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EULER PATHS AND CIRCUITS

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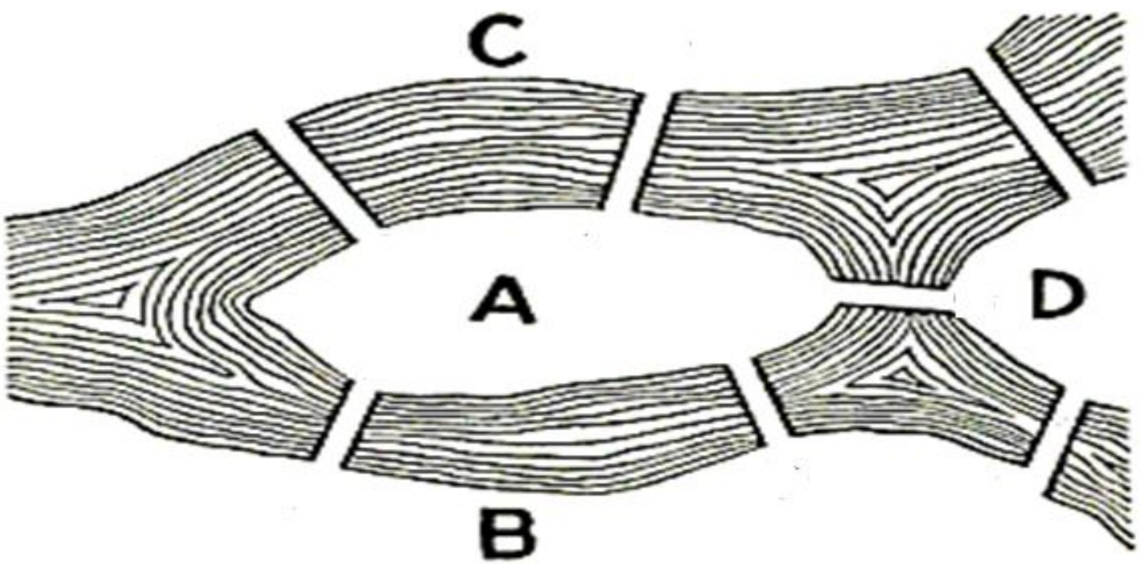
Reference Book: Kenneth H. Rosen, Discrete Mathematics and its Applications, 7th ed, McGraw Hill

THE SEVEN BRIDGES OF KONIGSBERG

In Konigsberg, Germany, was divided into 04 sections by a river ran through the city such that in its center was an island. Seven bridges were built so that the people of the city could get from one part to another.



The original problem

- A resident of Konigsberg wrote to Leonard Euler saying that a popular pastime for couples was to try to cross each of the seven beautiful bridges in the city exactly once -- without crossing any bridge more than once.
- It was believed that it was impossible to do – but why?

- Euler explain the reason.

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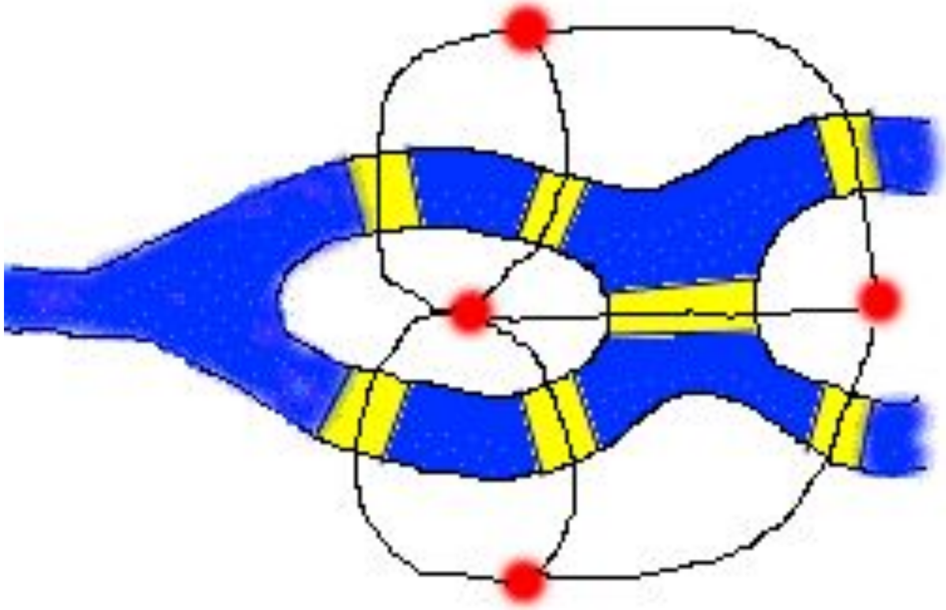
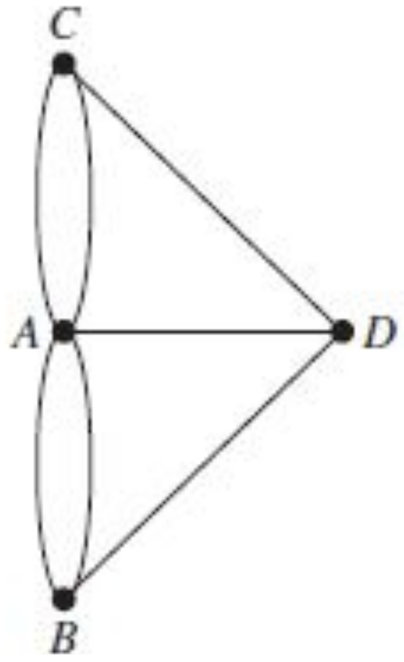
Euler Invents Graph Theory

Euler realized that all problems of this form could be represented by replacing areas of land by points (what we call nodes), and the bridges to and from them by arcs.



- Euler studied this problem using the multigraph obtained when the four regions are represented by vertices and the bridges by edges. This multigraph is shown



- The problem now becomes one of drawing this picture without retracing any line and without picking your pencil up off the paper.

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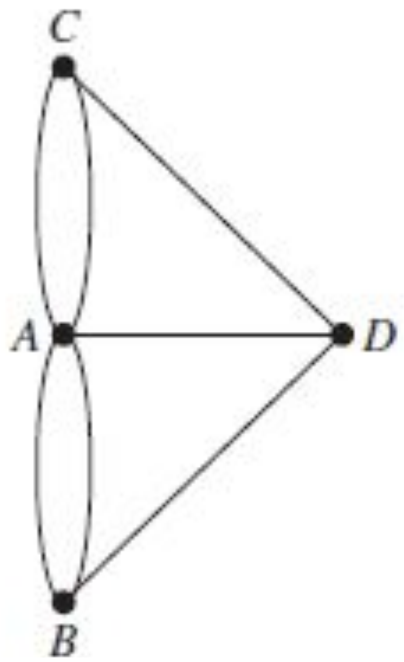
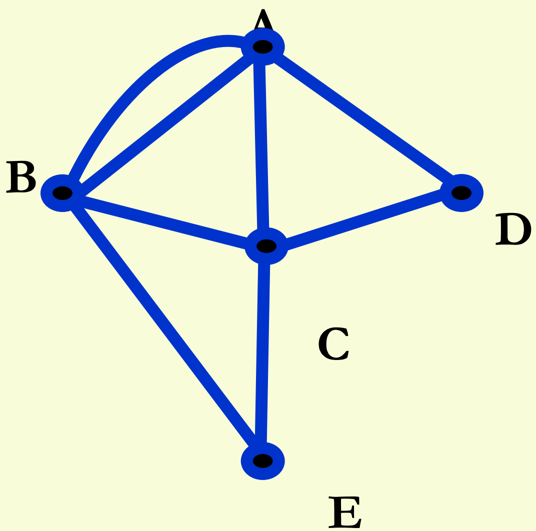


Paths and Circuits

- **Eulerian path-** Eulerian path in a graph is path that traverses each edge in the graph once and only once.
- **Eulerian circuit-** when a Eulerian path begins and ends at the same vertex.

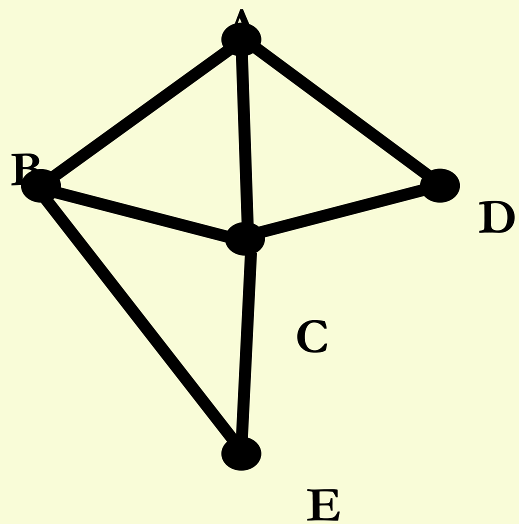
Circuit?

Start at A
End at A



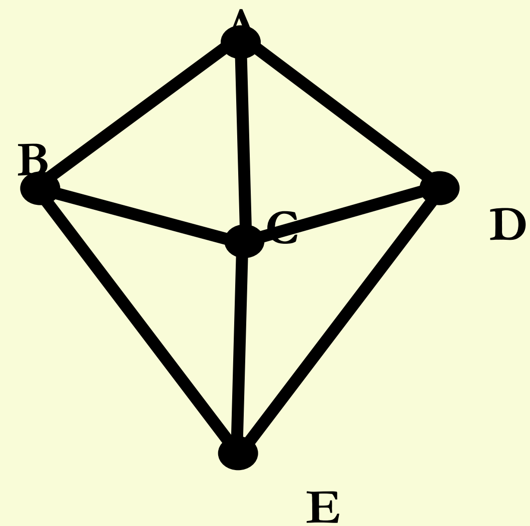
Path?

Start at A
End at B

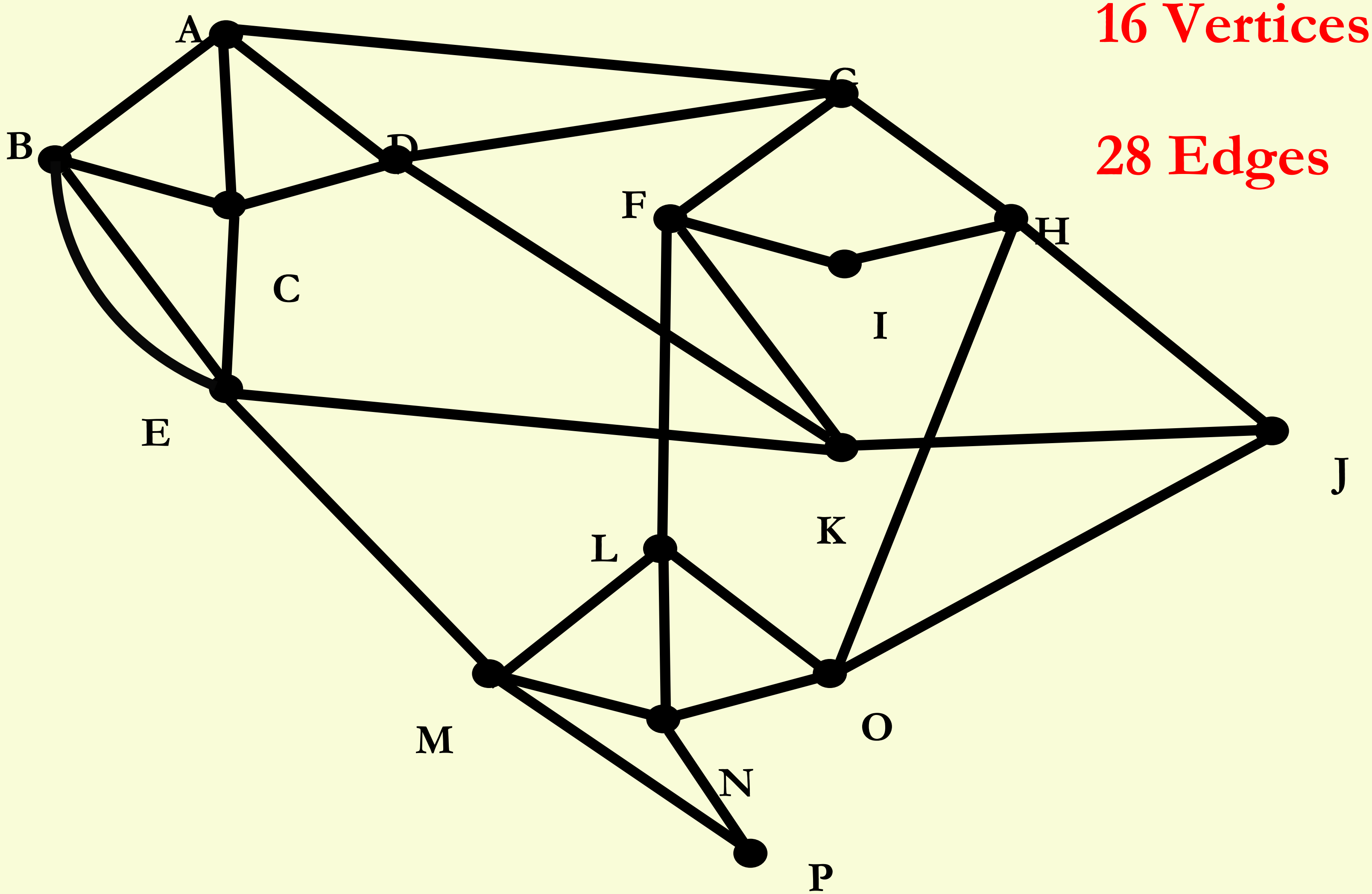


Non-traversable?

Start at A
Miss an edge



Circuit? Path? Non-traversable?



16 Vertices

28 Edges

Theorem 1: A undirected graph has an Euler path/circuit if and only if it is connected and has either two or zero vertices of odd degree.

Proof: First suppose that graph posses an Euler path. (Necessary Condition)

Graph must be connected (otherwise path will not exist)

Second observation is that every time path meets an vertex it goes through two edges (through one enter the vertex and through other leave the vertex) which are incident with vertex that have not be traced before.

Thus except two vertices at the two ends of path, degree of all vertices in the graph should be even.

If the end points of the path coincides, then it becomes circuit and all vertices of the graph have even degree.

To prove sufficient part, we construct an eulerian path by staring at one of the two vertices of odd degree (In case all vertices are of even degree, we can start from any vertex)

For a vertex of even degree, whenever path enters the vertex through an edge it leaves through another edge that is not traced before.

There for eventually we comes to an vertex of odd degree.

If all edges of graph were traced we get the eulerian path.

If not all edges of the graph were traced, we can get a subgraph by removing all the traced edges form G.

The degree of all the vertices of subgraph will be again even. Therefore by applying the same logic as above we will get an simple circuit.

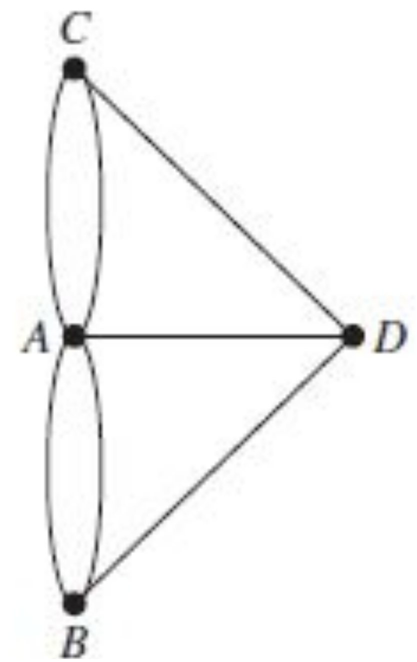
Moreover this graph must touch the path (obtained earlier) as graph G is connected.

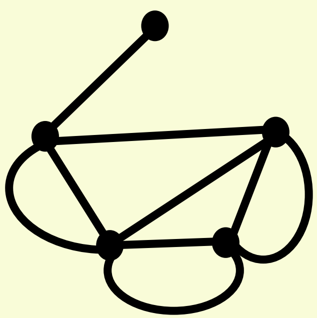
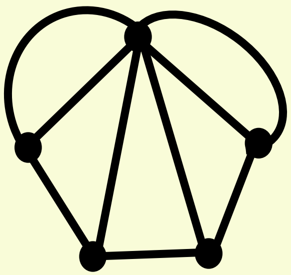
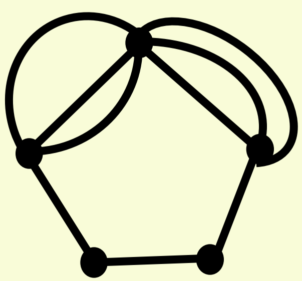
We can combine this circuit with path earlier obtained .

If necessary the argument is repeated until we obtain a path that transverse all the edges of the graph.

Solution of Konigsberg Problem

- All the vertices are of odd degree
- Therefore neither Eulerian path nor circuit will exists
- Therefore its not possible to cross each of the seven beautiful bridges in the city exactly once



Graph	Vertices	Degree	Type
	5	02 vertices of odd degree	Path
	5	03 vertices of odd degree	Non-travers able
	5	All vertices of even degree	Circuit