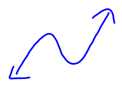
Connecting Equations and Graphs

Analyze Graphs of Polynomial Functions

Consider the polynomial function $f(x) = x^3 - x^2 - 10x - 8$

a) The graph of $f(x)$ will have a y-intercept at $(0, \underline{-8})$ and a maximum of $\underline{3}$ x-intercepts as indicated by degree $\underline{3}$.

b) Since $f(x)$ has an odd degree along with a positive leading coefficient, the graph of the polynomial will extend downward in quadrant $\underline{III}$ and upward in quadrant $\underline{I}$.



c) Determine the factors for $f(x)$ and use these factors to determine the zeros of $f(x)$.

x-intercepts.

① PIZ: $\pm1, \pm2, \pm4, \pm8$

② $f(-1)=0$ $\therefore$ $(x+1)$ is a factor

③
$$\begin{array}{r|rrrr} -1 & 1 & -1 & -10 & -8 \\ & \downarrow & -1 & 2 & 8 \\ \hline & 1 & -2 & -8 & \boxed{0} \end{array}$$

$f(x) = (x+1)(x^2-2x-8)$

④ $f(x) = (x+1)(x+2)(x-4)$

⑤ $\boxed{x = -1, -2, 4}$ x-intercepts $(x,0)$

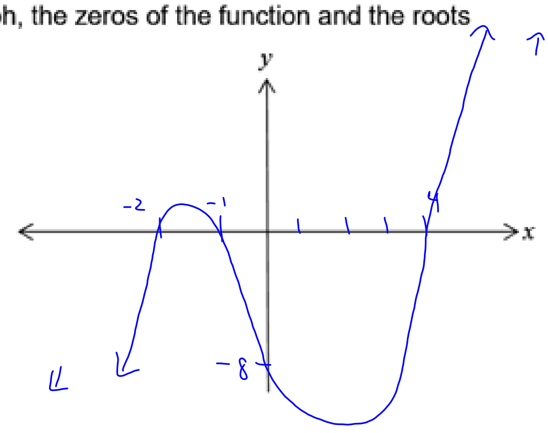
d) Solve the equation $x^3 - x^2 - 10x - 8 = 0$ to determine the roots.

zeros
x-int.
solns.

e) What do you notice about the x-intercepts of the graph, the zeros of the function and the roots of the equation?

They are the same.

f) Use the information from parts a) – e) to sketch the graph of $f(x)$. Verify with a graphing calculator.



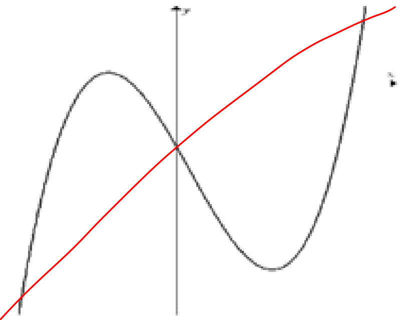
In the previous example, we noticed that:

- The **function** $f(x)=x^3-x^2-10x-8$ has **zeros** of -2 , -1 , 4 .
- The **equation** $0=x^3-x^2-10x-8$ has **roots** (or solutions) of -2 , -1 , 4 .
- The **graph** of $y=x^3-x^2-10x-8$ has ***x*-intercepts** of -2 , -1 , 4 .

To find all of the above features, we needed to factor the polynomial.

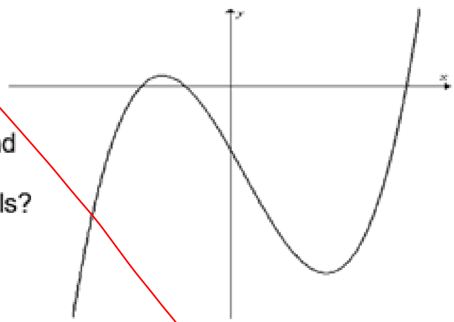
Intervals

Consider the graph of the function $f(x)=(x+2)(x+1)(x-4)$.



i) The ***x*-intercepts** of the graph divide the *x*-axis into four **intervals**. Indicate whether the function is positive (above the *x*-axis) or negative (below the *x*-axis) for each interval.

Set Notation				
Interval Notation				
Sign of $f(x)$				



ii) As you read the graph from left to right, the function values (*y*-values) increase and decrease. Using the graph, state the intervals of increase and decrease. Round answers to the nearest hundredth, if necessary.
What point(s) on the graph help you determine the intervals?

	Interval of Increase/Decrease	**Interval of Increase/Decrease**	**Interval of Increase/Decrease**
Set Notation			
Interval Notation			

Multiplicity of a Zero

$(x-a)^m$

If a polynomial has a factor of $(x - a)$ that is repeated m times, then $x = a$ is a zero with a **multiplicity** of m. Consider the two examples:

$= (x-1)(x-1)(x+3)$

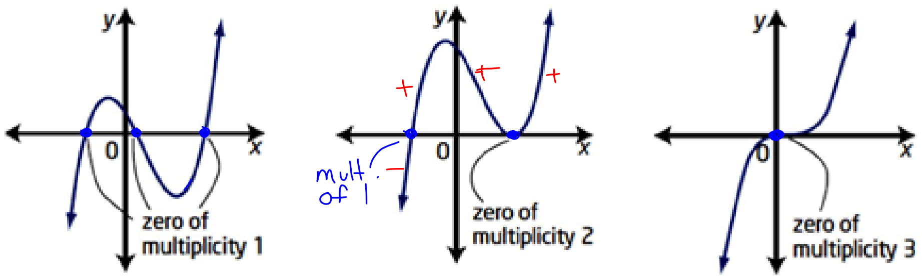
i) $f(x) = (x-1)^2(x+3)^1$ — zero at $x = 1$ with multiplicity 2

degree: 3 — zero at $x = -3$ with multiplicity 1

ii) $f(x) = (x-2)^3(x+3)^2$ — zero at $x = 2$ with multiplicity 3

degree: 5 — zero at $x = -3$ with multiplicity 2

The behaviour of the graph at a zero depends on its multiplicity.



In general, when a zero has

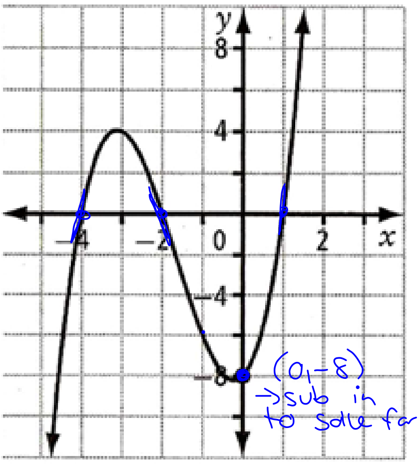
- **even multiplicity**, the graph touches the *x*-axis at the *x*-intercept, but does not cross it; we say that the graph is tangent ("bounces off" the *x*-axis) at these intercepts. The sign of $f(x)$ DOES NOT change at the *x*-intercepts.
- **odd multiplicity**, the graph crosses the *x*-axis at the *x*-intercept. The sign of $f(x)$ DOES change at the *x*-intercept.

Use your calculator to investigate multiplicities of 4 and higher.

Example 1: Use the following features of the graphs on the next page to write the equation of each polynomial function.

a) The *x*-intercepts and their least possible multiplicity
b) The factors of the function with least possible degree
c) The least possible degree of the function
d) The sign of the leading coefficient
e) The value of the leading coefficient
f) The equation of the function

i.



(0, -8)
→ sub in to solve for a

$y = a(x+4)(x+2)(x-1)$

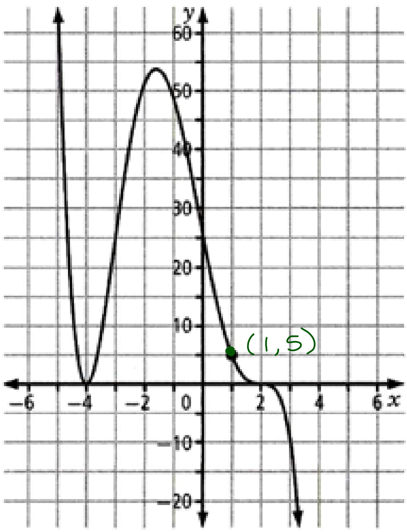
$-8 = a(4)(2)(-1)$

$-8 = -8a$

$a = 1$

a)	*x*-ints with least possible multiplicity $x=-4$ mult. 1 $x=-2$ mult. 1 $x=1$ mult. 1
b)	factors with least possible degree $(x+4)^1$ $(x+2)^1$ $(x-1)^1$
c)	least possible degree 3
d)	sign of leading coefficient +
e)	value of leading coefficient → must solve! $a = 1$
f)	equation of function $y = (x+4)(x+2)(x-1)$

ii.



$y = a(x+4)^2(x-2)^3$

$5 = a(1+4)^2(1-2)^3$

$5 = a(5)^2(-1)^3$

$5 = -25a$

$\boxed{a = -\frac{1}{5}}$

a)	*x*-ints with least possible multiplicity $x=-4$ mult. 2 $x=2$ mult. 3
b)	factors with least possible degree $(x+4)^2$ $(x-2)^3$
c)	least possible degree 5
d)	sign of leading coefficient −
e)	value of leading coefficient $-\frac{1}{5}$
f)	equation of function $y = -\frac{1}{5}(x+4)^2(x-2)^3$

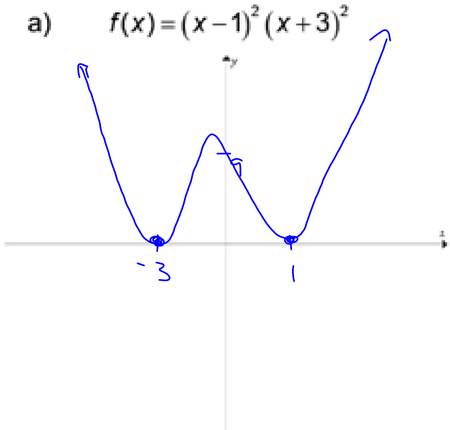
Sketching a Graph Given a Polynomial Equation

To sketch the graph of a polynomial function from its equation, determine the following:

- degree
- *y*-intercept
- sign of the leading coefficient
- end behavior
- zeros and corresponding multiplicity
  ↳ fully factor

Example 2: Without using technology, sketch the graph of each function. Label all intercepts.

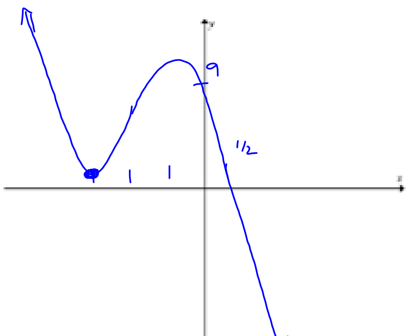
a) $f(x)=(x-1)^2(x+3)^2$



Degree	4
Y-intercept	Set $x=0$ $y=9$ $(0,9)$
Sign of Leading Coefficient	+
End Behaviour	Up in QII ↑ ↗ Up in QI
Zeros with Multiplicities	$x=1$ mult. 2 $x=-3$ mult. 2

Positive Interval(s)	
Negative Interval(s)	

b) $f(x)=-2x^3-11x^2-12x+9$



Degree	3
Y-intercept	9
Sign of Leading Coefficient	—
End Behaviour	Up in Q II ↖ ↘ Down in Q IV
Zeros with Multiplicities	→ fully factor to determine $x=-3$ mult. 2 $x=1/2$ mult. 1

Positive Interval(s)	
Negative Interval(s)	

① PIZ: ±1, ±3, ±9

② $f(-3)=0$ ∴ $(x+3)$ is a factor

③
$$\begin{array}{r|rrrr} -3 & -2 & -11 & -12 & 9 \\ & \downarrow & 6 & 15 & -9 \\ \hline & -2 & -5 & 3 & \boxed{0} \end{array}$$

$f(x)=(x+3)(-2x^2-5x+3)$

$f(x)=-(x+3)(2x^2+5x-3)$

④ $f(x)=-(x+3)(2x-1)(x+3)$

$f(x)=-(x+3)^2(2x-1)$

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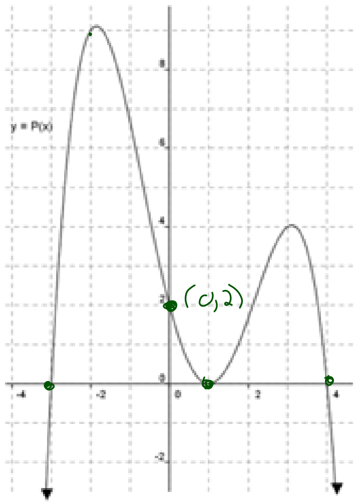
Example 3: The partial graph of a polynomial function $P(x)$ is shown below.



a) The y–intercept of $P(x)$ is 2.

b) The sign of the leading coefficient is neg.

c) The minimum degree of $P(x)$ is 4.

d) The zeros of $P(x)$ are $-3, 1, 4$.

e) The minimum multiplicity at $x = 1$ is 2.

f) Write the equation of the polynomial in factored form.

$y=a(x+3)(x-1)^2(x-4)$

$2=a(3)(-1)^2(-4)$

$2=-12a$

$a=-\frac{1}{6}$ $\rightarrow$ $\boxed{y=-\frac{1}{6}(x+3)(x-1)^2(x-4)}$

g) Did you make any assumptions about the graph when completing this question?

Example 4: Determine the equation, in factored form, of the cubic function whose graph is tangent to the x-axis at the origin and passes though the points (2,0) and (4,–16).

deg. 3

→ mult. of 2 (0,0)
(tangent to x-axis)

$\therefore$ $x=0$
mult. of 2

x^2

$x=2$
mult. 1

$(x-2)^1$

→ use to solve for a

$y=ax^2(x-2)$

$-16=a(4)^2(4-2)$

$-16=32a$

$a=-\frac{1}{2}$

$\boxed{\therefore y=-\frac{1}{2}x^2(x-2)}$

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Example 5: Practice Numerical Response

Column A below contains descriptions of functions in the form $f(x) = a(x-b)(x-c)(x-d)$, where $a > 0$, and Column B shows possible graphs of those functions.

Match the graph from Column B to the function described in Column A by placing the graph number in the blank provided.

Column A

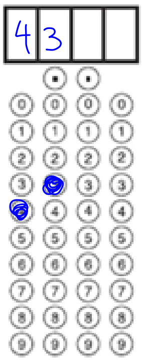
x=b mult. 2

_4_ $b = d, b \neq c$ $y=a(x-b)^2(x-c)$

x=c mult. 1

_3_ $b = c = d$ $y=a(x-b)^3$

x=b mult. 3



Column B