EE 242 Lab 2a – Frequency Domain Representation of Signals - Fourier Series

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This lab has 2 exercises to be completed as a team. Each should be given a separate code cell in your Notebook, followed by a markdown cell with report discussion. Your notebook should start with a markdown title and overview cell, which should be followed by an import cell that has the import statements for all assignments. For this assignment, you will need to import: numpy, the wavfile package from scipy.io, and matplotlib.pyplot.

In [1]:
```
# We'll refer to this as the "import cell." Every module you import should be imported here.
import numpy as np
import matplotlib.pyplot as plt
from scipy.io import wavfile
from IPython.display import display, Audio
```

Summary

In this lab, we will learn how to build periodic signals from component sinusoids and how to transform signals from the time domain to the frequency domain. The concepts we'll focus on include: implementation of the Fourier Series synthesis equation, using a discrete implementation of the Fourier Transform (DFT) with a digitized signal, and understanding the relationship between the discrete DFT index k and frequency ω for both the original continuous signal x(t). This is a two-week lab. You should plan on completing the first 2 assignments in the first week.

Lab 2a turn in checklist

• Lab 2a Jupyter notebook with code for the 2 exercises assignment in separate cells. Each assignment cell should contain markdown cells (same as lab overview cells) for the responses to lab report questions. Include your lab members' names at the top of the notebook.

Please submit the report as PDF

Assignment 1 -- Generating simple periodic signals

In the first assignment, you will develop an understanding of how some periodic signals are easier to approximate than others with a truncated Fourier Series. In this lab, we'll work with real signals and use the synthesis equation:

$$x(t) = a_0 + \sum_{k=1}^{N} 2|a_k|cos(k\omega_0 t + \angle a_k)$$

In lecture, you saw that you get ripples at transition points in approximating a square wave (**Gibbs phenomenon**). This happens for any signals with sharp edges. This assignment will involve approximating two signals (a sawtooth and a triangle wave) that have the same fundamental frequency (20Hz).

A. Write a function for generating a real-valued periodic time signal given the Fourier series coefficients [a_0 a_1 $\cdots$ a_N], the sampling frequency, and the fundamental frequency. You may choose to have complex input coefficients or have separate magnitude and phase vectors for describing a_k.

B. Define variables for the sampling frequency (8kHz) and the fundamental frequency (20Hz). Using this sampling frequency, create a time vector for a length of 200ms.

C. The sawtooth signal has coefficients as follows:

$$a_0 = 0.5, a_k = 1/(j2k\pi)$$

Using the function from part A, create three approximations of this signal with N = 2,5,20 and plot together in a 3×1 comparison.

D. A triangle signal has coefficients:

$$a_0 = 0.5, a_k = \frac{2sin(k\pi/2)}{j(k\pi)^2} e^{-j2k\pi/2}$$

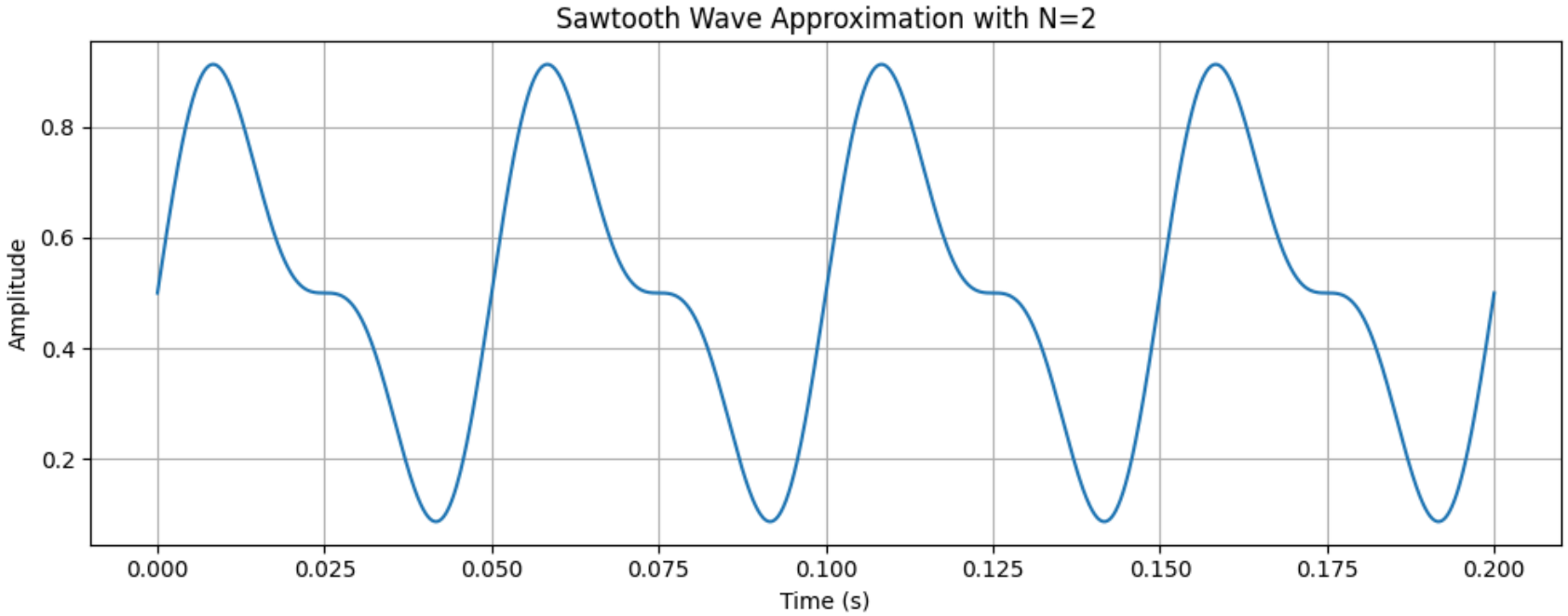
Create three approximations of this signal with N = 2,5,20 and plot together in a 3×1 comparison.

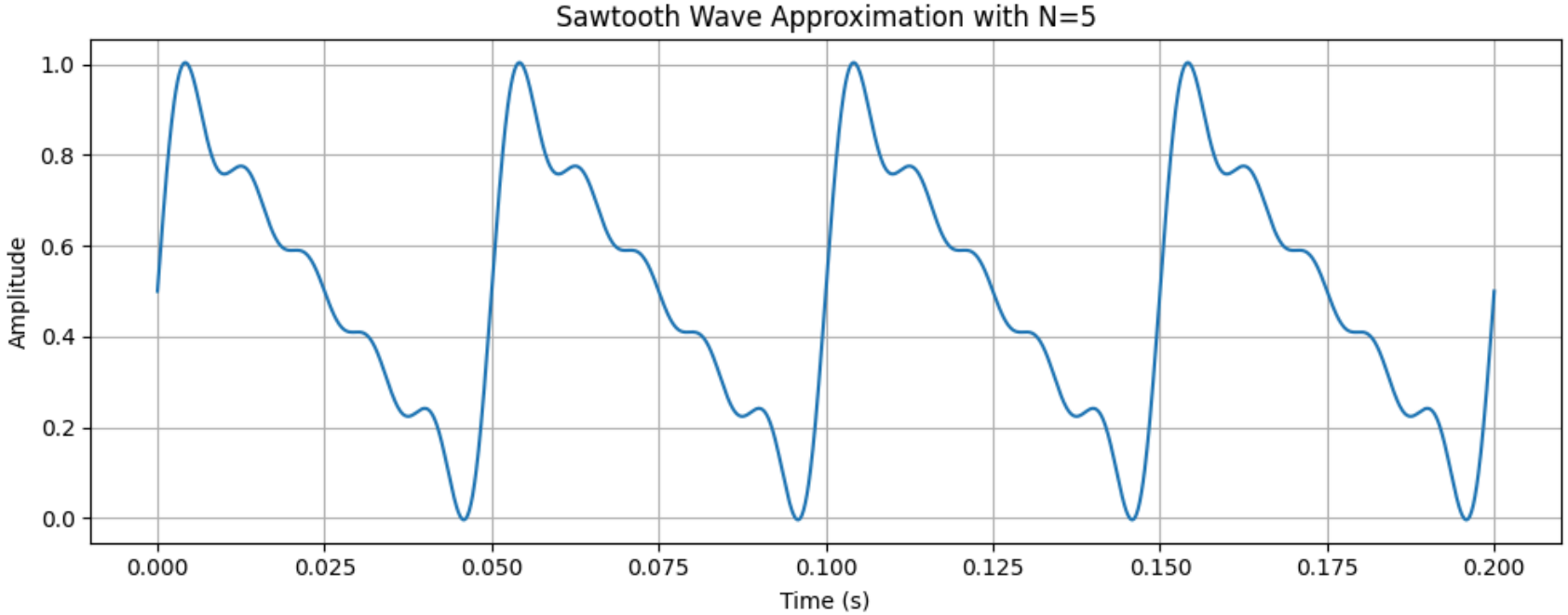
In [2]:
```
# Assignment 1 - Generating Periodic Signals with Fourier Series

##############################################################################################
# Part A - Writing a periodic signal generator function
```

```
#################################################################################################

# Input:
# t = time in seconds
# fs = sampling rate
# a = coefficients above
# w = fundamental frequency (in Hz)
# Output: x = periodic signal
def fourier_series(t, fs, a, w):
    samples = int(t * fs) # Number of samples
    length = len(a) # Length of the coefficients

    x = np.zeros(samples) # Initializes the signale
    s = np.linspace(0, t, samples) # Time vector

    amps = np.abs(a) # Amplitudes
    phases = np.angle(a) # Phases

    x += amps[0] # Adds the DC component

    for i in range(1, length):
        x += 2 * amps[i] * np.cos(i * w * 2 * np.pi * s + phases[i]) # Adds the harmonics

    return x

#################################################################################################
# Part B - Initialize the parameters
#################################################################################################

samp_freq = 8000
fund_freq = 20
duration = 0.2
time = np.linspace(0, duration, int(samp_freq * duration)) # Time vector

#################################################################################################
# Part C - Sawtooth Curve
#################################################################################################

# Creates a vector of a values as shown above for N = 2, 5, 20
# Uses the function above to find the approximations
# Plots the 3 approximations
```

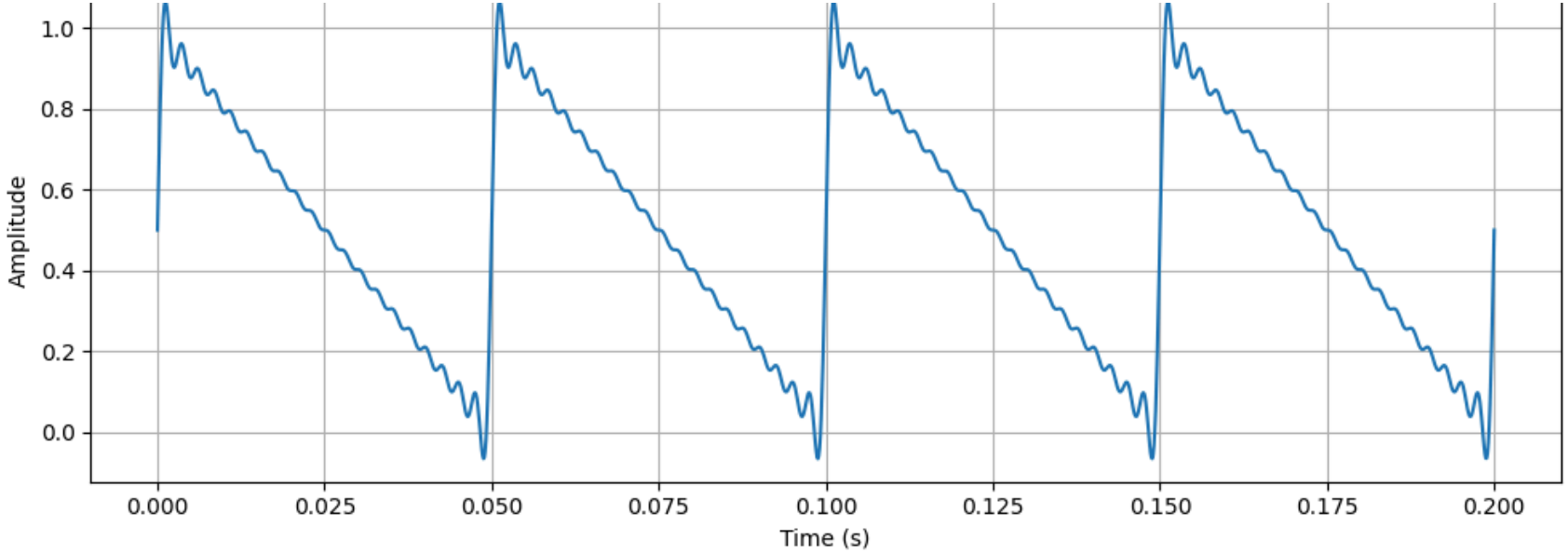
```
approximations = [2, 5, 20] # Number of harmonics to use
fig, axes = plt.subplots(3, 1, figsize=(10, 12))  # Create a 3x1 subplot layout

for index, length in enumerate(approximations):
    a = np.linspace(1, length, length)  # Initializes the coefficients
    a = 1 / (2 * 1j * a * np.pi)  # Calculates the coefficients
    a = np.concatenate((np.array([0.5]), a)) # Adds the DC component

    x = fourier_series(duration, samp_freq, a, fund_freq)  # Generates the signal

    # Formats the graphs nicely
    axes[index].plot(time, x)
    axes[index].set_title(f'Sawtooth Wave Approximation with N={length}')
    axes[index].set_xlabel('Time (s)')
    axes[index].set_ylabel('Amplitude')
    axes[index].grid()

plt.tight_layout() # Adjust layout to prevent overlap
plt.show()

##########################################################################################################
# Part D - Triangle Curve
##########################################################################################################

# Creates a vector of a values as shown above for N = 2, 5, 20
# Uses the function above to find the approximations
# Plots the 3 approximations

fig, axes = plt.subplots(3, 1, figsize=(10, 12))  # Create a 3x1 subplot layout

for index, length in enumerate(approximations):
    a = np.linspace(1, length, length)  # Initializes the coefficients
    a = (2 * np.sin(a * np.pi/2))/(1j * (a * np.pi) ** 2) * np.exp(-2j * a * np.pi/2) # Calculates coefficients
    a = np.concatenate((np.array([0.5]), a)) # Adds the DC component

    x = fourier_series(duration, samp_freq, a, fund_freq)  # Generates the signal

    # Formats the graphs nicely
    axes[index].plot(time, x)
    axes[index].set_title(f'Triangle Wave Approximation with N={length}')
    axes[index].set_xlabel('Time (s)')
    axes[index].set_ylabel('Amplitude')
```

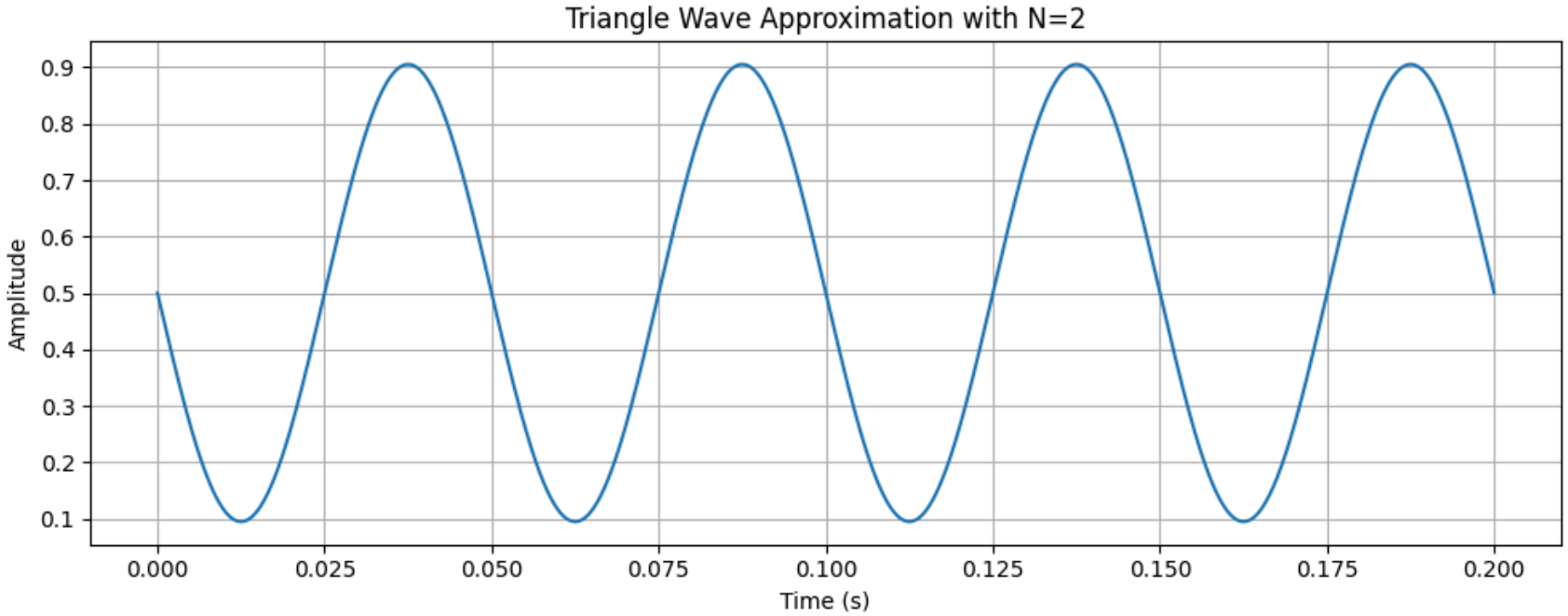
```
    axes[index].grid()

plt.tight_layout() # Adjust layout to prevent overlap
plt.show()
```

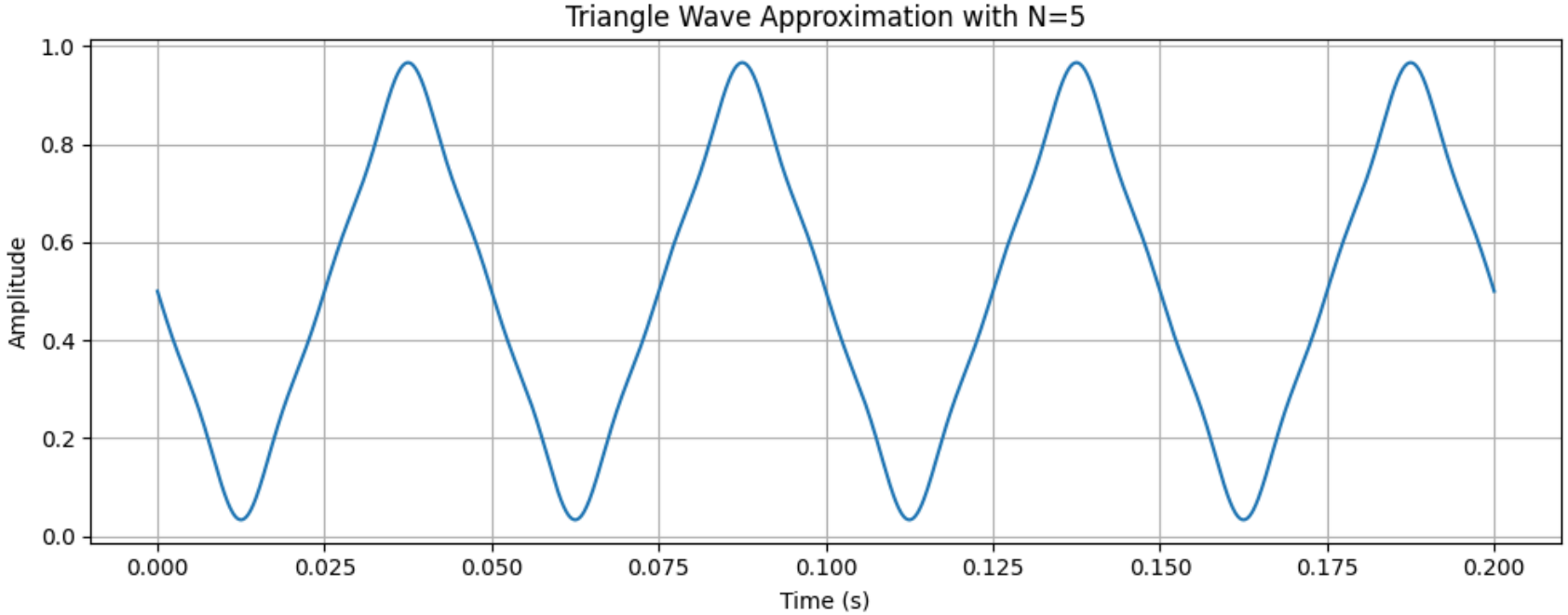

Sawtooth Wave Approximation with N=2
Amplitude
0.8
0.6
0.4
0.2
0.000
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200
Time (s)


Sawtooth Wave Approximation with N=5
Amplitude
1.0
0.8
0.6
0.4
0.2
0.0
0.000
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200
Time (s)

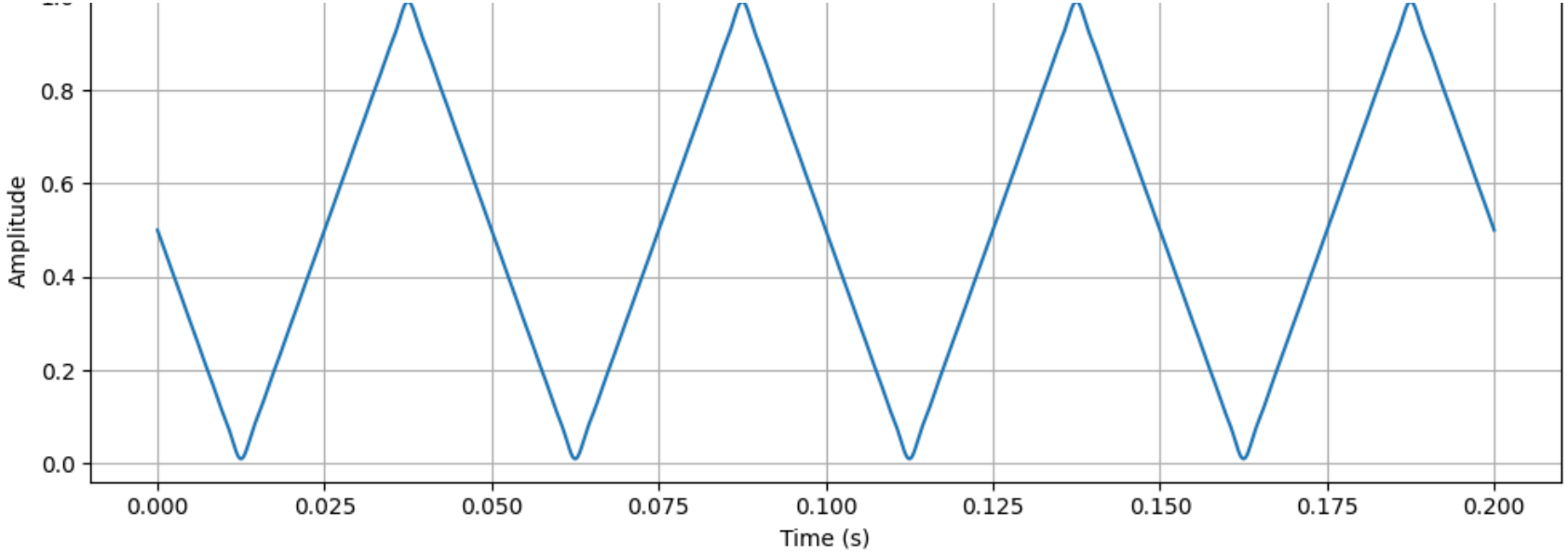

Sawtooth Wave Approximation with N=20


1.0
0.8
0.6
0.4
0.2
0.0
Amplitude
0.000
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200
Time (s)


Triangle Wave Approximation with N=2
Amplitude
Time (s)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.000
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200


Triangle Wave Approximation with N=5
Amplitude
Time (s)
1.0
0.8
0.6
0.4
0.2
0.0
0.000
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200


Triangle Wave Approximation with N=20
1.0



Discussion

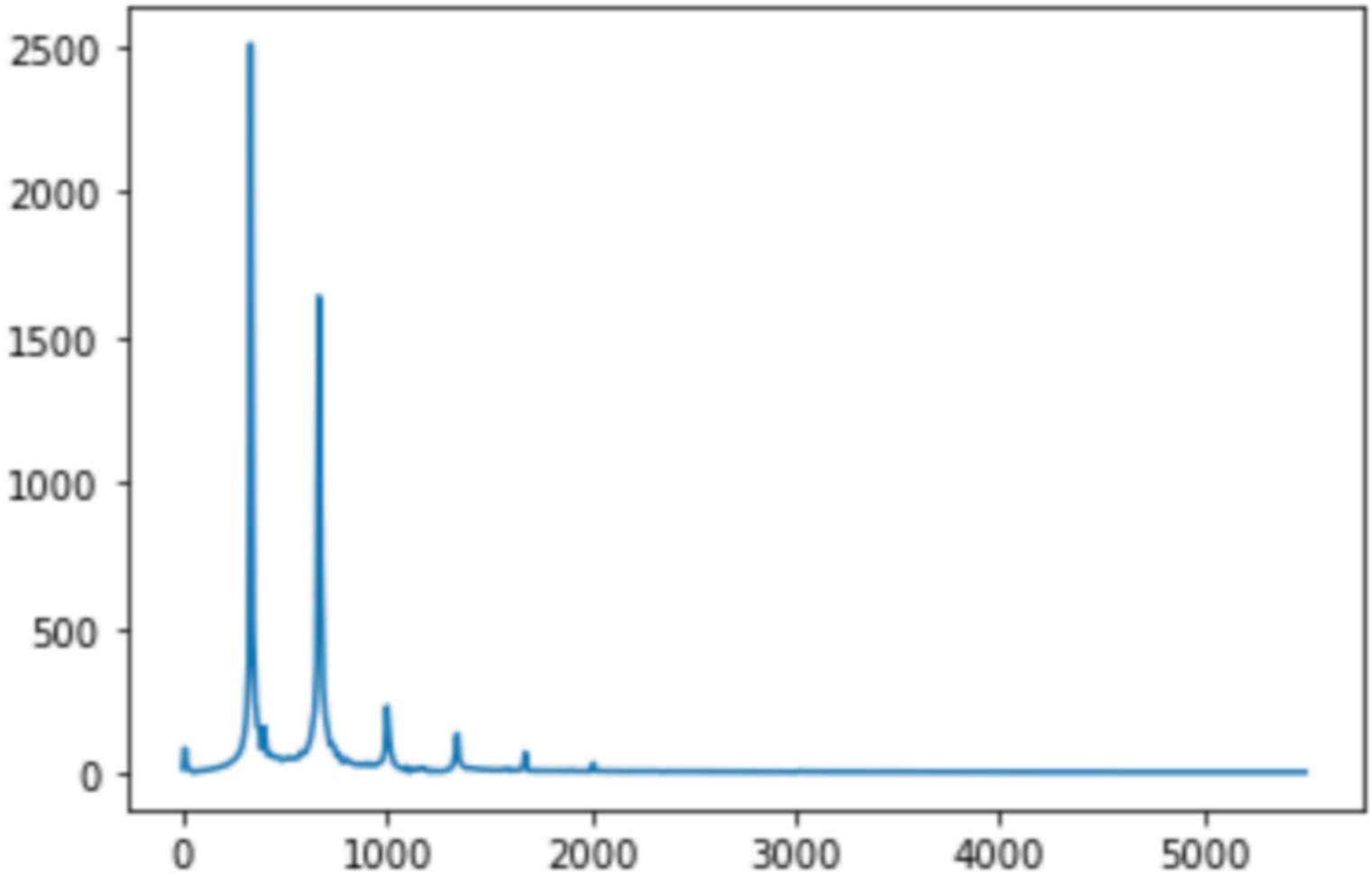
You should have noticed that the second signal converges more quickly. Discuss the two reasons for this.

- The first reason why the second signal converges more quickly is because of the **Gibbs phenomenon**. The Gibbs phenomenon describes how Fourier series for signals with discontinuous regions show oscillatory behavior at the discontinuity and converge more slowly those without them. As a result the Fourier series for the sawtooth wave converges much more slowly then the triangle wave due to the fact that the sawtooth wave contains discontinuous regions.
- The second reason why the second signal converges more quickly is because it is much more similar to a sin/cos wave. Because Fourier series represent signals using the sum of sin/cos terms, it is able to converge on signals that have shapes similar to that of sin/cos much more quickly. The triangle wave is much more similar to a cos/sin wave then a sawtooth wave - therefore the Fourier series is able to converge to its shape much more quickly. Specifically the Fourier coefficients for the triangle wave decay as 1/n^2 while the Fourier coefficients for the sawtooth wave decay at 1/n - leading to much faster convergence.

Assignment 2 -- Synthesizing a musical note

In this assignment, you will use the same synthesis equations to try to approximate a single note from a horn, which has the frequency characteristics illustrated below. Download the file horn11short.wav from the google drive to compare your synthesized version to the original.

Figure below shows the frequency component of a note played by a horn.



A. Read in the horn signal, and use the sampling rate f_s that you read in to create a time vector of length 100ms. Define the fundamental frequency to be f_0 = 335Hz. Create a signal that is a sinusoid at that frequency, and save it as a wav file.

B. Create a vector (or two) to characterize a_k using:

$$|a_k| : [2688, 1900, 316, 178, 78, 38]$$

$$\angle a_k : [-1.73, -1.45, 2.36, 2.30, -2.30, 1.13]$$

assuming $a_0 = 0$ and the first element of the vectors correspond to a_1. Use the function you created in part 1 to synthesize a signal, with f_s and f_0 above, and save it as a wav file. Because the phase and magnitude are now hard coded, you may need to modify your function from above to apply here, so it is recommended that you copy and rename the function into another cell to make your debugging easier.

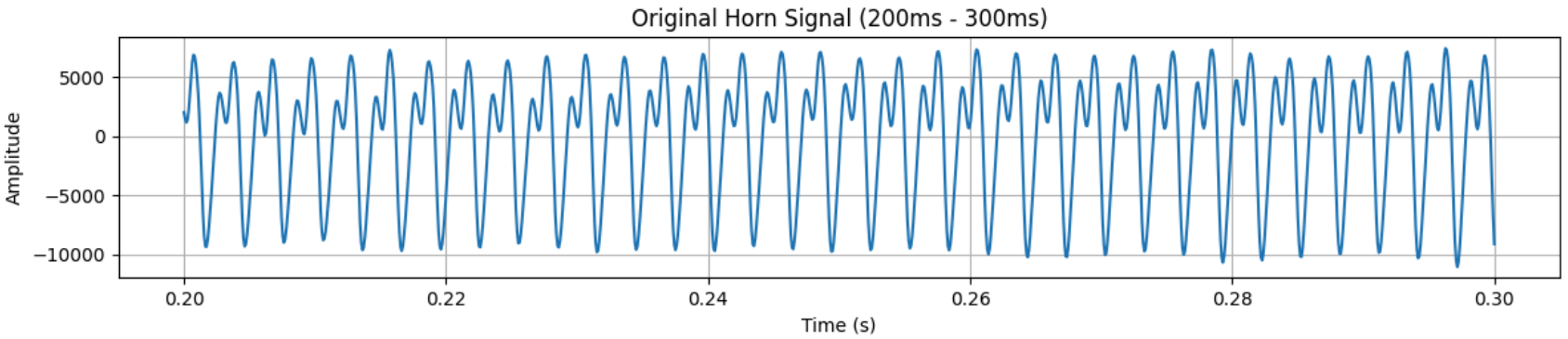
C. Plot the 100ms section of the original file starting at 200ms with a plot of the synthesized signal in a 2×1 plot.

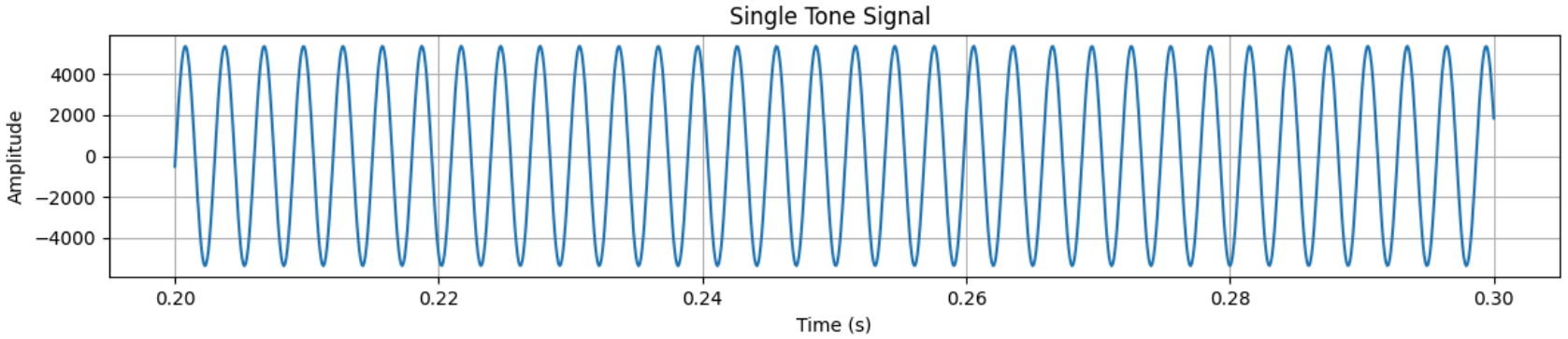
D. Play the original file, the single tone, and the 6-tone approximation in series.

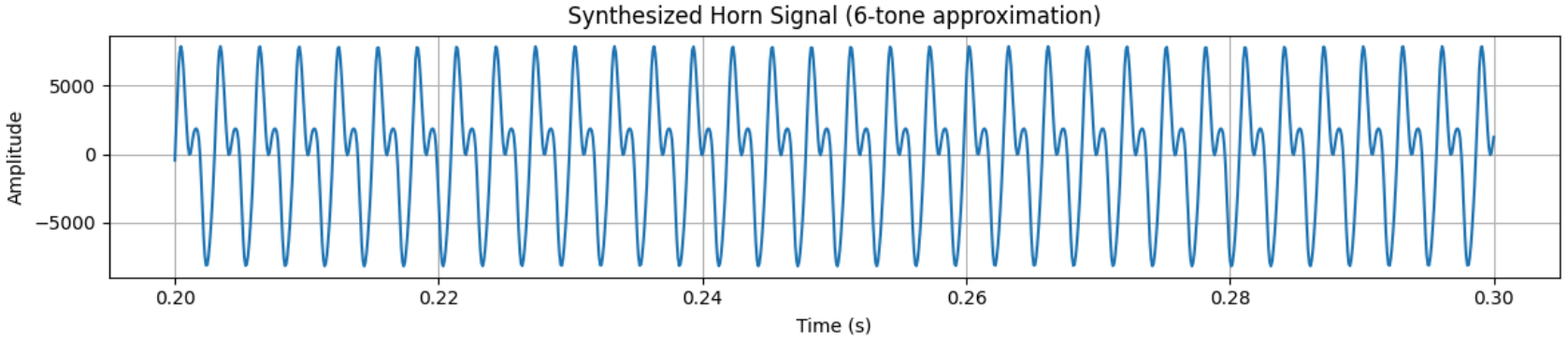
In [6]:
```
##################################################################################################
# Part A - Reading Base Signal and Creating Single Tone
##################################################################################################

# Defines the fundamental frequency (HZ)
f0 = 335

# Reads the horn signal
fs, horn_signal = wavfile.read('horn11short.wav')

# Uses function from first part to create s1 siganl
s1 = fourier_series(0.9, fs, np.array([0, 2688 * np.exp(1j * -1.73)]), f0)

# Saves the single tone as a WAV file
wavfile.write('single_tone.wav', fs, s1.astype(np.float32))

##################################################################################################
# Part B - Create Synthesized Signal Using Fourier Series
##################################################################################################

# Defines list of coeff amplitude and phase values
s2_magnitudes = np.array([0, 2688, 1900, 316, 178, 78, 38])
```

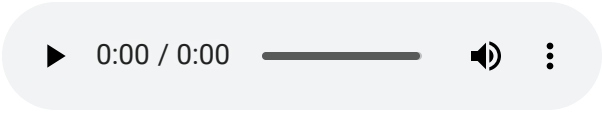
```
s2_phases = np.array([0, -1.73, -1.45, 2.36, 2.30, -2.30, 1.13])

# Converts magnitudes and phases to complex coefficients
s2_coeff = s2_magnitudes * np.exp(1j * s2_phases)

# Generates the synthesized signal using function from Assignment 1
s2 = fourier_series(0.9, fs, s2_coeff,  f0)

# Saves the synthesized signal as a WAV file
wavfile.write('synthesized_horn.wav', fs, s2.astype(np.float32))

##########################################################################################################
# Part C - Plot The Signals
##########################################################################################################

# Calculates sample indices for 100ms of audio starting at 200ms
start_sample = int(0.2 * fs)
end_sample = start_sample + int(0.1 * fs)

# Create a time vector for the plots
time = np.linspace(0.2, 0.3, int(fs * 0.1))

# Creates a 2×1 subplot
plt.figure(figsize=(12, 8))

# Plots the original signal
plt.subplot(3, 1, 1)
plt.plot(time, horn_signal[start_sample:end_sample])
plt.title('Original Horn Signal (200ms - 300ms)')
plt.xlabel('Time (s)')
plt.ylabel('Amplitude')
plt.grid(True)

# Single tone signal plot
plt.subplot(3, 1, 2)
plt.plot(time, s1[start_sample:end_sample])
plt.title("Single Tone Signal ")
plt.xlabel("Time (s)")
plt.ylabel("Amplitude")
plt.grid(True)

# Plots the synthesized signal
```

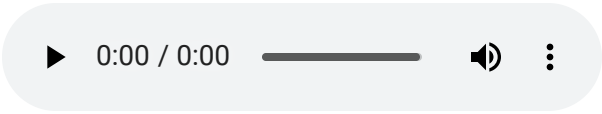
```
plt.subplot(3, 1, 3)
plt.plot(time, s2[start_sample:end_sample])
plt.title('Synthesized Horn Signal (6-tone approximation)')
plt.xlabel('Time (s)')
plt.ylabel('Amplitude')
plt.grid(True)

# Display plots
plt.tight_layout()
plt.show()

##########################################################################################################
# Part D - Play The Audio
##########################################################################################################

# Plays the audio files (original, single tone, and 6-tone approximation)
display(Audio(horn_signal, rate=fs))
display(Audio(s1, rate=fs))
display(Audio(s2, rate=fs))
```

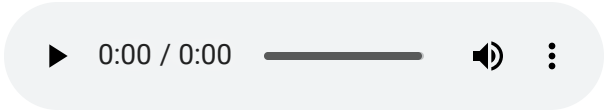

Original Horn Signal (200ms - 300ms)
Amplitude
5000
0
−5000
−10000
0.20
0.22
0.24
0.26
0.28
0.30
Time (s)


Single Tone Signal
Amplitude
4000
2000
0
−2000
−4000
0.20
0.22
0.24
0.26
0.28
0.30
Time (s)


Synthesized Horn Signal (6-tone approximation)
Amplitude
5000
0
−5000
0.20
0.22
0.24
0.26
0.28
0.30
Time (s)


0:00 / 0:00


0:00 / 0:00



Discussion

The approximation does not sound quite like the original signal and the plot should look pretty different. The difference in sound is in part due to multiple factors, including the truncated approximation, imperfect estimate of the parameters, and the fact that the original signal is not perfectly periodic. Try adjusting some parameters and determine what you think is the main source of distortion.

- The primary source of distortion in the synthesized signal is the **limited number of coefficients used**. To perfectly synthesize the original sound, it is likely that hundreds if not thousands of individual coefficients must be used to accurately produce all of the harmonics and non-periodic sounds made by the trumpet. In this case, because only 6 coefficients are used the resulting sound is significantly distorted compared to the original. We can see this even more clearly by lowering the number of coefficients used from 6 to 3 - when this is done the synthesized sound is even worse. Therefore, we can conclude that the primary cause of the distortion/inaccuracies in the output signal is due to the limited number of coefficients used for the Fourier series when synthesizing the sound. Note that given enough coefficients (infinite) the original sound could be perfectly reproduced using a Fourier series!