The University of Texas at Austin
Aerospace Engineering
and Engineering Mechanics
Cockrell School of Engineering

22 OCTOBER 2024

ASE 367K: FLIGHT DYNAMICS

TTH 09:30-11:00
CMA 2.306

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Topics for Today

< 250 knots

- Topic(s):
  - Videos of Longitudinal Dynamics
  - Thoughts on HW6
  - Linearized Lateral Equations of Motion

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USEFUL LONGITUDINAL DYNAMICS VIDEOS

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Useful Videos

- Phugoid
  - https://www.youtube.com/watch?v=9DnSG4H2UjE
  - https://www.youtube.com/watch?v=vbKaCqF_E1M

- Short Period
  - https://www.youtube.com/watch?v=1O7ZqBS0_B8

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THOUGHTS ON HW6

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u_1 = equilibrium speed

θ_1 =

Problem 1

Given the linear system model provided on Slide 8 of Lecture 15 and the dimensional stability derivatives for the B747 at “low cruise” provided on Slide 13 of Lecture 15:

a. Determine the damped frequency, natural frequency, damping ratio, the time to damp to half the initial amplitude, and the number of cycles to damp to half the initial amplitude for BOTH the short period and the Phugoid (long period) modes.

b. Develop a simulation (using the either MATLAB or Python) for this model and use it to determine and plot the values of the state variables Δu, $\Delta\alpha$, Δq, and $\Delta\theta$ (as functions of time) in response to the step and impulse elevator deflections of maximum amplitude.

Clean Config.
$\Rightarrow$ No flaps

747-737

200 ~ 210 knots
minimum speed

Problem 1b

$u_1 \rightarrow X_\delta$ ~ free stream velocity

- How do you determine u_1 and θ_1 for your simulation?
  - Assume level flight
  - Select altitude and weight ~ pick 747 weight ~ 10,000 ft
  - Set u_1 equal to the velocity of your choice
  - Compute C_L then α
  - Set θ_1 equal to α

Speed limit below 10,000 ft

$\theta_1 =$

Set init = 0

$\delta u, \Delta\alpha, \Delta q, \Delta\theta$

$L = W = \frac{1}{2}\rho V^2 S C_L$

C_L

$TF = C(sI - A)^{-1}B$ ✓

A ⇒ TF

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LINEARIZED LATERAL EOM

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Starting with the nonlinear lateral equations of motion…

- Because the lateral position equation is uncoupled and can be integrated independently… We get the following 5 nonlinear lateral equations of motion

$$Y = m(\dot{v} + ru - pw - g\cos\theta\sin\phi)$$

$$L = I_{xx}\dot{p} - I_{xz}(\dot{r} + pq) - (I_{yy} - I_{zz})qr$$

$$N = I_{zz}\dot{r} - I_{xz}(\dot{p} - qr) - (I_{xx} - I_{yy})pq$$

$$p = \dot{\phi} - \dot{\psi}\sin\theta$$

$$r = -\dot{\theta}\sin\phi + \dot{\psi}\cos\theta\cos\phi$$

Substituting speed and angular rate perturbations…

- We can represent the quantities of interest as the sum of the nominal operating condition and a perturbation from nominal

$$
\begin{aligned}
u &= u_1 + \Delta u \qquad & p &= p_1 + \Delta p \qquad & \phi &= \phi_1 + \Delta \phi \\
v &= v_1 + \Delta v \qquad & q &= q_1 + \Delta q \qquad & \theta &= \theta_1 + \Delta \theta \\
w &= w_1 + \Delta w \qquad & r &= r_1 + \Delta r \qquad & \psi &= \psi_1 + \Delta \psi
\end{aligned}
$$

Focusing in on the effects of these perturbations...

$$\Delta Y = m(\Delta\dot{v} + r_1\Delta u - p_1\Delta w - w_1\Delta p + u_1\Delta r - g\cos\theta_1\cos\phi_1\Delta\phi + g\sin\theta_1\sin\phi_1\Delta\theta)$$

$$\Delta L = I_{xx}\Delta\dot{p} - I_{xz}\Delta\dot{r} - I_{xz}p_1\Delta q - I_{xz}q_1\Delta p - (I_{yy} - I_{zz})q_1\Delta r - (I_{yy} - I_{zz})r_1\Delta q$$

$$\Delta N = I_{zz}\Delta\dot{r} - I_{xz}\Delta\dot{p} + I_{xz}r_1\Delta q + I_{xz}q_1\Delta r - (I_{xx} - I_{yy})q_1\Delta p - (I_{xx} - I_{yy})p_1\Delta q$$

$$\Delta p = \Delta\dot{\phi} - \dot{\psi}_1\cos\theta_1\Delta\theta - \sin\theta_1\Delta\dot{\psi}$$

$$\Delta r = -\sin\phi_1\Delta\dot{\theta} + \cos\theta_1\cos\phi_1\Delta\dot{\psi} - \dot{\theta}_1\cos\phi_1\Delta\phi - \dot{\psi}_1\cos\theta_1\sin\phi_1\Delta\phi - \dot{\psi}_1\sin\theta_1\cos\phi_1\Delta\theta$$

Eliminating steady-state values that are zero…

- If we linearize w.r.t. a constant velocity flight with <u>zero beta</u> and we choose a body-fixed frame that is aligned with the nominal velocity vector, then …

$$w_1 = 0 \qquad v_1 = 0\,, \qquad p_1 = 0\,, \qquad q_1 = 0\,, \qquad r_1 = 0\,, \qquad \text{and} \qquad \phi_1 = 0$$

$$\dot{\phi}_1 = 0\,, \qquad \dot{\theta}_1 = 0\,, \qquad \text{and} \qquad \dot{\psi}_1 = 0$$

- and…

$$\Delta Y = m(\Delta\dot{v} + \cancel{r_1}\Delta u - \cancel{p_1}\Delta w - \cancel{w_1}\Delta p + u_1\Delta r - g\cos\theta_1 \overbrace{\cos\phi_1}^{=1}\Delta\phi + g\sin\theta_1 \cancel{\sin\phi_1}\Delta\theta)$$

$$\Delta L = I_{xx}\Delta\dot{p} - I_{xz}\Delta\dot{r} - I_{xz}\cancel{p_1}\Delta q - I_{xz}\cancel{q_1}\Delta p - (I_{yy} - I_{zz})\cancel{q_1}\Delta r - (I_{yy} - I_{zz})\cancel{r_1}\Delta q$$

$$\Delta N = I_{zz}\Delta\dot{r} - I_{xz}\Delta\dot{p} + I_{xz}\cancel{r_1}\Delta q + I_{xz}\cancel{q_1}\Delta r \cancel{-} (I_{xx} - I_{yy})\cancel{q_1}\Delta p - (I_{xx} - I_{yy})\cancel{p_1}\Delta q$$

$$\Delta p = \Delta\dot{\phi} - \cancel{\dot{\psi}_1}\cos\theta_1\Delta\theta - \sin\theta_1\Delta\dot{\psi}$$

$$\Delta r = -\cancel{\sin\phi_1}\Delta\dot{\theta} + \cos\theta_1 \underbrace{\cos\phi_1}_{=1}\Delta\dot{\psi} - \cancel{\dot{\theta}_1}\cos\phi_1\Delta\phi - \cancel{\dot{\psi}_1}\cos\theta_1\sin\phi_1\Delta\phi - \cancel{\dot{\psi}_1}\sin\theta_1\cos\phi_1\Delta\theta$$

Gives us the linearized lateral equations of motion

$$\Delta Y = m(\Delta\dot{v} + u_1\Delta r - g\cos\theta_1\Delta\phi)$$

$$\Delta L = I_{xx}\Delta\dot{p} - I_{xz}\Delta\dot{r}$$

$$\Delta N = I_{zz}\Delta\dot{r} - I_{xz}\Delta\dot{p}$$

$$\Delta p = \Delta\dot{\phi} - \sin\theta_1\Delta\dot{\psi}$$

$$\Delta r = \cos\theta_1\Delta\dot{\psi}$$

Then replacing Δv with $\Delta \beta$ gives us...

$v_1 = 0$, therefore $\beta_1 = 0$, hence $\quad \Delta \beta = \dfrac{\Delta v}{u_1} \quad$ or $\quad \Delta v = u_1 \Delta \beta$

Furthermore, since $\dot{u}_1 = 0$, it follows that $\Delta \dot{v} = u_1 \Delta \dot{\beta}$

$$\Delta Y = m(u_1 \Delta \dot{\beta} + u_1 \Delta r - g \cos \theta_1 \Delta \phi)$$
$$\Delta L = I_{xx} \Delta \dot{p} - I_{xz} \Delta \dot{r}$$
$$\Delta N = I_{zz} \Delta \dot{r} - I_{xz} \Delta \dot{p}$$
$$\Delta p = \Delta \dot{\phi} - \sin \theta_1 \Delta \dot{\psi}$$
$$\Delta r = \cos \theta_1 \Delta \dot{\psi}$$

Which can be reorganized as...

$$\begin{aligned}
u_1\Delta\dot{\beta} &= \Delta Y/m - u_1\Delta r + g\cos\theta_1\Delta\phi \\
\Delta\dot{p} - (I_{xz}/I_{xx})\Delta\dot{r} &= \Delta L/I_{xx} \\
\Delta\dot{r} - (I_{xz}/I_{zz})\Delta\dot{p} &= \Delta N/I_{zz} \\
\Delta\dot{\phi} &= \Delta p + \tan\theta_1\Delta r \\
\Delta\dot{\psi} &= \sec\theta_1\Delta r
\end{aligned}$$

Applying the chain rule to the force and moments...

$$\Delta Y = \left(\frac{\partial Y}{\partial \beta}\right)_1 \Delta\beta + \left(\frac{\partial Y}{\partial p}\right)_1 \Delta p + \left(\frac{\partial Y}{\partial r}\right)_1 \Delta r + \left(\frac{\partial Y}{\partial \delta_a}\right)_1 \Delta\delta_a + \left(\frac{\partial Y}{\partial \delta_r}\right)_1 \Delta\delta_r$$

$$\Delta L = \left(\frac{\partial L}{\partial \beta}\right)_1 \Delta\beta + \left(\frac{\partial L}{\partial p}\right)_1 \Delta p + \left(\frac{\partial L}{\partial r}\right)_1 \Delta r + \left(\frac{\partial L}{\partial \delta_a}\right)_1 \Delta\delta_a + \left(\frac{\partial L}{\partial \delta_r}\right)_1 \Delta\delta_r$$

$$\begin{aligned}\Delta N = & \left(\frac{\partial N_A}{\partial \beta} + \frac{\partial N_T}{\partial \beta}\right)_1 \Delta\beta + \left(\frac{\partial N_A}{\partial p} + \frac{\partial N_T}{\partial p}\right)_1 \Delta p + \left(\frac{\partial N_A}{\partial r} + \frac{\partial N_T}{\partial r}\right)_1 \Delta r \\ & + \left(\frac{\partial N_A}{\partial \delta_a} + \frac{\partial N_T}{\partial \delta_a}\right)_1 \Delta\delta_a + \left(\frac{\partial N_A}{\partial \delta_r} + \frac{\partial N_T}{\partial \delta_r}\right)_1 \Delta\delta_r\end{aligned}$$

Neglecting effects that are insignificant...

Typically, the contribution to the thrust yaw moment due to rolling angular velocity and the lateral controls is zero, such that

$$\Delta Y = \left(\frac{\partial Y}{\partial \beta}\right)_1 \Delta\beta + \left(\frac{\partial Y}{\partial p}\right)_1 \Delta p + \left(\frac{\partial Y}{\partial r}\right)_1 \Delta r + \left(\frac{\partial Y}{\partial \delta_a}\right)_1 \Delta\delta_a + \left(\frac{\partial Y}{\partial \delta_r}\right)_1 \Delta\delta_r$$

$$\Delta L = \left(\frac{\partial L}{\partial \beta}\right)_1 \Delta\beta + \left(\frac{\partial L}{\partial p}\right)_1 \Delta p + \left(\frac{\partial L}{\partial r}\right)_1 \Delta r + \left(\frac{\partial L}{\partial \delta_a}\right)_1 \Delta\delta_a + \left(\frac{\partial L}{\partial \delta_r}\right)_1 \Delta\delta_r$$

$$\Delta N = \left(\frac{\partial N_A}{\partial \beta} + \frac{\partial N_T}{\partial \beta}\right)_1 \Delta\beta + \left(\frac{\partial N_A}{\partial p}\right)_1 \Delta p + \left(\frac{\partial N_A}{\partial r} + \frac{\partial N_T}{\partial r}\right)_1 \Delta r + \left(\frac{\partial N_A}{\partial \delta_a}\right)_1 \Delta\delta_a + \left(\frac{\partial N_A}{\partial \delta_r}\right)_1 \Delta\delta_r$$

Substituting the stability derivatives...

Let the dimensional stability derivatives be defined as

$$
\begin{aligned}
Y_\beta &= \frac{\bar{q}_1 S}{m} C_{y_\beta} & L_p &= \frac{\bar{q}_1 S b^2}{2 I_{xx} u_1} C_{\ell_p} & N_p &= \frac{\bar{q}_1 S b^2}{2 I_{zz} u_1} C_{n_p} \\
Y_p &= \frac{\bar{q}_1 S b}{2 m u_1} C_{y_p} & L_r &= \frac{\bar{q}_1 S b^2}{2 I_{xx} u_1} C_{\ell_r} & N_r &= \frac{\bar{q}_1 S b^2}{2 I_{zz} u_1} C_{n_r} \\
Y_r &= \frac{\bar{q}_1 S b}{2 m u_1} C_{y_r} & L_{\delta_a} &= \frac{\bar{q}_1 S b}{I_{xx}} C_{\ell_{\delta_a}} & N_{T_r} &= \frac{\bar{q}_1 S b^2}{2 I_{zz} u_1} C_{n_{T_r}} \\
Y_{\delta_a} &= \frac{\bar{q}_1 S}{m} C_{y_{\delta_a}} & L_{\delta_r} &= \frac{\bar{q}_1 S b}{I_{xx}} C_{\ell_{\delta_r}} & N_{\delta_a} &= \frac{\bar{q}_1 S b}{I_{zz}} C_{n_{\delta_a}} \\
Y_{\delta_r} &= \frac{\bar{q}_1 S}{m} C_{y_{\delta_r}} & N_\beta &= \frac{\bar{q}_1 S b}{I_{zz}} C_{n_\beta} & N_{\delta_r} &= \frac{\bar{q}_1 S b}{I_{zz}} C_{n_{\delta_r}} \\
L_\beta &= \frac{\bar{q}_1 S b}{I_{xx}} C_{\ell_\beta} & N_{T_\beta} &= \frac{\bar{q}_1 S b}{I_{zz}} C_{n_{T_\beta}} & &
\end{aligned}
$$

Gives the following accelerations...

$$\Delta Y/m = Y_\beta \Delta\beta + Y_p \Delta p + Y_r \Delta r + Y_{\delta_a} \Delta\delta_a + Y_{\delta_r} \Delta\delta_r$$
$$\Delta L/I_{xx} = L_\beta \Delta\beta + L_p \Delta p + L_r \Delta r + L_{\delta_a} \Delta\delta_a + L_{\delta_r} \Delta\delta_r$$
$$\Delta N/I_{zz} = \left(N_\beta + N_{T_\beta}\right) \Delta\beta + N_p \Delta p + \left(N_r + N_{T_r}\right) \Delta r + N_{\delta_a} \Delta\delta_a + N_{\delta_r} \Delta\delta_r$$

Which after putting it all together becomes...

$$
\begin{aligned}
u_1 \Delta \dot{\beta} &= Y_\beta \Delta\beta + Y_p \Delta p + Y_r \Delta r + Y_{\delta_a} \Delta\delta_a + Y_{\delta_r} \Delta\delta_r - u_1 \Delta r + g \cos\theta_1 \Delta\phi \\
\Delta\dot{p} - (I_{xz}/I_{xx}) \Delta\dot{r} &= L_\beta \Delta\beta + L_p \Delta p + L_r \Delta r + L_{\delta_a} \Delta\delta_a + L_{\delta_r} \Delta\delta_r \\
\Delta\dot{r} - (I_{xz}/I_{zz}) \Delta\dot{p} &= \left(N_\beta + N_{T_\beta}\right) \Delta\beta + N_p \Delta p + (N_r + N_{T_r}) \Delta r + N_{\delta_a} \Delta\delta_a + N_{\delta_r} \Delta\delta_r \\
\Delta\dot{\phi} &= \Delta p + \tan\theta_1 \Delta r \\
\Delta\dot{\psi} &= \sec\theta_1 \Delta r
\end{aligned}
$$

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LATERAL MODES

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The linearized lateral EOM in Matrix Form is…

$$\begin{bmatrix} u_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -(I_{xz}/I_{xx}) & 0 & 0 \\ 0 & -(I_{xz}/I_{zz}) & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\phi} \\ \Delta\dot{\psi} \end{bmatrix} = \begin{bmatrix} Y_\beta & Y_p & (Y_r - u_1) & g\cos\theta_1 & 0 \\ L_\beta & L_p & L_r & 0 & 0 \\ (N_\beta + N_{T_\beta}) & N_p & (N_r + N_{T_r}) & 0 & 0 \\ 0 & 1 & \tan\theta_1 & 0 & 0 \\ 0 & 0 & \sec\theta_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \end{bmatrix} + \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix}$$

Therefore, the lateral dynamics are given by the linear matrix equation

$$\boldsymbol{M}\dot{\boldsymbol{x}} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{F}\boldsymbol{\delta}$$

where

$$\boldsymbol{M} = \begin{bmatrix} u_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -(I_{xz}/I_{xx}) & 0 & 0 \\ 0 & -(I_{xz}/I_{zz}) & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} Y_\beta & Y_p & (Y_r - u_1) & g\cos\theta_1 & 0 \\ L_\beta & L_p & L_r & 0 & 0 \\ (N_\beta + N_{T_\beta}) & N_p & (N_r + N_{T_r}) & 0 & 0 \\ 0 & 1 & \tan\theta_1 & 0 & 0 \\ 0 & 0 & \sec\theta_1 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{F} = \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \end{bmatrix}, \quad \dot{\boldsymbol{x}} = \begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\phi} \\ \Delta\dot{\psi} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{\delta} = \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix}$$

In standard linear systems notation, this is $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$ where $\boldsymbol{A} = \boldsymbol{M}^{-1}\boldsymbol{R}$ and $\boldsymbol{B} = \boldsymbol{M}^{-1}\boldsymbol{F}$

A Boeing 747 airplane has the following characteristics

$$W = 636,636 \text{ [lb]}, \quad I^b_{xx} = 1.82 \times 10^7 \text{ [slugs}\cdot\text{ft}^2], \quad I^b_{zz} = 4.97 \times 10^7 \text{ [slugs}\cdot\text{ft}^2],$$
$$I^b_{xz} = 9.70 \times 10^5 \text{ [slugs}\cdot\text{ft}^2], \quad S = 5,500 \text{ [ft}^2], \quad \text{and} \quad b = 195.7 \text{ [ft]}$$

The aircraft is in trim at $u_1 = 399$ [knots], $\theta_1 = 2.4$[deg], and $\alpha_1 = 2.4$ [deg] when flying at an atmospheric density of $\rho = 1.2673 \times 10^{-3}$ [slugs/ft^3]. The aerodynamic coefficients relevant to lateral dynamic stability of the 747 are given by

$$\begin{aligned}
C_{y_\beta} &= -0.9000 & C_{\ell_p} &= -0.3400 & C_{n_p} &= -0.0260 \\
C_{y_p} &= 0.0000 & C_{\ell_r} &= 0.1300 & C_{n_r} &= -0.2800 \\
C_{y_r} &= 0.0000 & C_{\ell_{\delta_a}} &= 0.0130 & C_{n_{T_r}} &= 0.0000 \\
C_{y_{\delta_a}} &= 0.0000 & C_{\ell_{\delta_r}} &= 0.0080 & C_{n_{\delta_a}} &= 0.0018 \\
C_{y_{\delta_r}} &= 0.1200 & C_{n_\beta} &= 0.1600 & C_{n_{\delta_r}} &= -0.1000 \\
C_{\ell_\beta} &= -0.1600 & C_{n_{T_\beta}} &= 0.0000 & &
\end{aligned}$$

The A and B matrices are:

$$\mathbf{A} = \begin{bmatrix} -0.1067 & 0 & -1.0000 & 0.0477 & 0 \\ -2.7427 & -0.8404 & 0.3264 & 0 & 0 \\ 1.0146 & -0.0176 & -0.2554 & 0 & 0 \\ 0 & 1.0000 & 0.0419 & 0 & 0 \\ 0 & 0 & 1.0009 & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0.0142 \\ 0.2211 & 0.1482 \\ 0.0096 & -0.6231 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

With the matrix $\boldsymbol{A}$ built, we can compute the eigenvalues and eigenvectors of the matrix in order to determine stability and some modal characteristics of the system. For this case, we find that the eigenvalues are given by

Roll subsidence   Spiral   Dutch roll

$$\lambda_1 = 0\,, \quad \lambda_2 = -0.9388\,, \quad \lambda_3 = -0.0171\,, \quad \lambda_{4,5} = -0.1234 \pm 1.0416i$$

and that the normalized non-dimensional eigenvector magnitudes are

$$\|\tilde{\boldsymbol{v}}\|_1 = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 1.0000 \end{bmatrix}, \quad \|\tilde{\boldsymbol{v}}\|_2 = \begin{bmatrix} 0.0311 \\ 0.1364 \\ 0.0032 \\ 1.0000 \\ 0.0234 \end{bmatrix}, \quad \|\tilde{\boldsymbol{v}}\|_3 = \begin{bmatrix} 0.0039 \\ 0.0009 \\ 0.0025 \\ 0.3648 \\ 1.0000 \end{bmatrix}, \quad \|\tilde{\boldsymbol{v}}\|_{4,5} = \begin{bmatrix} 0.4859 \\ 0.1524 \\ 0.0699 \\ 1.0000 \\ 0.4589 \end{bmatrix}$$

Remember the 5 components of the state vector:

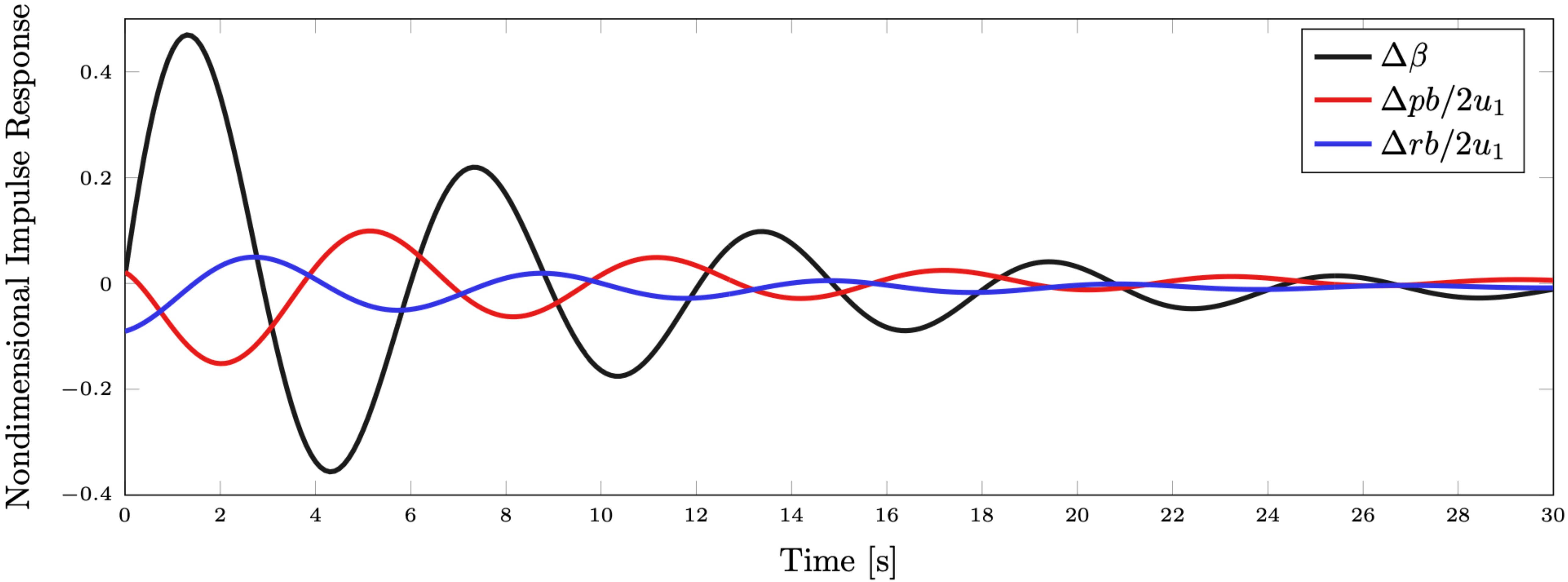
$$\boldsymbol{x} = \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \end{bmatrix}$$

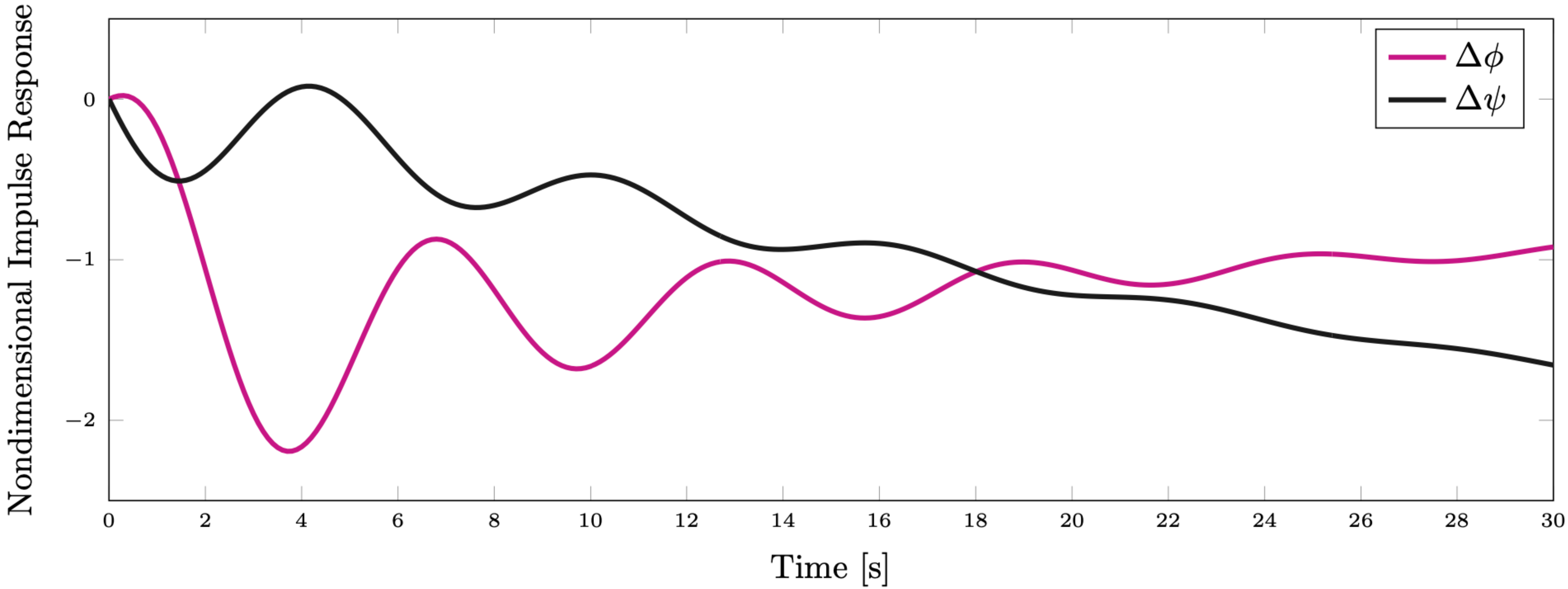
where normalization is done such that the largest element is one. Based on the above data, complete the following table:

Eigenvalue	Stable?	Oscillatory?	Mode?	Dominant Motion?
λ_1	Neutrally stable			
λ_2	✓			
λ_3	✓			
$\lambda_{4,5}$		✓		

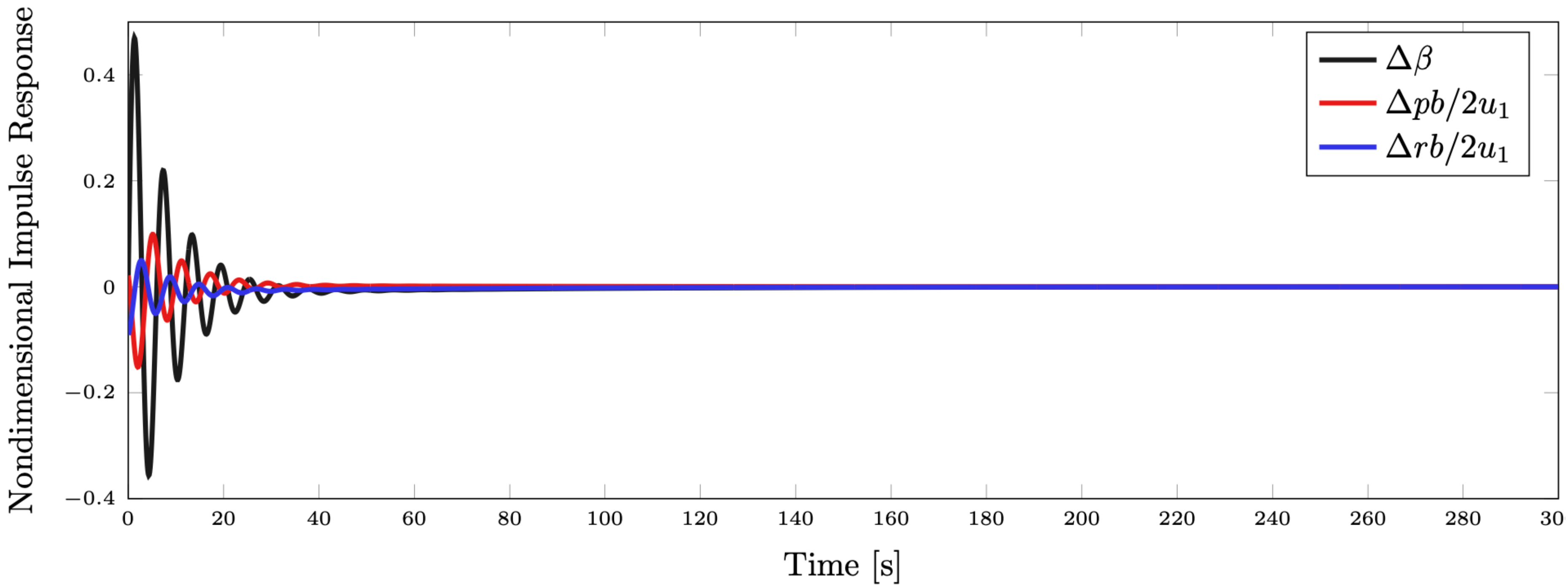
Rudder Impulse Response of the Linear System:

rolling ⟷ yawing
out of phase

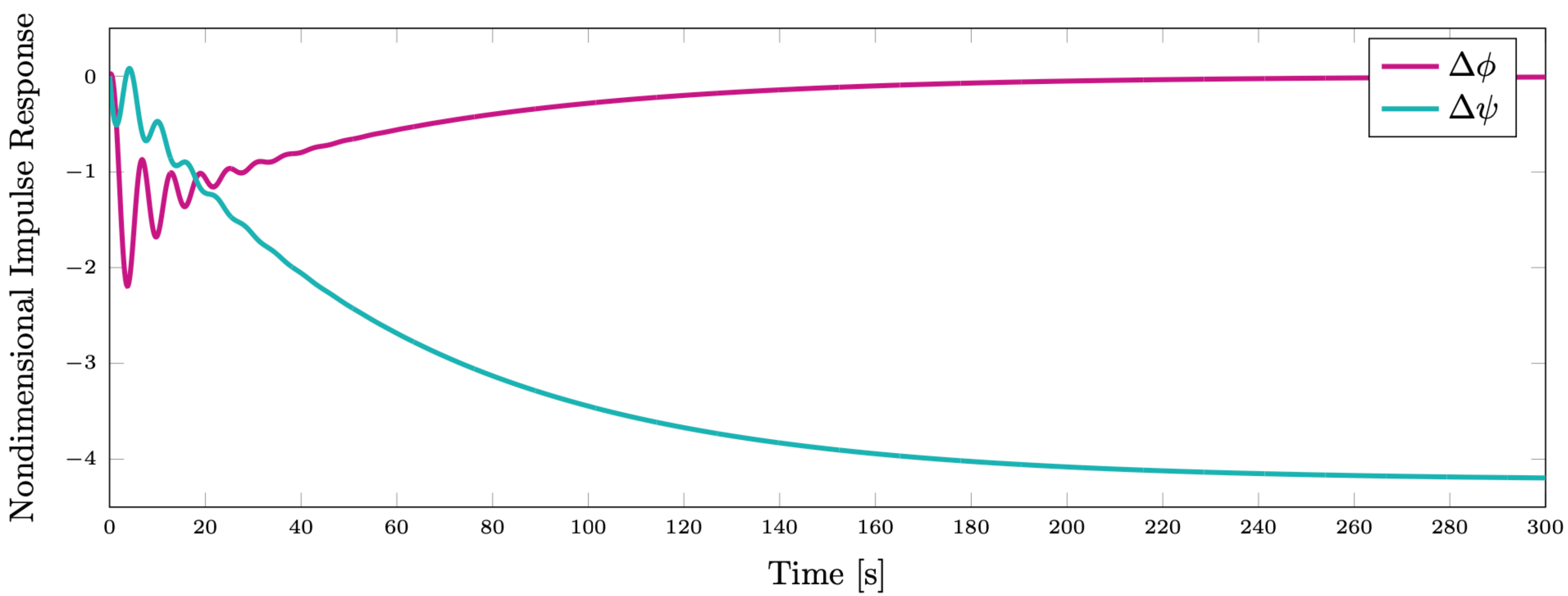




$\Delta\phi$
$\Delta\psi$
0
−1
−2
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
Nondimensional Impulse Response
Time [s]



Nondimensional Impulse Response
0.4
0.2
0
−0.2
−0.4
0 20 40 60 80 100 120 140 160 180 200 220 240 260 280 30
Time [s]
$\Delta\beta$
$\Delta pb/2u_1$
$\Delta rb/2u_1$



Nondimensional Impulse Response
0
−1
−2
−3
−4
0 20 40 60 80 100 120 140 160 180 200 220 240 260 280 300
Time [s]
$\Delta\phi$
$\Delta\psi$

Oscillating one is yaw and roll out of phase

Dutch-roll motion is primarily described by stability-axis yaw rate and sideslip angle

Yaw damper to kill this

Roll and spiral motions are primarily described by stability-axis roll rate and roll angle

Roll Approximation

We start from the equation that contains the Roll torque

$$\Delta\dot{p} - (I_{xz}/I_{xx})\Delta\dot{r} = L_\beta\Delta\beta + L_p\Delta p + L_r\Delta r + L_{\delta_a}\Delta\delta_a + L_{\delta_r}\Delta\delta_r$$

The roll mode occurs at approximately constant (and zero) sideslip and yaw, such that $\Delta r = \Delta\beta = 0$ and $\Delta\dot{r} = \Delta\dot{\beta} = 0$, which gives

$$\Delta\dot{p} = L_p\Delta p + L_{\delta_a}\Delta\delta_a + L_{\delta_r}\Delta\delta_r$$

The primary control action is through the aileron deflection,

$$\Delta\dot{p} = L_p\Delta p + L_{\delta_a}\Delta\delta_a$$

let the control term be set to zero, it can be shown that

$$\Delta p(t) = \Delta p_0 e^{L_p t}, \quad t \geq 0$$

Spiral Approximation

The spiral mode is characterized by changes in the bank angle, ϕ, and the heading angle, ψ. The sideslip angle is usually quite small, but cannot be fully neglected. Typically, the spiral mode is very slow to develop following a disturbance, so it is usually assumed that $\Delta\beta$, Δp, and Δr are quasi-steady relative to the time scale of the mode, hence $\Delta\dot{\beta} = \Delta\dot{p} = \Delta\dot{r} = 0$. Making these substitutions

$$0 = -u_1\Delta r + g\Delta\phi + Y_{\delta_a}\Delta\delta_a \tag{11.24a}$$

$$0 = L_\beta\Delta\beta + L_p\Delta p + L_r\Delta r + L_{\delta_a}\Delta\delta_a \tag{11.24b}$$

$$0 = N_\beta\Delta\beta + N_p\Delta p + N_r\Delta r + N_{\delta_a}\Delta\delta_a \tag{11.24c}$$

$$\Delta\dot{\phi} = \Delta p \tag{11.24d}$$

Note that the yaw angle does not influence the above equations, so the last equation of Eqs. (11.23) has been omitted in Eqs. (11.24). Multiplying Eq. (11.24b) by N_β and Eq. (11.24c) by L_β, subtracting the resulting equations, and solving for Δr gives

$$\Delta r = \frac{L_\beta N_p - N_\beta L_p}{N_\beta L_r - L_\beta N_r}\Delta p + \frac{L_\beta N_{\delta_a} - N_\beta L_{\delta_a}}{N_\beta L_r - L_\beta N_r}\Delta \delta_a \tag{11.25}$$

Substituting Eq. (11.25) into Eq. (11.24a) and clearing the fractions yields

$$0 = -u_1(L_\beta N_p - N_\beta L_p)\Delta p + g(N_\beta L_r - L_\beta N_r)\Delta\phi + (Y_{\delta_a} - u_1(L_\beta N_{\delta_a} - N_\beta L_{\delta_a}))\Delta\delta_a$$

Finally, from Eq. (11.24d), $\Delta\dot{\phi} = \Delta p$, such that

$$0 = -u_1(L_\beta N_p - N_\beta L_p)\Delta\dot{\phi} + g(N_\beta L_r - L_\beta N_r)\Delta\phi + (Y_{\delta_a} - u_1(L_\beta N_{\delta_a} - N_\beta L_{\delta_a}))\Delta\delta_a$$

or

$$u_1(L_\beta N_p - N_\beta L_p)\Delta\dot{\phi} + g(L_\beta N_r - N_\beta L_r)\Delta\phi = (Y_{\delta_a} - u_1(L_\beta N_{\delta_a} - N_\beta L_{\delta_a}))\Delta\delta_a$$

In the presence of no control action

$$\Delta\dot{\phi} + \frac{g(L_\beta N_r - N_\beta L_r)}{u_1(L_\beta N_p - N_\beta L_p)}\Delta\phi = 0$$

Applying this solution to the zero-input governing equation for the spiral approximation yields

$$\Delta\phi(t) = \Delta\phi_0 e^{-(g/u_1)[(L_\beta N_r - N_\beta L_r)/(L_\beta N_p - N_\beta L_p)]t}, \quad t \geq 0$$

It is therefore clear that $\Delta\phi(t) \to \infty$ as $t \to \infty$ if

$$\frac{g(L_\beta N_r - N_\beta L_r)}{u_1(L_\beta N_p - N_\beta L_p)} < 0$$

In the case that this term is identically zero, the roll angle remains fixed at its initial value, but the response does not decay to zero. Thus, to obtain a stable spiral model, we require

$$\frac{g(L_\beta N_r - N_\beta L_r)}{u_1(L_\beta N_p - N_\beta L_p)} > 0 \tag{11.27}$$

Typically, $(L_\beta N_p - N_\beta L_p) > 0$, which means that we get the classical requirement that

$$L_\beta N_r > N_\beta L_r$$

Want much more aggressive roll than yaw.

Dutch Roll Approximation

In the prior two modes we focused on the roll angle and roll rate. We study the Dutch Roll approximation by focusing the other two: sideslip angle and yaw rate. Neglecting roll in Dutch roll is clearly contradictory, but it is based on the fa that the mode is first a yawing oscillation and aerodynamic coupling causes rolling motion as a secondary effect.
It is generally true that, for most aircraft, the roll to yaw ratio in Dutch rolling motion is less that one; in some cases, it may be much less than one, lending the assumption some small credibility. Given that the Dutch roll mode consists o primarily sideslipping and yawing motions, the approximation is:

$$
\begin{aligned}
u_1 \Delta\dot{\beta} &= Y_\beta \Delta\beta + Y_p \Delta p + (Y_r - u_1)\Delta r + g\cos\theta_1 \Delta\phi + Y_{\delta_a}\Delta\delta_a + Y_{\delta_r}\Delta\delta_r \\
\Delta\dot{r} - (I_{xz}/I_{zz})\Delta\dot{p} &= \left(N_\beta + N_{T_\beta}\right)\Delta\beta + N_p \Delta p + \left(N_r + N_{T_r}\right)\Delta r + N_{\delta_a}\Delta\delta_a + N_{\delta_r}\Delta\delta_r
\end{aligned}
$$

From the assumption that no rolling motion is involved, setting Δp, $\Delta\dot{p}$, $\Delta\phi$, and $\Delta\dot{\phi}$ to zero leads to

$$
\begin{aligned}
u_1 \Delta\dot{\beta} &= Y_\beta \Delta\beta + (Y_r - u_1)\Delta r + Y_{\delta_a}\Delta\delta_a + Y_{\delta_r}\Delta\delta_r \\
\Delta\dot{r} &= \left(N_\beta + N_{T_\beta}\right)\Delta\beta + \left(N_r + N_{T_r}\right)\Delta r + N_{\delta_a}\Delta\delta_a + N_{\delta_r}\Delta\delta_r
\end{aligned}
$$

Finally, to arrive at the Dutch roll approximation, neglect the propulsive effects and note that the primary control action occurs through the rudder deflection, such that

$$
\begin{aligned}
u_1 \Delta\dot{\beta} &= Y_\beta \Delta\beta + (Y_r - u_1)\Delta r + Y_{\delta_r}\Delta\delta_r \\
\Delta\dot{r} &= N_\beta \Delta\beta + N_r \Delta r + N_{\delta_r}\Delta\delta_r
\end{aligned}
$$

or, in standard matrix form

$$\begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{r} \end{bmatrix} = \begin{bmatrix} Y_\beta/u_1 & Y_r/u_1 - 1 \\ N_\beta & N_r \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta r \end{bmatrix} + \begin{bmatrix} Y_{\delta_r}/u_1 \\ N_{\delta_r} \end{bmatrix} \Delta\delta_r$$

$$\begin{vmatrix} Y_\beta/u_1 - \lambda & Y_r/u_1 - 1 \\ N_\beta & N_r - \lambda \end{vmatrix} = 0$$

or

$$(Y_\beta/u_1 - \lambda)(N_r - \lambda) - (Y_r/u_1 - 1)N_\beta = 0$$

Reducing this to a standard degree two polynomial in λ gives the characteristic equation

$$\lambda^2 - \frac{Y_\beta + N_r u_1}{u_1}\lambda + \frac{N_r Y_\beta - N_\beta Y_r + N_\beta u_1}{u_1} = 0$$

$$\omega_n = \sqrt{\frac{N_r Y_\beta - N_\beta Y_r + N_\beta u_1}{u_1}} \qquad \zeta = -\frac{1}{2\omega_n}\frac{Y_\beta + N_r u_1}{u_1}$$

Dutch-roll motion is primarily described by stability-axis yaw rate and sideslip angle

Roll and spiral motions are primarily described by stability-axis roll rate and roll angle

The University of Texas at Austin
Aerospace Engineering
and Engineering Mechanics
Cockrell School of Engineering

BACKUP SLIDES

JOHN-PAUL CLARKE
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$$
\begin{aligned}
Y &= m(\dot{v} + ru - pw - g\cos\theta\sin\phi) \\
&= m\Big[(\dot{v}_1 + \Delta\dot{v}) + (r_1 + \Delta r)(u_1 + \Delta u) - (p_1 + \Delta p)(w_1 + \Delta w) && \text{substitute in perturbations} \\
&\qquad - g\cos(\theta_1 + \Delta\theta)\sin(\phi_1 + \Delta\phi)\Big] \\
&= m\Big[(\dot{v}_1 + \Delta\dot{v}) + (r_1 + \Delta r)(u_1 + \Delta u) - (p_1 + \Delta p)(w_1 + \Delta w) && \text{angle addition formula} \\
&\qquad - g(\cos\theta_1\cos\Delta\theta - \sin\theta_1\sin\Delta\theta)(\sin\phi_1\cos\Delta\phi + \cos\phi_1\sin\Delta\phi)\Big] \\
&= m\Big[(\dot{v}_1 + \Delta\dot{v}) + (r_1 + \Delta r)(u_1 + \Delta u) - (p_1 + \Delta p)(w_1 + \Delta w) && \text{small angle approximation} \\
&\qquad - g(\cos\theta_1 - \sin\theta_1\Delta\theta)(\sin\phi_1 + \cos\phi_1\Delta\phi)\Big] \\
&= m(\dot{v}_1 + r_1u_1 - p_1w_1 - g\cos\theta_1\sin\phi_1) && \text{neglect higher-order terms} \\
&\qquad + m\Big[(\Delta\dot{v} + r_1\Delta u - p_1\Delta w - w_1\Delta p + u_1\Delta r) \\
&\qquad - g\cos\theta_1\cos\phi_1\Delta\phi + g\sin\theta_1\sin\phi_1\Delta\theta\Big] \\
&= Y_1 + \Delta Y
\end{aligned}
$$

$$
\begin{aligned}
L &= I_{xx}\dot{p} - I_{xz}(\dot{r} + pq) - (I_{yy} - I_{zz})qr \\
&= I_{xx}(\dot{p}_1 + \Delta\dot{p}) - I_{xz}\big[(\dot{r}_1 + \Delta\dot{r}) + (p_1 + \Delta p)(q_1 + \Delta q)\big] && \text{substitute in perturbations} \\
&\qquad - (I_{yy} - I_{zz})(q_1 + \Delta q)(r_1 + \Delta r) \\
&= \big[I_{xx}\dot{p}_1 - I_{xz}(\dot{r}_1 + p_1 q_1) - (I_{yy} - I_{zz})q_1 r_1\big] && \text{neglect higher-order terms} \\
&\qquad + I_{xx}\Delta\dot{p} - I_{xz}\Delta\dot{r} - I_{xz}p_1\Delta q - I_{xz}q_1\Delta p \\
&\qquad - (I_{yy} - I_{zz})q_1\Delta r - (I_{yy} - I_{zz})r_1\Delta q \\
&= L_1 + \Delta L
\end{aligned}
$$

$$
\begin{aligned}
N &= I_{zz}\dot{r} - I_{xz}(\dot{p} - qr) - (I_{xx} - I_{yy})pq \\
&= I_{zz}(\dot{r}_1 + \Delta\dot{r}) - I_{xz}\big[(\dot{p}_1 + \Delta\dot{p}) - (q_1 + \Delta q)(r_1 + \Delta r)\big] && \text{substitute in perturbations} \\
&\qquad - (I_{xx} - I_{yy})(p_1 + \Delta p)(q_1 + \Delta q) \\
&= \big[I_{zz}\dot{r}_1 - I_{xz}(\dot{p}_1 - q_1 r_1) - (I_{xx} - I_{yy})p_1 q_1\big] && \text{neglect higher-order terms} \\
&\qquad + I_{zz}\Delta\dot{r} - I_{xz}\Delta\dot{p} + I_{xz} r_1 \Delta q + I_{xz} q_1 \Delta r \\
&\qquad - (I_{xx} - I_{yy}) q_1 \Delta p - (I_{xx} - I_{yy}) p_1 \Delta q \\
& N_1 + \Delta N
\end{aligned}
$$

$$
\begin{aligned}
p &= \dot{\phi} - \dot{\psi}\sin\theta \\
&= (\dot{\phi}_1 + \Delta\dot{\phi}) - (\dot{\psi}_1 + \Delta\dot{\psi})\sin(\theta_1 + \Delta\theta_1) && \text{substitute in perturbations} \\
&= (\dot{\phi}_1 + \Delta\dot{\phi}) - (\dot{\psi}_1 + \Delta\dot{\psi})(\sin\theta_1 \cos\Delta\theta + \cos\theta_1 \sin\Delta\theta) && \text{angle addition formula} \\
&= (\dot{\phi}_1 + \Delta\dot{\phi}) - (\dot{\psi}_1 + \Delta\dot{\psi})(\sin\theta_1 + \cos\theta_1 \Delta\theta) && \text{small angle approximation} \\
&= (\dot{\phi}_1 - \dot{\psi}_1 \sin\theta_1) + (\Delta\dot{\phi} - \dot{\psi}_1 \cos\theta_1 \Delta\theta - \sin\theta_1 \Delta\dot{\psi}) && \text{neglect higher-order terms} \\
&= p_1 + \Delta p
\end{aligned}
$$

$$
\begin{aligned}
r &= -\dot{\theta}\sin\phi + \dot{\psi}\cos\theta\cos\phi \\
&= -(\dot{\theta}_1 + \Delta\dot{\theta})\sin(\phi_1 + \Delta\phi) && \text{substitute in perturbations} \\
&\qquad + (\dot{\psi}_1 + \Delta\dot{\psi})\cos(\theta_1 + \Delta\theta)\cos(\phi_1 + \Delta\phi) \\
&= -(\dot{\theta}_1 + \Delta\dot{\theta})(\sin\phi_1\cos\Delta\phi + \cos\phi_1\sin\Delta\phi) && \text{angle addition formula} \\
&\qquad + (\dot{\psi}_1 + \Delta\dot{\psi})(\cos\theta_1\cos\Delta\theta - \sin\theta_1\sin\Delta\theta) \\
&\qquad \times (\cos\phi_1\cos\Delta\phi - \sin\phi_1\sin\Delta\phi) \\
&= -(\dot{\theta}_1 + \Delta\dot{\theta})(\sin\phi_1 + \cos\phi_1\Delta\phi) && \text{small angle approximation} \\
&\qquad + (\dot{\psi}_1 + \Delta\dot{\psi})(\cos\theta_1 - \sin\theta_1\Delta\theta)(\cos\phi_1 - \sin\phi_1\Delta\phi) \\
&= (-\dot{\theta}_1\sin\phi_1 + \dot{\psi}_1\cos\theta_1\cos\phi_1) && \text{neglect higher order terms} \\
&\qquad + (-\sin\phi_1\Delta\dot{\theta} + \cos\theta_1\cos\phi_1\Delta\dot{\psi} - \dot{\theta}_1\cos\phi_1\Delta\phi \\
&\qquad - \dot{\psi}_1\cos\theta_1\sin\phi_1\Delta\phi - \dot{\psi}_1\sin\theta_1\cos\phi_1\Delta\theta) \\
&= r_1 + \Delta r
\end{aligned}
$$

$$
\begin{aligned}
\beta &= \arctan \frac{v}{u} \\
&= \frac{v}{u} && \text{small angle approximation} \\
&= \frac{v_1 + \Delta v}{u_1 + \Delta u} && \text{substitute in perturbations} \\
&= (v_1 + \Delta v)(u_1 + \Delta u)^{-1} \\
&= \frac{v_1 + \Delta v}{u_1} \left(1 + \frac{\Delta u}{u_1} \right)^{-1} \\
&= \frac{v_1 + \Delta v}{u_1} \left(1 - \frac{\Delta u}{u_1} - \cdots \right) && \text{binomial expansion} \\
&= \frac{v_1}{u_1} + \frac{\Delta v}{u_1} - \frac{v_1 \Delta u}{u_1^2} && \text{neglect higher-order terms} \\
&= \frac{v_1}{u_1} + \frac{\Delta v}{u_1} && \text{trim conditions} \\
&= \beta_1 + \Delta \beta
\end{aligned}
$$

$$\Delta Y = m(u_1 \Delta\dot{\beta} + u_1 \Delta r - g\cos\theta_1 \Delta\phi) \tag{11.11a}$$

$$\Delta L = I_{xx}\Delta\dot{p} - I_{xz}\Delta\dot{r} \tag{11.11b}$$

$$\Delta N = I_{zz}\Delta\dot{r} - I_{xz}\Delta\dot{p} \tag{11.11c}$$

$$\Delta p = \Delta\dot{\phi} - \sin\theta_1 \Delta\dot{\psi} \tag{11.11d}$$

$$\Delta r = \cos\theta_1 \Delta\dot{\psi} \tag{11.11e}$$

Eq. (11.11e) can be solved for $\Delta\dot{\psi}$ as

$$\Delta\dot{\psi} = \sec\theta_1 \Delta r \tag{11.12}$$

Eq. (11.12) can then be substituted into Eq. (11.11d) to give

$$\begin{aligned} \Delta p &= \Delta\dot{\phi} - \sin\theta_1 \Delta\dot{\psi} \\ &= \Delta\dot{\phi} - \sin\theta_1 \sec\theta_1 \Delta r \\ &= \Delta\dot{\phi} - \tan\theta_1 \Delta r \end{aligned}$$

or, solving for $\Delta\dot{\phi}$,

$$\Delta\dot{\phi} = \Delta p + \tan\theta_1 \Delta r$$

Let’s look at the force and moments...

$$Y = \bar{q}SC_y$$

$$\left(\frac{\partial Y}{\partial \beta}\right)_1 = \bar{q}_1 S C_{y_\beta}$$
$$\left(\frac{\partial Y}{\partial p}\right)_1 = \frac{\bar{q}_1 S b}{2u_1} C_{y_p}$$
$$\left(\frac{\partial Y}{\partial r}\right)_1 = \frac{\bar{q}_1 S b}{2u_1} C_{y_r}$$
$$\left(\frac{\partial Y}{\partial \delta_a}\right)_1 = \bar{q}_1 S C_{y_{\delta_a}}$$
$$\left(\frac{\partial Y}{\partial \delta_r}\right)_1 = \bar{q}_1 S C_{y_{\delta_r}}$$

$$C_{y_\beta} = \left(\frac{\partial C_y}{\partial \beta}\right)_1$$
$$C_{y_p} = \left(\frac{\partial C_y}{\partial (pb/2u_1)}\right)_1$$
$$C_{y_r} = \left(\frac{\partial C_y}{\partial (rb/2u_1)}\right)_1$$
$$C_{y_{\delta_a}} = \left(\frac{\partial C_y}{\partial \delta_a}\right)_1$$
$$C_{y_{\delta_r}} = \left(\frac{\partial C_y}{\partial \delta_r}\right)_1$$

Let’s look at the force and moments...

$$L = \bar{q}SbC_\ell$$

$$\left(\frac{\partial L}{\partial \beta}\right)_1 = \bar{q}_1 SbC_{\ell_\beta} \qquad C_{\ell_\beta} = \left(\frac{\partial C_\ell}{\partial \beta}\right)_1$$

$$\left(\frac{\partial L}{\partial p}\right)_1 = \frac{\bar{q}_1 Sb^2}{2u_1} C_{\ell_p} \qquad C_{\ell_p} = \left(\frac{\partial C_\ell}{\partial (pb/2u_1)}\right)_1$$

$$\left(\frac{\partial L}{\partial r}\right)_1 = \frac{\bar{q}_1 Sb^2}{2u_1} C_{\ell_r} \qquad C_{\ell_r} = \left(\frac{\partial C_\ell}{\partial (rb/2u_1)}\right)_1$$

$$\left(\frac{\partial L}{\partial \delta_a}\right)_1 = \bar{q}_1 SbC_{\ell_{\delta_a}} \qquad C_{\ell_{\delta_a}} = \left(\frac{\partial C_\ell}{\partial \delta_a}\right)_1$$

$$\left(\frac{\partial L}{\partial \delta_r}\right)_1 = \bar{q}_1 SbC_{\ell_{\delta_r}} \qquad C_{\ell_{\delta_r}} = \left(\frac{\partial C_\ell}{\partial \delta_r}\right)_1$$

Let's look at the force and moments...

$$N_A = \bar{q}SbC_n$$

$$\left(\frac{\partial N_A}{\partial \beta}\right)_1 = \bar{q}_1 SbC_{n_\beta} \qquad C_{n_\beta} = \left(\frac{\partial C_n}{\partial \beta}\right)_1$$

$$\left(\frac{\partial N_A}{\partial p}\right)_1 = \frac{\bar{q}_1 Sb^2}{2u_1} C_{n_p} \qquad C_{n_p} = \left(\frac{\partial C_n}{\partial (pb/2u_1)}\right)_1$$

$$\left(\frac{\partial N_A}{\partial r}\right)_1 = \frac{\bar{q}_1 Sb^2}{2u_1} C_{n_r} \qquad C_{n_r} = \left(\frac{\partial C_n}{\partial (rb/2u_1)}\right)_1$$

$$\left(\frac{\partial N_A}{\partial \delta_a}\right)_1 = \bar{q}_1 SbC_{n_{\delta_a}} \qquad C_{n_{\delta_a}} = \left(\frac{\partial C_n}{\partial \delta_a}\right)_1$$

$$\left(\frac{\partial N_A}{\partial \delta_r}\right)_1 = \bar{q}_1 SbC_{n_{\delta_r}} \qquad C_{n_{\delta_r}} = \left(\frac{\partial C_n}{\partial \delta_r}\right)_1$$

Let's look at the force and moments...

$$N_T = \bar{q}SbC_{n_T}$$

$$\left(\frac{\partial N_T}{\partial \beta}\right)_1 = \bar{q}_1 SbC_{n_{T_\beta}}$$

$$\left(\frac{\partial N_T}{\partial r}\right)_1 = \frac{\bar{q}_1 Sb^2}{2u_1} C_{n_{T_r}}$$

$$C_{n_{T_\beta}} = \left(\frac{\partial C_{n_T}}{\partial \beta}\right)_1$$

$$C_{n_{T_r}} = \left(\frac{\partial C_{n_T}}{\partial (rb/2u_1)}\right)_1$$

Let's look at the force and moments...

$$\Delta Y = \bar{q}_1 S C_{y_\beta} \Delta\beta + \frac{\bar{q}_1 S b}{2u_1} C_{y_p} \Delta p + \frac{\bar{q}_1 S b}{2u_1} C_{y_r} \Delta r + \bar{q}_1 S C_{y_{\delta_a}} \Delta\delta_a + \bar{q}_1 S C_{y_{\delta_r}} \Delta\delta_r$$

$$\Delta L = \bar{q}_1 S b C_{\ell_\beta} \Delta\beta + \frac{\bar{q}_1 S b^2}{2u_1} C_{\ell_p} \Delta p + \frac{\bar{q}_1 S b^2}{2u_1} C_{\ell_r} \Delta r + \bar{q}_1 S b C_{\ell_{\delta_a}} \Delta\delta_a + \bar{q}_1 S b C_{\ell_{\delta_r}} \Delta\delta_r$$

$$\Delta N = \left(\bar{q}_1 S b C_{n_\beta} + \bar{q}_1 S b C_{n_{T_\beta}}\right) \Delta\beta + \frac{\bar{q}_1 S b^2}{2u_1} C_{n_p} \Delta p + \left(\frac{\bar{q}_1 S b^2}{2u_1} C_{n_r} + \frac{b^2}{\bar{q}_1 S 2u_1} C_{n_{T_r}}\right) \Delta r$$
$$+ \bar{q}_1 S b C_{n_{\delta_a}} \Delta\delta_a + \bar{q}_1 S b C_{n_{\delta_r}} \Delta\delta_r$$

Which gives the accelerations...

$$\Delta Y/m = \frac{\bar{q}_1 S}{m} C_{y_\beta} \Delta\beta + \frac{\bar{q}_1 S b}{2 m u_1} C_{y_p} \Delta p + \frac{\bar{q}_1 S b}{2 m u_1} C_{y_r} \Delta r + \frac{\bar{q}_1 S}{m} C_{y_{\delta_a}} \Delta\delta_a + \frac{\bar{q}_1 S}{m} C_{y_{\delta_r}} \Delta\delta_r$$

$$\Delta L/I_{xx} = \frac{\bar{q}_1 S b}{I_{xx}} C_{\ell_\beta} \Delta\beta + \frac{\bar{q}_1 S b^2}{2 I_{xx} u_1} C_{\ell_p} \Delta p + \frac{\bar{q}_1 S b^2}{2 I_{xx} u_1} C_{\ell_r} \Delta r + \frac{\bar{q}_1 S b}{I_{xx}} C_{\ell_{\delta_a}} \Delta\delta_a + \frac{\bar{q}_1 S b}{I_{xx}} C_{\ell_{\delta_r}} \Delta\delta_r$$

$$\Delta N/I_{zz} = \left(\frac{\bar{q}_1 S b}{I_{zz}} C_{n_\beta} + \frac{\bar{q}_1 S b}{I_{zz}} C_{n_{T_\beta}} \right) \Delta\beta + \frac{\bar{q}_1 S b^2}{2 I_{zz} u_1} C_{n_p} \Delta p + \left(\frac{\bar{q}_1 S b^2}{2 I_{zz} u_1} C_{n_r} + \frac{\bar{q}_1 S b^2}{2 I_{zz} u_1} C_{n_{T_r}} \right) \Delta r$$
$$+ \frac{\bar{q}_1 S b}{I_{zz}} C_{n_{\delta_a}} \Delta\delta_a + \frac{\bar{q}_1 S b}{I_{zz}} C_{n_{\delta_r}} \Delta\delta_r$$

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