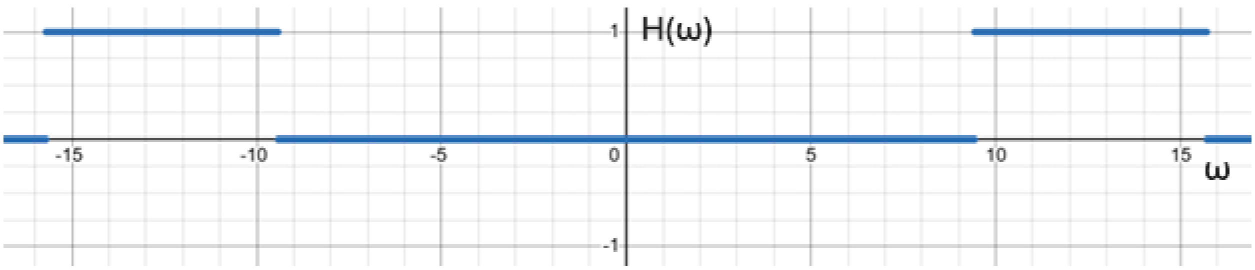
Homework #2

EE 242 **Yehoshua Luna**
Spring 2025 **2322458**

Problem #1

$\mathcal{F}(sinc(x)) = rect\left(\frac{\omega}{2\pi}\right)$, and multiplying by $2\cos(4\pi t)$ modulates this lowpass signal to be centered at ±4π on the frequency scale. Therefore, $H(\omega)$ should look like the graph below.



(a) $\mathcal{F}(x(t)) = \pi(2\delta(\omega) + \delta(\omega - \pi) + \delta(\omega + \pi) + \delta(\omega - 4\pi)/j - \delta(\omega + 4\pi)/j)$
Multiplying $H(\omega)$ and $X(\omega)$ gives $Y(\omega) = \frac{\pi}{j}(\delta(\omega - 4\pi) - \delta(\omega + 4\pi))$
Taking the inverse Fourier Transform them yields $y(t) = \sin(4\pi t)$

(b) $x(t)$ is simply an impulse train with sample period $T_s = 1$, so $\omega_s = 2\pi$ and $X(\omega) = 2\pi \sum_k \delta(\omega - 2\pi k)$. Multiplying by $H(\omega)$ gives $Y(\omega) = 2\pi\big(\delta(\omega - 4\pi) + \delta(\omega + 4\pi)\big)$.
Taking the inverse Fourier Transform then yields $y(t) = 2\cos(4\pi t)$.

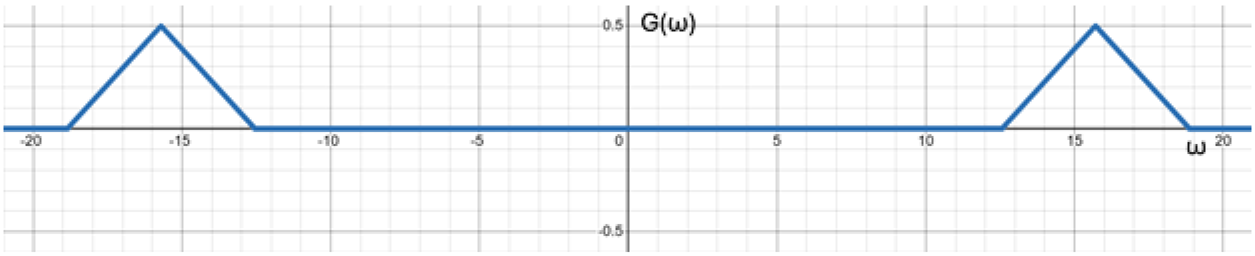
Problem #2

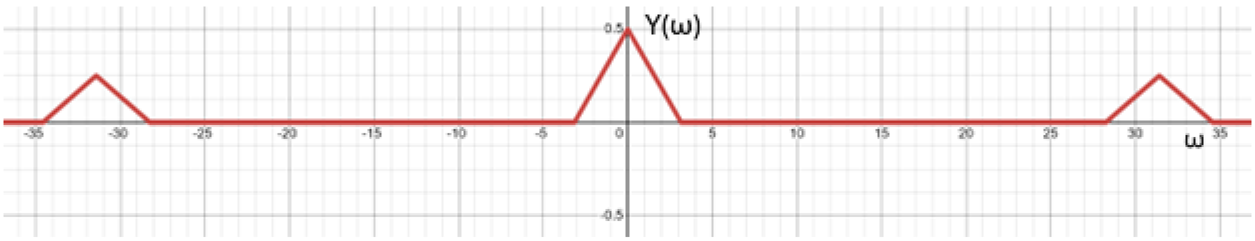
i. This is a low-pass filter with $h(t) = \frac{1}{2\pi}\int_{-3\pi}^{3\pi} j\omega e^{j\omega t} d\omega = \frac{6\pi t \cdot \cos(3\pi t) - 2\sin(3\pi t)}{2\pi t^2}$.

ii.

1. For signals with $\omega_{max} \leq 3\pi$, this filter simply calculates the derivative of the input, so $y(t) = \frac{d}{dt}x(t) = \frac{d}{dt}A \cdot \cos(2\pi t + \theta) = -2\pi A \cdot \sin(2\pi t + \theta)$. This is verified by looking at $X(\omega) = \frac{A}{2}\left[e^{j\theta}\delta(\omega - 2\pi) + e^{-j\theta}\delta(\omega + 2\pi)\right]$, which represents a single tone at ±2π. This is inside the range of our low-pass filter, so the signal is not attenuated.
2. $X(\omega) = \frac{A}{2}\left[e^{j\theta}\delta(\omega - 4\pi) + e^{-j\theta}\delta(\omega + 4\pi)\right]$, which is completely outside the range of the low-pass filter, so this signal is completely attenuated, and $y(t) = 0$.

Problem #3





Problem #4

The rectangular pulse train has period T and $\omega_0 = 2\pi/T$. It has Fourier coefficients $c_k = \frac{A}{k\pi}\sin(k\pi\frac{\tau}{T})$, so we have $C(\omega) = \sum_{k=-\infty}^{\infty}\frac{A}{k\pi}\sin(k\pi\frac{\tau}{T})\cdot\delta(\omega - 2\pi k/T)$.

(a) Multiplying by a rectangular pulse train with period T and width τ in the time domain is equivalent to convolving with a sampled sinc function in the frequency domain with sample period $\frac{2\pi}{T}$. To recover $x(t)$ from $y(t)$, we must not have any overlap between each distorted replica of $x(t)$ in the frequency domain. Therefore $X(\omega)$ cannot contain any frequencies greater than $\frac{\omega_0}{2} = \frac{\pi}{T}$.

(b) The cutoff frequency should be $\omega_c = \frac{\pi}{T}$ and the gain should be $\frac{T}{\tau}$ because $C(0) = A \cdot \frac{\tau}{T}$.

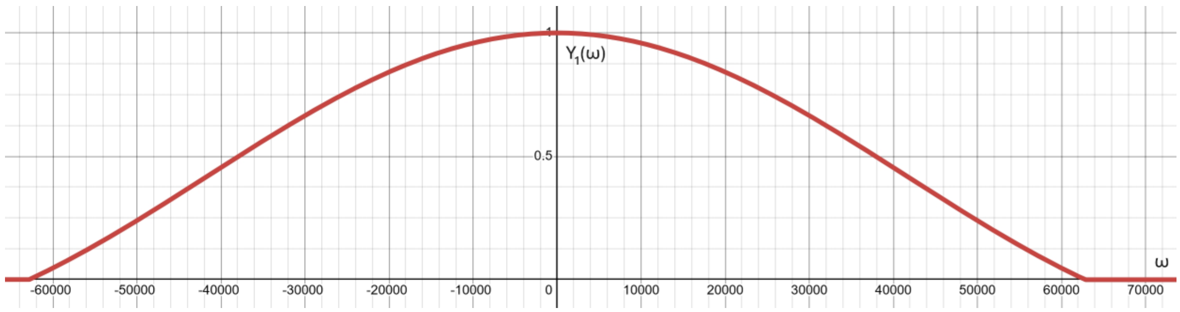
Problem #5

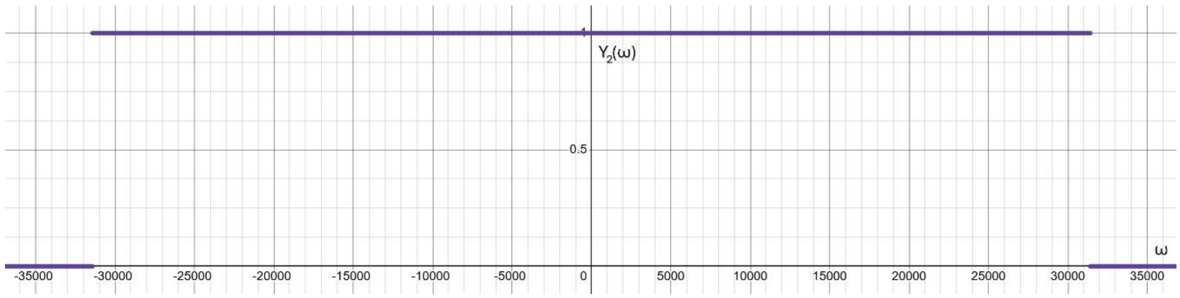
$$y(t)\cdot\cos(\omega_c t + \phi) = x(t)\cdot\cos(\omega_c t)\cdot\cos(\omega_c t + \phi) = x(t)\cdot\frac{\cos(2\omega_c t + \phi) + \cos(\phi)}{2}$$

The ideal low-pass filter then removes the $\cos(2\omega_c t + \phi)$ modulator because $\omega_c > \omega_m$ and amplifies the resulting signal by 2, giving $z(t) = x(t)\cdot\cos(\phi)$. This shows how phase offset impacts the retrieved signal. When $\phi = 0$, there is no phase offset, and the resulting signal is an exact replica of $x(t)$. If $\frac{\pi}{2} > \phi > 0$, then the recovered signal is attenuated by $\cos(\phi)$. When $\phi = \frac{\pi}{2}$, the original signal is totally lost because $\cos\left(\frac{\pi}{2}\right) = 0$. Once $\phi > \frac{\pi}{2}$, the signal becomes inverted, eventually peaking at $\phi = \pi$ and then begins oscillating back towards the initial condition when $\phi = 0$. This cycle repeats indefinitely.

Problem #6

(a)





(b) $Y_2(\omega)$ is a low-pass filter with a single-sided bandwidth of 1000π, and $Y_1(\omega)$ has a max frequency of 2000π. Multiplication in the time domain is convolution in the frequency domain, so the bandwidths of $Y_1(\omega)$ and $Y_2(\omega)$ are added, giving a $y(t)$ a maximum bandwidth of 3000π.

Problem #7

i.
 (a) $x(t) \cdot x(t) \Leftrightarrow X(\omega) * X(\omega)$. This process effectively doubles the bandwidth of $x(t)$ to $(-2\omega_B, 2\omega_B)$, so we must have a Nyquist sampling rate that follows $\omega_s \geq 4\omega_B$.
 (b) $\frac{d}{dt}x(t) \Leftrightarrow j\omega X(\omega)$. This doesn't change the bandwidth of $x(t)$, so $\omega_s \geq 2\omega_B$.

ii. $x_1(t) \cdot x_2(t) \Leftrightarrow X_1(\omega) * X_2(\omega)$. This sums their bandwidths, creating a signal with new frequency range $(-\omega_1 - \omega_2, \omega_1 + \omega_2)$, so we must have sample rate $\omega_s \geq 2(\omega_1 + \omega_2)$.

iii. Some cursory analysis shows that $\omega_0 = 1000\pi$ produces the desired sequence with a sampling rate of $T_s = 10^{-3}sec$. This is because $1000\pi \cdot 10^{-3}sec = \pi\ rad/s$. Adding or subtracting multiples of 2000π from ω_0 shifts this angular frequency by $2\pi\ rad/s$, so other distinct values of ω_0 that work are $\omega_0 = 3000\pi$ and $\omega_0 = 5000\pi$.

Problem #8

(a) $r(t) + RC\frac{d}{dt}r(t) = x(t) \Leftrightarrow R(j\omega) + j\omega \cdot RC \cdot R(j\omega) = X(j\omega)$

$$H_{RC}(\omega) = \frac{R(j\omega)}{X(j\omega)} = \frac{1}{1 + j\omega \cdot RC}$$

(b) The rectangular pulse train has $T = 2$ and $\omega_0 = 2\pi/T = \pi$. It has Fourier coefficients $c_k = \frac{1}{k\pi}\sin(\frac{k\pi}{2})$, so we have $X(\omega) = \sum_{k=-\infty}^{\infty}\frac{1}{k\pi}\sin(\frac{k\pi}{2}) \cdot \delta(\omega - k\pi)$. $H(\omega)$ limits this to frequencies less than 2 and multiplies it by π, so $X(\omega) = \begin{cases} 1/2, & \omega = 0 \\ 1, & \omega = \pm\pi \end{cases}$. If $RC > 0.05$, then $\omega_c < \pi$, so only the case where $\omega = 0$ is left unattenuated by $H_{RC}(\omega)$. This means that the output would be $z(t) = \frac{1}{2}\delta(t)$. But if $RC > 0.05$, then $X(\omega)$ is completely left alone by $H_{RC}(\omega)$, so $z(t) = \frac{1}{2}\delta(t) + \frac{1}{\pi}\cos(\pi t)$. (Please be merciful, this was the best that I could do. *cry*)

Problem #9

(a) $\frac{\sin(1000\pi t)}{\pi t}$ has a maximum of 1000 at $t = 0$. Therefore, over modulation occurs for any values of $A < 1000$. So, in order to use envelope detection, we must have $A \geq 1000$, with a minimum of $A = 1000$.

(b) $m = \frac{max|m(t)|}{A} = \frac{1000}{1000} = 1 = 100\%$