_1 S b}{I_{zz}} C_{n_\beta} + \frac{\bar{q}_1 S b}{I_{zz}} C_{n_{T_\beta}} \right) \Delta \beta + \frac{\bar{q}_1 S b^2}{2I_{zz}u_1} C_{n_p} \Delta p + \left( \frac{\bar{q}_1 S b^2}{2I_{zz}u_1} C_{n_r} + \frac{\bar{q}_1 S b^2}{2I_{zz}u_1} C_{n_{T_r}} \right) \Delta r \\ & + \frac{\bar{q}_1 S b}{I_{zz}} C_{n_{\delta_a}} \Delta \delta_a + \frac{\bar{q}_1 S b}{I_{zz}} C_{n_{\delta_r}} \Delta \delta_r \end{aligned}$$



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