Laboratory activity 4: Longitudinal state–space control of the balancing robot
(Theoretical background)

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1 Activity goal

The purpose of this laboratory activity is to design and test a longitudinal state–space controller for the two–wheeled balancing robot (also referred as "two–wheeled inverted pendulum robot", or "Segway–like robot") available in laboratory. The controller is designed to simultaneously stabilize the robot body to its upward vertical position, and the robot base to a desired longitudinal position set-point. The design is performed by resorting to a simplified model of the robot dynamics, obtained by assuming that the motion occurs along a straight line (i.e. the lateral or heading–angle dynamics is ignored).

There is no challenge for this LAB.

2 Analytical model of the balancing robot

A mathematical model for the longitudinal dynamics of the balancing robot is derived in this section. The model is valid under the simplifying assumption that the robot moves along a straight line, namely no lateral motion occurs (i.e. the heading–angle is constant).

2.1 Mechanical model

The two–wheeled balancing robot can be represented as a multi–body system that comprises the following rigid bodies (see also Fig. 1):

↳ **Wheels** (left/right).

↳ **DC gearmotor rotors** (left/right).

↳ **Robot body**. It comprises the **robot chassis** (includes all the electronic boards and the motor supporting brackets), the **DC gearmotor stators** (left/right) and the **battery**.

The following reference frames are introduced to describe the robot structure and configuration (see also Fig. 1÷4):

↳ **Body frame**. The frame is located on the wheels rotation axis, and is centered in the midpoint between the two wheels. The frame is rigidly attached with the robot chassis. Its y–axis is directed along the wheel rotation axis, and points toward the left wheel. The z–axis passes

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through the body *Center–of–Mass* (CoM), and points toward it. The x–axis direction and orientation is determined by the right–hand rule. The body frame is denoted with $\mathcal{F}_b$.

↳ **Vehicle frame**. The frame center and y–axis are the same as the body frame. However, the frame is not rigidly attached with the robot body; instead, its z–axis is always oriented oppositely to the acceleration of gravity. The x–axis direction and orientation is determined by the right–hand rule. The vehicle frame is denoted with $\mathcal{F}_v$.

↳ **World (or earth) frame**. Its pose (position and orientation) is ground–fixed, and coincides with the initial pose of the vehicle frame. The world frame is denoted with $\mathcal{F}_o$.

↳ **Wheel frames** (left/right). Each frame is rigidly attached with the wheel body, and centered on its CoM. The axes are initially aligned with those of the vehicle frame. The left and right wheel frames are denoted with, respectively, $\mathcal{F}_{w,l}$ and $\mathcal{F}_{w,r}$.

↳ **Rotor frames** (left/right). Each frame is rigidly attached with the rotor body, and centered on its CoM. The axes are initially aligned with those of the vehicle frame. The left and right rotor frames are denoted with, respectively, $\mathcal{F}_{rot,l}$ and $\mathcal{F}_{rot,r}$.

The geometrical and inertial parameters of each body (and its possible subparts) are provided in Tab. 1. Note that:

- The masses of the DC gearmotor rotor and stator are estimated as, respectively, the 35% and 65% of the whole motor mass (equal to $m_{mot} = 215\,\mathrm{g}$).

- The total mass of the robot body is computed as the sum of the chassis, battery and motor stators masses, namely

$$m_b \;=\; m_c \;+\; m_{batt} \;+\; 2\,m_{stat} \tag{1}$$

- The stator CoM position has been determined after noting (experimentally) that the motor CoM is displaced by $6\,\mathrm{mm}$ off the geometrical center, in the direction of motor gearbox.

- The body CoM z–axis coordinate z_b^b in the body frame is determined as follows

$$z_b^b \;=\; \frac{1}{m_b}\left(m_c\, z_c^b \;+\; m_{batt}\, z_{batt}^b \;+\; 2\,m_{stat}\, z_{stat}^b\right) \tag{2}$$

  The other two coordinates, are equal to zero because of geometrical symmetry reasons.

- The *Moments–of–Inertia* (MoI) of both the robot chassis and battery are determined by assuming that the bodies are solid parallelepipeds with uniform mass distribution. They are computed with respect to a frame that is centered on the corresponding body CoM, and has axes aligned with, respectively, the parallelepiped width, height and depth directions, namely:

$$I_{xx} = \frac{m}{12}\left(w^2 + h^2\right)\,, \qquad I_{yy} = \frac{m}{12}\left(d^2 + h^2\right)\,, \qquad I_{zz} = \frac{m}{12}\left(w^2 + d^2\right) \tag{3}$$

  where w, h and d are the parallelepiped dimensions (width, height and depth), and m the body mass.

- The MoI of both the wheel and DC gearmotor rotor/stator are determined by assuming that the bodies are solid cylinders with uniform mass distribution. They are computed with respect to a

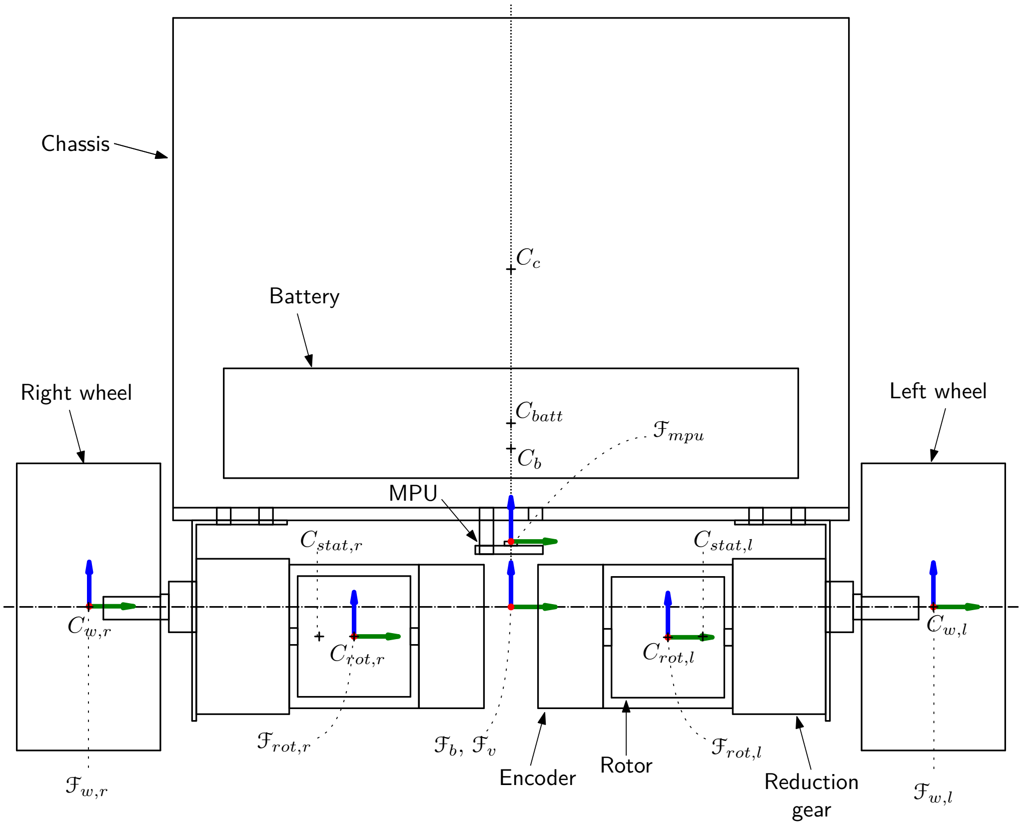
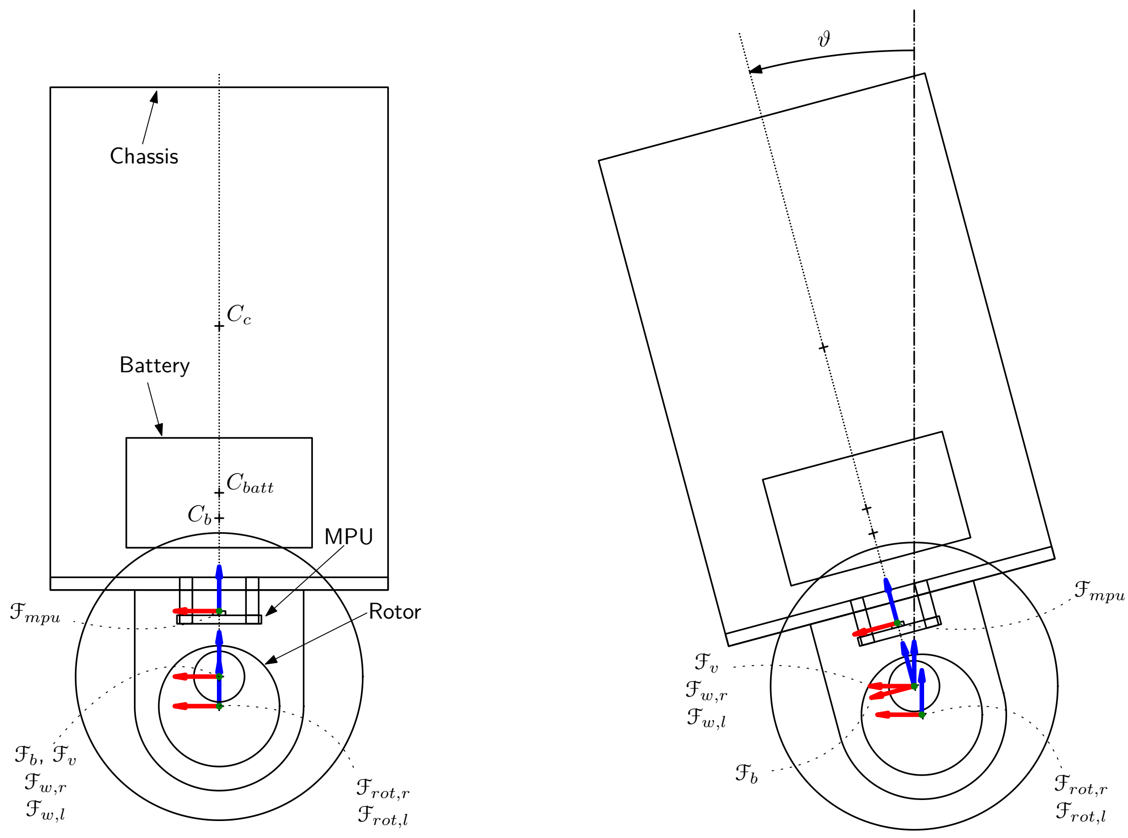


Figure 1: Front view.



(a) Straight position.

(b) Tilted position.

Figure 2: Side view.

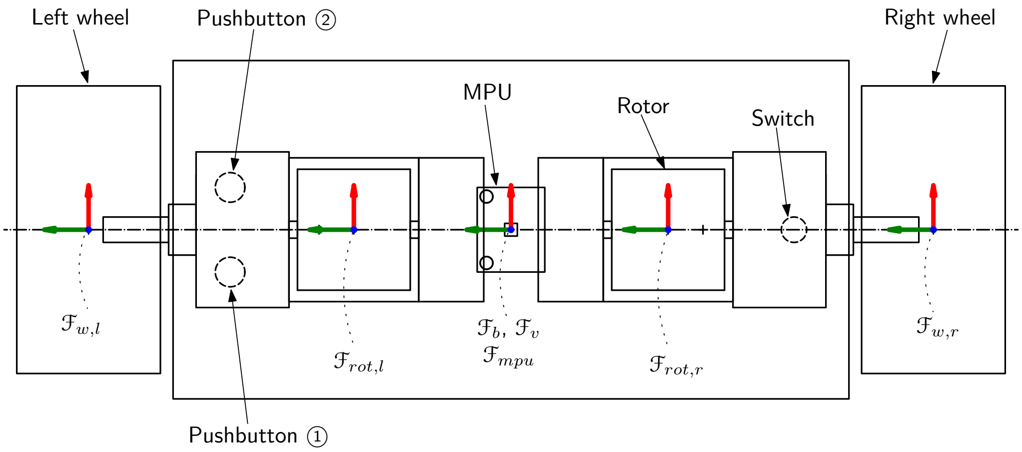


Figure 3: Top view.

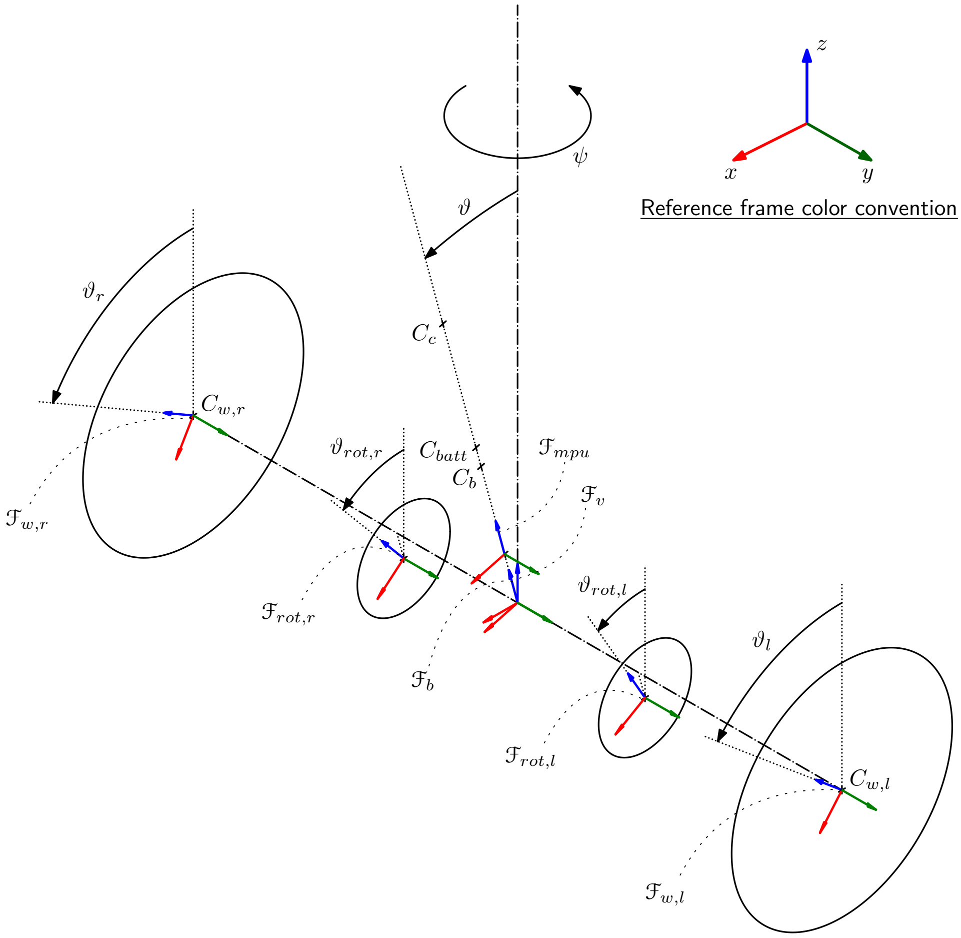


Figure 4: Simplified 3D view.

• **Robot body**			
Center–of–Mass coords wrt body frame $\mathcal{F}_b$	$x_b^b,\ y_b^b,\ z_b^b$	$0,\ 0,\ 46.05$	[mm]
Mass	m_b	1.06	[kg]
Principal Moments–of–Inertia	$I_{b,xx},\ I_{b,yy},\ I_{b,zz}$	$4.22,\ 2.20,\ 2.65$	[gm^2]
↳ **Robot chassis**			
Dimensions (width, height, depth)	$w_c,\ h_c,\ d_c$	$160,\ 119,\ 80$	[mm]
Center–of–Mass coords wrt body frame $\mathcal{F}_b$	$x_c^b,\ y_c^b,\ z_c^b$	$0,\ 0,\ 80$	[mm]
Mass	m_c	456	[g]
Principal Moments–of–Inertia	$I_{c,xx},\ I_{c,yy},\ I_{c,zz}$	$1.5,\ 0.78,\ 1.2$	[gm^2]
↳ **Battery**			
Dimensions (width, height, depth)	$w_{batt},\ h_{batt},\ d_{batt}$	$136,\ 26,\ 44$	[mm]
Center–of–Mass coords wrt body frame $\mathcal{F}_b$	$x_{batt}^b,\ y_{batt}^b,\ z_{batt}^b$	$0,\ 0,\ 44$	[mm]
Mass	m_{batt}	320	[g]
Principal Moments–of–Inertia	$I_{batt,xx},\ I_{batt,yy},\ I_{batt,zz}$	$0.51,\ 0.07,\ 0.06$	[gm^2]
↳ **DC gearmotor stator**			
Dimensions (height, radius)	$h_{stat},\ r_{stat}$	$68.1,\ 17$	[mm]
Center–of–Mass coords wrt body frame $\mathcal{F}_b$	$x_{stat}^b,\ y_{stat}^b,\ z_{stat}^b$	$0,\ \pm 52.1,\ -7$	[mm]
Mass	m_{stat}	139.75	[g]
Principal Moments–of–Inertia	$I_{stat,xx} = I_{stat,zz},\ I_{stat,yy}$	$0.064,\ 0.02$	[gm^2]

• **DC gearmotor rotor**			
Dimensions (height, radius)	$h_{rot},\ r_{rot}$	$30.7,\ 15.3$	[mm]
Center–of–Mass coords wrt body frame $\mathcal{F}_b$	$x_{rot}^b,\ y_{rot}^b,\ z_{rot}^b$	$0,\ \pm 42.7,\ -7$	[mm]
Mass	m_{rot}	75.25	[g]
Principal Moments–of–Inertia	$I_{rot,xx} = I_{rot,zz},\ I_{rot,yy}$	$0.01,\ 0.009$	[gm^2]

• **Wheels**			
Dimensions (height, radius)	$h_w,\ r_w$	$26,\ 34$	[mm]
Center–of–Mass coords wrt body frame $\mathcal{F}_b$	$x_w^b,\ y_w^b,\ z_w^b$	$0,\ \pm 100,\ 0$	[mm]
Mass	m_w	50	[g]
Principal Moments–of–Inertia	$I_{w,xx} = I_{w,zz},\ I_{w,yy}$	$0.017,\ 0.029$	[gm^2]

Table 1: Geometrical and inertial nominal parameters.

frame that is centered on the corresponding body CoM, and has axes aligned with, respectively, the cylinder height and radial directions, namely:

$$I_{xx} = I_{zz} = \frac{m}{12}\left(3r^2 + h^2\right), \qquad I_{yy} = \frac{m\,r^2}{2} \tag{4}$$

where h and r are the cylinder dimensions (height and radius), and m the body mass.

- The MoI of the robot body are determined with respect to a reference frame ($\mathcal{F}_b'$) centered on the robot body CoM, and aligned with the body frame $\mathcal{F}_b$. They are computed by resorting to the *Huygens–Steiner theorem* (or *parallel axis theorem*), which states that the moment of inertia I' of a body rotating about an axis z' displaced from the body CoM by a distance d is equal to $I' = I + md^2$, where I is the body moment of inertia with respect to an axis z parallel to z' and passing through the body CoM. It holds that:

$$\begin{aligned}
I_{b,xx} = I_{c,xx} \;&+\; m_c\,(z_b^b - z_c^b)^2 \;+\; I_{batt,xx} \;+\; m_{batt}\,(z_b^b - z_{batt}^b)^2 \;+\; \cdots \\
&\cdots \;+\; I_{stat,xx} \;+\; m_{stat}\left[(y_b^b - y_{stat,r}^b)^2 \;+\; (z_b^b - z_{stat,r}^b)^2\right] \;+\; \cdots \\
&\cdots \;+\; I_{stat,xx} \;+\; m_{stat}\left[(y_b^b - y_{stat,l}^b)^2 + (z_b^b - z_{stat,l}^b)^2\right] \\
I_{b,yy} = I_{c,yy} \;&+\; m_c\,(z_b^b - z_c^b)^2 \;+\; I_{batt,yy} \;+\; m_{batt}(z_b^b - z_{batt}^b)^2 \;+\; \cdots \\
&\cdots \;+\; 2\,I_{stat,yy} \;+\; 2\,m_{stat}\,(z_b^b - z_{stat}^b)^2 \\
I_{b,zz} = I_{c,zz} \;&+\; I_{batt,zz} \;+\; 2\,I_{stat,zz} \;+\; 2m_{stat}\,(y_{stat}^b)^2
\end{aligned} \tag{5}$$

To describe the robot configuration, the following set of **generalized coordinates** is introduced (see also Fig. 4):

↳ **robot position** $\boldsymbol{p}_v^o = (x_v,\, y_v,\, z_v)$: coordinates of the vehicle frame origin with respect to the world frame.

↳ **robot tilt angle** ϑ : pitch angle of the body frame with respect to the vehicle frame.

↳ **wheels angles** ϑ_l and ϑ_r : pitch angle of the wheel frame with respect to the vehicle frame.

Some additional coordinates are introduced to ease the derivation of the dynamical model:

↳ **robot heading angle** ψ : yaw angle of the vehicle frame with respect to the world frame.

↳ **rotor angles** $\vartheta_{rot,l}$ and $\vartheta_{rot,r}$: pitch angle of the rotor frames with respect to the vehicle frame.

However, these latter coordinates are not independent variables. In fact, they are directly related to the previously defined generalized coordinates because of the presence of the following **kinematic constraints**:

- **gearbox mechanical coupling**: let

$$\Delta\vartheta_l \;=\; \vartheta_l \;-\; \vartheta\,, \qquad \Delta\vartheta_{rot,l} \;=\; \vartheta_{rot,l} \;-\; \vartheta \tag{6}$$

denote the angular displacements of the left wheel and rotor with respect to the body frame. Since the motor stator is rigidly attached to the body frame, then $\Delta\vartheta_{rot,l}$ and $\Delta\vartheta_l$ are the angles by which the rotor and the output shaft of the gearmotor rotate with respect to the stator. The mechanical coupling due to the gearbox imposes that

$$\Delta\vartheta_{rot,l} \,=\, N\,\Delta\vartheta_l \tag{7}$$

where N denotes the gearbox ratio. By replacing (7) within (6), it follows that:

$$\vartheta_{rot,l} \,=\, \vartheta \,+\, N\,(\vartheta_l \,-\, \vartheta) \tag{8}$$

A similar expression holds for the right rotor angle $\vartheta_{rot,r}$.

- **pure rolling** and **no side–slip wheel constraints**: it is assumed that the velocity of the center of the wheel is parallel to the wheel sagittal plane (*no side–slip condition*) and is proportional to the wheel rotation velocity (*pure rolling condition*). It is immediate to verify that these conditions imply that:

$$v_l \,=\, r\,\dot{\vartheta}_l\,, \qquad v_r \,=\, r\,\dot{\vartheta}_r\,, \qquad w\,\dot{\psi} \,=\, v_r - v_l \tag{9}$$

  where $w = 2|y_w^b|$ is the distance between the two wheel CoM, r is the wheel radius, and v_l and v_r the left and right wheel velocities (parallel to the wheel sagittal plane, and hence orthogonal to the wheels axle). From (9) it follows that:

$$\dot{\psi} \,=\, \frac{r}{w}\left(\dot{\vartheta}_r - \dot{\vartheta}_l\right) \qquad \Rightarrow \qquad \psi \,=\, \psi(0) \,+\, \frac{r}{w}\,(\vartheta_r - \vartheta_l) \tag{10}$$

  where $\psi(0) = 0$ because the vehicle and world frame are aligned at $t = 0$.

In this handout, the analysis is restricted to the longitudinal dynamics only, provided that the robot moves along a straight line, i.e. the heading angle ψ is a constant. From (10), this is equivalent to state that the two wheels angles ϑ_l and ϑ_r are always identical: in the following, γ will be used to denote the common value of the two wheels angles. Without loss of generality, it can be assumed that $\psi = 0$, so that $y_v = 0$ and the motion remains confined on the xz–plane of the world frame $\mathcal{F}_o$. On such plane, the robot dynamics is equivalent to that of a planar robot with a single wheel and a single motor. With respect to the two–wheeled configuration, both the wheel/rotor inertial parameters and the motor torque are doubled (note that in order to keep a straight motion, it is required that the left and right motor torques are always identical). This **planar model** will be adopted in the following for the derivation of the *Equations–of–Motions* (EoMs) of the longitudinal dynamics. To further simplify the analysis, it will be assumed that the robot moves on a horizontal flat surface (i.e. the gravity vector is perpendicular to the ground surface), so that $z_v = 0$ and, because of the pure rolling condition imposed to the wheels, $x_v = r\gamma$.

The Equations–of–Motions are derived by using a **Lagrangian approach**. This requires first to obtain the kinematics equations for each body in the planar multi–body system. For such purpose, the following notation is introduced:

- $\boldsymbol{p}_a^b \,=\, \left[x_a^b,\, z_a^b\right]^T$: position vector of point a, expressed with respect to frame $\mathcal{F}_b$. If $\mathcal{F}_b$ is the world frame $\mathcal{F}_o$, then the superscripts in the position coordinates will be omitted.

- $\boldsymbol{r}_{a,b}^c \,=\, \left[x_{a,b}^c,\, z_{a,b}^c\right]^T$: displacement vector from point a to point b, expressed with respect to frame $\mathcal{F}_c$. It holds that:

$$\boldsymbol{r}_{a,b}^c \,=\, \boldsymbol{p}_b^c \,-\, \boldsymbol{p}_a^c \tag{11}$$

- $\boldsymbol{R}_a^b$: rotation matrix of frame $\mathcal{F}_a$ with respect to frame $\mathcal{F}_b$. If ϑ is the angle by which the frame $\mathcal{F}_a$ is rotated with respect to frame $\mathcal{F}_b$, then the rotation matrix is equal to

$$\boldsymbol{R}_a^b = \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix} \tag{12}$$

The derivation of the kinematics equations for each body in the planar multi–body system is reported below:

↳ **Robot body kinematics**. Let $\boldsymbol{p}_v^o = [r\,\gamma,\, 0]^T$ be the position of the vehicle frame with respect to world frame, $\boldsymbol{p}_{C_b}^b = [0,\, l]^T$ with $l \triangleq z_b^b$ the position of the robot body CoM with respect to the body frame, and

$$\boldsymbol{R}_b^o = \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix} \tag{13}$$

the rotation matrix of body frame with respect to world frame. Then, the position vector $\boldsymbol{p}_{C_b}^o = [x_{C_b},\, z_{C_b}]^T$ of the robot body CoM C_b with respect to the world frame is equal to:

$$\boldsymbol{p}_{C_b}^o \;=\; \boldsymbol{p}_v^o + \boldsymbol{r}_{v,C_b}^o \;=\; \boldsymbol{p}_v^o + \boldsymbol{R}_b^o\,\boldsymbol{p}_{C_b}^b \;=\; \begin{bmatrix} r\,\gamma + l\,\sin\vartheta \\ l\,\cos\vartheta \end{bmatrix} \tag{14}$$

The linear velocity vector $\dot{\boldsymbol{p}}_{C_b}^o = [\dot{x}_{C_b},\, \dot{z}_{C_b}]^T$ of the robot body CoM with respect to world frame is therefore equal to:

$$\dot{\boldsymbol{p}}_{C_b}^o \;=\; \dot{\boldsymbol{p}}_v^o + \frac{d\boldsymbol{R}_b^o}{dt}\,\boldsymbol{p}_{C_b}^b + \boldsymbol{R}_b^o\,\dot{\boldsymbol{p}}_{C_b}^b \;=\; \begin{bmatrix} r\,\dot{\gamma} + l\,\cos\vartheta\,\dot{\vartheta} \\ -l\,\sin\vartheta\,\dot{\vartheta} \end{bmatrix} \tag{15}$$

↳ **Wheel kinematics**. The position vector $\boldsymbol{p}_{C_l}^o = [x_{C_w},\, z_{C_w}]^T$ of the single–wheel CoM C_w with respect to the world frame is equal to

$$\boldsymbol{p}_{C_w}^o \;=\; \boldsymbol{p}_v^o + \boldsymbol{r}_{v,C_w}^o \;=\; \boldsymbol{p}_v^o \;=\; \begin{bmatrix} r\,\gamma \\ 0 \end{bmatrix} \tag{16}$$

since $\boldsymbol{r}_{v,C_w}^o = [0,\, 0]^T$. The linear velocity vector $\dot{\boldsymbol{p}}_{C_w}^o = [\dot{x}_{C_w},\, \dot{z}_{C_w}]^T$ is therefore equal to

$$\dot{\boldsymbol{p}}_{C_w}^o \;=\; \dot{\boldsymbol{p}}_v^o \;=\; \begin{bmatrix} r\,\dot{\gamma} \\ 0 \end{bmatrix} \tag{17}$$

↳ **Rotor kinematics**. Let $\boldsymbol{p}_{C_{rot}}^b = [0,\, z_{rot}^b]^T$ be the position of the single–rotor CoM C_{rot} with respect to the body frame. Then, the position vector $\boldsymbol{p}_{C_{rot}}^o = [x_{C_{rot}},\, z_{C_{rot}}]^T$ of the same CoM with respect to world frame is equal to:

$$\boldsymbol{p}_{C_{rot}}^o \;=\; \boldsymbol{p}_v^o + \boldsymbol{r}_{v,C_{rot}}^o \;=\; \boldsymbol{p}_v^o + \boldsymbol{R}_b^o\,\boldsymbol{p}_{C_{rot}}^b \;=\; \begin{bmatrix} r\,\gamma + z_{rot}^b\,\sin\vartheta \\ z_{rot}^b\,\cos\vartheta \end{bmatrix} \tag{18}$$

The linear velocity vector $\dot{\boldsymbol{p}}_{C_{rot}}^o = [\dot{x}_{C_{rot}}^o,\, \dot{z}_{C_{rot}}^o]^T$ of the rotor CoM with respect to world frame is therefore equal to:

$$\dot{\boldsymbol{p}}^o_{C_{rot}} \;=\; \dot{\boldsymbol{p}}^o_v \,+\, \frac{d\boldsymbol{R}^o_b}{dt}\,\boldsymbol{p}^b_{C_{rot}} \,+\, \boldsymbol{R}^o_b\,\dot{\boldsymbol{p}}^b_{C_{rot}} \;=\; \begin{bmatrix} r\,\dot\gamma \,+\, z^b_{rot}\,\cos\vartheta\,\dot\vartheta \\ -\,z^b_{rot}\,\sin\vartheta\,\dot\vartheta \end{bmatrix} \tag{19}$$

For the application of the Lagrangian approach, it is necessary to compute the *Lagrangian function*:

$$\mathcal{L} \;=\; T \,-\, U \tag{20}$$

which depends on the kinetic energy T and potential energy U of the whole multi–body system. These are in turn equal to the sum of the kinetic and potential energies of every single body composing the planar multi–body system, namely

$$T \;=\; T_b \,+\, T_w \,+\, T_{rot} \qquad \text{and} \qquad U \;=\; U_b \,+\, U_w \,+\, U_{rot} \tag{21}$$

The derivation of every single kinetic and potential energy contributions is reported below:

↳ **Robot body kinetic and potential energies**. The kinetic energy of the robot body is equal to:

$$\begin{aligned} T_b \;&=\; \frac{1}{2}\,m_b\left(\dot{\boldsymbol{p}}^o_{C_b}\right)^T\left(\dot{\boldsymbol{p}}^o_{C_b}\right) \,+\, \frac{1}{2}\,I_{b,yy}\,\dot\vartheta^2 \\ &=\; \frac{1}{2}\,m_b\,r^2\,\dot\gamma^2 \,+\, \frac{1}{2}\left(I_{b,yy} + m_b\,l^2\right)\dot\vartheta^2 \,+\, m_b\,r\,l\cos\vartheta\,\dot\gamma\,\dot\vartheta \end{aligned} \tag{22}$$

where both the translational kinetic energy of the CoM and the rotational kinetic energy of the body with respect to its CoM have been taken into account. The potential energy is equal to:

$$U_b \;=\; m_b\,g\,z_{C_b} \;=\; m_b\,g\,l\cos\vartheta \tag{23}$$

↳ **Wheel kinetic and potential energies**. The kinetic energy of the single–wheel is equal to:

$$\begin{aligned} T_w \;&=\; \frac{1}{2}\,(2m_w)\left(\dot{\boldsymbol{p}}^o_{C_w}\right)^T\left(\dot{\boldsymbol{p}}^o_{C_w}\right) \,+\, \frac{1}{2}\,(2I_{w,yy})\,\dot\gamma^2 \\ &=\; \left(I_{w,yy} + m_w r^2\right)\dot\gamma^2 \end{aligned} \tag{24}$$

The potential energy is instead equal to $U_w = (2m_w)\,g\,z_{C_w} = 0$.

↳ **Rotor kinetic and potential energies**. Let $\vartheta_{rot} = \vartheta + N\,(\gamma - \vartheta)$ denote the angular position of the single–rotor. Then, the kinetic energy is equal to:

$$\begin{aligned} T_{rot} \;&=\; \frac{1}{2}\,(2m_{rot})\left(\dot{\boldsymbol{p}}^o_{C_{rot}}\right)^T\left(\dot{\boldsymbol{p}}^o_{C_{rot}}\right) \,+\, \frac{1}{2}\,(2I_{rot,yy})\,\dot\vartheta^2_{rot} \\ &=\; \left(N^2 I_{rot,yy} + m_{rot}\,r^2\right)\dot\gamma^2 \,+\, \Big[(1-N)^2 I_{rot,yy} + m_{rot}(z^b_{rot})^2\Big]\,\dot\vartheta^2 \,+\, \cdots \\ &\quad \cdots + 2\Big[N(1-N) I_{rot,yy} \,+\, m_{rot}\,r\,z^b_{rot}\cos\vartheta\Big]\,\dot\gamma\,\dot\vartheta \end{aligned} \tag{25}$$

The potential energy is equal to:

$$U_{rot} \;=\; (2m_{rot})\,g\,z_{C_{rot}} \;=\; (2m_{rot})\,g\,z^b_{rot}\cos\vartheta \tag{26}$$

Gearbox viscous friction coefficient (at output shaft)	B	$25 \times 10^{-3}\,\mathrm{Nm/(rad/s)}$
Wheel viscous friction coefficient	B_w	$1.5 \times 10^{-3}\,\mathrm{Nm/(rad/s)}$

Table 2: Viscous friction nominal parameters.

The Lagrangian function is therefore equal to:

$$
\begin{aligned}
\mathcal{L} \;=\; & \left[I_{w,yy} \,+\, N^2\, I_{rot,yy} \,+\, \left(\frac{1}{2}\,m_b \,+\, m_w \,+\, m_{rot}\right) r^2\right] \dot{\gamma}^2 \,+\, \cdots \\
& \cdots + \left[\frac{1}{2}\, I_{b,yy} \,+\, (1-N)^2\, I_{rot,yy} \,+\, \frac{1}{2}\, m_b\, l^2 \,+\, m_{rot}(z^b_{rot})^2\right] \dot{\vartheta}^2 \,+\, \cdots \\
& \cdots + \left[2\,N\,(1-N)\,I_{rot,yy} \,+\, \left(m_b\, l \,+\, 2\, m_{rot}\, z^b_{rot}\right) r\cos\vartheta\right] \dot{\gamma}\,\dot{\vartheta} \,-\, \cdots \\
& \cdots - \left(m_b\, l \,+\, 2\, m_{rot}\, z^b_{rot}\right) g\, \cos\vartheta
\end{aligned}
\tag{27}
$$

The Equations-of–Motion describing the longitudinal dynamics of the balancing robot are obtained by evaluating the following *Lagrange equations*:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\gamma}} \,-\, \frac{\partial \mathcal{L}}{\partial \gamma} \;&=\; \xi_\gamma \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vartheta}} \,-\, \frac{\partial \mathcal{L}}{\partial \vartheta} \;&=\; \xi_\vartheta
\end{aligned}
\tag{28}
$$

where ξ_γ and ξ_ϑ are the **generalized forces** (**torques**) associated with the generalized coordinates γ and ϑ. The expression for computing the generic generalized force ξ_j associated with the generic generalized coordinate q_j is:

$$
\xi_j \;=\; \sum_{i=1}^{n} \boldsymbol{F}_i^T\, \frac{\partial \boldsymbol{p}_i}{\partial q_j} \,+\, \sum_{i=1}^{n} \boldsymbol{\tau}_i^T\, \frac{\partial \boldsymbol{\vartheta}_i}{\partial q_j}
\tag{29}
$$

where $\boldsymbol{F}_i$ and $\boldsymbol{\tau}_i$ are the generic force and torque acting on the system, while $\boldsymbol{p}_i$ and $\boldsymbol{\vartheta}_i$ are their point and axis of application (expressed in world frame). The forces/torques are either external (e.g. actuation forces/torques, external disturbances, etc.), or nonconservative (e.g. frictional force/torque). In the balancing robot example, the contributions to be considered are the motor torque τ generated at the output shaft, and the two dominant viscous friction torques

$$
\tau'_f \;=\; B\,(\dot{\gamma} \,-\, \dot{\vartheta}) \qquad \text{and} \qquad \tau''_f = B_w \dot{\gamma}
\tag{30}
$$

The former is due to the gearmotor internal friction, and is proportional to the rotation speed of the motor output shaft with respect to the stator. The latter is instead the viscous friction torque acting on the rolling wheel, and is proportional to the wheel speed. By considering that these contributions act identically on both the left and right sides, from (29) it follows that:

$$
\begin{aligned}
\xi_\gamma \;&=\; 2\,\tau \,-\, 2\,\tau'_f \,-\, 2\,\tau''_f \;=\; 2\tau \,-\, 2\,(B + B_w)\,\dot{\gamma} \,+\, 2B\dot{\vartheta} \\
\xi_\vartheta \;&=\; -2\tau + 2\tau'_f \;=\; -2\tau \,+\, 2B\dot{\gamma} \,-\, 2B\dot{\vartheta}
\end{aligned}
\tag{31}
$$

The estimated values of the viscous friction coefficients B and B_w are reported in Tab. 2.
After replacing (27) and (31) within (28), the following equations–of–motion finally result for the

longitudinal dynamics:

$$
\begin{aligned}
&\left[2\, I_{w,yy} \,+\, 2\, N^2\, I_{rot,yy} \,+\, (m_b \,+\, 2\, m_w \,+\, 2\, m_{rot})\, r^2\right] \ddot{\gamma} \,+\, 2\, (B + B_w)\, \dot{\gamma} \,+\, \cdots \\
&\qquad \cdots + \left[2\, N\, (1-N)\, I_{rot,yy} \,+\, \left(m_b\, l \,+\, 2\, m_{rot}\, z^b_{rot}\right) r \cos\vartheta\right] \ddot{\vartheta} \,-\, 2B\dot{\vartheta} \,-\, \cdots \\
&\qquad\qquad \cdots - \left(m_b\, l \,+\, 2\, m_{rot}\, z^b_{rot}\right) r\, \sin\vartheta\, \dot{\vartheta}^2 - 2\tau \;=\; 0
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
&\left[2\, N\, (1-N)\, I_{rot,yy} \,+\, \left(m_b\, l \,+\, 2\, m_{rot}\, z^b_{rot}\right) r \cos\vartheta\right] \ddot{\gamma} \,-\, 2B\dot{\gamma} \,+\, \cdots \\
&\qquad \cdots + \left[I_{b,yy} \,+\, 2\, (1-N)^2\, I_{rot,yy} \,+\, m_b\, l^2 \,+\, 2\, m_{rot} (z^b_{rot})^2\right] \ddot{\vartheta} \,+\, 2B\dot{\vartheta} \,-\, \cdots \\
&\qquad\qquad \cdots - \left(m_b\, l \,+\, 2\, m_{rot}\, z^b_{rot}\right) g\, \sin\vartheta \,+\, 2\tau \;=\; 0
\end{aligned} \tag{33}
$$

Let $\boldsymbol{q} = [\gamma,\, \vartheta]^T$ be the vector of the generalized variables. A compact matrix formulation of the dynamical model (32)–(33) is:

$$
\boldsymbol{M}(\boldsymbol{q})\, \ddot{\boldsymbol{q}} \,+\, \boldsymbol{C}(\boldsymbol{q},\, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}} \,+\, \boldsymbol{F}_v\, \dot{\boldsymbol{q}} \,+\, \boldsymbol{g}(\boldsymbol{q}) \;=\; \boldsymbol{\tau} \tag{34}
$$

where

$$
\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} M_{11}(\boldsymbol{q}) & M_{12}(\boldsymbol{q}) \\ M_{21}(\boldsymbol{q}) & M_{22}(\boldsymbol{q}) \end{bmatrix}, \quad \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{bmatrix} C_{11}(\boldsymbol{q}, \dot{\boldsymbol{q}}) & C_{12}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ C_{21}(\boldsymbol{q}, \dot{\boldsymbol{q}}) & C_{22}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{bmatrix}, \quad \boldsymbol{F}_v = \begin{bmatrix} F_{v,11} & F_{v,12} \\ F_{v,21} & F_{v,22} \end{bmatrix} \tag{35}
$$

with

$$
\begin{aligned}
M_{11}(\boldsymbol{q}) \;&=\; 2\, I_{w,yy} \,+\, 2\, N^2\, I_{rot,yy} \,+\, (m_b \,+\, 2\, m_w \,+\, 2\, m_{rot})\, r^2 \\
M_{12}(\boldsymbol{q}) \;&=\; M_{21}(\boldsymbol{q}) \;=\; 2\, N\, (1-N)\, I_{rot,yy} \,+\, \left(m_b\, l \,+\, 2\, m_{rot}\, z^b_{rot}\right) r \cos\vartheta \\
M_{22}(\boldsymbol{q}) \;&=\; I_{b,yy} \,+\, 2\, (1-N)^2\, I_{rot,yy} \,+\, m_b\, l^2 \,+\, 2\, m_{rot} (z^b_{rot})^2
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
C_{11}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \;&=\; C_{21}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \;=\; C_{22}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \;=\; 0 \\
C_{12}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \;&=\; -\left(m_b\, l \,+\, 2\, m_{rot}\, z^b_{rot}\right) r\, \sin\vartheta\, \dot{\vartheta}
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
F_{v,11} \;&=\; 2(B + B_w) \\
F_{v,12} \;&=\; F_{v,21} \;=\; -2B \\
F_{v,22} \;&=\; 2B
\end{aligned} \tag{38}
$$

are, respectively, the inertia matrix, the matrix of centrifugal and Coriolis–related coefficients (so that $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}}$ is the torque contribution due to the centrifugal and Coriolis accelerations), and the matrix of viscous friction coefficients $\boldsymbol{F}_v$, while

$$
\boldsymbol{g}(\boldsymbol{q}) \;=\; \left[0\,, \quad -\left(m_b\, l \,+\, 2\, m_{rot}\, z^b_{rot}\right) g \sin\vartheta\right]^T \tag{39}
$$

is the torque contribution due to gravity. The motor torque input is equal to $\boldsymbol{\tau} = [2\tau,\, -2\tau]^T$.

The dynamical model (32)–(33) is nonlinear; for the design of the linear state–space balancing controller, it has to be linearized about the unstable equilibrium configuration with the robot body in steady upward vertical position. Consider the unstable equilibrium point $P_0 = (\boldsymbol{q}_0,\, \dot{\boldsymbol{q}}_0,\, \ddot{\boldsymbol{q}}_0,\, \boldsymbol{\tau}_0)$ with $\boldsymbol{q}_0 = [\gamma_0,\, 0]^T$, $\dot{\boldsymbol{q}}_0 = \ddot{\boldsymbol{q}}_0 = [0,\, 0]^T$ and $\tau_0 = 0$ (note that P_0 is an equilibrium point for any choice of $\gamma_0 \in \mathbb{R}$). The linearization of (34) around P_0 is given by:

$$\boldsymbol{f}(P_0) \,+\, \frac{\partial \boldsymbol{f}\,(P_0)}{\partial \boldsymbol{q}}\,\delta\boldsymbol{q} \,+\, \frac{\partial \boldsymbol{f}\,(P_0)}{\partial \dot{\boldsymbol{q}}}\,\delta\dot{\boldsymbol{q}} \,+\, \frac{\partial \boldsymbol{f}\,(P_0)}{\partial \ddot{\boldsymbol{q}}}\,\delta\ddot{\boldsymbol{q}} \,+\, \frac{\partial \boldsymbol{f}\,(P_0)}{\partial \tau}\delta\tau \;=\; \boldsymbol{0} \tag{40}$$

where

$$\boldsymbol{f}\,(\boldsymbol{q},\, \dot{\boldsymbol{q}},\, \ddot{\boldsymbol{q}},\, \tau) \;=\; \boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} \,+\, \boldsymbol{C}(\boldsymbol{q},\, \dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} \,+\, \boldsymbol{F}_v\,\dot{\boldsymbol{q}} \,+\, \boldsymbol{g}(\boldsymbol{q}) \,-\, \boldsymbol{\tau} \tag{41}$$

and $\delta\boldsymbol{q}$, $\delta\dot{\boldsymbol{q}}$, $\delta\ddot{\boldsymbol{q}}$ and $\delta\tau$ denote small deviations around the equilibrium values specified in P_0 (e.g. $\delta\boldsymbol{q} = \boldsymbol{q} - \boldsymbol{q}_0$). Note that $\boldsymbol{f}\,(P_0) = \boldsymbol{0}$ in the Taylor's expansion of $\boldsymbol{f}$, since P_0 is an equilibrium point. The evaluation of (40) yields the following linearized model:

$$\begin{aligned} &\left[2\,I_{w,yy} \,+\, 2\,N^2\,I_{rot,yy} \,+\, (m_b \,+\, 2\,m_w \,+\, 2\,m_{rot})\,r^2\right]\ddot{\gamma} \,+\, 2\,(B + B_w)\,\dot{\gamma} \,+\, \cdots \\ &\cdots + \left[2\,N\,(1-N)\,I_{rot,yy} \,+\, \left(m_b\,l \,+\, 2\,m_{rot}\,z^b_{rot}\right)r\right]\ddot{\vartheta} \,-\, 2B\dot{\vartheta} \,-\, 2\tau \;=\; 0 \end{aligned} \tag{42}$$

$$\begin{aligned} &\left[2\,N\,(1-N)\,I_{rot,yy} \,+\, \left(m_b\,l \,+\, 2\,m_{rot}\,z^b_{rot}\right)r\right]\ddot{\gamma} \,-\, 2B\dot{\gamma} \,+\, \cdots \\ &\cdots + \left[I_{b,yy} \,+\, 2\,(1-N)^2\,I_{rot,yy} \,+\, m_b\,l^2 \,+\, 2\,m_{rot}(z^b_{rot})^2\right]\ddot{\vartheta} \,+\, 2B\dot{\vartheta} \,-\, \cdots \\ &\qquad \cdots - \left(m_b\,l \,+\, 2\,m_{rot}\,z^b_{rot}\right)g\,\vartheta \,+\, 2\tau \;=\; 0 \end{aligned} \tag{43}$$

where, with some abuse of notation, the variables γ, ϑ and τ have been reused (in place of $\delta\gamma$, $\delta\vartheta$ and $\delta\tau$) to denote the small deviations around the equilibrium values. A compact matrix formulation of (42)–(43) is:

$$\boldsymbol{M}\,\ddot{\boldsymbol{q}} \,+\, \boldsymbol{F}_v\,\dot{\boldsymbol{q}} \,+\, \boldsymbol{G}\,\boldsymbol{q} \;=\; \boldsymbol{\tau} \tag{44}$$

where

$$\boldsymbol{M} \;=\; \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \qquad \boldsymbol{G} \;=\; \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \tag{45}$$

with

$$\begin{aligned} M_{11} \;&=\; 2\,I_{w,yy} \,+\, 2\,N^2\,I_{rot,yy} \,+\, (m_b \,+\, 2\,m_w \,+\, 2\,m_{rot})\,r^2 \\ M_{12} \;&=\; M_{21} \;=\; 2\,N\,(1-N)\,I_{rot,yy} \,+\, \left(m_b\,l \,+\, 2\,m_{rot}\,z^b_{rot}\right)r \\ M_{22} \;&=\; I_{b,yy} \,+\, 2\,(1-N)^2\,I_{rot,yy} \,+\, m_b\,l^2 \,+\, 2\,m_{rot}(z^b_{rot})^2 \end{aligned} \tag{46}$$

$$\begin{aligned} G_{11} \;&=\; G_{12} \;=\; G_{21} \;=\; 0 \\ G_{22} \;&=\; -\left(m_b\,l \,+\, 2\,m_{rot}\,z^b_{rot}\right)g \end{aligned} \tag{47}$$

are the linearized versions around the equilibrium point P_0 of the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ and the gravity torque contribution $\boldsymbol{g}(\boldsymbol{q})$.

The linear model (44) can be alternatively rewritten in state–space form as follows. Define the vector of state variables as $\boldsymbol{x} = [\boldsymbol{q},\, \dot{\boldsymbol{q}}]^T = [\gamma,\, \vartheta,\, \dot{\gamma},\, \dot{\vartheta}]^T$; then, the resulting state–space model is:

$$\dot{\boldsymbol{x}} \;=\; \boldsymbol{A}\,\boldsymbol{x} \,+\, \boldsymbol{B}\,u \tag{48}$$

where

$$\boldsymbol{A} \;=\; \begin{bmatrix} \boldsymbol{0}_{2\times 2} & \boldsymbol{I}_{2\times 2} \\ -\boldsymbol{M}^{-1}\boldsymbol{G} & -\boldsymbol{M}^{-1}\boldsymbol{F}_v \end{bmatrix}, \qquad \boldsymbol{B} \;=\; \begin{bmatrix} \boldsymbol{0}_{2\times 2} \\ \boldsymbol{M}^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{49}$$

and $u = 2\,\tau$. Note that (48) has a scalar input; for the computation of the input matrix $\boldsymbol{B}$ in (49), it has been used the fact that $\boldsymbol{\tau} = [1,\, -1]^T(2\tau)$.

2.2 Model of the actuation system

The electromechanical torque generated by each DC gearmotor at the output shaft is equal to:

$$\tau \;=\; N\,k_t\,i_a \tag{50}$$

where i_a is the armature current, and k_t is the torque constant. The current i_a satisfies the electrical equation:

$$L_a\,\frac{di_a}{dt} \,+\, R_a\,i_a \,+\, k_e\,\frac{d\Delta\vartheta_{rot}}{dt} \;=\; u_a \tag{51}$$

where R_a and L_a are the resistance and inductance of the armature circuit, k_e is the electric (back electromotive force – BEMF) constant, $\Delta\vartheta_{rot} = \vartheta_{rot} - \vartheta$ is the angular displacement of the rotor with respect to the stator, and u_a is the supply voltage to the armature circuit. Note that the BEMF is proportional to how fast the rotor spins with respect to the stator. Suppose that the electrical time constant L_a/R_a is negligible (compared to the smallest mechanical time constant of the system): then, the differential equation (51) reduces to the following algebraic relation

$$i_a \;=\; \frac{1}{R_a}\left[u_a \,-\, k_e\,N\,(\dot{\gamma} - \dot{\vartheta})\right] \tag{52}$$

where it has been used the fact that $d(\Delta\vartheta_{rot})/dt = N(\dot{\gamma} - \dot{\vartheta})$. Correspondingly, the torque expression (50) becomes:

$$\tau \;=\; \frac{N k_t}{R_a}\,u_a \,-\, \frac{N^2 k_t k_e}{R_a}\,\left(\dot{\gamma} - \dot{\vartheta}\right) \tag{53}$$

Once the expression (53) is replaced within (34), the following nonlinear dynamical model results:

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} \,+\, \boldsymbol{C}(\boldsymbol{q},\,\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} \,+\, \boldsymbol{F}_v'\,\dot{\boldsymbol{q}} \,+\, \boldsymbol{g}(\boldsymbol{q}) \;=\; \boldsymbol{\tau}' \tag{54}$$

where

$$\boldsymbol{F}_v' \;=\; \boldsymbol{F}_v \,+\, \frac{2\,N^2\,k_t\,k_e}{R_a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \boldsymbol{\tau}' \;=\; \frac{2\,N\,k_t}{R_a}\begin{bmatrix} 1 \\ -1 \end{bmatrix} u_a \tag{55}$$

Similarly, by replacing (53) within (44), the following linear dynamical model results:

$$\boldsymbol{M}\,\ddot{\boldsymbol{q}} \,+\, \boldsymbol{F}_v'\,\dot{\boldsymbol{q}} \,+\, \boldsymbol{G}\,\boldsymbol{q} \;=\; \boldsymbol{\tau}' \tag{56}$$

Armature resistance	R_a	$2.4\,\Omega$
Armature inductance	L_a	n.a. (neglected)
Electric (BEMF) constant	k_e	$10.3 \times 10^{-3}\,\mathrm{V\,s/rad}$
Torque constant	k_m	$5.2 \times 10^{-3}\,\mathrm{Nm/A}$
Gearbox ratio	N	30

Table 3: DC gearmotor nominal parameters.

where $\boldsymbol{F}_v'$ and $\boldsymbol{\tau}'$ are defined as in (55). The corresponding state–space model is:

$$\dot{\boldsymbol{x}} \;=\; \boldsymbol{A}\,\boldsymbol{x} \,+\, \boldsymbol{B}\,u \tag{57}$$

with

$$\boldsymbol{A} \;=\; \begin{bmatrix} \boldsymbol{0}_{2\times 2} & \boldsymbol{I}_{2\times 2} \\ -\boldsymbol{M}^{-1}\boldsymbol{G} & -\boldsymbol{M}^{-1}\boldsymbol{F}_v' \end{bmatrix}, \qquad \boldsymbol{B} \;=\; \frac{2\,N\,k_t}{R_a} \begin{bmatrix} \boldsymbol{0}_{2\times 2} \\ \boldsymbol{M}^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{58}$$

and $u = u_a$. The nominal values of the DC gearmotor parameters are reported in Tab. 3.

2.3 Model of the inertial measurement system

The linear acceleration and angular velocity measurements provided by the *inertial measurement sensor* (also referred as **Motion Processing Unit** – MPU) are referred to a reference frame $\mathcal{F}_{mpu}$ whose origin is located on the sensor geometrical center C_{mpu}, and whose axes are aligned with those of the body frame $\mathcal{F}_b$. The coordinates of the sensor center with respect to the body frame are reported in Tab. 4. The acceleration measurement is provided by the onboard accelerometer, and is equal to the sum of the actual linear acceleration experienced by the robot body at the location of the sensor center, and the gravity acceleration vector. Instead, the angular velocity measurement is provided by the gyroscope, and it coincides with the actual angular velocity of the robot body.
To determine the expression of the accelerometer output as a function of the robot generalized coordinates (and their derivatives), it is first necessary to determine the position vector of the sensor center with respect to the world frame, and derive it twice to get the acceleration. Then, after adding the gravity contribution, the result has to be reprojected on the sensor frame to get the actual sensor output. Let $\boldsymbol{p}^b_{C_{mpu}} = [0,\, z^b_{mpu}]^T$ be the position vector of the sensor center with respect to the body frame. Then, the position vector $\boldsymbol{p}^o_{C_{mpu}} = [x_{C_{mpu}},\, z_{C_{mpu}}]^T$ of the sensor center with respect to the world frame is equal to:

$$\boldsymbol{p}^o_{C_{mpu}} \;=\; \boldsymbol{p}^o_v \,+\, \boldsymbol{R}^o_b\,\boldsymbol{p}^b_{C_{mpu}} \;=\; \begin{bmatrix} r\,\gamma \,+\, z^b_{mpu}\sin\vartheta \\ z^b_{mpu}\cos\vartheta \end{bmatrix} \tag{59}$$

The linear velocity and acceleration vectors are obtained by differentiation of the position vector, and are equal to:

$$\dot{\boldsymbol{p}}^o_{C_{mpu}} \;=\; \dot{\boldsymbol{p}}^o_v \,+\, \frac{d\boldsymbol{R}^o_b}{dt}\,\boldsymbol{p}^b_{C_{mpu}} \,+\, \boldsymbol{R}^o_b\,\dot{\boldsymbol{p}}^b_{C_{mpu}} \;=\; \begin{bmatrix} r\,\dot{\gamma} \,+\, z^b_{mpu}\cos\vartheta\,\dot{\vartheta} \\ -z^b_{mpu}\sin\vartheta\,\dot{\vartheta} \end{bmatrix} \tag{60}$$

Sensor center wrt body frame $\mathcal{F}_b$	$x^b_{mpu},\, y^b_{mpu},\, z^b_{mpu}$	$0,\, 0,\, 13.5$	[mm]

Table 4: Motion processing unit (MPU) geometrical parameters.

$$\ddot{\boldsymbol{p}}^o_{C_{mpu}} \;=\; \ddot{\boldsymbol{p}}^o_v + \frac{d^2\boldsymbol{R}^o_b}{dt^2}\,\boldsymbol{p}^b_{C_{mpu}} + 2\frac{d\,\boldsymbol{R}^o_b}{dt}\,\dot{\boldsymbol{p}}_{C_{mpu}} + \boldsymbol{R}^o_b\,\ddot{\boldsymbol{p}}^b_{C_{mpu}} \;=\; \begin{bmatrix} r\,\ddot{\gamma} \;+\; z^b_{mpu}\,(-\sin\vartheta\,\dot{\vartheta}^2 \;+\; \cos\vartheta\,\ddot{\vartheta}) \\ -z^b_{mpu}\,(\cos\vartheta\,\dot{\vartheta}^2 \;+\; \sin\vartheta\,\ddot{\vartheta}) \end{bmatrix} \tag{61}$$

The accelerometer output $\boldsymbol{y}_a = [x^{mpu}_a,\, z^{mpu}_a]^T$ is therefore equal to:

$$\boldsymbol{y}_a \;=\; \boldsymbol{R}^b_o\left(\ddot{\boldsymbol{p}}^o_{C_{mpu}} + \boldsymbol{g}^o\right) \tag{62}$$

where $\boldsymbol{g}^o = [0,\, -g]^T$ is the gravity acceleration vector with respect to the world frame, and

$$\boldsymbol{R}^b_o \;=\; \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \tag{63}$$

is the rotation matrix of the world frame $\mathcal{F}_o$ with respect to the body frame $\mathcal{F}_b$ (note that $\boldsymbol{R}^b_o = (\boldsymbol{R}^o_b)^{-1}$, and $(\boldsymbol{R}^o_b)^{-1} = (\boldsymbol{R}^o_b)^T$ because $\boldsymbol{R}^o_b$ is a rotation matrix). It finally holds that:

$$\boldsymbol{y}_a \;=\; \begin{bmatrix} r\,\ddot{\gamma}\cos\vartheta \;+\; z^b_{mpu}\,\ddot{\vartheta} \;+\; g\sin\vartheta \\ r\,\ddot{\gamma}\sin\vartheta \;-\; z^b_{mpu}\,\dot{\vartheta}^2 \;-\; g\cos\vartheta \end{bmatrix} \tag{64}$$

The gyroscope output is simply the rate of change of the robot body tilt angle, namely:

$$y_g \;=\; \dot{\vartheta} \tag{65}$$

3 Tilt estimation

The robot tilt angle ϑ can be estimated by using the data provided by the MPU. For such purpose, consider first the expression of the accelerometer output (64). Assume that the robot body motion is slow, so that the linear acceleration $r\ddot{\gamma}$, the tangential acceleration $z^b_{mpu}\ddot{\vartheta}$ and the centripetal acceleration $z^b_{mpu}\dot{\vartheta}^2$ can all be regarded as negligible quantities. Under this hypothesis, the measurement provided by the accelerometer is:

$$\boldsymbol{y}_a \;=\; \begin{bmatrix} x^{mpu}_a \\ z^{mpu}_a \end{bmatrix} \;\approx\; \begin{bmatrix} g\,\sin\vartheta \\ -g\cos\vartheta \end{bmatrix} \tag{66}$$

which consists of the projection of the gravity acceleration vector onto the two sensor axes. Therefore, the robot tilt angle can be estimated as follows:

$$\hat{\vartheta}_a \;=\; \mathrm{atan2}\,(x^{mpu}_a,\, -z^{mpu}_a) \tag{67}$$

where $\operatorname{atan2}$ is the four–quadrant arctangent function, namely:

$$\operatorname{atan2}(y,\,x) \;=\; \begin{cases} \operatorname{atan}(y/x) & \text{if} \quad x>0 \\ \operatorname{atan}(y/x)\,+\,\pi & \text{if} \quad x<0 \;\text{ and }\; y \geqslant 0 \\ \operatorname{atan}(y/x)\,-\,\pi & \text{if} \quad x<0 \;\text{ and }\; y<0 \\ +\pi/2 & \text{if} \quad x=0 \;\text{ and }\; y>0 \\ -\pi/2 & \text{if} \quad x=0 \;\text{ and }\; y<0 \\ \text{undefined} & \text{if} \quad x=0 \;\text{ and }\; y=0 \end{cases} \tag{68}$$

There are two drawbacks of using the estimate (67). Firstly, the estimate is sensitive to the robot body accelerations, and could be unreliable whenever the robot motion is not slow enough. Secondly, the estimate is corrupted by the accelerometer output noise, which is rather large, especially at high frequency. To overcome these issues, a different estimation method can be potentially exploited, which simply consists of integrating the gyroscope output (65) to obtain the desired tilt angle estimate, namely:

$$\hat{\vartheta}_g \;=\; \hat{\vartheta}_g(0) \,+\, \int_0^t y_g(\tau)\,d\tau \tag{69}$$

This estimate is certainly not affected by the robot body accelerations, and is also less noisy, since the gyroscope output noise is filtered by the operation of integration. Unfortunately, to be effective, the estimation procedure based on (69) requires to be properly initialized with the actual value of the tilt angle, i.e. the initial condition should be chosen as $\hat{\vartheta}_g(0) = \vartheta(0)$. Moreover, the gyroscope output is typically affected by both a bias (constant) and a drift (linear ramp) measurement error, which cause the integral in (69) to grow unbounded over time.

In practice, by noting that the accelerometer–based estimate $\hat{\vartheta}_a$ is more reliable at low frequency, while the gyroscope–based estimate $\hat{\vartheta}_g$ is more reliable at high frequency, one can resort to a **complementary filtering** approach for combining ("fusing") them together, thus yielding a final estimate $\hat{\vartheta}$ that is sufficiently accurate at every frequency. The approach can be briefly described as follows. Let $H(s)$ be a low–pass filter with cut–off frequency ω_c and unit DC gain, and $H'(s) = 1 - H(s)$ its complementary filter, namely the high–pass filter such that $H(s) + H'(s) = 1$. The low–pass filter can be used to remove most of the high–frequency noise affecting the accelerometer–based estimate $\hat{\vartheta}_a$, while the high–pass filter can be used to remove the low–frequency drift affecting the gyroscope–based estimate $\hat{\vartheta}_g$. In this sense, the filtered estimates:

$$\hat{\vartheta}_{a,f} \;=\; H(s)\,\hat{\vartheta}_a\,, \qquad \hat{\vartheta}_{g,f} \;=\; [1 - H(s)]\,\hat{\vartheta}_g \tag{70}$$

can be regarded as two reliable estimates of the tilt angle ϑ, which are however valid on two disjoint and complementary frequency ranges, namely the range $[0,\,\omega_c)$ for the former, and $[\omega_c,\,+\infty)$ for the latter. Consequently, their sum

$$\hat{\vartheta} \;=\; \hat{\vartheta}_{a,f} \,+\, \hat{\vartheta}_{g,f} \;=\; H(s)\,\hat{\vartheta}_a \,+\, [1 - H(s)]\,\hat{\vartheta}_g \tag{71}$$

can be regarded as a reliable estimate of the tilt angle ϑ over the whole frequency range $[0,\,+\infty)$. The choice of the order and cut–off frequency of the pair of complementary filters is necessarily obtained as a trade–off between two conflicting requirements, namely the attenuation of the accelerometer output noise at high frequency, and the rejection of the low frequency bias/drift due to the gyroscope.

In practice, such choice is performed by trial and error. The typical choices are:

- *first–order* complementary filters pair

$$H(s) \;=\; \frac{1}{T_c\, s+1}\,, \qquad 1-H(s) \;=\; \frac{T_c\, s}{T_c\, s+1} \tag{72}$$

This is the simplest choice, and is mostly motivated by the desired to reduce the implementation complexity of the complementary filters pair. However, note that the high–pass filter $1-H(s)$ is unable to effectively reject a bias or drift error that possibly affects the gyroscope output. In fact, suppose that the gyroscope output is affected by the error

$$\tilde{y}_g \;=\; d_g\, t \;+\; b_g \tag{73}$$

where $d_g\, t$ and b_g are, respectively, the drift and bias error components. Once integrated in (69), it gives rise to the position estimation error

$$\tilde{\vartheta}_g \;=\; \frac{1}{2}\, d_g\, t^2 \;+\; b_g\, t \tag{74}$$

The Laplace transform is

$$\tilde{\Theta}_g(s) \;=\; \frac{d_g}{s^3} \;+\; \frac{b_g}{s^2} \tag{75}$$

At the output of the high–pass filter it holds that

$$\begin{aligned} \tilde{\Theta}_{g,f}(s) \;&=\; [1 \,-\, H(s)]\, \tilde{\Theta}_g(s) \;=\; \frac{T_c\, d_g}{(T_c\, s+1)\, s^2} \;+\; \frac{T_c\, b_g}{(T_c\, s+1)\, s} \;=\; \cdots \\ \cdots \;&=\; -\,\frac{T_c^2\, d_g}{s} \;+\; \frac{T_c\, d_g}{s^2} \;+\; \frac{T_c^3 d_g}{T_c\, s+1} \;+\; \frac{T_c\, b_g}{s} \;-\; \frac{T_c^2\, b_g}{T_c\, s+1} \end{aligned} \tag{76}$$

where the last identity has been obtained by partial fraction expansion. For large values of the time t, it is immediate to verify that:

$$\tilde{\theta}_{g,f}(t) \;\approx\; (T_c\, d_g)\, t \;+\; \big(T_c\, b_g - T_c^2\, d_g\big) \tag{77}$$

namely the filtered position estimation error contains both a drift and a bias component. Therefore, an implementation based on (72) can be effectively used only when the gyroscope bias and drift are both negligible.

- *second–order* complementary filters pair

$$H(s) \;=\; \frac{2\, T_c\, s+1}{(T_c\, s+1)^2}\,, \qquad 1-H(s) \;=\; \frac{T_c^2\, s^2}{(T_c\, s+1)^2} \tag{78}$$

In this case, the filtered version of the position estimation error (74)–(75) is

$$\begin{aligned} \tilde{\Theta}_{g,f}(s) \;&=\; [1 \,-\, H(s)]\, \tilde{\Theta}_g(s) \;=\; \frac{T_c^2\, d_g}{(T_c\, s+1)^2\, s} \;+\; \frac{T_c^2\, b_g}{(T_c\, s+1)^2} \;=\; \cdots \\ \cdots \;&=\; \frac{T_c^2\, d_g}{s} \;-\; \frac{T_c^3 d_g}{T_c\, s+1} \;-\; \frac{T_c^3 d_g}{(T_c\, s+1)^2} \;+\; \frac{T_c^2\, b_g}{(T_c\, s+1)^2} \end{aligned} \tag{79}$$

where the last identity has been obtained by partial fraction expansion. For large values of the time t, it is immediate to verify that the filtered position estimation error is a constant:

$$\tilde{\theta}_{g,f}(t) \approx T_c^2 d_g \tag{80}$$

In particular, the error is insensitive to the gyroscope bias. Therefore, the implementation based on (72) is suitable for the cases where the gyroscope drift is negligible.

- *third–order* complementary filters pair

$$H(s) = \frac{3\,T_c^2\,s^2 + 3\,T_c\,s + 1}{(T_c\,s+1)^3}\,, \qquad 1 - H(s) = \frac{T_c^3\,s^3}{(T_c\,s+1)^3} \tag{81}$$

In this case, the filtered version of the position estimation error (74)–(75) is

$$\tilde{\Theta}_{g,f}(s) = [1 - H(s)]\,\tilde{\Theta}_g(s) = \frac{T_c^3\,d_g}{(T_c\,s+1)^3} + \frac{T_c^3\,b_g\,s}{(T_c\,s+1)^3} \tag{82}$$

For large values of the time t, it is immediate to verify that the filtered position estimation error is equal to zero, i.e. $\tilde{\theta}_{g,f}(t) \approx 0$. In practice, with a third–order complementary filters pair, the position estimation becomes insensitive to both the gyroscope bias and drift.

It is worth to point out here that complementary filtering is not the only method available for obtaining a reasonably accurate estimate of the robot tilt angle from the measurements provided by the MPU (accelerometer & gyroscope combo). An alternative approach, which is often used in practice, consists of resorting to a state observer, such as a Kalman filter[1]. Although appealing, this approach is however not considered in these notes.

[1] A nice tutorial paper on complementary and Kalman filtering techniques is: W. T. Higgins, "A Comparison of Complementary and Kalman Filtering", in IEEE Transactions on Aerospace and Electronic Systems, vol. AES-11, no. 3, pp. 321-325, May 1975.