The University of Texas at Austin
Aerospace Engineering and Engineering Mechanics
Cockrell School of Engineering

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ASE 367K: FLIGHT DYNAMICS

TTH 09:30-11:00
CMA 2.306

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Topics for Today

- Topic(s):
  - Linearized Longitudinal Equations of Motion

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LINEARIZED LONGITUDINAL EOM

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Turning our attention to longitudinal motion…

- If we neglect small change in air density due to the small variations in altitude during cruise flight, then…
  - Position equations become uncoupled and can be integrated independently
  - We are left with the following 4 nonlinear equation of Longitudinal motion

→ Velocity in rotating frame

→ velocity

longitudinal

$$X = m(\dot{u} + qw - rv + g \sin\theta)$$

→ rotation

$$Z = m(\dot{w} + pv - qu - g \cos\theta \cos\phi)$$

$$M = I_{yy}\dot{q} - I_{xz}(r^2 - p^2) - (I_{zz} - I_{xx})pr$$

$$q = \dot{\theta} \cos\phi + \dot{\psi} \cos\theta \sin\phi$$

Turning our attention to longitudinal motion…

- If we linearize the equations about some operating condition, which can be completely arbitrary, and …
  - Denote the operating condition by the subscript "1"
  - Assume the perturbations about the operating condition are small
- We can represent the quantities of interest as the sum of the nominal operating condition and a perturbation from nominal

$$
\begin{aligned}
u &= u_1 + \Delta u & \qquad p &= p_1 + \Delta p & \qquad \phi &= \phi_1 + \Delta \phi \\
v &= v_1 + \Delta v & \qquad q &= q_1 + \Delta q & \qquad \theta &= \theta_1 + \Delta \theta \\
w &= w_1 + \Delta w & \qquad r &= r_1 + \Delta r & \qquad \psi &= \psi_1 + \Delta \psi
\end{aligned}
$$

$$
\begin{aligned}
X &= m(\dot{u} + qw - rv + g\sin\theta) \\
&= m\big[(\dot{u}_1 + \Delta\dot{u}) + (q_1 + \Delta q)(w_1 + \Delta w) - (r_1 + \Delta r)(v_1 + \Delta v) && \text{substitute in perturbations} \\
&\qquad + g\sin(\theta_1 + \Delta\theta)\big] \\
&= m\big[(\dot{u}_1 + \Delta\dot{u}) + (q_1 + \Delta q)(w_1 + \Delta w) - (r_1 + \Delta r)(v_1 + \Delta v) && \text{angle addition formula} \\
&\qquad + g(\sin\theta_1 \cos\Delta\theta + \cos\theta_1 \sin\Delta\theta)\big] \\
&= m\big[(\dot{u}_1 + \Delta\dot{u}) + (q_1 + \Delta q)(w_1 + \Delta w) - (r_1 + \Delta r)(v_1 + \Delta v) && \text{small angle approximation} \\
&\qquad + g(\sin\theta_1 + \cos\theta_1 \Delta\theta)\big] \\
&= m(\dot{u}_1 + q_1 w_1 - r_1 v_1 + g\sin\theta_1) && \text{neglect higher-order terms} \\
&\qquad + m\big[(\Delta\dot{u} - r_1\Delta v + q_1\Delta w + w_1\Delta q - v_1\Delta r) + g\cos\theta_1\Delta\theta\big] && (8.2a) \\
&= X_1 + \Delta X && (8.2b)
\end{aligned}
$$

if equilibrium $X_1 = 0$

$$
\begin{aligned}
Z &= m(\dot{w} + pv - qu - g\cos\theta\cos\phi) \\
&= m\big[(\dot{w}_1 + \Delta\dot{w}) + (p_1 + \Delta p)(v_1 + \Delta v) - (q_1 + \Delta q)(u_1 + \Delta u) && \text{substitute in perturbations} \\
&\qquad - g\cos(\theta_1 + \Delta\theta)\cos(\phi_1 + \Delta\phi)\big] \\
&= m\big[(\dot{w}_1 + \Delta\dot{w}) + (p_1 + \Delta p)(v_1 + \Delta v) - (q_1 + \Delta q)(u_1 + \Delta u) && \text{angle addition formula} \\
&\qquad - g(\cos\theta_1\cos\Delta\theta - \sin\theta_1\sin\Delta\theta)(\cos\phi_1\cos\Delta\phi - \sin\phi_1\sin\Delta\phi)\big] \\
&= m\big[(\dot{w}_1 + \Delta\dot{w}) + (p_1 + \Delta p)(v_1 + \Delta v) - (q_1 + \Delta q)(u_1 + \Delta u) && \text{small angle approximation} \\
&\qquad - g(\cos\theta_1 - \sin\theta_1\Delta\theta)(\cos\phi_1 - \sin\phi_1\Delta\phi)\big] \\
&= m(\dot{w}_1 + p_1 v_1 - q_1 u_1 - g\cos\theta_1\cos\phi_1) && \text{neglect higher-order terms} \\
&\qquad + m\big[(\Delta\dot{w} - q_1\Delta u + p_1\Delta v + v_1\Delta p - u_1\Delta q) \\
&\qquad + g\cos\theta_1\sin\phi_1\Delta\phi + g\sin\theta_1\cos\phi_1\Delta\theta\big] && \text{(8.3a)} \\
&= Z_1 + \Delta Z && \text{(8.3b)}
\end{aligned}
$$

$$
\begin{aligned}
M &= I_{yy}\dot{q} - I_{xz}(r^2 - p^2) - (I_{zz} - I_{xx})pr \\
&= I_{yy}(\dot{q}_1 + \Delta\dot{q}) - I_{xz}\big[(r_1 + \Delta r)(r_1 + \Delta r) - (p_1 + \Delta p)(p_1 + \Delta p)\big] && \text{substitute in perturbations} \\
&\qquad - (I_{zz} - I_{xx})(p_1 + \Delta p)(r_1 + \Delta r) \\
&= \big[I_{yy}\dot{q}_1 - I_{xz}(r_1^2 - p_1^2) - (I_{zz} - I_{xx})p_1 r_1\big] && \text{neglect higher-order terms} \\
&\qquad + I_{yy}\Delta\dot{q} + (2I_{xz}p_1 - (I_{zz} - I_{xx})r_1)\Delta p \\
&\qquad - (2I_{xz}r_1 + (I_{zz} - I_{xx})p_1)\Delta r && \text{(8.4a)} \\
&= M_1 + \Delta M && \text{(8.4b)}
\end{aligned}
$$

$$
\begin{aligned}
q &= \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \\
&= (\dot{\theta}_1 + \Delta\dot{\theta})\cos(\phi_1 + \Delta\phi) && \text{substitute in perturbations} \\
&\quad + (\dot{\psi}_1 + \Delta\dot{\psi})\cos(\theta_1 + \Delta\theta)\sin(\phi_1 + \Delta\phi) \\
&= (\dot{\theta}_1 + \Delta\dot{\theta})(\cos\phi_1\cos\Delta\phi - \sin\phi_1\sin\Delta\phi) && \text{angle addition formula} \\
&\quad + (\dot{\psi}_1 + \Delta\dot{\psi})(\cos\theta_1\cos\Delta\theta - \sin\theta_1\sin\Delta\theta) \\
&\quad \times (\sin\phi_1\cos\Delta\phi + \cos\phi_1\sin\Delta\phi) \\
&= (\dot{\theta}_1 + \Delta\dot{\theta})(\cos\phi_1 - \sin\phi_1\Delta\phi) && \text{small angle approximation} \\
&\quad + (\dot{\psi}_1 + \Delta\dot{\psi})(\cos\theta_1 - \sin\theta_1\Delta\theta)(\sin\phi_1 + \cos\phi_1\Delta\phi) \\
&= (\dot{\theta}_1\cos\phi_1 + \dot{\psi}_1\cos\theta_1\sin\phi_1) && \text{neglect higher order terms} \\
&\quad + \cos\phi_1\Delta\dot{\theta} + \cos\theta_1\sin\phi_1\Delta\dot{\psi} + \dot{\psi}_1\cos\theta_1\cos\phi_1\Delta\phi \\
&\quad - \dot{\psi}_1\sin\theta_1\sin\phi_1\Delta\theta && \text{(8.5a)} \\
&= q_1 + \Delta q && \text{(8.5b)}
\end{aligned}
$$

Focusing in on the perturbations...

$$\Delta X = m(\Delta\dot{u} - r_1\Delta v + q_1\Delta w + w_1\Delta q - v_1\Delta r + g\cos\theta_1\Delta\theta)$$

$$\Delta Z = m(\Delta\dot{w} - q_1\Delta u + p_1\Delta v + v_1\Delta p - u_1\Delta q + g\cos\theta_1\sin\phi_1\Delta\phi + g\sin\theta_1\cos\phi_1\Delta\theta)$$

$$\Delta M = I_{yy}\Delta\dot{q} + (2I_{xz}p_1 - (I_{zz} - I_{xx})r_1)\Delta p - (2I_{xz}r_1 + (I_{zz} - I_{xx})p_1)\Delta r$$

$$\Delta q = \cos\phi_1\Delta\dot{\theta} + \cos\theta_1\sin\phi_1\Delta\dot{\psi} + \dot{\psi}_1\cos\theta_1\cos\phi_1\Delta\phi - \dot{\psi}_1\sin\theta_1\sin\phi_1\Delta\theta$$

Pitching moment

Stability frame doesn't consider β.

Focusing in on the perturbations…

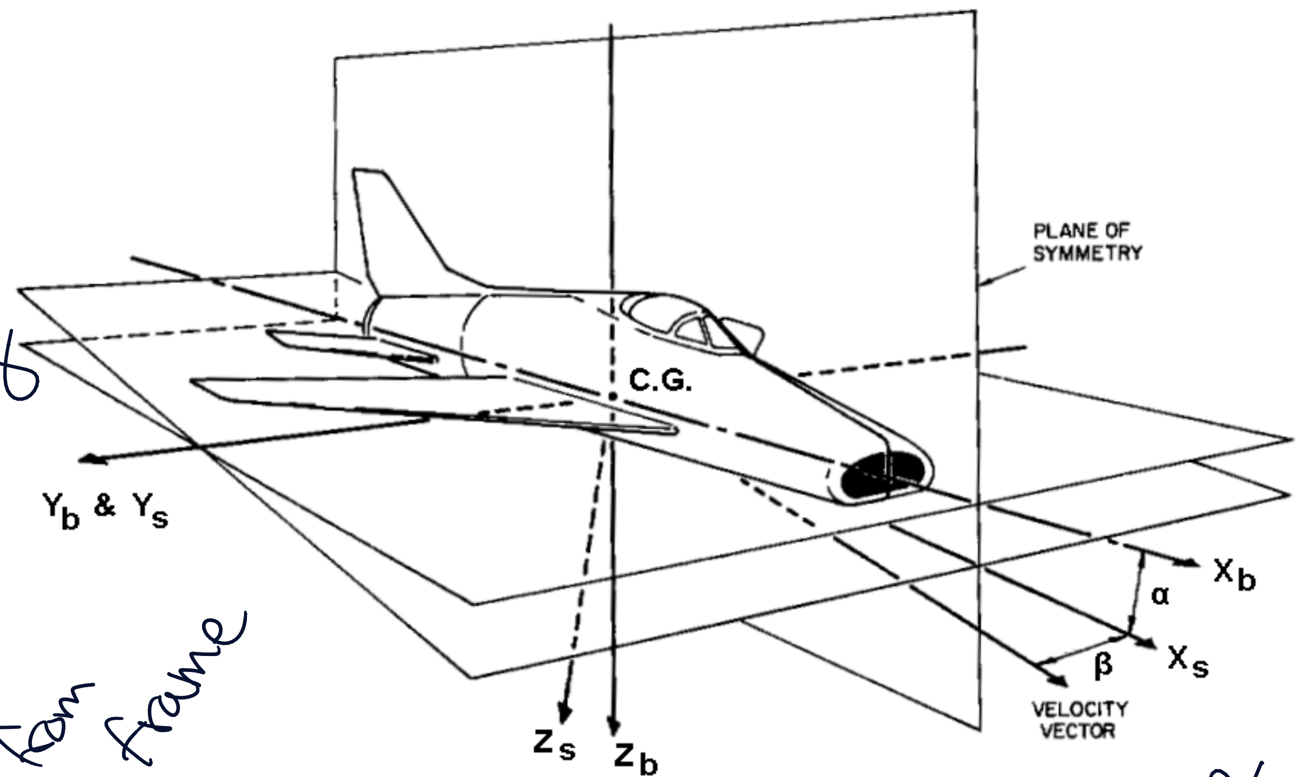
- If we linearize w.r.t. a constant velocity flight with <u>zero beta</u> and we choose a body-fixed frame that is aligned with the nominal velocity vector…

Y axis is the same.

S: Stability

x_s matches velocity

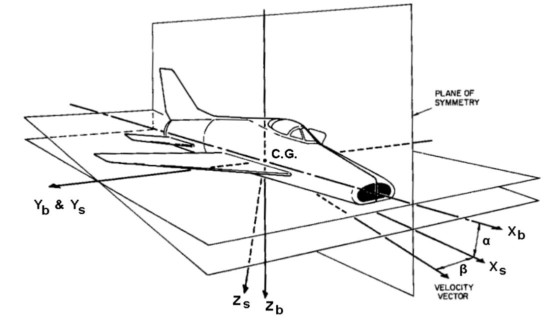
S frame is rotated by $-\alpha$ from body frame



Definition of Body and Stability Axes

$\alpha = 0$ for stability axis

Focusing in on the perturbations…



Definition of Body and Stability Axes

- If we linearize w.r.t. a constant velocity flight with <u>zero beta</u> and we choose a body-fixed frame that is aligned with the nominal velocity vector, then …

Stability axis

No side slipping

$$w_1 = 0 \qquad v_1 = 0\,, \qquad p_1 = 0\,, \qquad q_1 = 0\,, \qquad r_1 = 0\,, \qquad \text{and} \qquad \phi_1 = 0$$

$$\dot{\phi}_1 = 0\,, \qquad \dot{\theta}_1 = 0\,, \qquad \text{and} \qquad \dot{\psi}_1 = 0$$

- and…

$$\Delta X = m(\Delta\dot{u} - \cancel{r_1}\Delta v + \cancel{q_1}\Delta w + \cancel{w_1}\Delta q - \cancel{v_1}\Delta r + g\cos\theta_1\Delta\theta)$$

$$\Delta Z = m(\Delta\dot{w} - \cancel{q_1}\Delta u + \cancel{p_1}\Delta v + \cancel{v_1}\Delta p - u_1\Delta q + g\cos\theta_1\cancel{\sin\phi_1}\Delta\phi + g\sin\theta_1\overset{=1}{\cos\phi_1}\Delta\theta)$$

$$\Delta M = I_{yy}\Delta\dot{q} + (2I_{xz}\cancel{p_1} - (I_{zz} - I_{xx})\cancel{r_1})\Delta p - (2I_{xz}\cancel{r_1} + (I_{zz} - I_{xx})\cancel{p_1})\Delta r$$

$$\Delta q = \underset{=1}{\cos\phi_1}\Delta\dot{\theta} + \cos\theta_1\cancel{\sin\phi_1}\Delta\dot{\psi} + \cancel{\dot{\psi}_1}\cos\theta_1\cos\phi_1\Delta\phi - \dot{\psi}_1\sin\theta_1\cancel{\sin\phi_1}\Delta\theta$$

pqr

Focusing in on the perturbations...

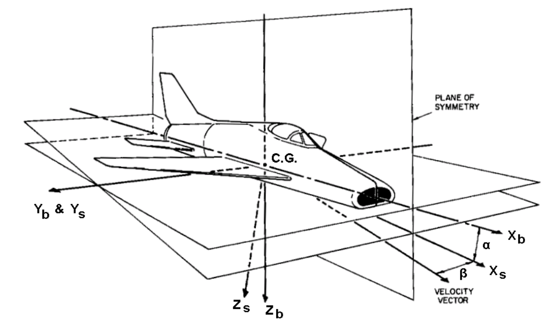
$$\Delta X = m(\Delta \dot{u} + g \cos \theta_1 \Delta \theta)$$

$$\Delta Z = m(\Delta \dot{w} - u_1 \Delta q + g \sin \theta_1 \Delta \theta)$$

$$\Delta M = I_{yy} \Delta \dot{q}$$

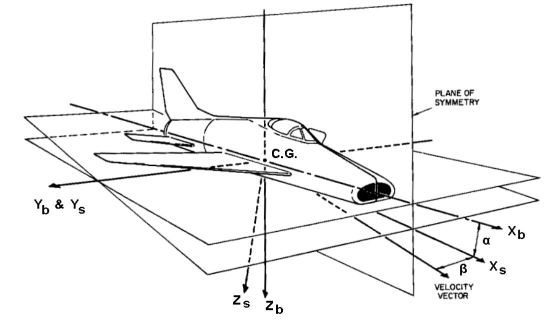
$$\Delta q = \Delta \dot{\theta}$$

Replacing Δw with $\Delta\alpha$ ...



Definition of Body and Stability Axes

$$
\begin{aligned}
\alpha &= \arctan\frac{w}{u} \\
&= \frac{w}{u} && \text{small angle approximation} \\
&= \frac{w_1 + \Delta w}{u_1 + \Delta u} && \text{substitute in perturbations} \\
&= (w_1 + \Delta w)(u_1 + \Delta u)^{-1} \\
&= \frac{w_1 + \Delta w}{u_1}\left(1 + \frac{\Delta u}{u_1}\right)^{-1} \\
&= \frac{w_1 + \Delta w}{u_1}\left(1 - \frac{\Delta u}{u_1} - \cdots\right) && \text{binomial expansion} \\
&= \frac{w_1}{u_1} + \frac{\Delta w}{u_1} - \frac{w_1 \Delta u}{u_1^2} && \text{neglect higher-order terms} \\
&= \alpha_1 + \Delta\alpha
\end{aligned}
$$



Definition of Body and Stability Axes

Replacing Δw with $\Delta\alpha$ ...

$w_1 = 0$, therefore $\alpha_1 = 0$, hence $\quad \Delta\alpha = \dfrac{\Delta w}{u_1} \quad$ or $\quad \Delta w = u_1 \Delta\alpha$

Since we chose to use stability axis

Furthermore, since $\dot{u}_1 = 0$, it follows that

$$\Delta\dot{w} = u_1 \Delta\dot{\alpha}$$

$$\Delta X = m(\Delta\dot{u} + g\cos\theta_1 \Delta\theta)$$

$$\Delta Z = m(u_1\Delta\dot{\alpha} - u_1\Delta q + g\sin\theta_1\Delta\theta)$$

$$\Delta M = I_{yy}\Delta\dot{q}$$

$$\Delta q = \Delta\dot{\theta}$$

Let's look at those forces...

$$X = T + L \sin \Delta\alpha - D \cos \Delta\alpha$$
$$Z = -L \cos \Delta\alpha - D \sin \Delta\alpha$$

$$T = T_1 + \Delta T\,, \quad L = L_1 + \Delta L\,, \quad \text{and} \quad D = D_1 + \Delta D$$

Let's look at those forces...

$$\begin{aligned}
X &= T + L\sin\Delta\alpha - D\cos\Delta\alpha \\
&= (T_1 + \Delta T) + (L_1 + \Delta L)\sin\Delta\alpha - (D_1 + \Delta D)\cos\Delta\alpha && \text{substitute in perturbations} \\
&= (T_1 + \Delta T) + (L_1 + \Delta L)\Delta\alpha - (D_1 + \Delta D) && \text{small angle approximation} \\
&= (T_1 - D_1) + (\Delta T + L_1\Delta\alpha - \Delta D) && \text{neglect higher-order terms} \\
&= X_1 + \Delta X
\end{aligned}$$

$$\begin{aligned}
Z &= -L\cos\alpha - D\sin\alpha \\
&= -(L_1 + \Delta L)\cos\Delta\alpha - (D_1 + \Delta D)\sin\Delta\alpha && \text{substitute in perturbations} \\
&= -(L_1 + \Delta L) - (D_1 + \Delta D)\Delta\alpha && \text{small angle approximation} \\
&= (L_1 - D_1) + (-\Delta L - D_1\Delta\alpha) && \text{neglect higher-order terms} \\
&= Z_1 + \Delta Z
\end{aligned}$$

Let's look at those forces...

$$\Delta X = \Delta T + L_1 \Delta\alpha - \Delta D$$
$$\Delta Z = -\Delta L - D_1 \Delta\alpha$$

$$\Delta\dot{u} = \Delta T/m + L_1/m\Delta\alpha - \Delta D/m - g\cos\theta_1\Delta\theta$$
$$u_1\Delta\dot{\alpha} = -\Delta L/m - D_1/m\Delta\alpha + u_1\Delta q - g\sin\theta_1\Delta\theta$$
$$\Delta\dot{q} = \Delta M/I_{yy}$$
$$\Delta\dot{\theta} = \Delta q$$

Forces are expressed as a product of the dynamic
pressure, a reference area, and a non-dimensional coefficient, such that $F = \bar{q} S C_F$

$$\left(\frac{\partial F}{\partial u}\right)_1 = \left(\frac{\partial \bar{q}}{\partial u}\right)_1 S C_{F_1} + \bar{q}_1 S \left(\frac{\partial C_F}{\partial u}\right)_1$$

Since the dynamic pressure is given by

$$\bar{q} = \frac{1}{2}\rho(u^2 + v^2 + w^2)$$

we have

$$\left(\frac{\partial \bar{q}}{\partial u}\right)_1 = \left(\frac{1}{2}\rho \cdot 2u\right)_1 = \rho u_1 = \frac{2\bar{q}_1}{u_1}$$

such that

$$\left(\frac{\partial F}{\partial u}\right)_1 = \bar{q}_1 S \left(\frac{\partial C_F}{\partial u}\right)_1 + \frac{2\bar{q}_1 S}{u_1} C_{F_1}$$

Applying the chain rule to the first term

$$\left(\frac{\partial C_F}{\partial u}\right)_1 = \left(\frac{\partial C_F}{\partial (u/u_1)}\right)_1 \frac{\partial (u/u_1)}{\partial u} = \frac{1}{u_1} C_{F_u}$$

which gives

$$\left(\frac{\partial F}{\partial u}\right)_1 = \frac{\bar{q}_1 S}{u_1}(C_{F_u} + 2C_{F_1}) \qquad \text{where} \qquad C_{F_u} = \left(\frac{\partial C_F}{\partial (u/u_1)}\right)_1$$

$$\begin{aligned}
\left(\frac{\partial F}{\partial \alpha}\right)_1 &= \bar{q}_1 S C_{F_\alpha} & C_{F_\alpha} &= \left(\frac{\partial C_F}{\partial \alpha}\right)_1 \\
\left(\frac{\partial F}{\partial q}\right)_1 &= \frac{\bar{q}_1 S \bar{c}}{2u_1} C_{F_q} & C_{F_q} &= \left(\frac{\partial C_F}{\partial (q\bar{c}/2u_1)}\right)_1 \\
\left(\frac{\partial F}{\partial \dot{\alpha}}\right)_1 &= \frac{\bar{q}_1 S \bar{c}}{2u_1} C_{F_{\dot{\alpha}}} & C_{F_{\dot{\alpha}}} &= \left(\frac{\partial C_F}{\partial (\dot{\alpha}\bar{c}/2u_1)}\right)_1 \\
\left(\frac{\partial F}{\partial \delta_e}\right)_1 &= \bar{q}_1 S C_{F_{\delta_e}} & C_{F_{\delta_e}} &= \left(\frac{\partial C_F}{\partial \delta_e}\right)_1
\end{aligned}$$

Let’s look at those forces again...

$$\Delta D = \frac{\bar{q}_1 S}{u_1}(C_{D_u} + 2C_{D_1})\Delta u + \bar{q}_1 S C_{D_\alpha}\Delta\alpha + \frac{\bar{q}_1 S\bar{c}}{2u_1}C_{D_q}\Delta q + \frac{\bar{q}_1 S\bar{c}}{2u_1}C_{D_{\dot{\alpha}}}\Delta\dot{\alpha} + \bar{q}_1 S C_{D_{\delta_e}}\Delta\delta_e$$
$$\Delta T = \frac{\bar{q}_1 S}{u_1}(C_{T_u} + 2C_{T_1})\Delta u + \bar{q}_1 S C_{T_\alpha}\Delta\alpha + \frac{\bar{q}_1 S\bar{c}}{2u_1}C_{T_q}\Delta q + \frac{\bar{q}_1 S\bar{c}}{2u_1}C_{T_{\dot{\alpha}}}\Delta\dot{\alpha} + \bar{q}_1 S C_{T_{\delta_e}}\Delta\delta_e$$
$$\Delta L = \frac{\bar{q}_1 S}{u_1}(C_{L_u} + 2C_{L_1})\Delta u + \bar{q}_1 S C_{L_\alpha}\Delta\alpha + \frac{\bar{q}_1 S\bar{c}}{2u_1}C_{L_q}\Delta q + \frac{\bar{q}_1 S\bar{c}}{2u_1}C_{L_{\dot{\alpha}}}\Delta\dot{\alpha} + \bar{q}_1 S C_{L_{\delta_e}}\Delta\delta_e$$

Variations in the pitch rate and rate of change of the angle of attack mainly contribute to effects at the tail through the lift. This means that

$$C_{D_q} = 0\,, \quad C_{D_{\dot{\alpha}}} = 0\,, \quad C_{T_q} = 0\,, \quad \text{and} \quad C_{T_{\dot{\alpha}}} = 0$$

Furthermore, variations in the angle of attack and elevator deflection do not significantly contribute to variations in the thrust, giving

$$C_{T_\alpha} = 0 \qquad \text{and} \qquad C_{T_{\delta_e}} = 0$$

Making the above substitutions, ΔD, ΔT, and ΔL become

$$\begin{aligned}
\Delta D &= \frac{\bar{q}_1 S}{u_1}(C_{D_u} + 2C_{D_1})\Delta u + \bar{q}_1 S C_{D_\alpha}\Delta\alpha + \bar{q}_1 S C_{D_{\delta_e}}\Delta\delta_e \\
\Delta T &= \frac{\bar{q}_1 S}{u_1}(C_{T_u} + 2C_{T_1})\Delta u \\
\Delta L &= \frac{\bar{q}_1 S}{u_1}(C_{L_u} + 2C_{L_1})\Delta u + \bar{q}_1 S C_{L_\alpha}\Delta\alpha + \frac{\bar{q}_1 S\bar{c}}{2u_1}C_{L_q}\Delta q + \frac{\bar{q}_1 S\bar{c}}{2u_1}C_{L_{\dot{\alpha}}}\Delta\dot{\alpha} + \bar{q}_1 S C_{L_{\delta_e}}\Delta\delta_e
\end{aligned}$$

Let's look at the moment...

$$M = \bar{q}S\bar{c}C_M$$

Following the same approach as taken for the force terms, it can be shown that

$$\left(\frac{\partial M}{\partial u}\right)_1 = \frac{\bar{q}_1 S\bar{c}}{u_1}(C_{M_u} + 2C_{M_1}) \qquad C_{M_u} = \left(\frac{\partial C_M}{\partial(u/u_1)}\right)_1$$

$$\left(\frac{\partial M}{\partial \alpha}\right)_1 = \bar{q}_1 S\bar{c}C_{M_\alpha} \qquad C_{M_\alpha} = \left(\frac{\partial C_M}{\partial \alpha}\right)_1$$

$$\left(\frac{\partial M}{\partial q}\right)_1 = \frac{\bar{q}_1 S\bar{c}^2}{2u_1}C_{M_q} \qquad C_{M_q} = \left(\frac{\partial C_M}{\partial(q\bar{c}/2u_1)}\right)_1$$

$$\left(\frac{\partial M}{\partial \dot{\alpha}}\right)_1 = \frac{\bar{q}_1 S\bar{c}^2}{2u_1}C_{M_{\dot{\alpha}}} \qquad C_{M_{\dot{\alpha}}} = \left(\frac{\partial C_M}{\partial(\dot{\alpha}\bar{c}/2u_1)}\right)_1$$

$$\left(\frac{\partial M}{\partial \delta_e}\right)_1 = \bar{q}_1 S\bar{c}C_{M_{\delta_e}} \qquad C_{M_{\delta_e}} = \left(\frac{\partial C_M}{\partial \delta_e}\right)_1$$

Let's look at the moment...

The aerodynamic and thrust pitch moment perturbations are then given by substituting the above relationships into Eq. (8.19), yielding

$$\Delta M_A = \frac{\bar{q}_1 S\bar{c}}{u_1}(C_{M_u} + 2C_{M_1})\Delta u + \bar{q}_1 S\bar{c}C_{M_\alpha}\Delta\alpha + \frac{\bar{q}_1 S\bar{c}^2}{2u_1}C_{M_q}\Delta q + \frac{\bar{q}_1 S\bar{c}^2}{2u_1}C_{M_{\dot\alpha}}\Delta\dot\alpha + \bar{q}_1 S\bar{c}C_{M_{\delta_e}}\Delta\delta_e$$

$$\Delta M_T = \frac{\bar{q}_1 S\bar{c}}{u_1}(C_{M_{T_u}} + 2C_{M_{T_1}})\Delta u + \bar{q}_1 S\bar{c}C_{M_{T_\alpha}}\Delta\alpha + \frac{\bar{q}_1 S\bar{c}^2}{2u_1}C_{M_{T_q}}\Delta q + \frac{\bar{q}_1 S\bar{c}^2}{2u_1}C_{M_{T_{\dot\alpha}}}\Delta\dot\alpha + \bar{q}_1 S\bar{c}C_{M_{T_{\delta_e}}}\Delta\delta_e$$

Let's look at the moment...

Variations in the pitch rate, rate of change of the angle of attack, and elevator deflection do not significantly contribute to the thrust pitch moment, i.e.

$$C_{M_{T_q}} = 0\,, \quad C_{M_{T_{\dot\alpha}}} = 0\,, \quad \text{and} \quad C_{M_{T_{\delta_e}}} = 0$$

which gives

$$\begin{aligned}
\Delta M_A &= \frac{\bar q_1 S\bar c}{u_1}(C_{M_u} + 2C_{M_1})\Delta u + \bar q_1 S\bar c C_{M_\alpha}\Delta\alpha + \frac{\bar q_1 S\bar c^2}{2u_1}C_{M_q}\Delta q \\
&\quad + \frac{\bar q_1 S\bar c^2}{2u_1}C_{M_{\dot\alpha}}\Delta\dot\alpha + \bar q_1 S\bar c C_{M_{\delta_e}}\Delta\delta_e \\
\Delta M_T &= \frac{\bar q_1 S\bar c}{u_1}(C_{M_{T_u}} + 2C_{M_{T_1}})\Delta u + \bar q_1 S\bar c C_{M_{T_\alpha}}\Delta\alpha
\end{aligned}$$

Putting it all together...

x-axis

$$\Delta D/m = \frac{\bar{q}_1 S}{m u_1}(C_{D_u} + 2C_{D_1})\Delta u + \frac{\bar{q}_1 S}{m} C_{D_\alpha}\Delta\alpha + \frac{\bar{q}_1 S}{m} C_{D_{\delta_e}}\Delta\delta_e$$

$$\Delta T/m = \frac{\bar{q}_1 S}{m u_1}(C_{T_u} + 2C_{T_1})\Delta u$$

Change in x-axis

Δ in z-axis

$$\Delta L/m = \frac{\bar{q}_1 S}{m u_1}(C_{L_u} + 2C_{L_1})\Delta u + \frac{\bar{q}_1 S}{m} C_{L_\alpha}\Delta\alpha + \frac{\bar{q}_1 S\bar{c}}{2 m u_1} C_{L_q}\Delta q + \frac{\bar{q}_1 S\bar{c}}{2 m u_1} C_{L_{\dot\alpha}}\Delta\dot\alpha + \frac{\bar{q}_1 S}{m} C_{L_{\delta_e}}\Delta\delta_e$$

$$\Delta M/I_{yy} = \frac{\bar{q}_1 S\bar{c}}{I_{yy} u_1}(C_{M_u} + 2C_{M_1})\Delta u + \frac{\bar{q}_1 S\bar{c}}{I_{yy} u_1}(C_{M_{T_u}} + 2C_{M_{T_1}})\Delta u + \frac{\bar{q}_1 S\bar{c}}{I_{yy}} C_{M_\alpha}\Delta\alpha + \frac{\bar{q}_1 S\bar{c}}{I_{yy}} C_{M_{T_\alpha}}\Delta\alpha + \frac{\bar{q}_1 S\bar{c}^2}{2 I_{yy} u_1} C_{M_q}\Delta q + \frac{\bar{q}_1 S\bar{c}^2}{2 I_{yy} u_1} C_{M_{\dot\alpha}}\Delta\dot\alpha + \frac{\bar{q}_1 S\bar{c}}{I_{yy}} C_{M_{\delta_e}}\Delta\delta_e$$

Putting it all together...

$$
\begin{aligned}
\Delta\dot{u} = &-\frac{\bar{q}_1 S}{m u_1}(C_{D_u} + 2C_{D_1})\Delta u + \frac{\bar{q}_1 S}{m u_1}(C_{T_u} + 2C_{T_1})\Delta u - \frac{\bar{q}_1 S}{m}(C_{D_\alpha} - C_{L_1})\Delta\alpha \\
&- g\cos\theta_1 \Delta\theta - \frac{\bar{q}_1 S}{m} C_{D_{\delta_e}} \Delta\delta_e \\
\left(u_1 + \frac{\bar{q}_1 S \bar{c}}{2 m u_1} C_{L_{\dot{\alpha}}}\right)\Delta\dot{\alpha} = &-\frac{\bar{q}_1 S}{m u_1}(C_{L_u} + 2C_{L_1})\Delta u - \frac{\bar{q}_1 S}{m}(C_{L_\alpha} + C_{D_1})\Delta\alpha + \left(u_1 - \frac{\bar{q}_1 S \bar{c}}{2 m u_1} C_{L_q}\right)\Delta q \\
&- g\sin\theta_1 \Delta\theta - \frac{\bar{q}_1 S}{m} C_{L_{\delta_e}} \Delta\delta_e \\
\Delta\dot{q} - \frac{\bar{q}_1 S \bar{c}^2}{2 I_{yy} u_1} C_{M_{\dot{\alpha}}} \Delta\dot{\alpha} = &\frac{\bar{q}_1 S \bar{c}}{I_{yy} u_1}(C_{M_u} + 2C_{M_1})\Delta u + \frac{\bar{q}_1 S \bar{c}}{I_{yy} u_1}(C_{M_{T_u}} + 2C_{M_{T_1}})\Delta u + \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_\alpha}\Delta\alpha \\
&+ \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_{T\alpha}}\Delta\alpha + \frac{\bar{q}_1 S \bar{c}^2}{2 I_{yy} u_1} C_{M_q}\Delta q + \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_{\delta_e}}\Delta\delta_e \\
\Delta\dot{\theta} = &\Delta q
\end{aligned}
$$

Stability derivatives

Putting it all together...

To make the equations more manageable, let the dimensional stability derivatives be defined as

$$X_u = -\frac{\bar{q}_1 S}{m u_1}\left(C_{D_u} + 2C_{D_1}\right) \qquad Z_q = -\frac{\bar{q}_1 S \bar{c}}{2 m u_1} C_{L_q} \qquad M_\alpha = \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_\alpha}$$

$$X_{T_u} = \frac{\bar{q}_1 S}{m u_1}\left(C_{T_u} + 2C_{T_1}\right) \qquad Z_{\dot{\alpha}} = -\frac{\bar{q}_1 S \bar{c}}{2 m u_1} C_{L_{\dot{\alpha}}} \qquad M_{T_\alpha} = \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_{T_\alpha}}$$

$$X_\alpha = -\frac{\bar{q}_1 S}{m}\left(C_{D_\alpha} - C_{L_1}\right) \qquad Z_{\delta_e} = -\frac{\bar{q}_1 S}{m} C_{L_{\delta_e}} \qquad M_q = \frac{\bar{q}_1 S \bar{c}^2}{2 I_{yy} u_1} C_{M_q}$$

$$X_{\delta_e} = -\frac{\bar{q}_1 S}{m} C_{D_{\delta_e}} \qquad M_u = \frac{\bar{q}_1 S \bar{c}}{I_{yy} u_1}\left(C_{M_u} + 2C_{M_1}\right) \qquad M_{\dot{\alpha}} = \frac{\bar{q}_1 S \bar{c}^2}{2 I_{yy} u_1} C_{M_{\dot{\alpha}}}$$

$$Z_u = -\frac{\bar{q}_1 S}{m u_1}\left(C_{L_u} + 2C_{L_1}\right) \qquad M_{T_u} = \frac{\bar{q}_1 S \bar{c}}{I_{yy} u_1}\left(C_{M_{T_u}} + 2C_{M_{T_1}}\right) \qquad M_{\delta_e} = \frac{\bar{q}_1 S \bar{c}}{I_{yy}} C_{M_{\delta_e}}$$

$$Z_\alpha = -\frac{\bar{q}_1 S}{m}\left(C_{L_\alpha} + C_{D_1}\right)$$

Putting it all together...

Then, the linear longitudinal dynamics can be found in their final form by substituting the dimensional stability derivatives into the dynamics equations to get

$$\Delta\dot{u} = (X_u + X_{T_u})\Delta u + X_\alpha \Delta\alpha - g\cos\theta_1 \Delta\theta + X_{\delta_e}\Delta\delta_e \tag{8.22a}$$

$$(u_1 - Z_{\dot{\alpha}})\Delta\dot{\alpha} = Z_u\Delta u + Z_\alpha\Delta\alpha + (u_1 + Z_q)\Delta q - g\sin\theta_1\Delta\theta + Z_{\delta_e}\Delta\delta_e \tag{8.22b}$$

$$\Delta\dot{q} - M_{\dot{\alpha}}\Delta\dot{\alpha} = (M_u + M_{T_u})\Delta u + (M_\alpha + M_{T_\alpha})\Delta\alpha + M_q\Delta q + M_{\delta_e}\Delta\delta_e \tag{8.22c}$$

$$\Delta\dot{\theta} = \Delta q \tag{8.22d}$$

The Linear Longitudinal Dynamics are...

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u_1 - Z_{\dot{\alpha}} & 0 & 0 \\ 0 & -M_{\dot{\alpha}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u + X_{T_u} & X_\alpha & 0 & -g\cos\theta_1 \\ Z_u & Z_\alpha & u_1 + Z_q & -g\sin\theta_1 \\ M_u + M_{T_u} & M_\alpha + M_{T_\alpha} & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} X_{\delta_e} \\ Z_{\delta_e} \\ M_{\delta_e} \\ 0 \end{bmatrix} \Delta\delta_e \tag{8.23}$$

Therefore, the longitudinal dynamics are given by the linear matrix equation

$$\boldsymbol{M}\dot{\boldsymbol{x}} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{F}\boldsymbol{\delta}$$

where

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u_1 - Z_{\dot{\alpha}} & 0 & 0 \\ 0 & -M_{\dot{\alpha}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} X_u + X_{T_u} & X_\alpha & 0 & -g\cos\theta_1 \\ Z_u & Z_\alpha & u_1 + Z_q & -g\sin\theta_1 \\ M_u + M_{T_u} & M_\alpha + M_{T_\alpha} & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\boldsymbol{F} = \begin{bmatrix} X_{\delta_e} \\ Z_{\delta_e} \\ M_{\delta_e} \\ 0 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \\ \Delta\theta \end{bmatrix}, \quad \dot{\boldsymbol{x}} = \begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{\delta} = \Delta\delta_e$$

In standard linear systems notation, this is

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$\text{where} \qquad \boldsymbol{A} = \boldsymbol{M}^{-1}\boldsymbol{R} \qquad \text{and} \qquad \boldsymbol{B} = \boldsymbol{M}^{-1}\boldsymbol{F}$$

Example: Boeing 747 in High Cruise

A Boeing 747 airplane has the following physical characteristics:

$$W = 636,636 \text{ [lb]}, \quad I_{yy} = 3.31 \times 10^7 \text{ [slugs}\cdot\text{ft}^2\text{]}, \quad S = 5,500 \text{ [ft}^2\text{]}, \quad \text{and} \quad \bar{c} = 27.3 \text{ [ft]}$$

The aircraft is in trim at $u_1 = 516$ [knots] and $\theta_1 = 0$ [deg] when flying at an atmospheric density of $\rho = 5.8727 \times 10^{-4}$ [slugs/ft^3], which corresponds to an altitude of $h = 40,000$ [ft]. The aerodynamic coefficients relevant to longitudinal dynamics of the Boeing 747 are given by

$$\begin{array}{llll}
C_{L_1} = 0.52 & C_{L_{\delta_e}} = 0.3 & C_{T_1} = 0.045 & C_{M_{\dot\alpha}} = -9.0 \\
C_{L_u} = -0.23 & C_{D_1} = 0.045 & C_{T_u} = -0.95 & C_{M_q} = -25.0 \\
C_{L_\alpha} = 5.5 & C_{D_u} = 0.22 & C_{M_1} = 0.0 & C_{M_{\delta_e}} = -1.2 \\
C_{L_{\dot\alpha}} = 8.0 & C_{D_\alpha} = 0.50 & C_{M_u} = -0.09 & C_{M_{T_1}} = 0.0 \\
C_{L_q} = 7.8 & C_{D_{\delta_e}} = 0.0 & C_{M_\alpha} = -1.6 & C_{M_{T_u}} = 0.0
\end{array}$$

Using $g = 32.2$ [ft/s^2], we wish to determine the matrices governing the linear longitudinal dynamics for the Boeing 747. First of all, we compute the dimensional stability derivatives:

$$\begin{array}{lll}
X_u = -0.0221 & Z_q = -7.5742 & M_\alpha = -1.6165 \\
X_{T_u} = -0.0612 & Z_{\dot\alpha} = -7.7684 & M_{T_\alpha} = 0.0000 \\
X_\alpha = 1.2391 & Z_{\delta_e} = -18.5867 & M_q = -0.3959 \\
X_{\delta_e} = 0.0000 & M_u = -0.0001 & M_{\dot\alpha} = -0.1425 \\
Z_u = -0.0576 & M_{T_u} = 0.0000 & M_{\delta_e} = -1.2124 \\
Z_\alpha = -343.5450 & &
\end{array}$$

These values can then be used to assemble the $\boldsymbol{M}$ and $\boldsymbol{R}$ matrices, yielding

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 878.6787 & 0 & 0 \\ 0 & 0.1425 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{R} = \begin{bmatrix} -0.0832 & 1.2391 & 0 & -32.2000 \\ -0.0576 & -343.5450 & 863.3361 & 0 \\ -0.0001 & -1.6165 & -0.3959 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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