Homework #1

EE 242 **Yehoshua Luna**
Spring 2025 **2322458**

Problem #1

a) $3e^{j5t}$ is defined for all $t < 0$ and is therefore non-causal. We can also determine that it is neither even nor odd, since $3e^{-j5t} \neq 3e^{j5t}$ and $3e^{-j5t} \neq -3e^{j5t}$. However, $3e^{j5t}$ behaves periodically since it expands to $3\cos(5t) + j \cdot 3\sin(5t)$, and these are both periodic functions with the same frequency. Lastly, because $3e^{j5t}$ is periodic, defined for all $t \in \mathbb{R}$, and has nonzero values, it cannot be an energy signal. Integration further backs this: $\int_{-\infty}^{\infty} \left|3e^{j5t}\right|^2 dt = \infty$, but $\frac{1}{T}\int_0^T \left|3e^{j5t}\right|^2 dt = 9$. Therefore, it is a power signal.

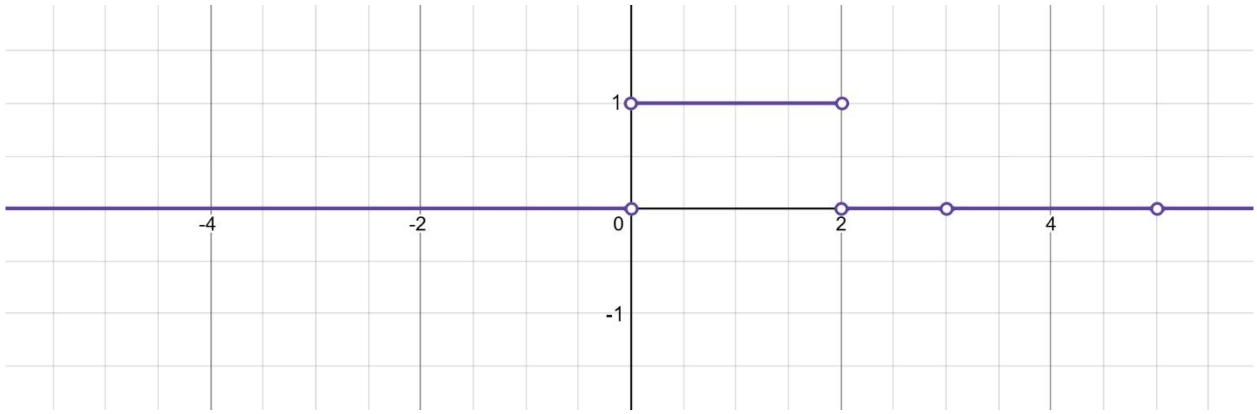
b) $t \cdot u(t)$ is defined for all $t < 0$ (though it is zero for those values) and is therefore non-causal. We also can see that it is neither even nor odd, since $-t \cdot u(-t) \neq t \cdot u(t)$ and $-t \cdot u(-t) \neq -t \cdot u(t)$. This function is not periodic because there is no value T such that $(t + kT) \cdot u(t + kT) = t \cdot u(t)$ for $k \in \mathbb{Z}$. Lastly, since $t \cdot u(t)$ is not periodic and grows unbounded for $t > 0$, it is neither an energy nor a power signal. The integration again shows this: $\int_{-\infty}^{\infty} |t \cdot u(t)|^2 dt = \infty$ and $\lim_{T\to\infty} \frac{1}{T}\int_0^T |t \cdot u(t)|^2 dt = \infty$.

Problem #2

The first term of $x(t)$ is $\cos(t)$, which we know is an even function, and the second term is $\sin(t)$, which is an odd function. The last term can be expanded to $\frac{1}{2}\sin(2t)$ using the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, which is also an odd function. $x(t)$ is a power signal because it is defined for all $t \in \mathbb{R}$, periodic, and has nonzero values. It has a fundamental period of 2π, so the power of $x(t)$ is $\frac{1}{2\pi}\int_0^{2\pi} \left|\cos(t) + \sin(t) + \frac{1}{2}\sin(2t)\right|^2 dt = 1.125$.

Problem #3

We know that $\frac{d}{dt}r(t) = 1$ for $t > 0$, $\frac{d}{dt}r(t) = 0$ for $t < 0$, and $\frac{d}{dt}u(t) = \delta(t)$. Now we are ready to make our graph of $y(t) = \frac{d}{dt}x(t)$, which can be seen below. From this graph, we can also see that $\int_{-\infty}^{\infty} y(t)\, dt = 2$. Our result is validated by the simple fact that $y(t) = \frac{d}{dt}x(t)$, which yields the following calculation: $\int_{-\infty}^{\infty} y(t)\, dt = x(\infty) - x(-\infty) = 2 - 0 = 2$.



Problem #4

From the question description, we have $x(n) = e^{j\omega_0 n T_s}$ and want to determine the condition for T_s such that $x(n)$ is periodic. Combining ω_0 and T_s into a new frequency gives $\omega_s = \omega_0 T_s$, so $x(n) = e^{j\omega_s n}$. This discrete function is only periodic when $\frac{2\pi}{\omega_s} = \frac{2\pi}{\omega_0 T_s} = \frac{T_0}{T_s}$ is rational.

Problem #5

a) From the complex Fourier Spectrum, we can see harmonics at $\omega = 3.6\pi$, $\omega = 8.4\pi$, $\omega = -3.6\pi$, and $\omega = -8.4\pi$. The largest fundamental frequency that can equal all of these values via some integer scaler is $\omega_0 = 1.2\pi$, so our fundamental period T_0 must therefore be $T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{1.2\pi} = \frac{5}{3}$.

b) All of the coefficients for the frequencies shown in the complex Fourier Spectrum are nonzero. It then follows that $c_{-7} = 3e^{-j\pi t/4}$, $c_{-3} = 4e^{j\pi t/3}$, $c_0 = 7e^{j\pi}$, $c_3 = 4e^{-j\pi t/3}$, and $c_7 = 3e^{j\pi t/4}$. We can then express $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ as the following sum: $3e^{-j\pi t/4} \cdot e^{-8.4\pi \cdot jx} + 4e^{j\pi t/3} \cdot e^{-3.6\pi \cdot jx} + 7e^{j\pi} + 4e^{-j\pi t/3} \cdot e^{3.6\pi \cdot jx} + 3e^{j\pi t/4} \cdot e^{8.4\pi \cdot jx}$. This can be further simplified to $x(t) = 8\sin\left(3.6\pi t + \frac{\pi}{6}\right) + 6\cos\left(8.4\pi t + \frac{\pi}{4}\right) - 7$.

Problem #6

We must have $\omega_0 = \frac{2\pi}{T_0} = 2\pi$ and $A_0 = A_1 = 0$ because $x(t)$ is real and odd. This means that our expression for $x(t)$ can be condensed to $x(t) = B_1 \sin(2\pi t)$, since $c_k = 0$ for all $|k > 1|$. Solving for B_1 requires some calculus: $\int_0^1 |B_1 \sin(2\pi t)|^2\, dt = 1 \rightarrow \int_0^1 (B_1 \sin(2\pi t))^2 dt = 1 \rightarrow {B_1}^2 \int_0^1 sin^2(2\pi t)\, dt = 1 \rightarrow {B_1}^2 \int_0^1 \frac{1-cos(4\pi t)}{2}\, dt = 1 \rightarrow \frac{1}{2}{B_1}^2 = 1 \rightarrow B_1 = \mp\sqrt{2}$. Therefore, our final expression for the function is $x(t) = \mp\sqrt{2}\sin(2\pi t)$.

Problem #7

It can easily be shown that $y(t) = x(t-a) + x(a-t)$ is periodic through simple algebraic expansion: $y(t+T) = x(t+T-a) + x(a-t-T) = x(t-a) + x(a-t) = y(t)$. This logic works because $x(t)$ is periodic, thus $y(t)$ is also periodic.

Finding an expression for the Fourier Series coefficients of $y(t)$ is a bit trickier. We know that $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$, therefore $x(t-a) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 (t-a)} = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} e^{-jn\omega_0 a}$ and $x(a-t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 (a-t)} = \sum_{n=-\infty}^{\infty} c_n e^{-jn\omega_0 t} e^{jn\omega_0 a}$. $y(t)$ is simply a combination of both expressions, so we can add them together, giving $y(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} e^{-jn\omega_0 a} + \sum_{n=-\infty}^{\infty} c_n e^{-jn\omega_0 t} e^{jn\omega_0 a}$. Combining and rearranging both terms then reduces the expression to a sum of coefficients $y(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}(e^{-jn\omega_0 a} + e^{jn\omega_0 a})$. These coefficients can then be simplified using $e^{j\theta} + e^{-j\theta} = 2\cos(\theta)$, thus $y(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \cdot 2\cos(n\omega_0 a)$. Now it becomes obvious that $d_n = 2c_n \cos(n\omega_0 a)$.

Problem #8

We can express the Fourier Transform function in terms of sine and cosine instead of polar form, which gives $X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot \cos(\omega t)\, dt + j\int_{-\infty}^{\infty} x(t) \cdot \sin(\omega t)\, dt$. Cosine is even, and sine is odd. Furthermore, we know that all the cosine terms of a Fourier Series or Transform are zero when operating on an odd function, and all the sine terms are zero when operating on an even function. In other words, an even function can never have an odd function in its decomposition, and an odd function never has an even function in its decomposition. So if $x(t) = x_e(t) + x_o(t)$, then we know that the Fourier Transform of $x_e(t)$ will consist of the only the cosine term, and $x_o(t)$ will consist of just the sine term. But by definition, the cosine term is real, and the sine term is imaginary! Therefore, we have:

$$X(j\omega) = \int_{-\infty}^{\infty} \big(x_e(t) + x_o(t)\big) \cdot \cos(\omega t)\, dt + j\int_{-\infty}^{\infty} \big(x_e(t) + x_o(t)\big) \cdot \sin(\omega t)\, dt$$

$$= \int_{-\infty}^{\infty} x_e(t) \cdot \cos(\omega t)\, dt + j\int_{-\infty}^{\infty} x_o(t) \cdot \sin(\omega t)\, dt = F\{x_e\} + F\{x_o\}$$

Given the previous logic, it then follows that $Re[X(\omega)] = \int_{-\infty}^{\infty} x_e(t) \cdot \cos(\omega t)\, dt = F\{x_e\}$ and $Im[X(\omega)] = j\int_{-\infty}^{\infty} x_o(t) \cdot \sin(\omega t)\, dt = F\{x_o\}$.

Problem #9

a) $F\{x(t)\} = \int_{-\infty}^{\infty} e^{-\alpha|t-1|} e^{-j\omega t}\, dt = \int_{-\infty}^{1} e^{-\alpha(1-t)} e^{-j\omega t}\, dt + \int_{1}^{\infty} e^{-\alpha(t-1)} e^{-j\omega t}\, dt$. Each integral can now be solved after splitting:

$$\int_{-\infty}^{1} e^{-\alpha(1-t)}e^{-j\omega t}\,dt = e^{-\alpha}\int_{-\infty}^{1} e^{(\alpha-j\omega)t}\,dt = e^{-\alpha}\frac{e^{(\alpha-j\omega)t}}{\alpha-j\omega}\begin{matrix} t=1 \\ t=-\infty \end{matrix} = \frac{e^{-\alpha}e^{(\alpha-j\omega)}}{\alpha-j\omega}$$

$$\int_{1}^{\infty} e^{-\alpha(t-1)}e^{-j\omega t}\,dt = e^{-\alpha}\int_{1}^{\infty} e^{(j\omega-\alpha)t}\,dt = e^{-\alpha}\frac{e^{(j\omega-\alpha)t}}{j\omega-\alpha}\begin{matrix} t=\infty \\ t=1 \end{matrix} = \frac{e^{-\alpha}e^{(j\omega-\alpha)}}{j\omega-\alpha}$$

So $X(j\omega) = e^{-\alpha}\left(\frac{e^{(\alpha-j\omega)}}{\alpha-j\omega} + \frac{e^{(j\omega-\alpha)}}{j\omega-\alpha}\right)$.

b) $F\{y(t)\} = \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(\omega_0 t)e^{-j\omega t}\,dt = \frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)e^{-j\omega t}\,dt$. We again can then just split the integral and solve:

$$\frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j\omega_0 t}e^{-j\omega t}\,dt = \frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j(\omega_0-\omega)t}\,dt = \frac{1}{2}\cdot\frac{e^{j(\omega_0-\omega)t}}{j(\omega_0-\omega)}\begin{matrix} t=\frac{T}{2} \\ t=-\frac{T}{2} \end{matrix}$$

$$\frac{1}{2}\cdot\frac{e^{j(\omega_0-\omega)T/2}-e^{-j(\omega_0-\omega)T/2}}{j(\omega_0-\omega)} = \frac{\sin\left(\frac{T}{2}(\omega_0-\omega)\right)}{\omega_0-\omega} = \frac{T}{2}\cdot sinc\left(\frac{T}{2}(\omega_0-\omega)\right)$$

$$\frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\omega_0 t}e^{-j\omega t}\,dt = \frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j(-\omega_0-\omega)t}\,dt = \frac{1}{2}\cdot\frac{e^{j(-\omega_0-\omega)t}}{j(-\omega_0-\omega)}\begin{matrix} t=\frac{T}{2} \\ t=-\frac{T}{2} \end{matrix}$$

$$\frac{1}{2}\cdot\frac{e^{j(-\omega_0-\omega)T/2}-e^{-j(-\omega_0-\omega)T/2}}{j(-\omega_0-\omega)} = \frac{\sin\left(\frac{T}{2}(-\omega_0-\omega)\right)}{-\omega_0-\omega} = \frac{T}{2}\cdot sinc\left(\frac{T}{2}(-\omega_0-\omega)\right)$$

So $X(j\omega) = \frac{T}{2}\left(sinc\left(\frac{T}{2}(\omega_0-\omega)\right) + sinc\left(\frac{T}{2}(-\omega_0-\omega)\right)\right)$.

Problem #10

a) $x(t) = \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} 2\pi\delta(\omega)e^{j\omega t}d\omega + \int_{-\infty}^{\infty}\pi\delta(\omega-4\pi)e^{j\omega t}d\omega + \int_{-\infty}^{\infty}\pi\delta(\omega+4\pi)e^{j\omega t}d\omega\right)$

$$= \frac{1}{2\pi}\left(2\pi e^0 + \pi e^{j4\pi t} + \pi e^{-j4\pi t}\right) = 1 + \frac{1}{2}e^{j4\pi t} + \frac{1}{2}e^{-j4\pi t} = 1 + \cos(4\pi t)$$

b) $\cos(2\omega)\cdot\sin(\omega) = \frac{1}{2}(\sin(3\omega)-\sin(\omega)) = \frac{1}{4j}\left(e^{j3\omega}-e^{-j3\omega}-e^{j\omega}+e^{-j\omega}\right)$. Taking the inverse Fourier Transform then gives:

$$x(t) = \frac{1}{4j}(\delta(t+3)-\delta(t-3)-\delta(t+1)+\delta(t-1))$$

Problem #11

a) $sgn(t) = 2u(t)-1$, so $F\{sgn(t)\} = F\{2u(t)-1\} = F\{2u(t)\} - F\{1\} = 2\pi\cdot\delta(\omega) + \frac{2}{j\omega} - 2\pi\cdot\delta(\omega)$. This then gives $X(j\omega) = \frac{2}{j\omega}$.

b) $X(j\omega) = \frac{2}{j\omega}$, and $u(t) = \frac{sgn(t)+1}{2}$, so $F\{u(t)\} = F\left\{\frac{sgn(t)+1}{2}\right\} = \pi\cdot\delta(\omega) + \frac{1}{j\omega}$.