The University of Texas at Austin
Aerospace Engineering
and Engineering Mechanics
Cockrell School of Engineering

3 OCTOBER 2024

ASE 367K: FLIGHT DYNAMICS

TTH 09:30-11:00
CMA 2.306

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Topics for Today

- Topic(s):
  - Translational Position
  - Rotational Orientation
  - Angular Momentum
  - Rate of Change of Angular Momentum

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TRANSLATIONAL POSITION

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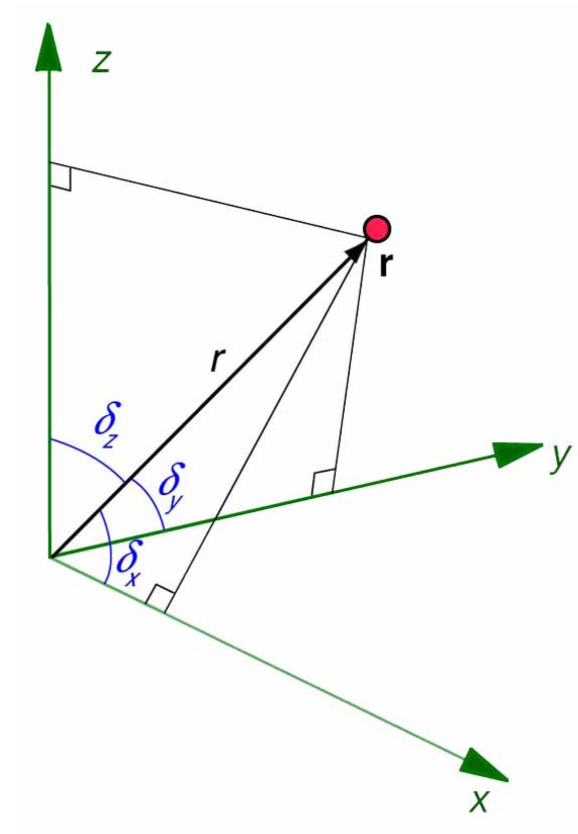
Position of a Particle

Projections of vector magnitude on three axes

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} \cos\delta_x \\ \cos\delta_y \\ \cos\delta_z \end{bmatrix}$$

$$\begin{bmatrix} \cos\delta_x \\ \cos\delta_y \\ \cos\delta_z \end{bmatrix} = \textbf{Direction cosines}$$



Measurement of Position in Alternative Frames - 1

1. what ref you are looking from
2. what reference do I note it in

- **Two reference frames of interest**
  - *I*: Inertial frame (fixed to inertial space)
  - *B*: Body frame (fixed to body)

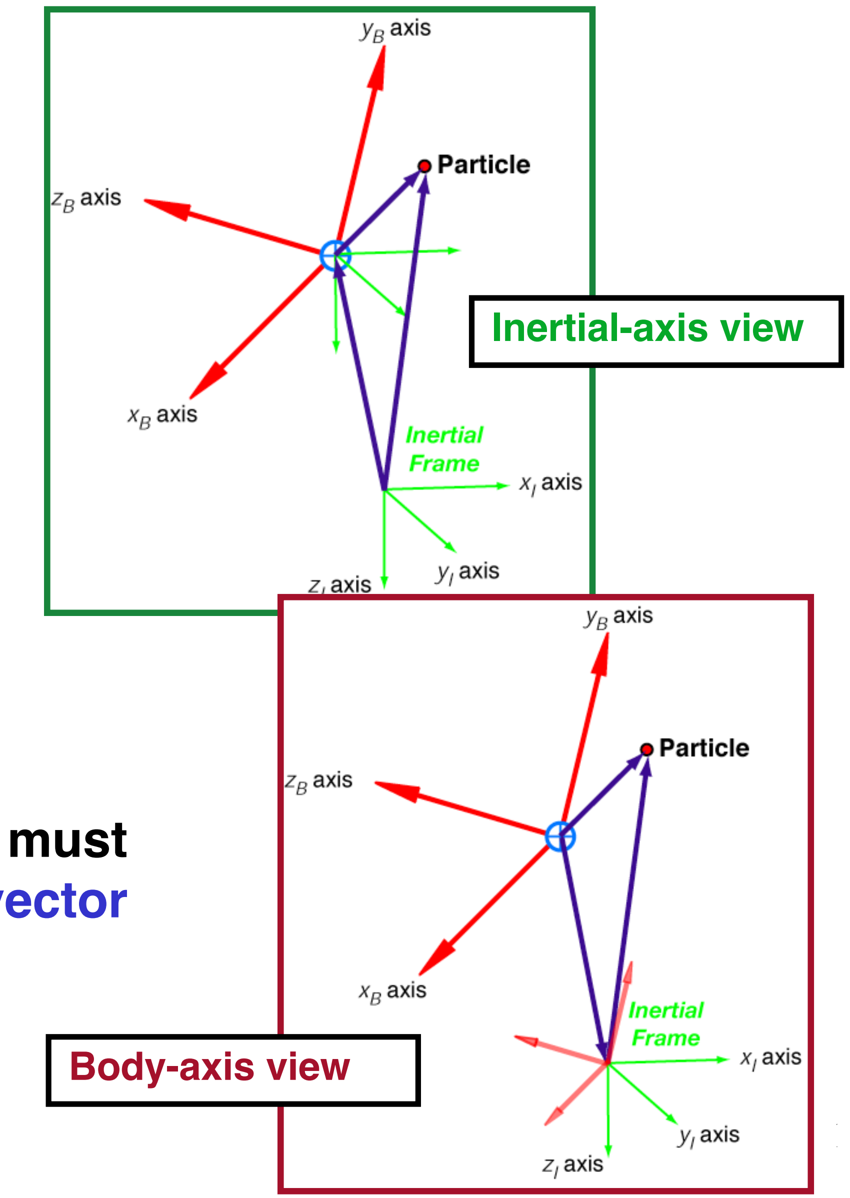
1. What reference is it in
2. Notation Reference

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathbf{r}_{particle} = \mathbf{r}_{origin} + \Delta\mathbf{r}_{w.r.t.origin}$$

Body

- **Differences in frame orientations** must be **taken into account** in **adding vector components**



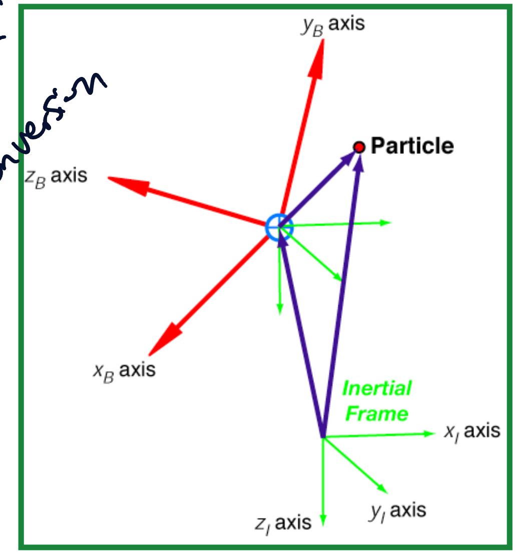
Measurement of Position in Alternative Frames - 2

Inertial-axis view

transforming Vector Body to I conversion

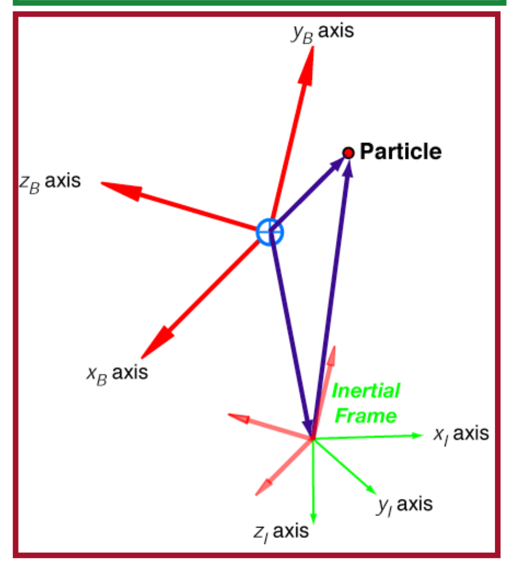
$$\mathbf{r}_{particle_I} = \mathbf{r}_{origin-B_I} + \mathbf{H}_B^I \Delta\mathbf{r}_B$$

captures orientation.



Body-axis view

$$\mathbf{r}_{particle_B} = \mathbf{r}_{origin-I_B} + \mathbf{H}_I^B \Delta\mathbf{r}_I$$



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ROTATIONAL ORIENTATION

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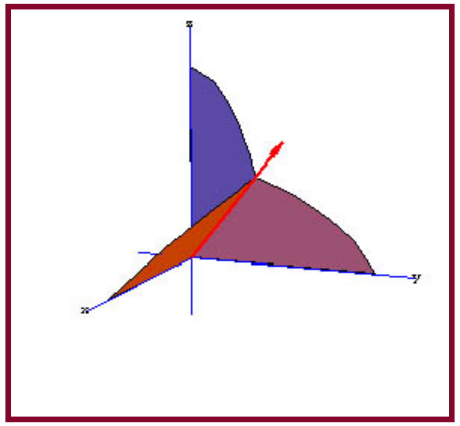
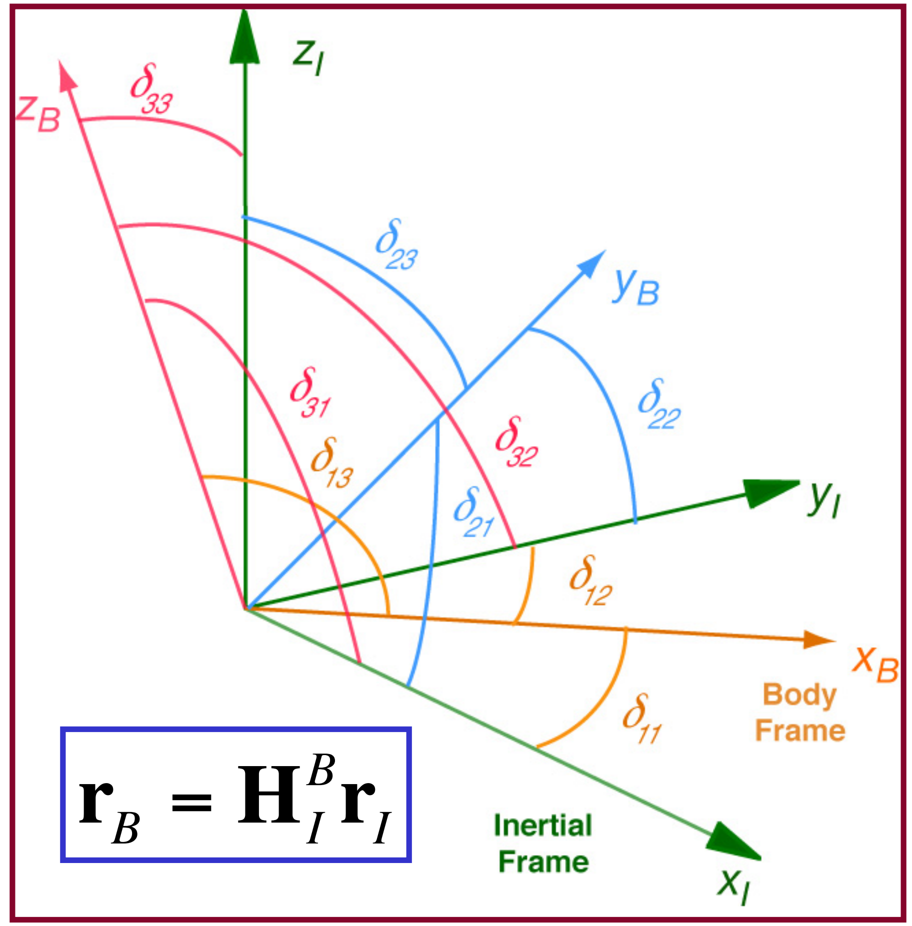
or i.e. δ_{31} z-axis on inertial looks like on x-axis on Body.

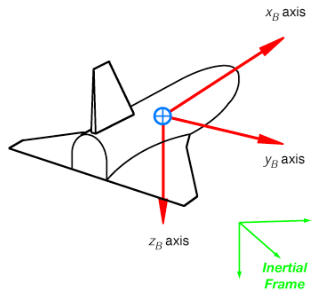


Direction Cosine Matrix

$$\mathbf{H}_I^B = \begin{bmatrix} \cos\delta_{11} & \cos\delta_{21} & \cos\delta_{31} \\ \cos\delta_{12} & \cos\delta_{22} & \cos\delta_{32} \\ \cos\delta_{13} & \cos\delta_{23} & \cos\delta_{33} \end{bmatrix}$$

- **Projections of unit vector components of one reference frame on another**
- **Rotational orientation of one reference frame with respect to another**
- **Cosines of angles between each *I* axis and each *B* axis**



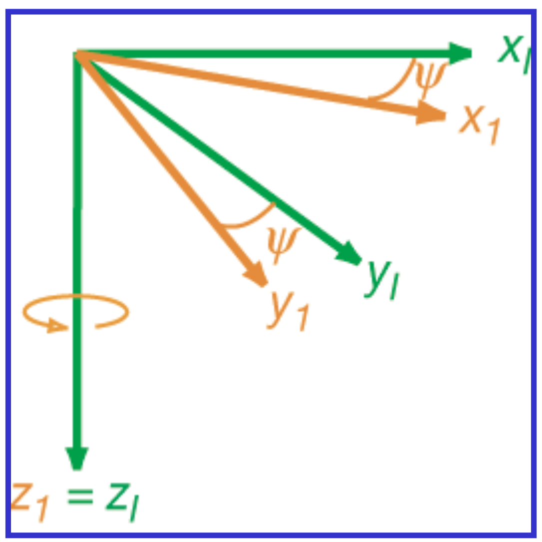


Euler Angles Measure the Orientation of One Frame with Respect to the Other

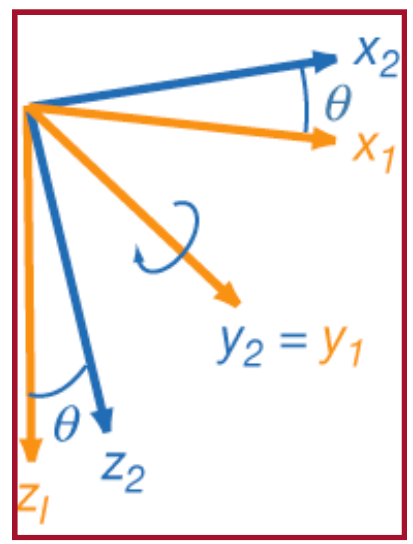
NED

- Conventional sequence of rotations from inertial to body frame
  - Each rotation is about a single axis
  - Right-hand rule
  - Yaw, then pitch, then roll
  - These are called Euler Angles

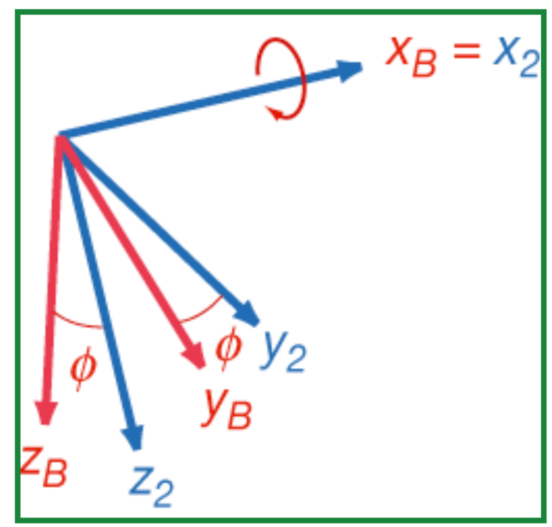
In airplane → 3 - 1 - 2
z y x



Yaw rotation (ψ) about z_I



Pitch rotation (θ) about y_1

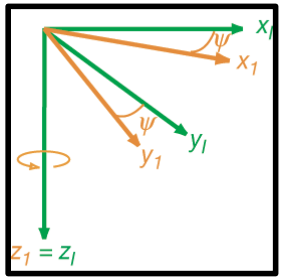


Roll rotation (ϕ) about x_2

Other sequences of 3 rotations can be chosen; however, once sequence is chosen, it must be retained

Reference Frame Rotation from Inertial to Body: Aircraft Convention (3-2-1)

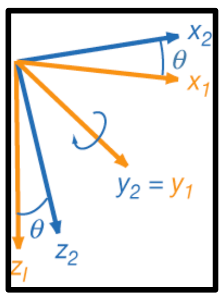
Yaw rotation (ψ) about z_I axis



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_1 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_I = \begin{bmatrix} x_I \cos\psi + y_I \sin\psi \\ -x_I \sin\psi + y_I \cos\psi \\ z_I \end{bmatrix}$$

$$\mathbf{r}_1 = \mathbf{H}_I^1 \mathbf{r}_I$$

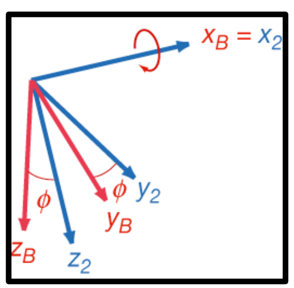
Pitch rotation (θ) about y_1 axis



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_1$$

$$\mathbf{r}_2 = \mathbf{H}_1^2 \mathbf{r}_1 = \left[\mathbf{H}_1^2 \mathbf{H}_I^1 \right] \mathbf{r}_I = \mathbf{H}_I^2 \mathbf{r}_I$$

Roll rotation (ϕ) about x_2 axis



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_2$$

$$\mathbf{r}_B = \mathbf{H}_2^B \mathbf{r}_2 = \left[\mathbf{H}_2^B \mathbf{H}_1^2 \mathbf{H}_I^1 \right] \mathbf{r}_I = \mathbf{H}_I^B \mathbf{r}_I$$

The Rotation Matrix

The three-angle rotation matrix is the product of 3 single-angle rotation matrices:

$$\boxed{\mathbf{H}_I^B(\phi,\theta,\psi) = \mathbf{H}_2^B(\phi)\mathbf{H}_1^2(\theta)\mathbf{H}_I^1(\psi)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \hline -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta \\ \hline \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{array}\right]$$

Exam

an expression of the ***Direction Cosine Matrix***

Rotation Matrix Inverse

Inverse relationship: interchange sub- and superscripts

$$\boxed{\begin{aligned}\mathbf{r}_B &= \mathbf{H}_I^B \mathbf{r}_I \\ \mathbf{r}_I &= \left(\mathbf{H}_I^B\right)^{-1} \mathbf{r}_B = \mathbf{H}_B^I \mathbf{r}_B\end{aligned}}$$

Because transformation is orthonormal

Inverse = transpose

Rotation matrix is always non-singular

$$\boxed{\left[\mathbf{H}_I^B(\phi,\theta,\psi)\right]^{-1} = \left[\mathbf{H}_I^B(\phi,\theta,\psi)\right]^T = \mathbf{H}_B^I(\psi,\theta,\phi)}$$

$$\boxed{\mathbf{H}_B^I = \left(\mathbf{H}_I^B\right)^{-1} = \left(\mathbf{H}_I^B\right)^T = \mathbf{H}_1^I \mathbf{H}_2^1 \mathbf{H}_B^2}$$

$$\boxed{\mathbf{H}_B^I \mathbf{H}_I^B = \mathbf{H}_I^B \mathbf{H}_B^I = \mathbf{I}}$$

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ANGULAR MOMENTUM

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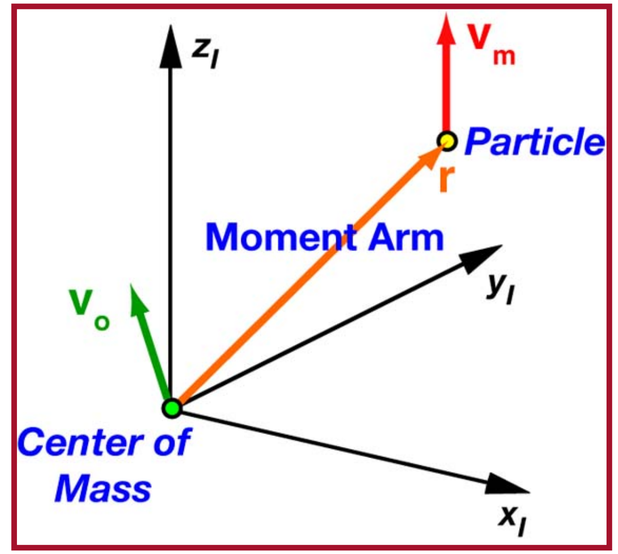
Ang momentum perpendicular was angular momentum

Angular Momentum of a Particle



- Moment of linear momentum of differential particles that make up the body
  - (Differential masses) x components of the velocity that are perpendicular to the moment arms

Angular momentum

$$d\mathbf{h} = \left(\mathbf{r} \times dm\ \mathbf{v}\right) = \left(\mathbf{r} \times \mathbf{v}_m\right) dm$$
$$= \left[\mathbf{r} \times \left(\mathbf{v}_o + \boldsymbol{\omega} \times \mathbf{r}\right)\right] dm$$

Any object

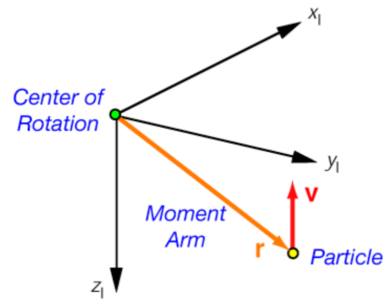
$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

can be decomposed into two motion

Any momentum

- Cross Product: Evaluation of a determinant with unit vectors (*i, j, k*) along axes, (*x, y, z*) and (v_x, v_y, v_z) projections on to axes

$$\mathbf{r} \times \mathbf{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} = \left(yv_z - zv_y\right)\boldsymbol{i} + \left(zv_x - xv_z\right)\boldsymbol{j} + \left(xv_y - yv_x\right)\boldsymbol{k}$$



Cross-Product-Equivalent Matrix

Cross product

$$\mathbf{r} \times \mathbf{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} = \left(yv_z - zv_y\right)\boldsymbol{i} + \left(zv_x - xv_z\right)\boldsymbol{j} + \left(xv_y - yv_x\right)\boldsymbol{k}$$

$$= \begin{bmatrix} \left(yv_z - zv_y\right) \\ \left(zv_x - xv_z\right) \\ \left(xv_y - yv_x\right) \end{bmatrix} = \tilde{\mathbf{r}}\mathbf{v} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Cross-product-equivalent matrix

$$\tilde{\mathbf{r}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

Angular Momentum of the Aircraft

- **Integrate moment of linear momentum of differential particles over the body**

$$\mathbf{h} = \int_{Body} \left[\mathbf{r} \times \left(\mathbf{v}_o + \boldsymbol{\omega} \times \mathbf{r} \right) \right] dm = \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \int_{z_{\min}}^{z_{\max}} (\mathbf{r} \times \mathbf{v}) \rho(x,y,z)\, dx\, dy\, dz = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}$$

$\rho(x,y,z) =$ Density of the body

- **Choose the center of mass as the rotational center**

$$\mathbf{h} = \int_{Body} (\mathbf{r} \times \mathbf{v}_o)\, dm + \int_{Body} \left[\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) \right] dm$$

$$= 0 - \int_{Body} \left[\mathbf{r} \times (\mathbf{r} \times \boldsymbol{\omega}) \right] dm$$

$$= -\int_{Body} (\mathbf{r} \times \mathbf{r})\, dm \times \boldsymbol{\omega} \equiv -\int_{Body} (\tilde{\mathbf{r}}\tilde{\mathbf{r}})\, dm\, \boldsymbol{\omega}$$

$r = 0$, since V_o happens at Body center

It's b/c $(r \times \omega r)$

typo?

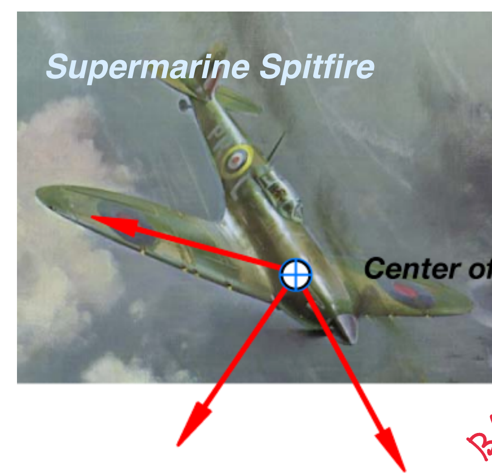
Not zero because we are integrating

this is more like

$\omega \times r = -(r \times \omega)$

BAC-CAB Rule

$r \times (\omega \times r) = \omega(r \cdot r) - r(r \cdot \omega)$



$h = r \times V_{total}$
$= r \times (\omega \times r)$

The Inertia Matrix

$$\mathbf{h} = -\int_{Body} \tilde{\mathbf{r}}\,\tilde{\mathbf{r}}\,\boldsymbol{\omega}\,dm = -\int_{Body} \tilde{\mathbf{r}}\,\tilde{\mathbf{r}}\,dm\,\boldsymbol{\omega} = \mathbb{I}\boldsymbol{\omega}$$

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

where

$$\mathbb{I} = -\int_{Body} \tilde{\mathbf{r}}\,\tilde{\mathbf{r}}\,dm = -\int_{Body} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} dm$$

$$= \int_{Body} \begin{bmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{bmatrix} dm$$

moment of inertia around x

around y

around z

Inertia matrix derives from equal effect of angular rate on all particles of the aircraft

effect around x only affects around z-axis

Moments and Products of Inertia

ω_y affect x

rotation around one axis affecting around other axis

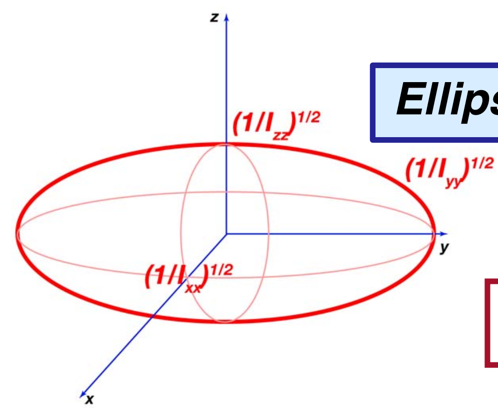
$$\mathbb{I} = \int_{Body} \begin{bmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{bmatrix} dm = \begin{bmatrix} \mathbb{I}_{xx} & -\mathbb{I}_{xy} & -\mathbb{I}_{xz} \\ -\mathbb{I}_{xy} & \mathbb{I}_{yy} & -\mathbb{I}_{yz} \\ -\mathbb{I}_{xz} & -\mathbb{I}_{yz} & \mathbb{I}_{zz} \end{bmatrix}$$

Inertia matrix

- **Moments of inertia on the diagonal**
- **Products of inertia off the diagonal**

- **If products of inertia are zero, (x, y, z) are principal axes --->**
- **All rigid bodies have a set of principal axes**

$$\begin{bmatrix} \mathbb{I}_{xx} & 0 & 0 \\ 0 & \mathbb{I}_{yy} & 0 \\ 0 & 0 & \mathbb{I}_{zz} \end{bmatrix}$$



Ellipsoid of Inertia

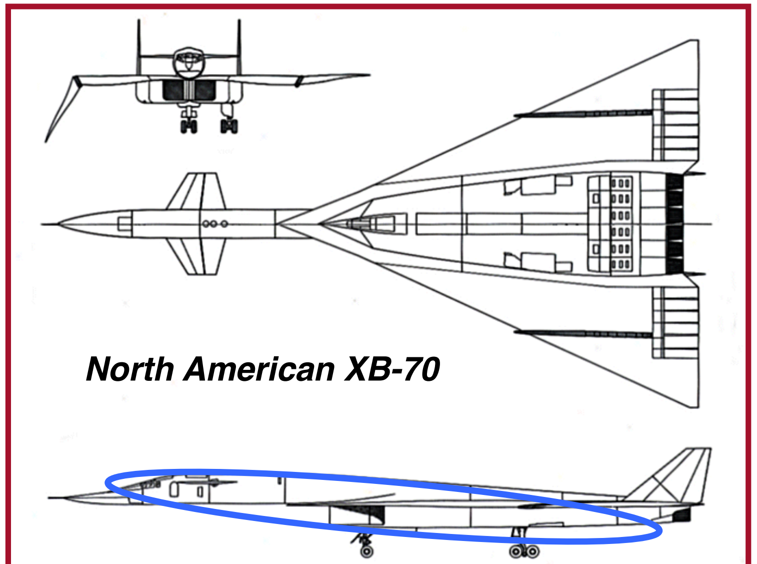
$$\mathbb{I}_{xx}x^2 + \mathbb{I}_{yy}y^2 + \mathbb{I}_{zz}z^2 = 1$$

for aircraft
y and z axe
symmetric

Inertia Matrix of an Aircraft with Mirror Symmetry

$$\mathbb{I} = \int_{Body} \begin{bmatrix} (y^2+z^2) & 0 & -xz \\ 0 & (x^2+z^2) & 0 \\ -xz & 0 & (x^2+y^2) \end{bmatrix} dm = \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix}$$

Nose high/low product of inertia, I_{xz}



Nominal Configuration

Tips folded, 50% fuel, W = 38,524 lb

x_{cm} @0.218 $\bar{c}$

$I_{xx} = 1.8 \times 10^6$ slug-ft^2

$I_{yy} = 19.9 \times 10^6$ slug-ft^2

$I_{xx} = 22.1 \times 10^6$ slug-ft^2

$I_{xz} = -0.88 \times 10^6$ slug-ft^2

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RATE OF CHANGE OF ANGULAR MOMENTUM

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Newton's 2nd Law, Applied to Rotational Motion

In inertial frame, rate of change of angular momentum = applied moment (or torque), M

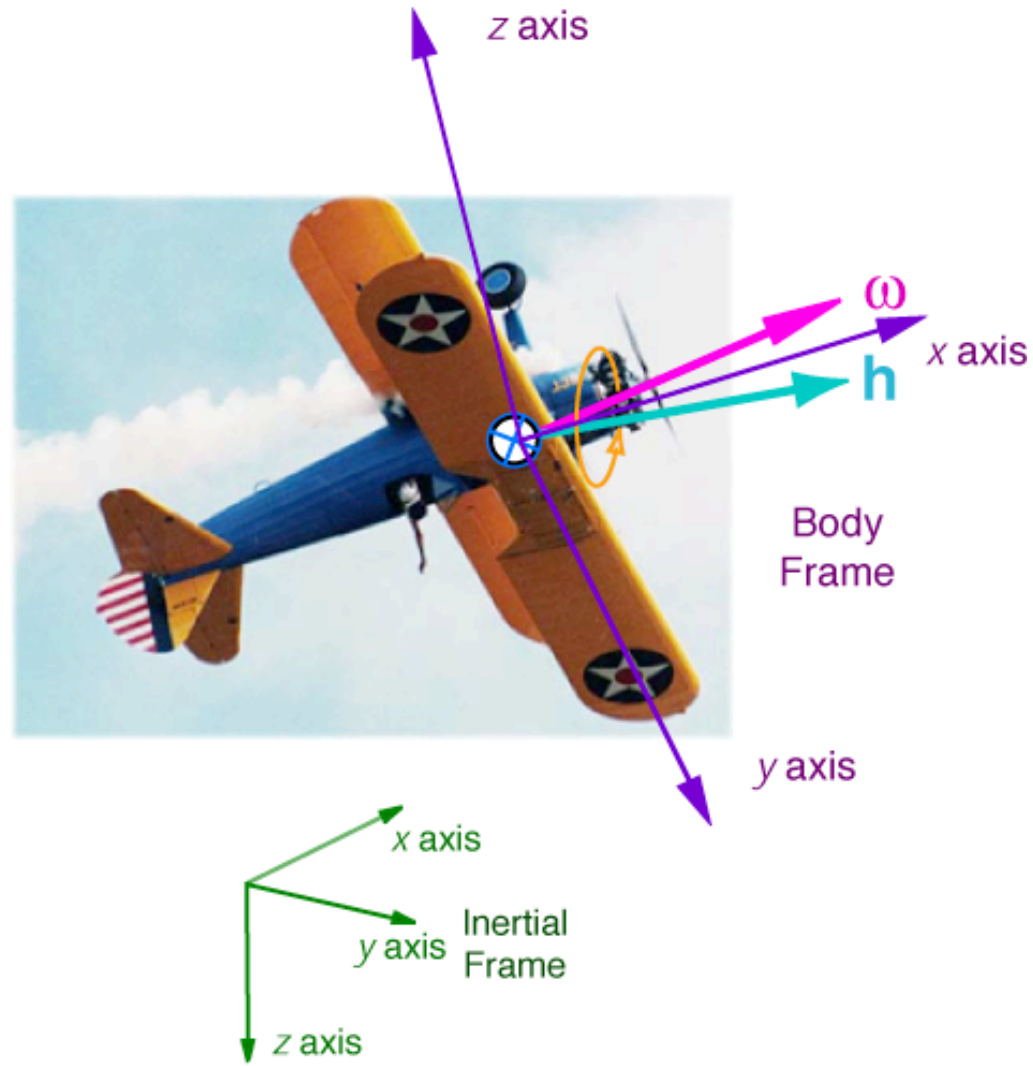
$$\boxed{\frac{d\mathbf{h}}{dt} = \frac{d\left(\mathbb{I}\boldsymbol{\omega}\right)}{dt} = \frac{d\mathbb{I}}{dt}\boldsymbol{\omega} + \mathbb{I}\frac{d\boldsymbol{\omega}}{dt}}$$

$$\boxed{= \dot{\mathbb{I}}\boldsymbol{\omega} + \mathbb{I}\dot{\boldsymbol{\omega}} = \mathbf{M} = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}}$$

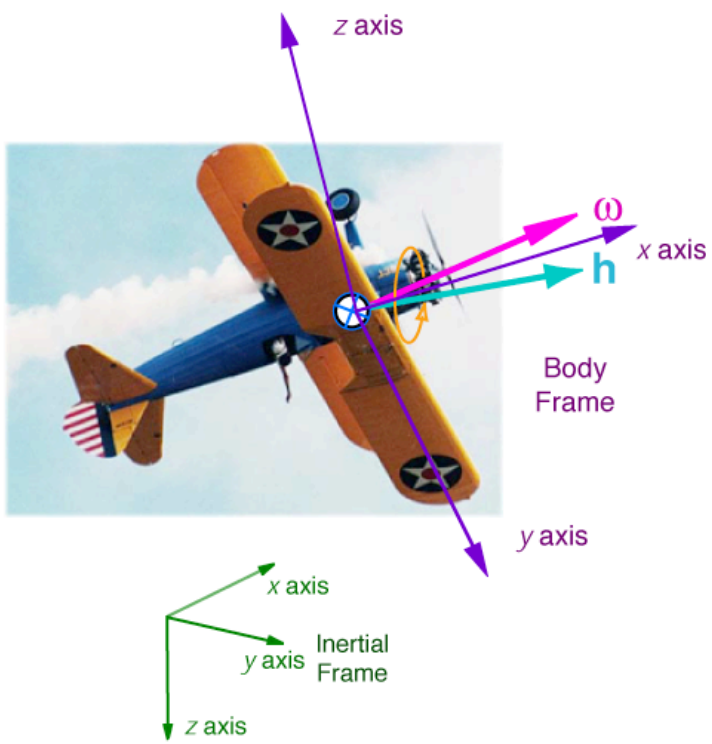
Angular Momentum and Rate

Angular momentum and rate vectors are not necessarily aligned

$$\boxed{\mathbf{h} = \mathbb{I}\boldsymbol{\omega}}$$



Angular Momentum Expressed in Two Frames of Reference



- Angular momentum and rate are vectors
  - Expressed in either the inertial or body frame
  - Two frames related algebraically by the rotation matrix

$$\mathbf{h}_B(t) = \mathbf{H}_I^B(t)\mathbf{h}_I(t); \qquad \mathbf{h}_I(t) = \mathbf{H}_B^I(t)\mathbf{h}_B(t)$$

$$\boldsymbol{\omega}_B(t) = \mathbf{H}_I^B(t)\boldsymbol{\omega}_I(t); \qquad \boldsymbol{\omega}_I(t) = \mathbf{H}_B^I(t)\boldsymbol{\omega}_B(t)$$

Vector Derivative Expressed in a Rotating Frame

Chain Rule

Effect of body-frame rotation

$$\dot{\mathbf{h}}_I = \mathbf{H}_B^I \dot{\mathbf{h}}_B + \dot{\mathbf{H}}_B^I \mathbf{h}_B$$

Rate of change expressed in body frame

Alternatively

$$\dot{\mathbf{h}}_I = \mathbf{H}_B^I \dot{\mathbf{h}}_B + \boldsymbol{\omega}_I \times \mathbf{h}_I = \mathbf{H}_B^I \dot{\mathbf{h}}_B + \tilde{\boldsymbol{\omega}}_I \mathbf{h}_I$$

Consequently, the 2nd term is

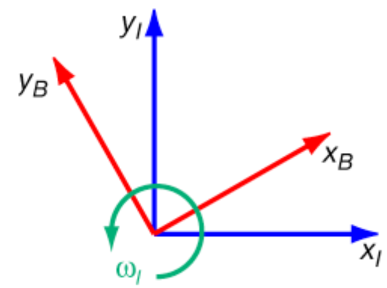
$$\dot{\mathbf{H}}_B^I \mathbf{h}_B = \tilde{\boldsymbol{\omega}}_I \mathbf{h}_I = \tilde{\boldsymbol{\omega}}_I \mathbf{H}_B^I \mathbf{h}_B$$

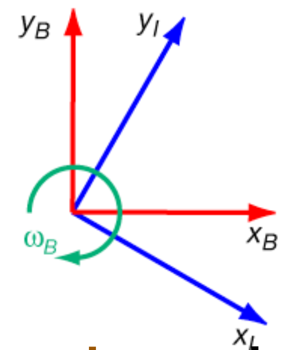
... where the cross-product equivalent matrix of angular rate is

$$\tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

External Moment Causes Change in Angular Rate

Positive rotation of Frame B w.r.t. Frame A is a **negative** rotation of Frame A w.r.t. Frame B



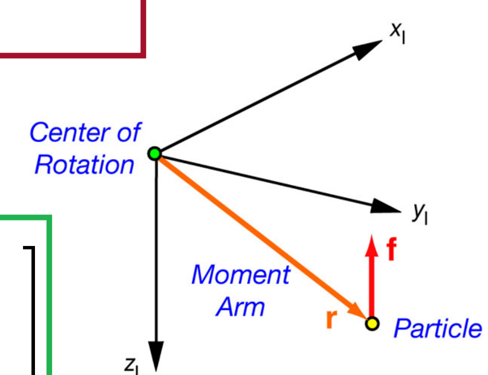


In the body frame of reference, the angular momentum change is

$$
\begin{aligned}
\dot{\mathbf{h}}_B &= \mathbf{H}_I^B \dot{\mathbf{h}}_I + \dot{\mathbf{H}}_I^B \mathbf{h}_I = \mathbf{H}_I^B \dot{\mathbf{h}}_I - \boldsymbol{\omega}_B \times h_B = \mathbf{H}_I^B \dot{\mathbf{h}}_I - \tilde{\boldsymbol{\omega}}_B h_B \\
&= \mathbf{H}_I^B \mathbf{M}_I - \tilde{\boldsymbol{\omega}}_B \mathbb{I}_B \boldsymbol{\omega}_B = \mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B \mathbb{I}_B \boldsymbol{\omega}_B
\end{aligned}
$$

Moment = torque = force x moment arm

$$
\mathbf{M}_I = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}_I ; \quad \mathbf{M}_B = \mathbf{H}_I^B \mathbf{M}_I = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}
$$



Rate of Change of Body-Referenced Angular Rate due to External Moment

In the body frame of reference, the angular momentum change is

$$\boxed{\begin{aligned}\dot{\mathbf{h}}_B &= \mathbf{H}_I^B\dot{\mathbf{h}}_I + \dot{\mathbf{H}}_I^B\mathbf{h}_I = \mathbf{H}_I^B\dot{\mathbf{h}}_I - \boldsymbol{\omega}_B \times h_B \\ &= \mathbf{H}_I^B\dot{\mathbf{h}}_I - \tilde{\boldsymbol{\omega}}_B h_B = \mathbf{H}_I^B\mathbf{M}_I - \tilde{\boldsymbol{\omega}}_B\mathbb{I}_B\boldsymbol{\omega}_B \\ &= \mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B\mathbb{I}_B\boldsymbol{\omega}_B\end{aligned}}$$

For constant body-axis inertia matrix

$$\boxed{\dot{\mathbf{h}}_B = \mathbb{I}_B\dot{\boldsymbol{\omega}}_B = \mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B\mathbb{I}_B\boldsymbol{\omega}_B}$$

Consequently, the differential equation for angular rate of change is

$$\boxed{\dot{\boldsymbol{\omega}}_B = \mathbb{I}_B^{-1}\left(\mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B\mathbb{I}_B\boldsymbol{\omega}_B\right)}$$

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