The University of Texas at Austin
Aerospace Engineering
and Engineering Mechanics
Cockrell School of Engineering

8 OCTOBER 2024

ASE 367K: FLIGHT DYNAMICS

TTH 09:30-11:00
CMA 2.306

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Topics for Today

- Topic(s):
  - Rate of Change of Angular Momentum
  - Euler Angle Rates

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RATE OF CHANGE OF ANGULAR MOMENTUM

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Newton's 2^{nd} Law, Applied to Rotational Motion

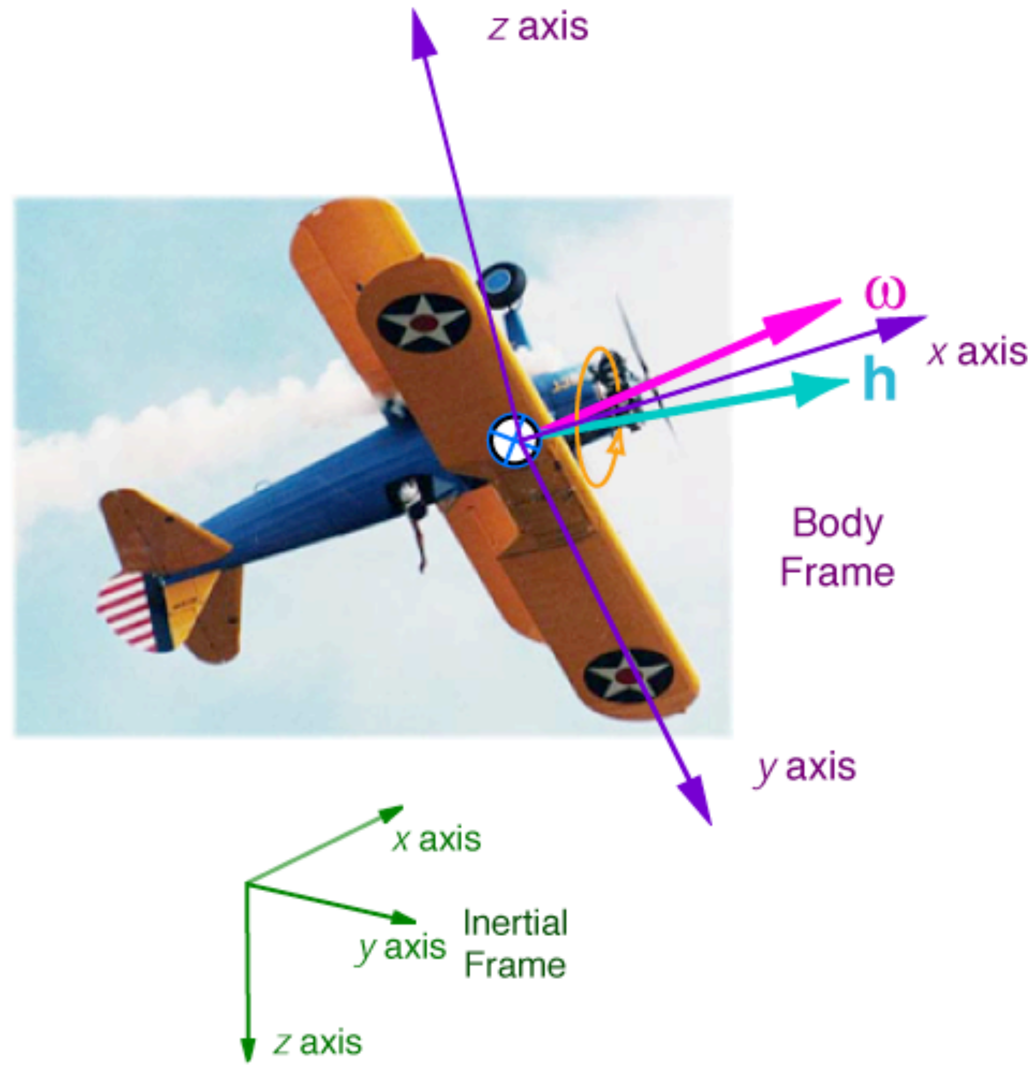
In inertial frame, rate of change of angular momentum = applied moment (or torque), M

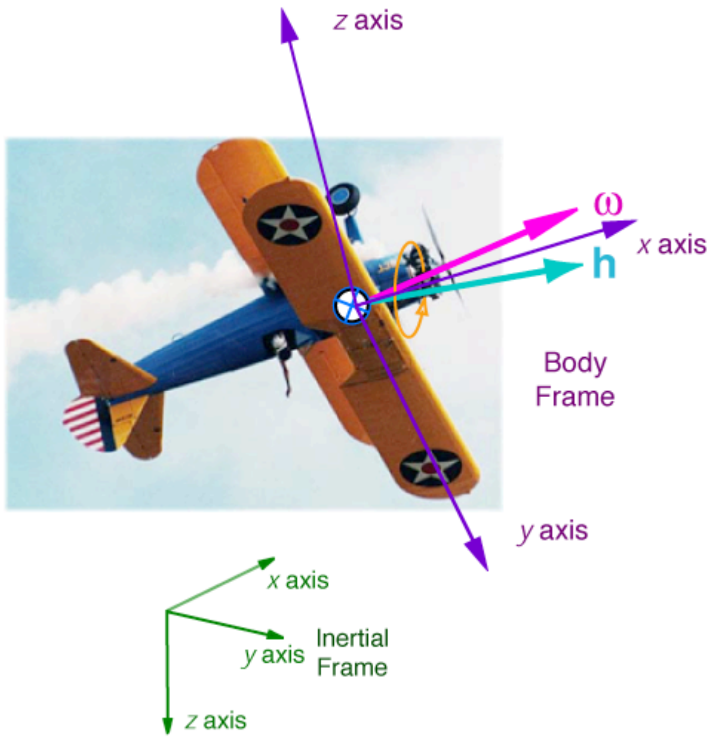
$$\boxed{\frac{d\mathbf{h}}{dt} = \frac{d\left(\mathbb{I}\boldsymbol{\omega}\right)}{dt} = \frac{d\mathbb{I}}{dt}\boldsymbol{\omega} + \mathbb{I}\frac{d\boldsymbol{\omega}}{dt}} \quad \boxed{= \dot{\mathbb{I}}\boldsymbol{\omega} + \mathbb{I}\dot{\boldsymbol{\omega}} = \mathbf{M} = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}}$$

Angular Momentum and Rate

Angular momentum and rate vectors are not necessarily aligned

$$\boxed{\mathbf{h} = \mathbb{I}\boldsymbol{\omega}}$$





Angular Momentum Expressed in Two Frames of Reference

- Angular momentum and rate are vectors
  - Expressed in either the inertial or body frame
  - Two frames related algebraically by the rotation matrix

$$\mathbf{h}_B(t) = \mathbf{H}_I^B(t)\mathbf{h}_I(t); \qquad \mathbf{h}_I(t) = \mathbf{H}_B^I(t)\mathbf{h}_B(t)$$

$$\boldsymbol{\omega}_B(t) = \mathbf{H}_I^B(t)\boldsymbol{\omega}_I(t); \qquad \boldsymbol{\omega}_I(t) = \mathbf{H}_B^I(t)\boldsymbol{\omega}_B(t)$$

Vector Derivative Expressed in a Rotating Frame

Chain Rule

$$\dot{\mathbf{h}}_I = \mathbf{H}_B^I \dot{\mathbf{h}}_B + \dot{\mathbf{H}}_B^I \mathbf{h}_B$$

Effect of body-frame rotation

Rate of change expressed in body frame

Alternatively

$$\dot{\mathbf{h}}_I = \mathbf{H}_B^I \dot{\mathbf{h}}_B + \boldsymbol{\omega}_I \times \mathbf{h}_I = \mathbf{H}_B^I \dot{\mathbf{h}}_B + \tilde{\boldsymbol{\omega}}_I \mathbf{h}_I$$

Consequently, the 2nd term is

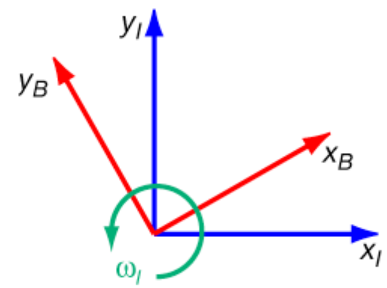
$$\dot{\mathbf{H}}_B^I \mathbf{h}_B = \tilde{\boldsymbol{\omega}}_I \mathbf{h}_I = \tilde{\boldsymbol{\omega}}_I \mathbf{H}_B^I \mathbf{h}_B$$

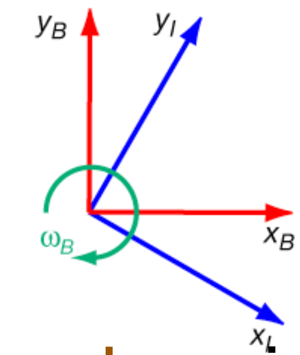
... where the cross-product equivalent matrix of angular rate is

$$\tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

External Moment Causes Change in Angular Rate

Positive rotation of Frame B w.r.t. Frame A is a **negative** rotation of Frame A w.r.t. Frame B



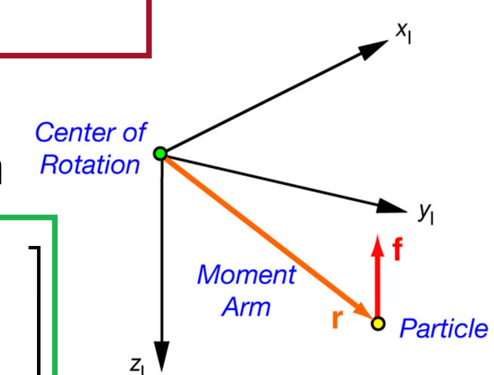


In the body frame of reference, the angular momentum change is

$$\boxed{\begin{aligned}\dot{\mathbf{h}}_B &= \mathbf{H}_I^B \dot{\mathbf{h}}_I + \dot{\mathbf{H}}_I^B \mathbf{h}_I = \mathbf{H}_I^B \dot{\mathbf{h}}_I - \boldsymbol{\omega}_B \times h_B = \mathbf{H}_I^B \dot{\mathbf{h}}_I - \tilde{\boldsymbol{\omega}}_B h_B \\ &= \mathbf{H}_I^B \mathbf{M}_I - \tilde{\boldsymbol{\omega}}_B \mathbb{I}_B \boldsymbol{\omega}_B = \mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B \mathbb{I}_B \boldsymbol{\omega}_B\end{aligned}}$$

Moment = torque = force x moment arm

$$\boxed{\mathbf{M}_I = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}_I ; \quad \mathbf{M}_B = \mathbf{H}_I^B \mathbf{M}_I = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}}$$



Rate of Change of Body-Referenced Angular Rate due to External Moment

In the body frame of reference, the angular momentum change is

$$\boxed{\begin{aligned}\dot{\mathbf{h}}_B &= \mathbf{H}_I^B\dot{\mathbf{h}}_I + \dot{\mathbf{H}}_I^B\mathbf{h}_I = \mathbf{H}_I^B\dot{\mathbf{h}}_I - \boldsymbol{\omega}_B \times h_B \\ &= \mathbf{H}_I^B\dot{\mathbf{h}}_I - \tilde{\boldsymbol{\omega}}_B h_B = \mathbf{H}_I^B\mathbf{M}_I - \tilde{\boldsymbol{\omega}}_B\mathbb{I}_B\boldsymbol{\omega}_B \\ &= \mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B\mathbb{I}_B\boldsymbol{\omega}_B\end{aligned}}$$

For constant body-axis inertia matrix

$$\boxed{\dot{\mathbf{h}}_B = \mathbb{I}_B\dot{\boldsymbol{\omega}}_B = \mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B\mathbb{I}_B\boldsymbol{\omega}_B}$$

Consequently, the differential equation for angular rate of change is

$$\boxed{\dot{\boldsymbol{\omega}}_B = \mathbb{I}_B^{-1}\left(\mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B\mathbb{I}_B\boldsymbol{\omega}_B\right)}$$

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EULER ANGLE RATES

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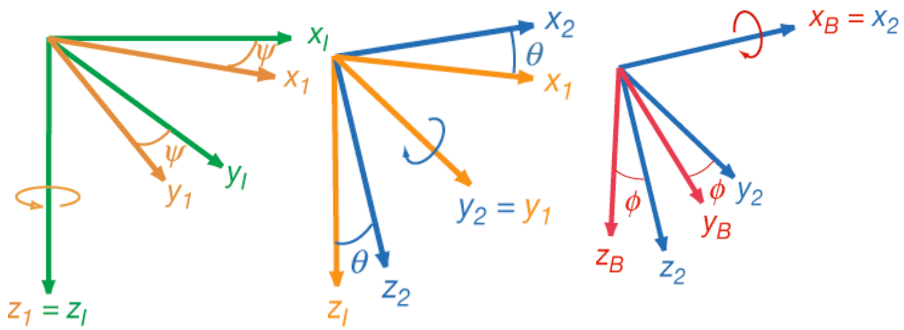
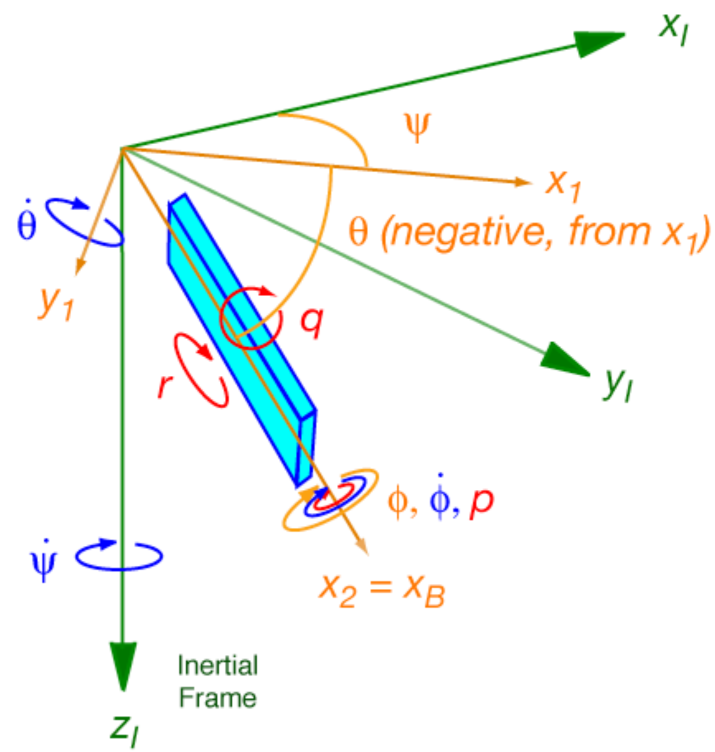
Euler-Angle Rates and Body-Axis Rates



Body-axis angular rate vector orthogonal

Euler angles form a non-orthogonal vector

Euler-angle rate vector is not orthogonal



Relationship Between Euler-Angle Rates and Body-Axis Rates

- $\dot{\psi}$ is measured in the Inertial Frame
- $\dot{\theta}$ is measured in Intermediate Frame #1
- $\dot{\phi}$ is measured in Intermediate Frame #2
- **... which is**

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{I}_3 \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{H}_2^B \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{H}_2^B \mathbf{H}_1^2 \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \mathbf{L}_I^B \dot{\mathbf{\Theta}}$$

Can the inversion become singular?
What does this mean?

Inverse transformation $[(.)^{-1} \neq (.)^T]$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{L}_B^I \boldsymbol{\omega}_B$$

Avoiding the Euler Angle Singularity at $\theta = \pm 90^\circ$

- **Alternatives to Euler angles**
  - Direction cosine (rotation) matrix
  - Quaternions

Propagation of direction cosine matrix (9 parameters)

$$\boxed{\dot{\mathbf{H}}_B^I \mathbf{h}_B = \tilde{\boldsymbol{\omega}}_I \mathbf{H}_B^I \mathbf{h}_B}$$

Consequently

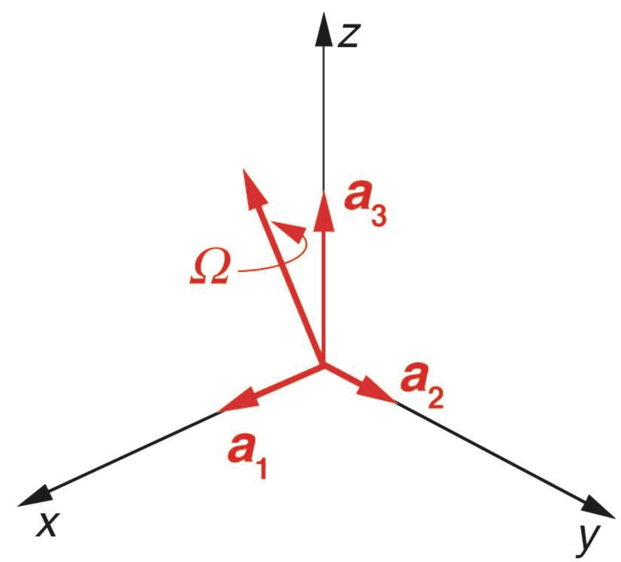
$$\dot{\mathbf{H}}_I^B(t) = -\tilde{\boldsymbol{\omega}}_B(t)\mathbf{H}_I^B(t) = -\begin{bmatrix} 0 & -r(t) & q(t) \\ r(t) & 0 & -p(t) \\ -q(t) & p(t) & 0(t) \end{bmatrix}_B \mathbf{H}_I^B(t)$$

$$\boxed{\mathbf{H}_I^B(0) = \mathbf{H}_I^B(\phi_0, \theta_0, \psi_0)}$$

Avoiding the Euler Angle Singularity at $\theta = \pm 90°$

Propagation of quaternion vector: single rotation from inertial to body frame (4 parameters)

- Rotation from one axis system, *I*, to another, *B*, represented by
  - Orientation of axis vector about which the rotation occurs (3 parameters of a unit vector, a_1, a_2, and a_3)
  - Magnitude of the rotation angle, Ω, rad



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