⑥

We want to find...

$$\Lambda(z) = \frac{p(z|\theta \in \chi_1)}{p(z|\theta \in \chi_0)} \underset{H_0}{\overset{H_1}{\gtrless}} v$$

$$p(z|\theta \in \chi_0) = \frac{1}{2\pi\sigma^2}\left[-\frac{1}{2\sigma^2}(z-\mu)^T(z-\mu)\right]$$

$$= \frac{1}{2\pi\sigma^2}\left[e^{-\frac{1}{2\sigma^2}(z-\mu)^T(z-\mu)}\right]$$

$$= \frac{1}{2\pi\sigma^2}\left[e^{-\frac{1}{2\sigma^2}z^Tz}\right]$$

$$p(z|\theta \in \chi_0) = \int_0^{2\pi} p(z|\theta_1 = A, \theta_2 = \xi)\, p_2(\xi)\, d\xi$$

$$= \frac{1}{4\pi^2\sigma^2}\int_0^{2\pi} e^{-\frac{1}{2\sigma^2}[z-\mu(A,\xi)]^T[z-\mu(A,\xi)]}\, d\xi$$

$$\frac{p(z|\theta \in \chi_1)}{p(z|\theta \in \chi_0)} = \frac{\frac{1}{\cancel{4}\pi^2\sigma^2}\int_0^{2\pi} e^{-\frac{1}{2\sigma^2}[z-\mu(A,\xi)]^T[z-\mu(A,\xi)]}\, d\xi}{\frac{1}{\cancel{2}\pi\sigma^2}\, e^{-\frac{1}{2\sigma^2}z^Tz}}$$

$$\mu = \mu(\theta_1,\theta_2) = \begin{bmatrix} A\cos\theta_2 \\ A\sin\theta_2 \end{bmatrix}$$

$$= \frac{e^{\frac{1}{2\sigma^2}z^Tz}}{2\pi}\int_0^{2\pi} e^{-\frac{1}{2\sigma^2}[z-\mu(A,\xi)]^T[z-\mu(A,\xi)]}\, d\xi$$

$$\underline{z} \triangleq [z_1, z_2]$$

$$z_1 = A\cos\theta_2 + w_1$$

$$z_2 = A\sin\theta_2 + w_2$$

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$z^Tz - 2z^T\mu(A,\xi) + \mu(A,\xi)^T\mu(A,\xi)$$

$$z^Tz = [A\cos\theta_2 + w_1 \quad A\sin\theta_2 + w_2]\begin{bmatrix} A\cos\theta_2 + w_1 \\ A\sin\theta_2 + w_2 \end{bmatrix}$$

$$= [\mu + w]^T[\mu + w]$$

$$= \mu^T\mu + \underbrace{\cancel{\mu^Tw + w^T\mu}}_{0,\text{ since } w \sim N(0,\sigma^2)} + \cancel{w^Tw}^{\,0}$$

$$= [A\cos\theta_2 \quad A\sin\theta_2]\begin{bmatrix} A\cos\theta_2 \\ A\sin\theta_2 \end{bmatrix}$$

$$= A^2\cos^2\theta_2 + A^2\sin^2\theta_2$$

$[z_1 z_2]$

$$z^T z = A^2$$

$$z^T z - 2z^T\begin{bmatrix} A\cos\theta_2 \\ A\sin\theta_2 \end{bmatrix} + [A\cos\theta_2 \quad A\sin\theta_2]\begin{bmatrix} A\cos\theta_2 \\ A\sin\theta_2 \end{bmatrix}$$

$$= A^2 - 2Az_1\cos\theta_2 - 2Az_2\sin\theta_2 + A^2$$

$$= 2A^2 - 2A(z_1\cos\theta_2 - z_2\sin\theta_2)$$

$$\Lambda(z) = \frac{e^{\frac{A^2}{2\sigma^2}}}{2\pi}\int_0^{2\pi} \exp\left[-\frac{1}{2\sigma^2}(2A^2 - 2A(z_1\cos\theta_2 - z_2\sin\theta_2)\right] d\varphi$$

$$= \frac{1}{2\pi} e^{\frac{A^2}{2\sigma^2}} e^{-\frac{A^2}{\sigma^2}} \int_0^{2\pi} \exp\left[\frac{A}{\sigma^2}(z_1\cos\theta_2 + z_2\sin\theta_2)\right] d\varphi$$

$$= \cancel{\frac{e^{\frac{-A^2}{2\sigma^2}}}{2\pi}} \int_0^{2\pi} \exp\left[\frac{A}{\sigma^2}(z_1\cos\xi + z_2\sin\xi)\right] d\xi$$

divide this term over to υ

$$\Rightarrow \int_0^{2\pi} \exp\left[\frac{A}{\sigma^2}(z_1\cos\xi + z_2\sin\xi)\right] d\xi \gtrless \upsilon' = \frac{2\pi\upsilon}{e^{\frac{-A^2}{2\sigma^2}}}$$

$$z_1\cos\xi = \sum x(j)\cos\xi \qquad z_2 = \sum x(j)\sin\xi$$

Collection of observations

$$\Lambda'(z) = \int_0^{2\pi} \exp\left[\frac{A}{\sigma^2}\left(\sum_j x(j)\cos\xi + \sum_j x(j)\sin\xi\right)\right] d\xi \underset{H_0}{\overset{H_1}{\gtrless}} \upsilon'$$

Since...

$$S(j,\xi) \triangleq D[\tau_j - t_d(\tau_j)]\,C[\tau_j - t_s(\tau_j)]\cos[2\pi f_{IF}\tau_j + \xi]$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$S(j,\xi) \triangleq D\;C\left(\cos[2\pi f_{IF}\tau_j]\cos[\xi] - \sin[2\pi f_{IF}\tau_j]\sin[\xi]\right)$$

$$= S_c(j)\cos\xi - S_s(j)\sin\xi$$

$$\Lambda'(z) = \int_0^{2\pi} \exp\left[\frac{A}{\sigma^2}\left(\sum x(j)\left(\cos\xi + \sin\xi\right)\right]\right]d\xi$$

$$= \int_0^{2\pi} \exp\left[\frac{A}{\sigma^2}\left(\sum x(j)\,S(j,\xi)\right)\right]d\xi$$

$$= \int_0^{2\pi} \exp\left[\frac{A}{\sigma^2}\left(\sum x(j)\left[S_c(j)\cos\xi - S_s(j)\sin\xi\right]\right)\right]d\xi$$

A is not necessary because it is again a scaling term that can be incorporated into threshold instead of forming detection statistic.