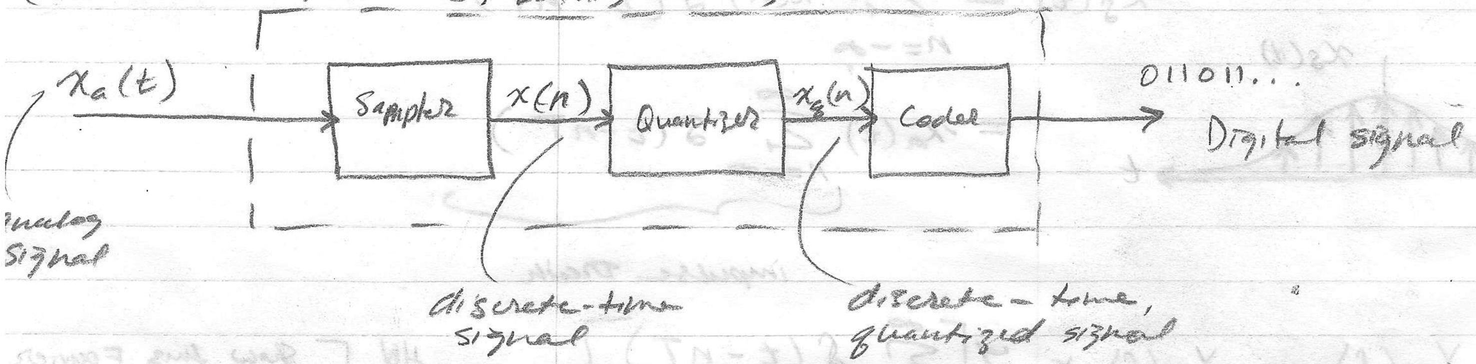
Lecture 9: Analog-to-Digital Conversion

The signal exiting the final mixing stage in the RF front end must be converted from an analog to a digital signal: (notes from Proakis, Lathi, others)



We are only concerned with uniform sampling, where the sampling period T is constant.

Intuitively, if we wish to faithfully represent an analog signal as a discrete-time signal, then we should expect to have to sample faster (smaller T) the higher the original analog signal ~~has higher frequencies~~ frequency content.

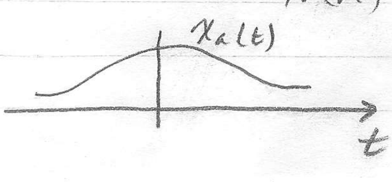
Sampling theorem is a precise statement of this reasoning:

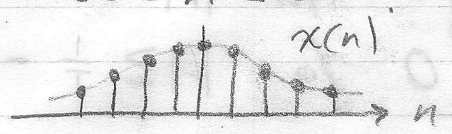
1. If $x_a(t)$ is bandlimited to W ($X_a(f) = 0$ for $|f| \geq W$) then it is sufficient to sample at $T = \frac{1}{2W}$

2. If we are allowed to employ a sophisticated interpolation algorithm (not just linear interpolation), then we can recover $x_a(t)$ exactly from its samples if (1) is satisfied.

To prove this we only have to look at the relationship between $X_a(f) = \mathcal{F}[x_a(t)]$ and $X(f) = \mathcal{F}[x(n)]$, where

$$x(n) = x_a(nT), \quad -\infty < n < \infty.$$





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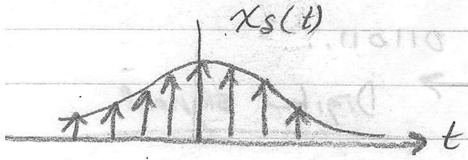
[Delta fcn. is identity for convolution. Also called impulse fcn.]

Finding $X(f)$ is somewhat involved.

We'll leave this to homework. Meanwhile, the principles of sampling can be illustrated by considering "impulse" sampling:

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(n)\, \delta(t-nT)$$



$$= x_a(t) \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-nT)}_{\text{impulse train}}$$

$$X_\delta(f) = X_a(f) * \mathcal{F}\left[\sum_{n=-\infty}^{\infty} \delta(t-nT)\right]$$

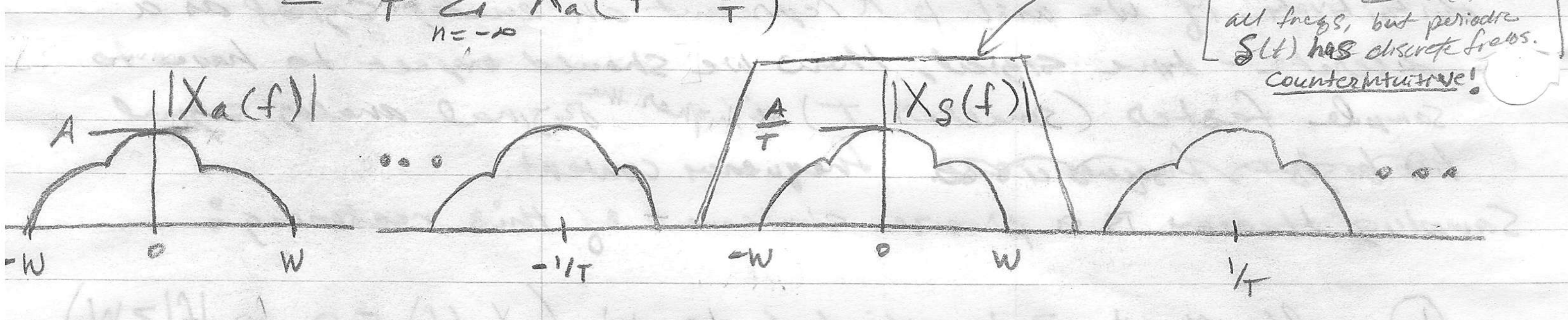
HW [Show this Fourier trans. or something related.]

$$= X_a(f) * \frac{1}{T}\sum_{n=-\infty}^{\infty} \delta\left(f-\frac{n}{T}\right)$$

[Impulse train is periodic w/ period T, so it will give rise to a Fourier series w/ period $1/T$.]

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_a\left(f-\frac{n}{T}\right)$$

[Note that one $\delta(t)$ has all freqs, but periodic $\delta(t)$ has discrete freqs. Counterintuitive!]



Q: For what T do the replicated spectra in $X_\delta(f)$ overlap?

A: For $T > 1/2W$. 2W is called the Nyquist sampling rate.

Q: What are the consequences of overlap?

A: Impossible to recover $X_a(f)$ from $X_\delta(f)$; we introduce aliasing distortion.

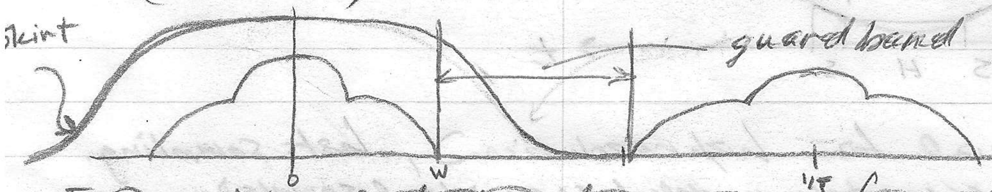
If $T \le \frac{1}{2W}$, then there is no overlap. Theoretically, then, we could recover the signal $x_a(t)$ by proper filtering of $x_\delta(t)$.

Q: What is the frequency response $H(f)$ of a filter that would allow us to recover $x_a(t)$?

A: $$H(f) = \begin{cases} T & \text{for } |f| < W \\ 0 & \text{for } |f| \ge \frac{1}{T} - W \end{cases}$$

Note that if $T = \frac{1}{2W}$, our filter would have to be an ideal lowpass filter $H(f) = T \cdot \Pi\left(\frac{f}{2W}\right)$.

This isn't practical, so we typically sample at a rate above Nyquist ($1/T > 2W$) to make room for the "skirts" of a practical filter:



~~Ex: National Instruments uses $f_s = \frac{1}{T} = \frac{2W}{0.8}$ Hz~~

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But, supposing we could find an ideal filter with BW W', $W \le W' < \frac{1}{T} - W$

then:

$$X_a(f) = X_s(f)\, T \cdot \Pi\left(\frac{f}{2W'}\right)$$

$$x_a(t) = x_s(t) * 2W'T \operatorname{sinc}(2W't)$$

$$\boxed{x_a(t) = \sum_{n=-\infty}^{\infty} 2W'T\, x(n) \operatorname{sinc}[2W'(t - nT)]}$$

Looks infinitely before and after sample.

(ideal reconstruction formula)

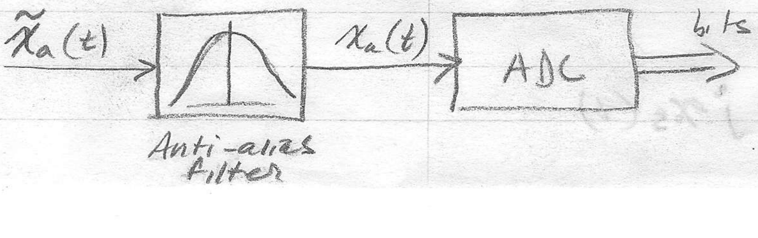
HW [Might explore this in Matlab. Actually use the sinc support functions. compare against linear interp.

Periodicity in time domain → impulses in freq dom

Periodic impulses in time domain → periodic impulses in freq.

Practical Sampling

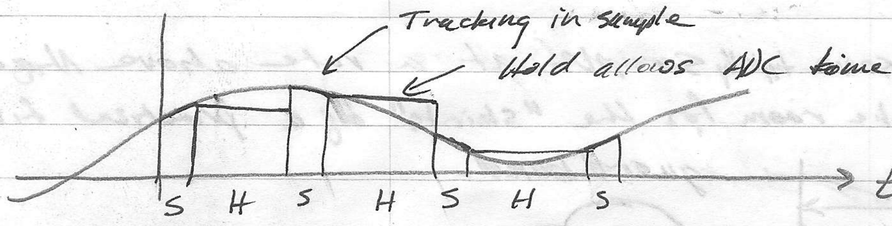
Rarely do we get signals in practice that are perfectly band-limited. ~~such that $X_a(f) = 0$ for $|f| > W$~~ We therefore generally ~~usually identify an effective bandwidth beyond which signal~~ filter an analog signal before sampling so that it obeys $X_a(f) = 0$ for $|f| > W$. approximately.



~~Ex: For NATIONAL INST. Equipment, the width of the~~

HW [Anti-aliasing filters are the key to understanding the conversion between continuous-time and discrete-time <u>noise</u> representations. Explore in HW.

Sampling by impulses is hardly practical. Instead, ADCs employ a sample + hold circuit:



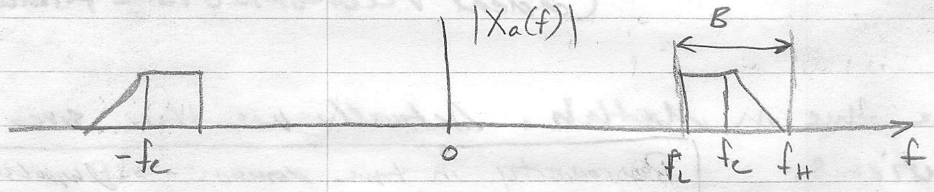
Sample and Hold is crucial for high resolution, fast sampling. A good S/H circuit is accurately modeled as an ideal sampler.

HW [Explore effects of sampling jitter. Look at Psiaki's paper.

<u>Bandpass</u> <u>Sampling</u>

Recall that radionav signals are all bandpass signals, which can be represented as

$$x_a(t) = a(t)\cos\left[2\pi f_c t + \theta(t)\right]$$
$$= x_c(t)\cos 2\pi f_c t - x_s(t)\sin 2\pi f_c t$$



Let $B = f_H - f_L$ be the BW of the BP signal. Assume $f_c = \frac{1}{2}(f_L + f_H)$. [BW of BP signals refers to single-sided BW]

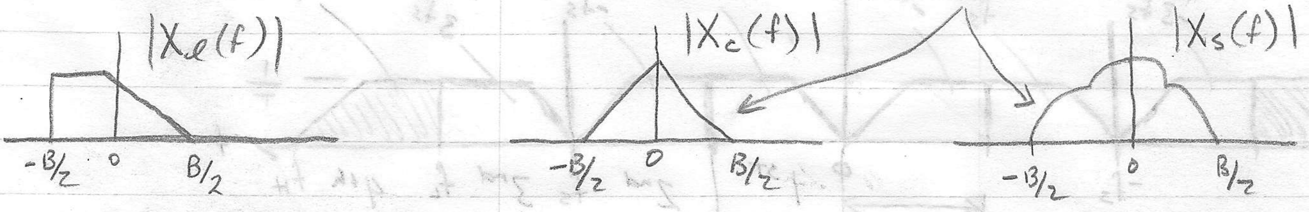
Q: How fast must we sample $x_a(t)$ to represent it as a discrete-time signal without aliasing distortion?

A: A straightforward application of the sampling thm. would require sampling at $T \leq \frac{1}{2f_H}$. But we can be more clever. Recall that the baseband representation of $x_a(t)$ is

$$x_\ell(t) = x_c(t) + j x_s(t)$$

$x_\ell(t)$ contains all the information in $x_a(t)$. Both the in-phase component $x_c(t)$ and the quadrature component $x_s(t)$ are low-pass real-valued signals with one-sided bandwidth $B/2$ Hz:

symmetrical about $f=0$



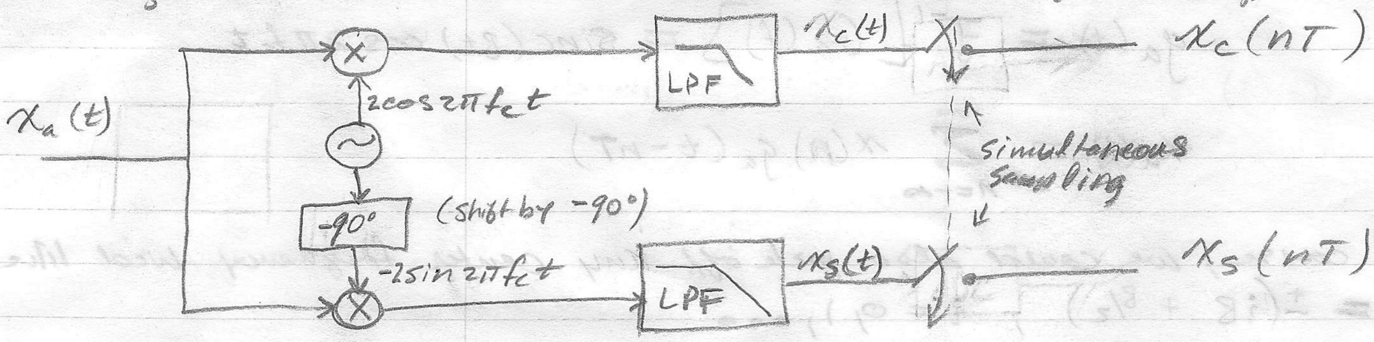
(stylized shapes just for illustration)

HW [Look at symmetry of real signals, asymmetry of complex signals. Also autocorrelation of the same.

Thus, we could uniquely represent $x_c(t)$ and $x_s(t)$ by the sequences $x_c(nT)$ and $x_s(nT)$ with $T \le \frac{1}{B}$. Results in a total of $2B$ real samples per second, or when viewed as $x_c(n)+jx_s(n)$, B complex samples per second.

recall: $\left[\begin{array}{l} 2\cos^2(x) = 1+\cos 2x \\ 2\sin^2(x) = 1-\cos 2x \end{array} \right.$ or $\left. \begin{array}{l} 2\cos x\cos y = \cos(x-y)+\cos(x+y) \\ 2\sin x\sin y = \cos(x-y)-\cos(x+y) \end{array} \right]$

Mechanization:

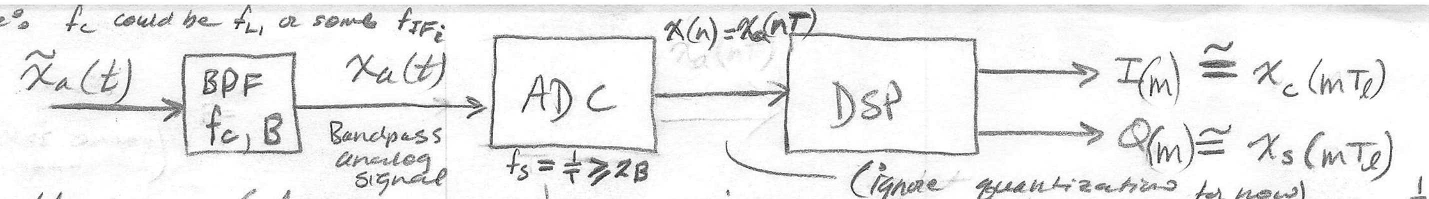


This is the "quadrature approach" to bandpass sampling. [Widely used ~~in the past~~ in radar + comm. systems].

Two observations:

1. Balancing the two processing paths is a challenge: Phase errors can be 2–3°. DSP mitigation works well.
2. As ADCs get faster and cheaper, it becomes attractive to take another approach...

Note: f_c could be f_L, or some f_{IF_i}

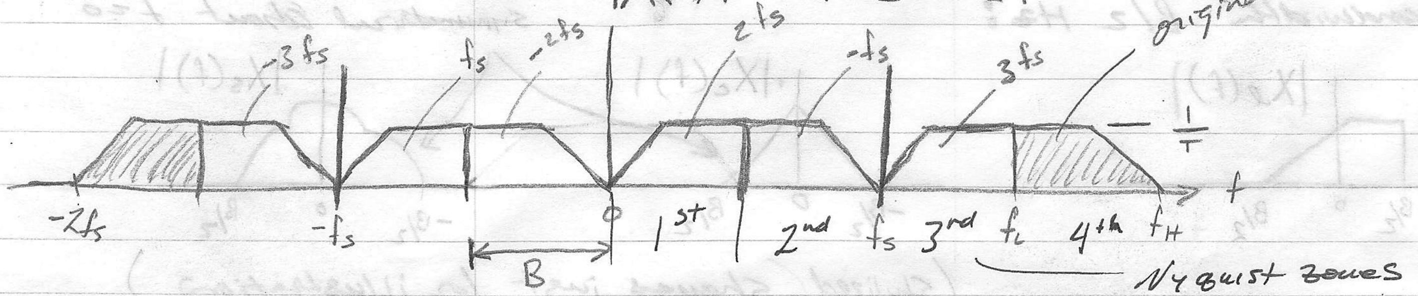


Uniform (first-order) sampling and digital down conversion. $\frac{1}{T_\ell} \geq B$

Suppose $f_H = mB$ with m an integer.

Ex: for $m = 4$

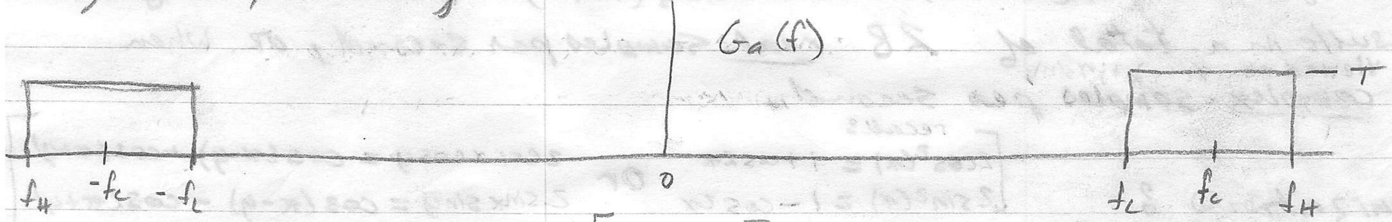
$$|X(f)| = |\mathcal{F}[x(n)]|$$



Now suppose we choose a sampling rate $f_s = \frac{1}{T} = 2B$. [Fill in the spectrum with all the repetitions]

Notice: No aliasing distortion!

We could recover the original bandpass signal from the sampled signal by filtering with:



$$g_a(t) = \mathcal{F}^{-1}\left[G_a(f)\right] = \operatorname{sinc}(Bt)\cos 2\pi f_c t$$

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n)\, g_a(t - nT)$$

Of course, we could also pick off any center frequency we'd like:

$f_c = \pm(iB + B/2)$, $i = 0, 1, \ldots$

For $i = 0$ we get a signal near baseband (inverted!)

this is a clever way of doing frequency conversion without actually mixing the analog signal. We let the sampling do the work for us!

For arbitrary band positioning ($\frac{f_H}{B}$ not an integer)

we can show that the range of acceptable uniform sampling rates is determined by $\frac{2f_H}{K} \leq f_s \leq \frac{2f_L}{K-1}$

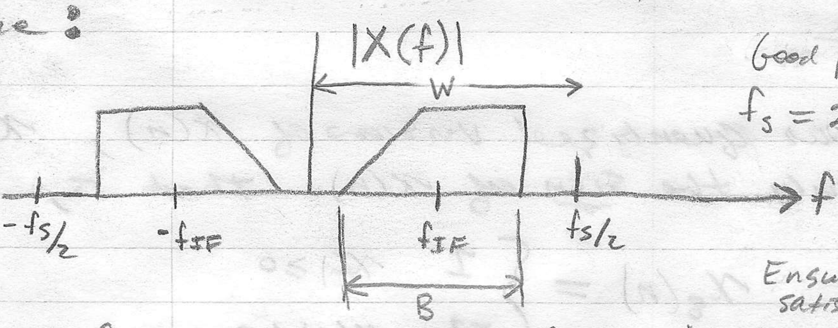
where k is an integer given by $1 \le k \le \left\lfloor \frac{f_H}{B} \right\rfloor$
where $\lfloor b \rfloor$ denotes the integer part ("floor") of b.

Note that not all frequencies above 2B are allowed. [counterintuitive!]
[See the "spiky hair" graph from Vaughan et al.]

HW — [HW - Work with this graph for some real systems.

Once we have done uniform sampling to get $x(n)$, what do we do in the DSP to get $I(m)$ and $Q(m)$? There are several techniques (see papers on Canvas). The most straightforward is digital quadrature demodulation; that is, a discrete-time implementation of the continuous-time quadrature approach discussed earlier. This approach follows:

Now that we have converted $x_a(t)$ to a discrete-time signal $x(n)$, we can, for convenience, focus on the 1st Nyquist zone:



Good practice: sample at $f_s = 2W = \frac{2B}{0.8}$ to provide guard bands that prevent aliasing. Ensure $f_H = mW$ or otherwise satisfy spikey hair plot.

The intermediate frequency that results from uniform sampling is

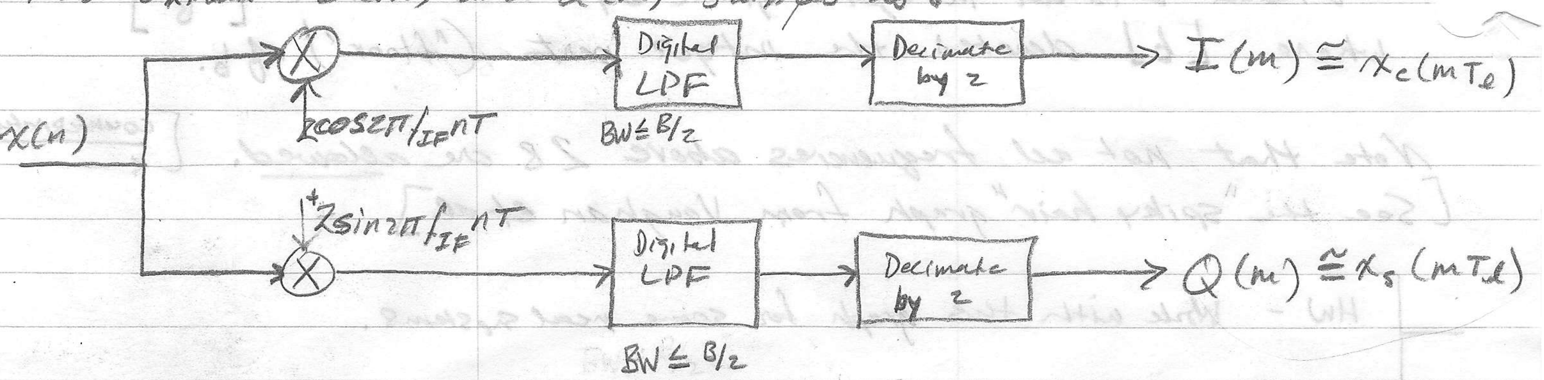
HW now this [
$$f_{IF} = \left| f_c - f_s \cdot \left[\frac{f_c}{f_s} \right] \right|$$

$[\cdot]$ rounds to the nearest integer. If the operand of $|\cdot|$ is negative, then this indicates a carrier phase reversal has occurred. This is the digital downconversion equivalent of high-side mixing. (High-side mixing occurred in above figure.)

We can now model the signal as (ignoring replicas in other Nyq. zones):
$$x(n) = x_c(nT)\cos 2\pi f_{IF} nT + x_s(nT)\sin 2\pi f_{IF} nT$$
(plus due to high side mixing)

Sampling interval here is $T \geq 2B$, which is twice as fast as we now need.

And extract $I(m)$ and $Q(m)$ samples as:



This assumes that $T_d = 2T$.

~~Summary of simplest approach: (1) Sample at $f_s = 2W \cong \frac{2B}{0.8}$ (2) Assume $f_H \cong m \cdot W$, $m \in \mathbb{Z}$ (3) Work with alias in 1st Nyquist zone~~

HW [Do this with Plessey data to get $I(m)$ & $Q(m)$.
Then go the other way with NI data to get sampling compatible with the GRID receiver.

[Insert here section on next page under ~~ΔΔ~~]

Quantization

Q: Suppose our quantized version of $x(n)$, $x_q(n) = Q[x(n)]$ retains only the sign of $x(n)$. That is,

$$x_q(n) = \begin{cases} 1 & x(n) \geq 0 \\ -1 & x(n) < 0 \end{cases}$$

By how much will this coarse quantization degrade C/N_0 in subsequent GNSS processing?

A: By less than 3 dB! [Surprising!]

To properly investigate the effects of quantization, one must take into account the signal structure, the front-end filtering, and the quantization scheme.

HW [See the paper by Curran (2009)
Maybe re-create his simulation?

Most important points about GNSS quantization:

(From Hegarty 2011)

- Loss $\cong$ for narrowband signal

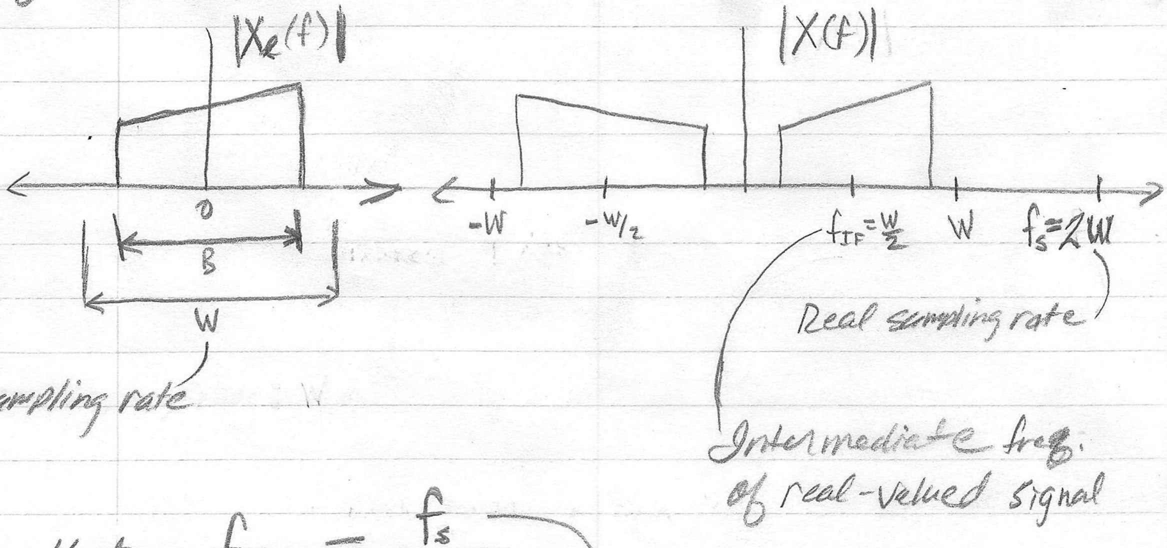
2.43 dB	1 bit	($N = 2$ levels)
1.01 dB	2 bit	($N = 4$ levels)
0.63 dB	3 bit	($N = 8$ levels)
0.48 dB	5 bit	($N = 32$ levels)

- We can get away with coarse quantization in GNSS because the received C/N_0 are fairly similar (within ~20 dB). Strong in-band interference (jamming) can "capture" the quantization process and make tracking of 1-bit quantized signals impossible. Military RX quantize at 8-12 bits. Large dynamic range enables post-quantization interference cancellation (e.g., Near-Far, Notch)

~~Also a severe mismatch in C/N_0 between 2 GNSS signals (maybe due to multipath tracking) can be a problem with coarse quantization.~~ [See my book chapter for more on quant.]

~~AA~~

Note that for up-conversion from IQ samples to real-valued samples w/ intermediate freq f_{IF}, simplest to set f_{IF} as follows:



Note that $f_{IF} = \frac{f_s}{4}$ real sampling rate