Assume that the likelihood function is Binomial with parameter N, θ and the prior is beta with parameter a, b. We want to prove that the posterior is also a beta distribution.

① likelihood: $P(X=m \mid N, \theta) = \binom{N}{m} \theta^{m} (1-\theta)^{N-m}$

② prior: $P(\theta \mid a, b) = \frac{1}{B(a,b)} \theta^{a-1} (1-\theta)^{b-1}$

③ marginal: Since the support of a beta distribution is $[0,1]$,

$$P(X \mid \theta \in [0,1]) = \int_0^1 P(X=m \mid N, \theta) \cdot P(\theta \mid a, b)\, d\theta$$

$$= \frac{\binom{N}{m}}{B(a,b)} \int_0^1 \theta^{m+a-1} (1-\theta)^{N-m+b-1} d\theta = \frac{\binom{N}{m}}{B(a,b)} \cdot B(m+a, N-m+b)$$

$$\left(\because \int_0^1 \frac{1}{B(m+a, N-m+b)} \theta^{m+a-1} (1-\theta)^{N-m+b-1} = 1\right)$$

By ①~③, the posterior of θ is $P(\theta \mid X) = \dfrac{P(X=m \mid N, \theta) \cdot P(\theta \mid a, b)}{P(X \mid \theta \in [0,1])}$

$$= \frac{\cancel{\binom{N}{m}} \theta^{m} (1-\theta)^{N-m} \frac{1}{\cancel{B(a,b)}} \theta^{a-1} (1-\theta)^{b-1}}{\frac{\cancel{\binom{N}{m}}}{\cancel{B(a,b)}} B(m+a, N-m+b)}$$

$$= \frac{1}{B(m+a, N-m+b)} \theta^{m+a-1} (1-\theta)^{N-m+b-1}$$

$\Rightarrow P(\theta \mid X=m) \sim \text{Beta}(m+a, N-m+b)$