The University of Texas at Austin
Aerospace Engineering
and Engineering Mechanics
Cockrell School of Engineering

17 OCTOBER 2024

ASE 367K: FLIGHT DYNAMICS

TTH 09:30-11:00
CMA 2.306

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Topics for Today

- Topic(s):
  - Second Order Dynamic Systems
  - Long and Short Period Longitudinal Modes

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SECOND ORDER DYNAMIC SYSTEMS

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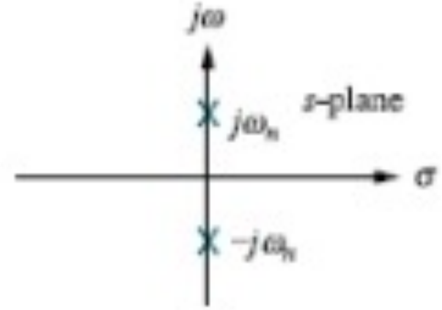
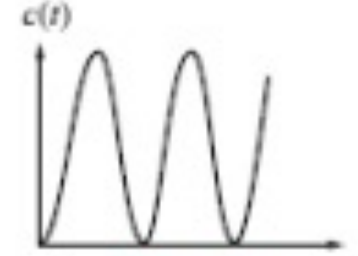
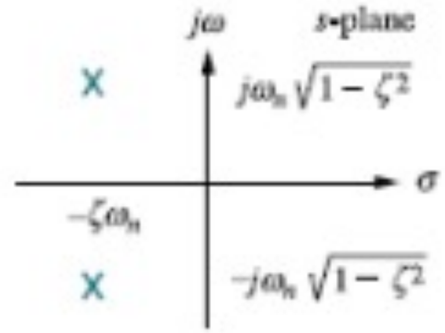
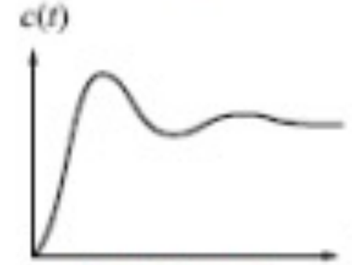
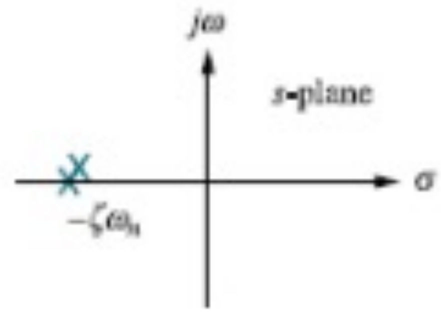
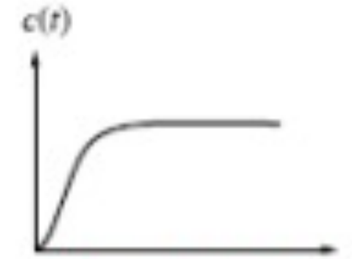
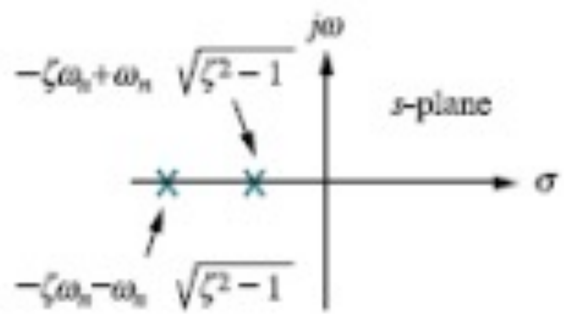
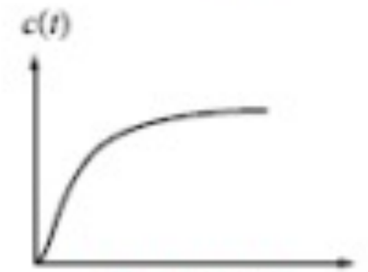
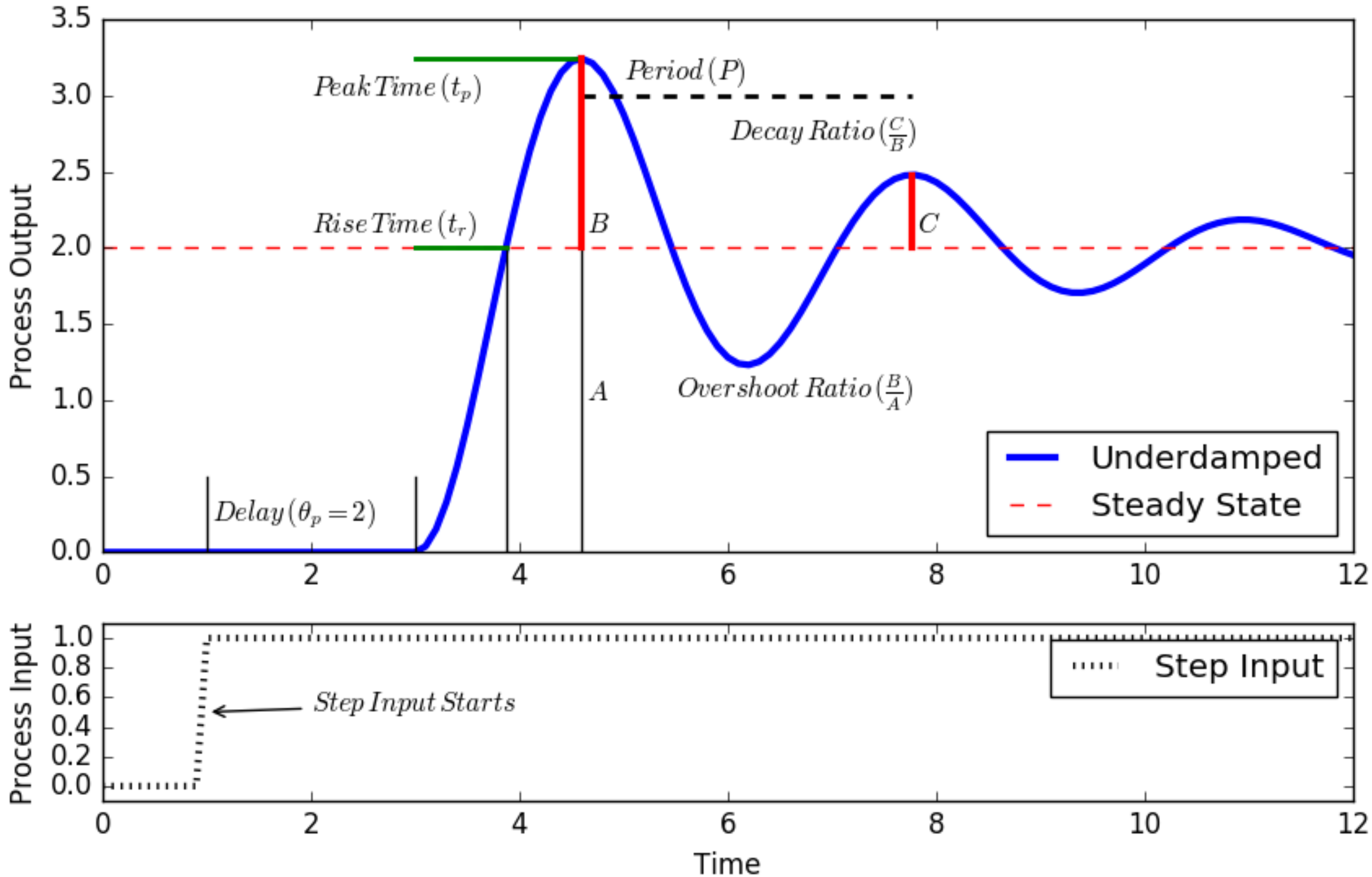
ζ	Poles	Step response
0		 Undamped
$0 < \zeta < 1$		 Underdamped
$\zeta = 1$		 Critically damped
$\zeta > 1$		 Overdamped



Process Output
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
2
4
6
8
10
12
Peak Time (t_p)
Period (P)
Decay Ratio $(\frac{C}{B})$
Rise Time (t_r)
B
C
A
Overshoot Ratio $(\frac{B}{A})$
Delay $(\theta_p = 2)$
Underdamped
Steady State
Process Input
1.0
0.8
0.6
0.4
0.2
0.0
Step Input Starts
Step Input
Time

Useful properties of an under damped second order system

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

Rise time (from 0 to 100%): $t_r = \frac{\pi - \beta}{\omega_d}$, where $\beta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

Peak time: $t_p = \frac{\pi}{\omega_d}$

Maximum over shoot: $M_p = e^{-\zeta\pi / \sqrt{1-\zeta^2}}$

Settling time (2% criterion): $t_s = \frac{4}{\zeta\omega_n}$

Unit step response: $x(t) = \frac{1}{m\omega_n^2}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(w_d t + \phi_0)\right]$

where $\phi_0 = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

Unit impulse response: $x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin w_d t$

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SHORT AND LONG PERIOD LONGITUDINAL MODES

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The Linear Longitudinal Dynamics are...

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u_1 - Z_{\dot{\alpha}} & 0 & 0 \\ 0 & -M_{\dot{\alpha}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u + X_{T_u} & X_\alpha & 0 & -g\cos\theta_1 \\ Z_u & Z_\alpha & u_1 + Z_q & -g\sin\theta_1 \\ M_u + M_{T_u} & M_\alpha + M_{T_\alpha} & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} X_{\delta_e} \\ Z_{\delta_e} \\ M_{\delta_e} \\ 0 \end{bmatrix} \Delta\delta_e \tag{8.23}$$

Therefore, the longitudinal dynamics are given by the linear matrix equation

$$\boldsymbol{M}\dot{\boldsymbol{x}} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{F}\boldsymbol{\delta}$$

where

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u_1 - Z_{\dot{\alpha}} & 0 & 0 \\ 0 & -M_{\dot{\alpha}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} X_u + X_{T_u} & X_\alpha & 0 & -g\cos\theta_1 \\ Z_u & Z_\alpha & u_1 + Z_q & -g\sin\theta_1 \\ M_u + M_{T_u} & M_\alpha + M_{T_\alpha} & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\boldsymbol{F} = \begin{bmatrix} X_{\delta_e} \\ Z_{\delta_e} \\ M_{\delta_e} \\ 0 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \\ \Delta\theta \end{bmatrix}, \quad \dot{\boldsymbol{x}} = \begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{\delta} = \Delta\delta_e$$

In standard linear systems notation, this is

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$\text{where} \qquad \boldsymbol{A} = \boldsymbol{M}^{-1}\boldsymbol{R} \qquad \text{and} \qquad \boldsymbol{B} = \boldsymbol{M}^{-1}\boldsymbol{F}$$

Omitting the control term from the full linear equations of motion for the longitudinal dynamics...

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u_1 - Z_{\dot{\alpha}} & 0 & 0 \\ 0 & -M_{\dot{\alpha}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u + X_{T_u} & X_\alpha & 0 & -g\cos\theta_1 \\ Z_u & Z_\alpha & u_1 + Z_q & -g\sin\theta_1 \\ M_u + M_{T_u} & M_\alpha + M_{T_\alpha} & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

or, in shorthand notation

$$\boldsymbol{M}\dot{\boldsymbol{x}} = \boldsymbol{R}\boldsymbol{x} \qquad \text{or} \qquad \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

Determining the "modes" of the system

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$$

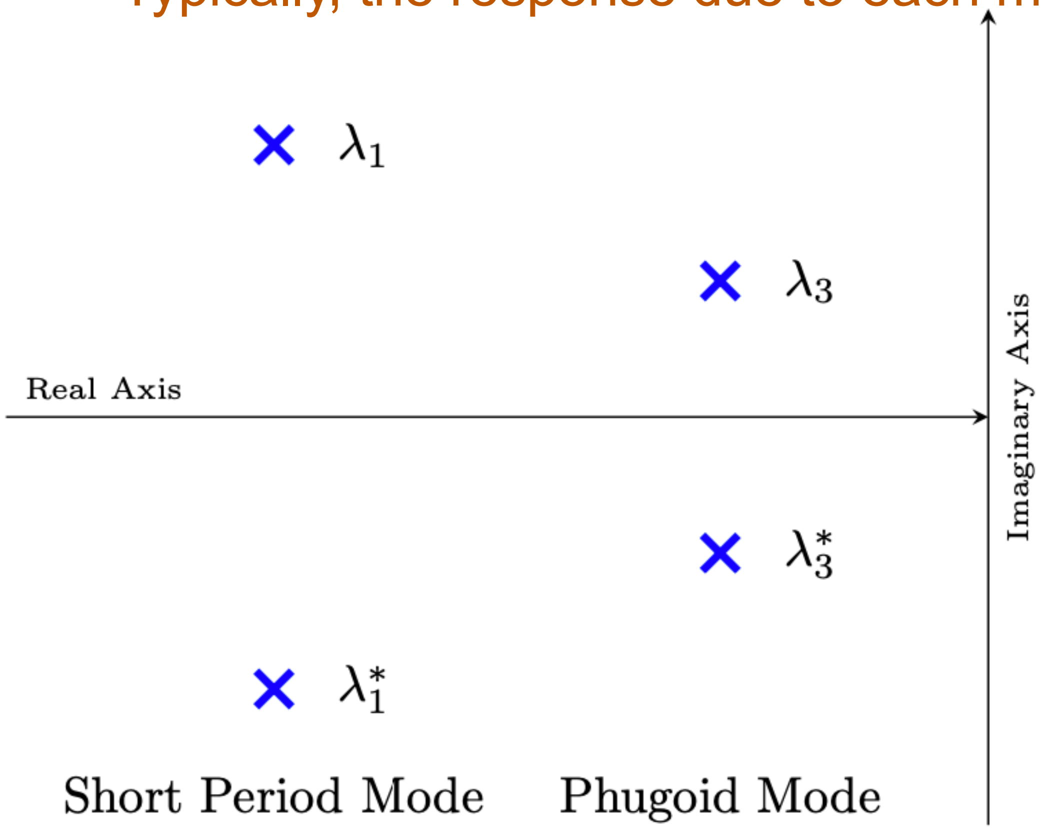
which yields four eigenvalues λ_1, λ_2, λ_3, and λ_4 along with four associated eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, and $\boldsymbol{v}_4$ which each satisfies the eigenvalue equation above.

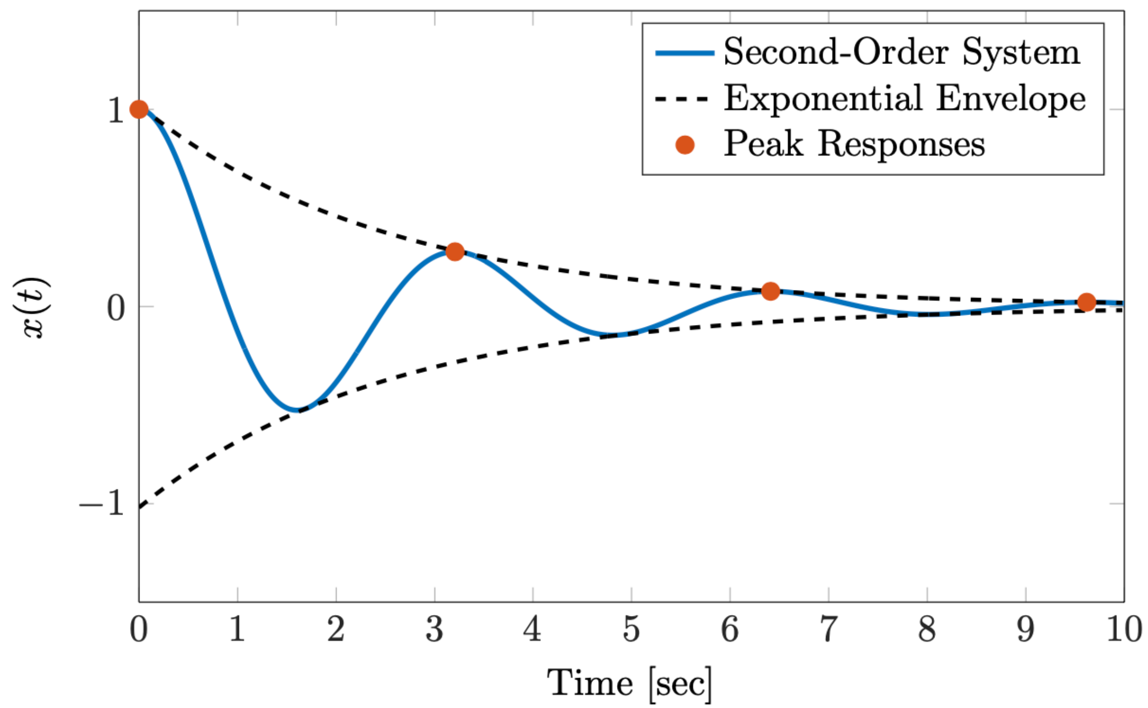
There are three possibilities for the eigenvalues:

1. four real eigenvalues (unlikely)
2. two real eigenvalues and one complex conjugate pair (happens occasionally)
3. two complex conjugate pair (most common)

Focusing on case 3…

- There are four total roots, but $\lambda_2 = \lambda_1^*$ and $\lambda_4 = \lambda_3^*$ (i.e., the complex conjugate).
- Typically, the response due to each mode is considered separately.





What else do we do with the eigenvectors?

- The eigenvectors can be used to determine which degrees of freedom dominate the response in each of the modes (short period and phugoid)
- In order to make a comparison on the dominant degrees of freedom, the units of the eigenvectors must be removed
- To assess the dominant modes, it is customary to normalize by one of the values

Boeing 747 in low cruise (at sea level)

$$
\begin{aligned}
X_u &= -0.0188 & Z_u &= -0.1862 & Z_{\delta_e} &= -8.7058 & M_{T_\alpha} &= 0.0000 \\
X_{T_u} &= 0.0000 & Z_\alpha &= -149.4408 & M_u &= 0.0001 & M_q &= -0.4275 \\
X_\alpha &= 11.5905 & Z_q &= -6.8045 & M_{T_u} &= 0.0000 & M_{\dot\alpha} &= -0.0658 \\
X_{\delta_e} &= 0.0000 & Z_{\dot\alpha} &= -8.4426 & M_\alpha &= -0.5294 & M_{\delta_e} &= -0.5630
\end{aligned}
$$

- We calculate matrices M and R and from them we calculate A

$$
\boldsymbol{A} = \begin{bmatrix} -0.0188 & 11.5905 & 0 & -32.2000 \\ -0.0006 & -0.5197 & 0.9470 & 0 \\ 0.0001 & -0.4952 & -0.4898 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}
$$

Boeing 747 in low cruise (at sea level)

- The eigenvalues of A are

$$\lambda_{1,2} = -0.5125 \pm 0.6830i \qquad \text{and} \qquad \lambda_{3,4} = -0.0017 \pm 0.1322i$$

- The eigenvectors (after being made unitless) are

$$\bar{\boldsymbol{v}}_{1,2} = \begin{bmatrix} 0.0036 \pm 0.0000i \\ 0.0428 \pm 0.0048i \\ -0.0001 \pm 0.0015i \\ 0.0307 \mp 0.0194i \end{bmatrix} \qquad \text{and} \qquad \bar{\boldsymbol{v}}_{3,4} = \begin{bmatrix} -0.0036 \pm 0.0000i \\ 0.0003 \pm 0.0001i \\ -0.0000 \pm 0.0000i \\ 0.0006 \pm 0.0041i \end{bmatrix}$$

Short Period mode

The short period mode is characterized by the complex-conjugate eigenvalues

$$\lambda_{1,2} = -0.5125 \pm 0.6830i$$

and the non-dimensional, normalized (with respect to $\Delta\theta$) eigenvector magnitudes

$$\|\tilde{\boldsymbol{v}}\|_{1,2} = \begin{bmatrix} 0.0984 \\ 1.1862 \\ 0.0418 \\ 1.0000 \end{bmatrix}$$

From the eigenvector, it is clear that the motion involves mostly the α and θ degrees of freedom. From the imaginary part of the eigenvalues, the damped natural frequency is

$$\omega_d = 0.6830 \text{ [rad/s]} = 0.1087 \text{ [Hz]}$$

Short Period mode

From the eigenvector, it is clear that the motion involves mostly the α and θ degrees of freedom. From the imaginary part of the eigenvalues, the damped natural frequency is

$$\omega_d = 0.6830 \text{ [rad/s]} = 0.1087 \text{ [Hz]}$$

Aircraft rapidly changes angle of attack with a highly damped pitch rate:

Short Period mode

From the real part of the eigenvalues, the product of the natural frequency and the damping ratio is

$$\zeta\omega_n = 0.5125 \text{ [rad/s]}$$

The natural frequency is found as the magnitude of the vector in the complex plane, giving

$$\omega_n = \sqrt{(\zeta\omega_n)^2 + \omega_d^2} = \sqrt{0.5125^2 + 0.6830^2} \text{ [rad/s]} = 0.8539 \text{ [rad/s]} = 0.1359 \text{ [Hz]}$$

The damping ratio can then be found from the natural frequency and the real part of the eigenvalue as

$$\zeta = \frac{\zeta\omega_n}{\omega_n} = \frac{0.5125 \text{ [rad/s]}}{0.8539 \text{ [rad/s]}} = 0.6002$$

Short Period mode

Using the logarithmic decrement, the time to damp to half of the initial amplitude is

$$\Delta T = \frac{\ln 2}{\zeta\omega_n} = \frac{\ln 2}{0.5125\ [\mathrm{rad/s}]} = 1.3525\ [\mathrm{s}]$$

Similarly, the number of cycles required to damp to half amplitude is

$$N = \frac{\ln 2}{\zeta\omega_n T_d} = \frac{\ln 2}{2\pi}\frac{\omega_d}{\zeta\omega_n} = \frac{\ln 2}{2\pi}\frac{0.6830\ [\mathrm{rad/s}]}{0.5125\ [\mathrm{rad/s}]} = 0.1470$$

Phugoid (Long Period) mode

The phugoid mode is characterized by the complex-conjugate eigenvalues

$$\lambda_{3,4} = -0.0017 \pm 0.1322i$$

and the non-dimensional, normalized (with respect to $\Delta\theta$) eigenvector magnitudes

$$\|\tilde{\boldsymbol{v}}\|_{3,4} = \begin{bmatrix} 0.8576 \\ 0.0664 \\ 0.0065 \\ 1.0000 \end{bmatrix}$$

From the eigenvector, it is clear that the motion involves mostly the u and θ degrees of freedom. From the imaginary part of the eigenvalues, the damped natural frequency is

$$\omega_d = 0.1322 \text{ [rad/s]} = 0.0210 \text{ [Hz]}$$

Phugoid (Long Period) mode

From the eigenvector, it is clear that the motion involves mostly the u and θ degrees of freedom. From the imaginary part of the eigenvalues, the damped natural frequency is

$$\omega_d = 0.1322 \text{ [rad/s]} = 0.0210 \text{ [Hz]}$$

Aircraft pitches and changes velocity at an almost constant angle of attack:

1. pitches up and climbs
2. pitches down and descends

Phugoid (Long Period) mode

From the real part of the eigenvalues, the product of the natural frequency and the damping ratio is

$$\zeta\omega_n = 0.0017 \text{ [rad/s]}$$

The natural frequency is found as the magnitude of the vector in the complex plane, giving

$$\omega_n = \sqrt{(\zeta\omega_n)^2 + \omega_d^2} = \sqrt{0.0017^2 + 0.1322^2} \text{ [rad/s]} = 0.1322 \text{ [rad/s]} = 0.0210 \text{ [Hz]}$$

The damping ratio can then be found from the natural frequency and the real part of the eigenvalue as

$$\zeta = \frac{\zeta\omega_n}{\omega_n} = \frac{0.0017 \text{ [rad/s]}}{0.1322 \text{ [rad/s]}} = 0.0127$$

Phugoid (Long Period) mode

Using the logarithmic decrement, the time to damp to half of the initial amplitude is

$$\Delta T = \frac{\ln 2}{\zeta\omega_n} = \frac{\ln 2}{0.0017\ [\mathrm{rad/s}]} = 412.4617\ [\mathrm{s}]$$

Similarly, the number of cycles required to damp to half amplitude is

$$N = \frac{\ln 2}{\zeta\omega_n T_d} = \frac{\ln 2}{2\pi}\frac{\omega_d}{\zeta\omega_n} = \frac{\ln 2}{2\pi}\frac{0.1322\ [\mathrm{rad/s}]}{0.0017\ [\mathrm{rad/s}]} = 8.6757$$

Short Period approximation

- Typically occurs so quickly that it proceeds at essentially constant vehicle speed, so a useful approximation is to set $\Delta u = 0$, which eliminates the first equation resulting in…

$$\begin{bmatrix} u_1 - Z_{\dot{\alpha}} & 0 & 0 \\ -M_{\dot{\alpha}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} Z_\alpha & u_1 + Z_q & -g\sin\theta_1 \\ M_\alpha + M_{T_\alpha} & M_q & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

- Assuming that the trim conditions are for level flight such that $\sin\theta_1 \approx 0$ results in…

$$\begin{bmatrix} u_1 - Z_{\dot{\alpha}} & 0 & 0 \\ -M_{\dot{\alpha}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} Z_\alpha & u_1 + Z_q & 0 \\ M_\alpha + M_{T_\alpha} & M_q & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

Short Period approximation

- The third equation is superfluous and can be eliminated…

$$\begin{bmatrix} u_1 - Z_{\dot{\alpha}} & 0 \\ -M_{\dot{\alpha}} & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\dot{q} \end{bmatrix} = \begin{bmatrix} Z_{\alpha} & u_1 + Z_q \\ M_{\alpha} + M_{T_{\alpha}} & M_q \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta q \end{bmatrix}$$

- This is the basic approximated Short Period mode, but further approximations can be made as appropriate
  - Sometimes $M_{\dot{\alpha}}$ is negligible (the book assumes this… but we will keep this term)
  - Typically we assume $\|Z_{\dot{\alpha}}\| \ll \|u_1\|$ and $\|Z_q\| \ll \|u_1\|$
  - We can also assume that $M_{T_{\alpha}} = 0$

Short Period approximation

- This short period approximation is useful for homework and quizzes

$$\begin{bmatrix} u_1 & 0 \\ -M_{\dot{\alpha}} & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\dot{q} \end{bmatrix} = \begin{bmatrix} Z_\alpha & u_1 \\ M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta q \end{bmatrix}$$

- This can be written in standard linear form as

$$\begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\dot{q} \end{bmatrix} = \begin{bmatrix} Z_\alpha/u_1 & 1 \\ M_{\dot{\alpha}}(Z_\alpha/u_1) + M_\alpha & M_{\dot{\alpha}} + M_q \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta q \end{bmatrix}$$

- The corresponding eigenvalues are given by the solution of

$$\lambda^2 - ((Z_\alpha/u_1) + M_{\dot{\alpha}} + M_q)\,\lambda + ((Z_\alpha/u_1)M_q - M_\alpha) = 0$$

Phugoid (Long Period) approximation

- The phugoid mode is characterized by a slow exchange of kinetic and potential energy that occurs at nearly constant angle of attack, hence…
  - $\Delta\alpha$ = 0 and its derivative is also zero
  - We can zero-out the second column of M and R

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u + X_{T_u} & 0 & 0 & -g\cos\theta_1 \\ Z_u & 0 & u_1 + Z_q & -g\sin\theta_1 \\ M_u + M_{T_u} & 0 & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

Phugoid (Long Period) approximation

- The first two equations from the reduced linear system are

$$\Delta\dot{u} = (X_u + X_{T_u})\Delta u - g\cos\theta_1\Delta\theta$$
$$0 = Z_u\Delta u + (u_1 + Z_q)\Delta q - g\sin\theta_1\Delta\theta$$

- Substituting $\Delta\dot{\theta} = \Delta q$ and assuming $\theta_1 \approx 0$ and $\|Z_q\| \ll \|u_1\|$

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u + X_{T_u} & -g \\ -Z_u/u_1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix}$$

General approximation

- The characteristic equation is $\lambda^2 - (X_u + X_{T_u})\,\lambda - g(Z_u/u_1) = 0$
- Hence… $$2\zeta\omega_n = -(X_u + X_{T_u})$$ $$\omega_n^2 = -g(Z_u/u_1)$$
- Or... $$\omega_n = \sqrt{-g(Z_u/u_1)}$$ $$\zeta = -\frac{X_u + X_{T_u}}{2\sqrt{-g(Z_u/u_1)}}$$
- The definition… $$Z_u = -\frac{\bar{q}_1 S(C_{L_u} + 2C_{L_1})}{mu_1} \Rightarrow \omega_n = \sqrt{\frac{\rho S g}{2m}(C_{L_u} + 2C_{L_1})}$$
- Similarly… $$X_u = -\frac{\bar{q}_1 S(C_{D_u} + 2C_{D_1})}{mu_1}$$ $$X_{T_u} = \frac{\bar{q}_1 S(C_{T_u} + 2C_{T_1})}{mu_1}$$ $$\Rightarrow \zeta = \sqrt{\frac{\bar{q}_1 S}{4W}}\frac{(C_{D_u} + 2C_{D_1}) - (C_{T_u} + 2C_{T_1})}{\sqrt{C_{L_u} + 2C_{L_1}}}$$

Low Subsonic approximation

- If we assume… $C_{L_u} \ll C_{L_1}$
- The natural frequency reduces… $\omega_n = \sqrt{\dfrac{\rho S g C_{L_1}}{m}}$
- If the aircraft is in level flight then...

$$W = mg = \bar{q}_1 S C_{L_1} \quad \Rightarrow \quad \omega_n = \sqrt{2}(g/u_1)$$

- Making the additional assumptions…

$$C_{L_u} \ll C_{L_1} \; X_{T_u} \text{ is negligible, } C_{T_u} = 0,\; C_{T_1} = 0,\; C_{D_u} = 0, \text{ and } C_{L_u} = 0$$

- We get… $\zeta = \sqrt{\dfrac{\bar{q}_1 S}{W}} \dfrac{C_{D_1}}{\sqrt{2 C_{L_1}}}$

- Finally, substituting the level flight assumption… $\zeta = \dfrac{1}{\sqrt{2}} \dfrac{C_{D_1}}{C_{L_1}} \quad \Rightarrow$ High L/D results in low damping

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