Homework #2

EE 242 **Yehoshua Luna**
Spring 2025 **2322458**

Problem #1

a)

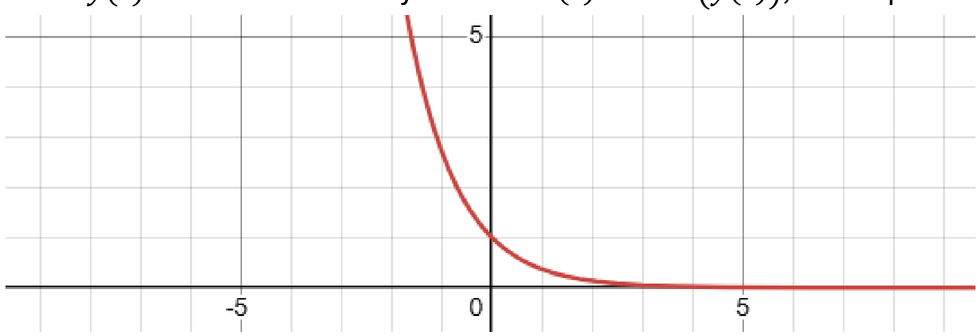
i. $y(t)$ is nonlinear. $T(x_1(t)+x_2(t)) = e^{-(x_1(t)+x_2(t))} = e^{-x_1(t)} \cdot e^{-x_2(t)} \neq e^{-x_1(t)} + e^{-x_2(t)} = T\big(x_1(t)\big) + T(x_2(t))$

ii. $y(t)$ is time-invariant. $y(t-t_0) = e^{-(t-t_0)} = T(x(t-t_0))$

iii. $y(t)$ is stable. $|x(t)| < M \rightarrow 0 \leq e^{-x(t)} = y(t) \leq e^{M}$

iv. $y(t)$ is causal because it only depends on $x(t_0)$.

v. $y(t)$ is invertible. As the graph below shows, for each $x(t)$, there is a unique value $y(t)$. We can also easily see that $x(t) = -\ln\big(y(t)\big)$, which proves this.



b)

i. $y(t)$ is linear. $T\big(ax_1(t)+bx_2(t)\big) = \int_{-\infty}^{2t} ax_1(u) + bx_2(u)\, du = a\int_{-\infty}^{2t} x_1(u)\, du + b\int_{-\infty}^{2t} x_2(u)\, du = a \cdot T\big(x_1(t)\big) + b \cdot T\big(x_2(t)\big)$

ii. $y(t)$ is time-variant. $y(t-t_0) = \int_{-\infty}^{2(t-t_0)} x(u)\, du \neq \int_{-\infty}^{2t-t_0} x(u)\, du = \int_{-\infty}^{2t} x(u-t_0)\, du = T(x(t-t_0))$

iii. $y(t)$ is unstable. If $x(t) = 1$, then $y(t) = \int_{-\infty}^{2t} 1\, du = \infty$ for all $t \neq -\infty$. This clearly violates BIBO stability, so $y(t)$ is therefore unstable.

iv. $y(t)$ is not causal. It depends on $\int x(u)\, du$ at $u = 2t_0$ from integration bounds.

v. $y(t)$ is not invertible because it loses information about the input $x(t)$ through integration over the given bounds. For example, if $x_1(t) = u(t) - u(t-2)$ and $x_2(t) = u(t+2) - u(t)$, then $y_1(2) = y_2(2) = 2$. Both of the input functions produced the same result, so this system cannot be invertible. It is impossible to mathematically characterize the input function from this system's output.

Problem #2

Using the associative property, we can add e^{-2t} and $\delta(t+1)$ together into one response, namely, $e^{-2t} + \delta(t+1)$. This can then be convolved with $u(t)$ to produce the equivalent system response. The calculations are as follows:

$$H(t) = (e^{-2t} + \delta(t+1)) * u(t) = \int_{-\infty}^{\infty} \left(e^{-2\tau} + \delta(\tau+1)\right) \cdot u(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{t} e^{-2\tau} + \delta(\tau+1)\, d\tau = \infty$$

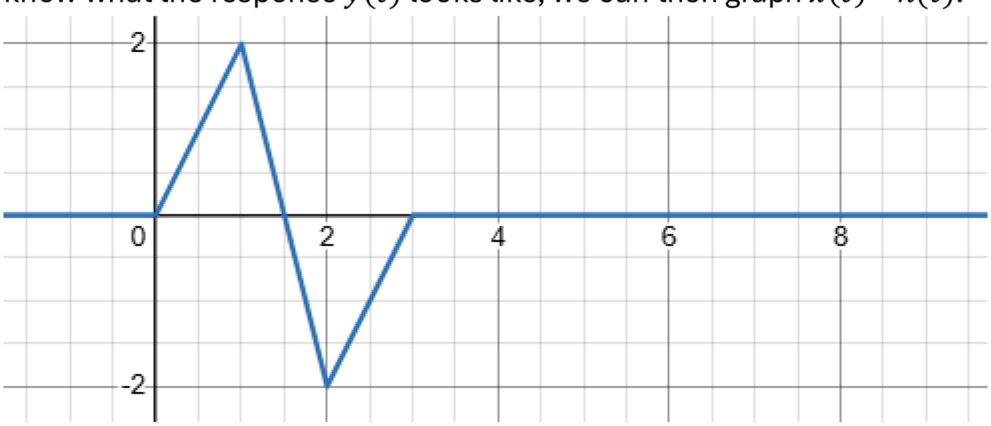
The equivalent system response is $H(t) = \infty$. (Was there supposed to be another $u(t)$?)

Problem #3

Because $x(n)$ is only nonzero at $n = 0$ and $n = 1$, we can simplify the discrete convolution to $y(n) = \sum_{k=0}^{1} x(k) \cdot h(n-k) = x(0) \cdot h(n-0) + x(1) \cdot h(n-1) = h(n) - h(n-1)$. We must have $h(0) - h(-1) = 1$, but $h(n)$ is causal, so $h(-1) = 0$ and $h(0) = 1$. Increasing n by one gives $h(1) - h(0) = 0$, so $h(1) = 1$. $h(2) - h(1) = 0$, so $h(2) = 2$. $h(3) - h(2) = 0$, so $h(3) = 3$. $h(4) - h(3) = -1$, so we finally have $h(4) = 0$. Thus, $h(n) = [\ldots 0, 1, 1, 1, 0, \ldots]$.

Problem #4

$x(t) = u(t) - 2u(t-1) + u(t-2)$, so $x(t) * h(t) = u(t) * h(t) - 2u(t-1) * h(t) + u(t-2) * h(t) = y(t) - 2y(t-1) + y(t-2)$ by the distributive property. Because we know what the response $y(t)$ looks like, we can then graph $x(t) * h(t)$.



Problem #5

$x(t - t_0)$ shifts the input signal $x(t)$ to the left by t_0. $h(t + t_0)$ shifts the impulse response $h(t)$ to the right by t_0. However, these two shifts cancel during the process of convolution, so the output is simply $y(t)$.

Problem #6

According to the textbook, $h(n) = s(n) - s(n-1)$, so $h(n) = \alpha^n u(n) - \alpha^{n-1}u(n-1)$.

Problem #7

$\mathrm{x(t)} = \cos(t) = \frac{e^{jt} - e^{-jt}}{2} = \frac{e^{jt}}{2} + \frac{e^{-jt}}{2}$ and $H(j\omega) = \frac{1}{1+j\omega}$. Using the given Eigenfunction properties of the system, we must have $y(t) = \frac{e^{jt}}{2} \cdot H(j) - \frac{e^{-jt}}{2} \cdot H(-j) = \frac{e^{jt}}{2} \cdot \frac{1}{1+j} + \frac{e^{-jt}}{2} \cdot \frac{1}{1-j} = \frac{e^{jt}}{2} \cdot \frac{1-j}{2} + \frac{e^{-jt}}{2} \cdot \frac{1+j}{2} = \frac{1}{4}\left(e^{jt} - je^{jt} + e^{-jt} + je^{-jt}\right) = \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t) = y(t)$.

Problem #8

a)
   i. $h(t) = 0$ for $t < 0$, so the system is causal.
   ii. $h(t) = \cos\,(\omega_0 t)$ for $t > 0$. $\int_0^\infty |\cos\,(\omega_0 t)|\ dt = \infty$, so the system is unstable.

b)
   i. $h(n) \neq 0$ for $n < 0$, so the system is not causal.
   ii. $h(n) = 0.8^n$ for $n \geq -2$, so $\sum_{n=-2}^{\infty} |0.8^n| = 7.8125$ and the system is stable.

Problem #9

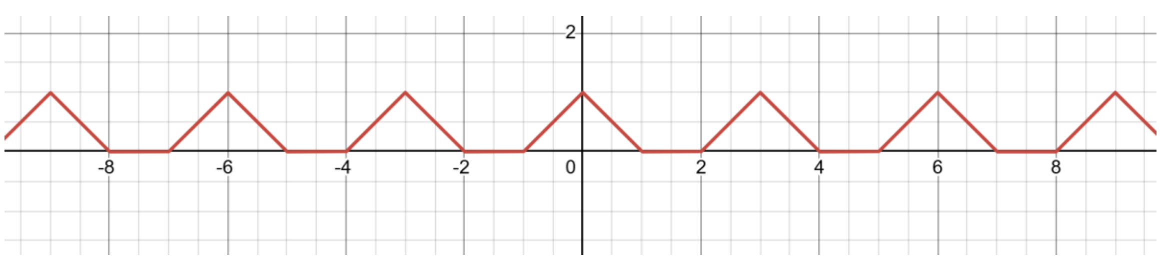
$$y(t) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau) \cdot e^{-b(t-\tau)}u(t-\tau)\, d\tau = e^{-bt}\int_0^t e^{(b-a)\tau}\, d\tau = e^{-bt} \cdot \frac{e^{(b-a)t} - 1}{b-a}$$

When $a = b$, $e^{(b-a)\tau}$ evaluates to 1, so $\int_0^t e^{(b-a)\tau}\, d\tau = t$. Therefore $y(t) = te^{-bt}$ for $a = b$.

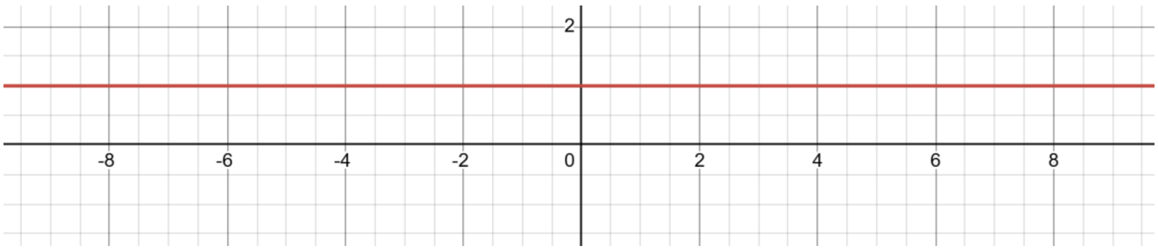
Problem #10

$$y(n) = \sum_{k=-\infty}^{\infty} \alpha^k u(k) \cdot \beta^{n-k}u(n-k) = \sum_{k=0}^{n} \alpha^k \cdot \beta^{n-k} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

Problem #11



a)



b)

Problem #12

a) $H(j\omega)$ is just the Fourier Transform of $h(t)$, so we can find $h(t)$ by calculating the inverse Fourier Transform of $H(j\omega)$. Using the tables provided in my textbook and the distributive property of integrals, I can easily see that $h(t) = \delta(t) + \alpha\delta(t - t_0)$.

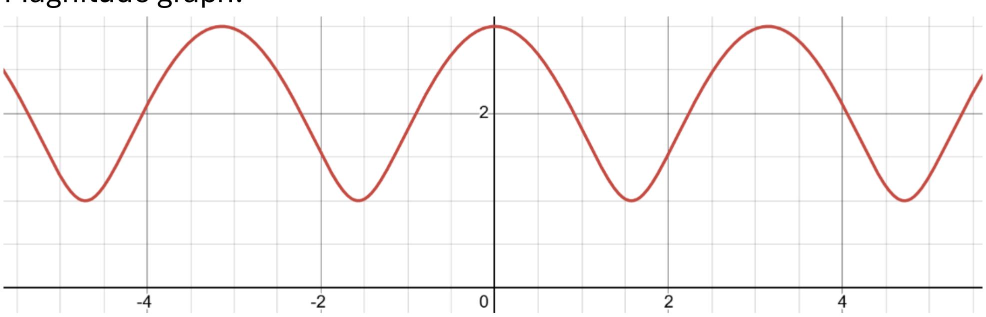
b) We need to separate the real and imaginary parts of $H(j\omega)$ to find the magnitude and phase of the transfer function. Expanding $H(j\omega)$ gives $1 + \alpha\cos(\omega t_0) - j\alpha\sin(\omega t_0)$. Now we can find the magnitude and phase:

$$
\begin{aligned}
|H(j\omega)| = \sqrt{Re^2 + Im^2} &= \sqrt{(1 + \alpha\cos(\omega t_0))^2 + (\alpha\sin(\omega t_0))^2} \\
&= \sqrt{1 + 2\alpha\cos(\omega t_0) + \alpha^2\cos^2(\omega t_0) + \alpha^2\sin^2(\omega t_0)} \\
&= \sqrt{1 + 2\alpha\cos(\omega t_0) + \alpha^2}
\end{aligned}
$$

$$\angle H(j\omega) = \tan^{-1}\left(\frac{Im}{Re}\right) = \tan^{-1}\left(\frac{-\alpha\sin(\omega t_0)}{1 + \alpha\cos(\omega t_0)}\right)$$

We can then graph the magnitude and phase of $H(j\omega)$ as functions of ω for fixed values $\alpha = t_0 = 2$ to get an understanding of how the output varies with ω.

Magnitude graph:



Phase graph: