Graph Theory

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Reference Book: Kenneth H. Rosen, Discrete Mathematics and its Applications, 7th ed, McGraw Hill

Varying Applications (examples)

- We can determine whether it is possible to walk down all the streets in a city without going down a street twice.
- Graphs can be used to determine whether a circuit can be implemented on a planar circuit board
- Solve shortest path problems between cities
- Scheduling exams and assign channels to television stations

Topics Covered

- Definitions
- Types
- Terminology
- Representation
- Sub-graphs
- Connectivity
- Hamilton and Euler definitions
- Shortest Path
- Planar Graphs
- Graph Coloring

DEFINITIONS - GRAPH

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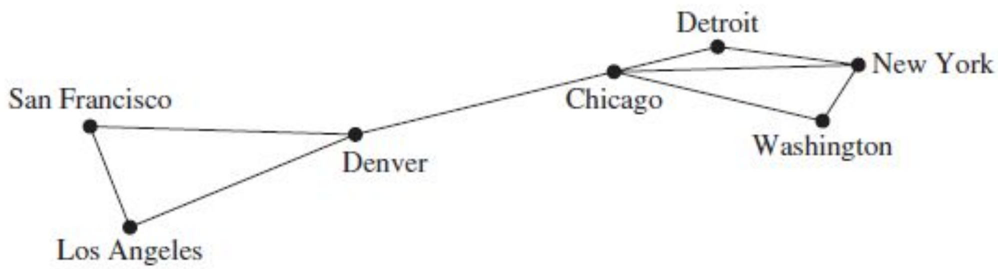
A graph G = (V ,E) consists of V , a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

A graph with an infinite vertex set or an infinite number of edges is called an infinite graph.

A graph with a finite vertex set and a finite edge set is called a finite graph.

Now suppose that a network is made up of data centers and communication links between computers. We can represent the location of each data center by a point and each communications

link by a line segment, as below

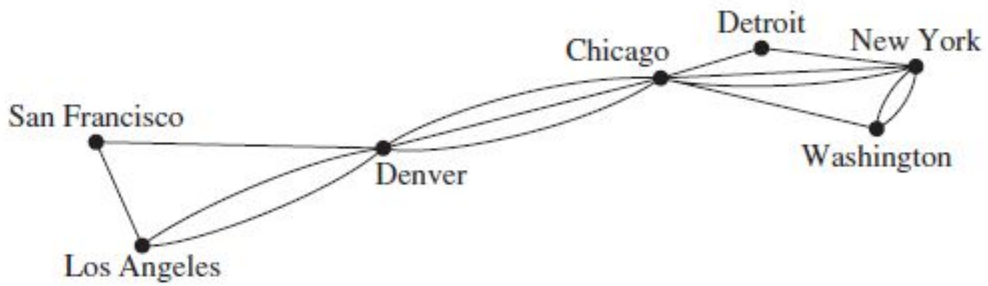


Note that each edge of the graph representing this computer network connects two different vertices.

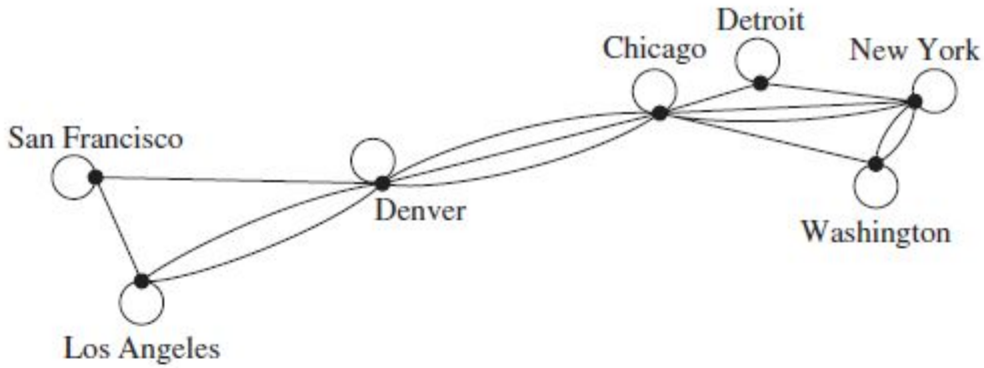
Simple Graph : A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

Multigraphs: Graphs that may have multiple edges connecting the same vertices are called multigraphs. When there are m different edges associated to the same unordered pair of vertices

{u, v}, we also say that {u, v} is an edge of multiplicity m.



Sometimes a communications link connects a data center with itself, perhaps a feedback loop for diagnostic purposes.



Pseudographs: Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices or a vertex to itself **(Loops)**, are sometimes called pseudographs.

So far the graphs we have introduced are **undirected graphs**. Their edges are also said

to be undirected.

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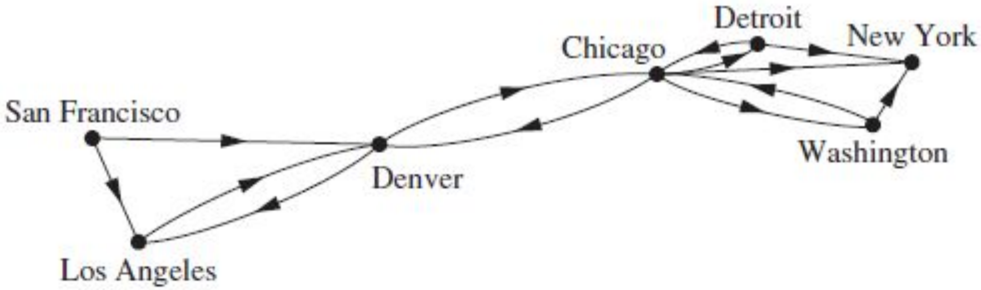
In a computer network, some links may operate in only one direction (such links are called single duplex lines). This may be the case if there is a large amount of traffic sent to some data centers, with little or no traffic going in the opposite direction.



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A directed graph (or digraph) (V ,E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices.

The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

A directed graph may also contain directed edges that connect vertices u and v in both directions; that is, when a digraph contains an edge from u to v, it may also contain one or more edges from v to u.

When a directed graph has no loops and has no multiple directed edges, it is called a simple directed graph.

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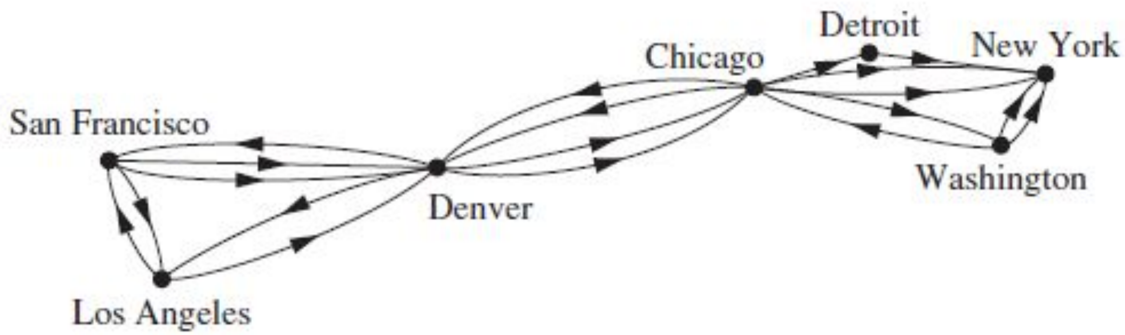
In some computer networks, multiple communication links between two data centers may be present.

Directed graphs that may have multiple directed edges from a vertex to a second (possibly the same) vertex are used to model such networks. We called such graphs directed **multigraphs**.

When there are m directed edges, each associated to an ordered pair of vertices (u, v), we say that (u, v) is an edge of multiplicity m.



For some models we may need a graph where some edges are undirected, while others are directed. A graph with both directed and undirected edges is called a mixed graph.

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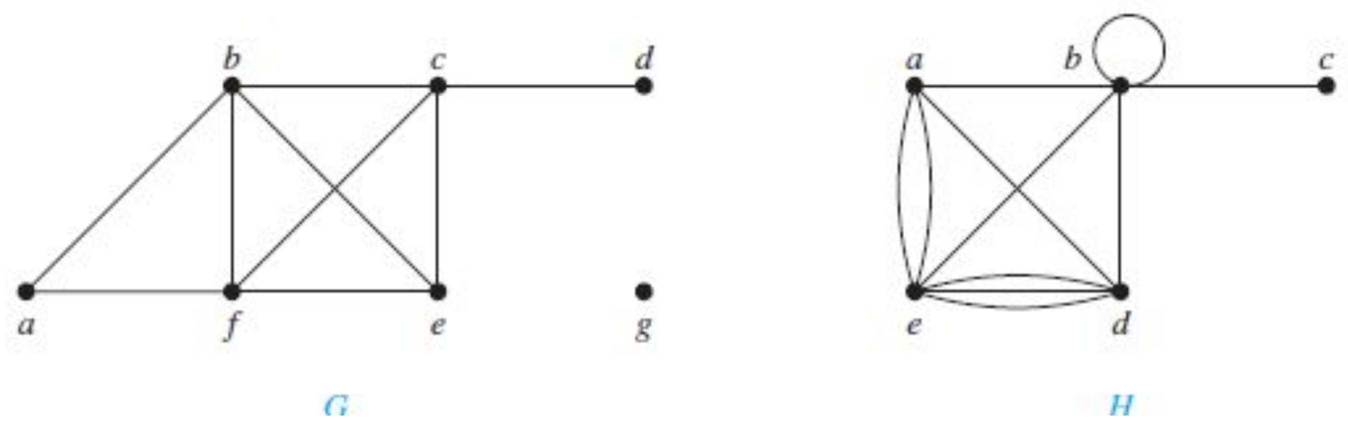
TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

- Two vertices u and v in an undirected graph G are called **adjacent** (or neighbors) in G if u and v are endpoints of an edge e of G. Such an edge e is called **incident** with the vertices u and v and e is said to connect u and v.

- The **degree** of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by deg(v).

Exercise: What are the degrees and what are the neighborhoods of the vertices in the graphs G and H?



THE HANDSHAKING THEOREM Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} \deg(v).$$

(Note that this applies even if multiple edges and loops are present.)

Proof: Each edge contributes twice to the total degree count of all vertices. Thus, both sides of the equation equal to twice the number of edges.

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Example : How many edges are there in a graph with 10 vertices each of degree six?

Solution:

Sum of the degrees of the vertices is $6 \cdot 10 = 60$

Therefore, $2m = 60$ (m is the number of edges)

Therefore, $m = 30$.

Theorem : An undirected graph has an even number of vertices of odd degree.

Proof : Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph $G = (V, E)$ with m edges.

Then

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V2} \deg(v) = \sum_{v \in V2} \deg(v)$$

Because $\deg(v)$ is even for $v \in V_1$, the $\sum_{v \in V1} \deg(v)$ in the right-hand side of the last equality is even.

Furthermore, the sum of the two terms on the right-hand side of the last equality is even,

because this sum is 2m.

Hence, the $\sum_{v \in V2} \deg(v)$ in the sum is also even.

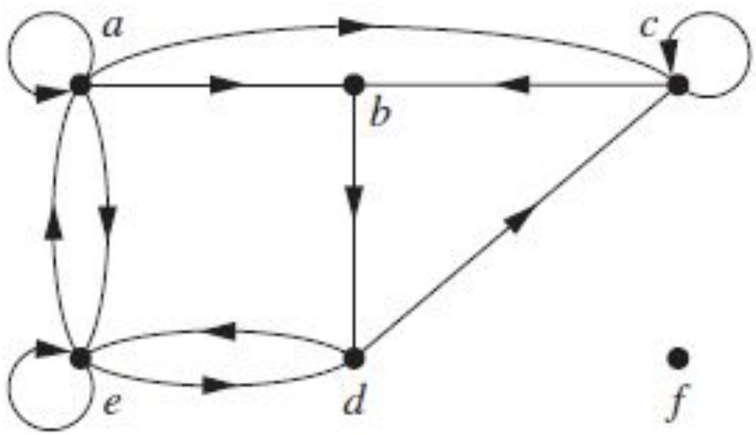
Because all the terms in this sum are odd, there must be an even number of such terms.

Thus, there are an even number of vertices of odd degree.

When *(u, v)* is an edge of the graph *G* with directed edges, *u* is said to be ***adjacent to v*** **and *v*** is said to be *adjacent from u*. The vertex *u* is called the *initial vertex* of *(u, v)*, and *v* is called the *terminal* or *end vertex* of *(u, v)*. The initial vertex and terminal vertex of a loop are the same.

In a graph with directed edges the **in-degree of a vertex v**, denoted by deg− (v), is the number of edges with v as their terminal vertex. The **out-degree of v**, denoted by deg+(v), is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

Find in-degree and out-degree of the following graph:



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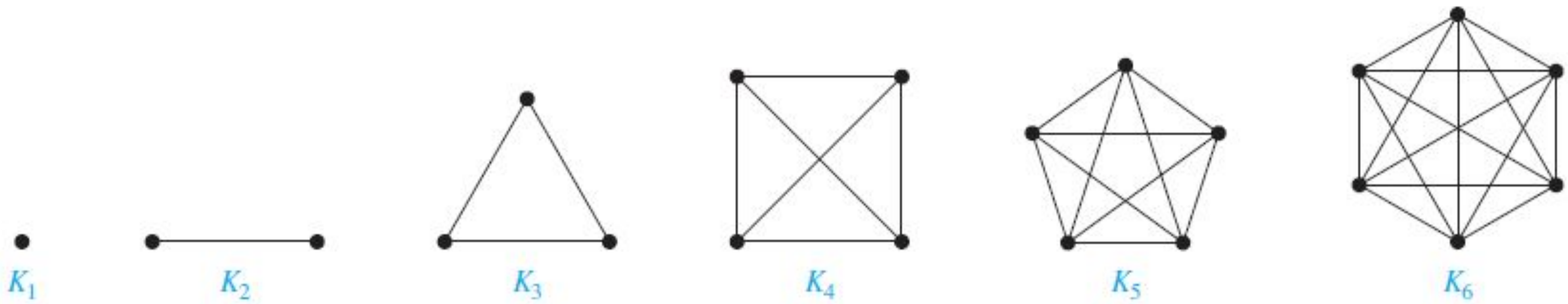
Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

Some Special Simple Graphs

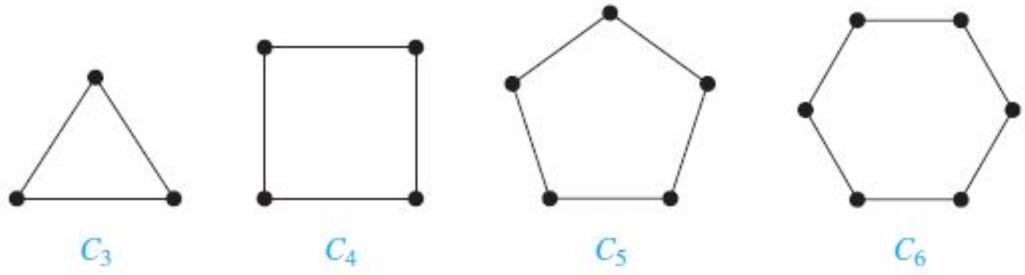
- **Complete Graphs :** A **complete graph on *n* vertices** (denoted by K_n) is a simple graph that contains exactly one edge between each pair of distinct vertices. The graphs K_n, for $n = 1, 2, 3, 4, 5, 6$, are displayed below:



- Number of edges in $K_n = n_{C_2}$

- **Cycles** A **cycle** $\boldsymbol{C_n}$, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.

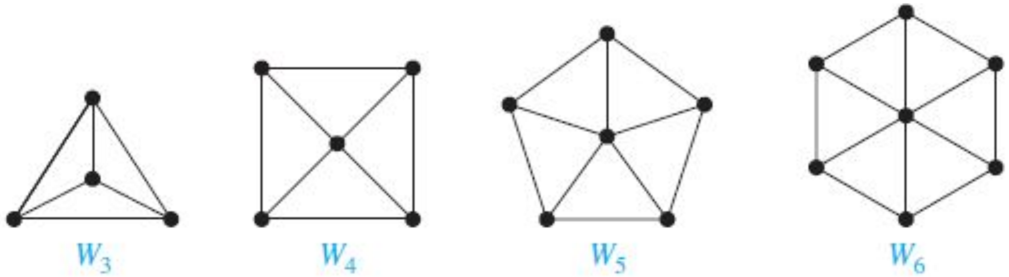
The cycles C_3, C_4, C_5, and C_6 are displayed below



Number of edges in C_n = n

- **Wheels :** We obtain a **wheel** W_n when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.

The wheels are displayed below:

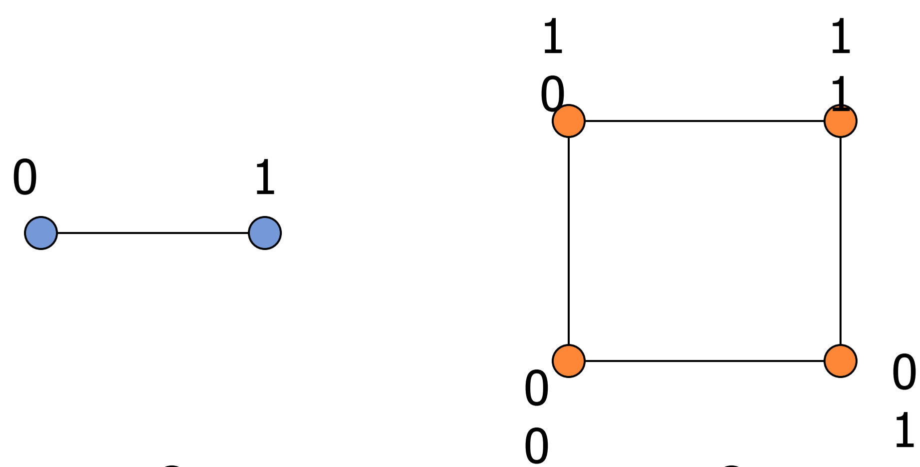


Number of vertices in **wheel** W_n = n+1

Number of edges in **wheel** W_n = n+n =2n

- **N-cubes:** N- Cube graphs Q_n, vertices represented by 2n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ by exactly one bit positions

Representation Example: Q1, Q2

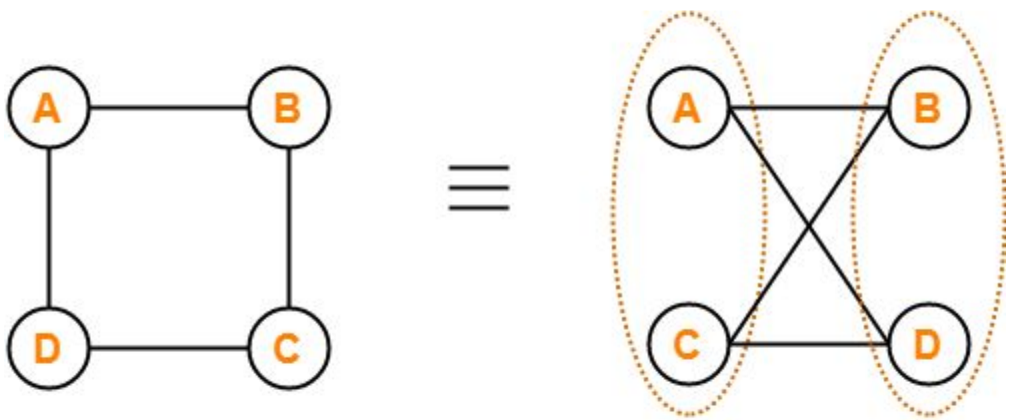


Q_1 Q_2

Bipartite Graphs

A simple graph G is called ***bipartite*** if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (no edge in G connects either two vertices in V_1 or two vertices in V_2).

When this condition holds, we call the pair (V_1, V_2) a ***bipartition*** of the vertex set V of G.

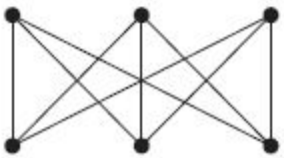


Example of Bipartite Graph

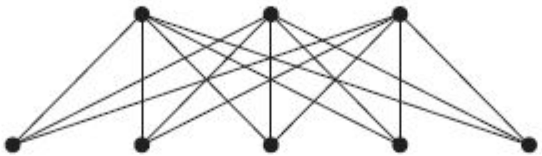
- **Complete Bipartite Graphs** A **complete bipartite graph** $\boldsymbol{K_{m,n}}$ is bipartite graph where every vertex of the first set is connected to every vertex of the second set.

The complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,5}$, and $K_{2,6}$ are displayed

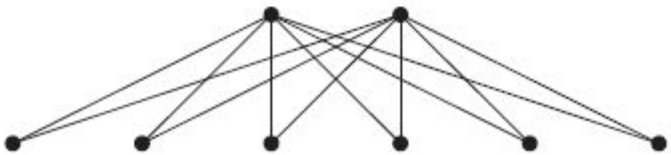
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$



$K_{2,6}$

Theorem : A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Proof: First, suppose that $G = (V, E)$ is a bipartite simple graph.

Then $V = V_1 \cup V_2$, where V_1 and V_2 are disjoint sets and every edge in E connects a vertex in V_1 and a vertex in V_2.

If we assign one color to each vertex in V_1 and a second color to each vertex in V_2, then no two adjacent vertices are assigned the same color.

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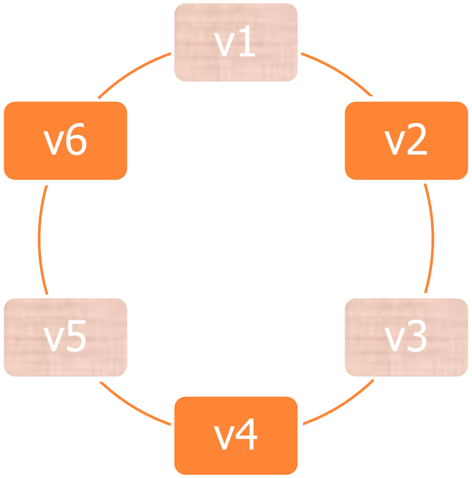
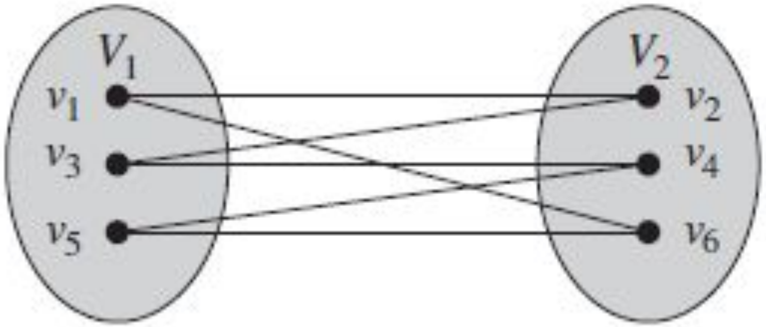
Now suppose that it is possible to assign colors to the vertices of the graph using just two colors so that no two adjacent vertices are assigned the same color.

Let V_1 be the set of vertices assigned one color and V_2 be the set of vertices assigned the other color. Then, V_1 and V_2 are disjoint and $V = V_1 \cup V_2$.

Therefore, every edge connects a vertex in V_1 and a vertex in V_2. Consequently, G is bipartite.

Example : Show that C_6 is bipartite.

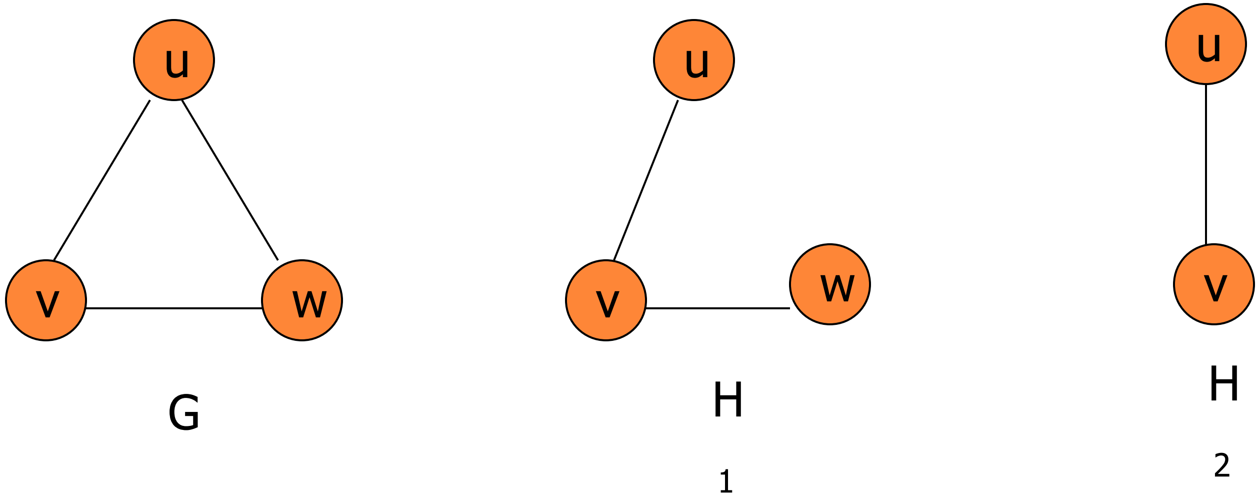




Sub Graph

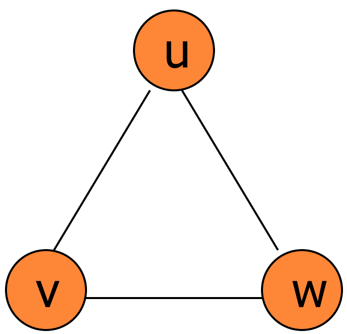
- A ***subgraph*** *of a graph* $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H \neq G$.

- Here H1 and H2 are sub graphs of Graph G



REMOVING EDGES OF A GRAPH

- Given a graph $G = (V, E)$ and an edge $e \in E$, we can produce a subgraph of G by removing the edge e. The resulting subgraph, denoted by $G - e$, has the same vertex set V as G. Its edge set is $E - e$. Hence, $G - e = (V, E - \{e\})$.



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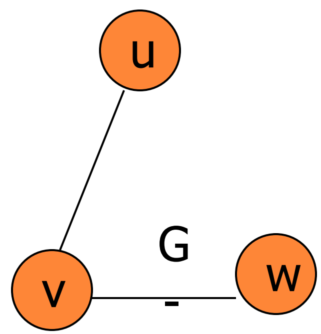
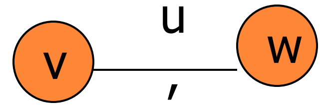


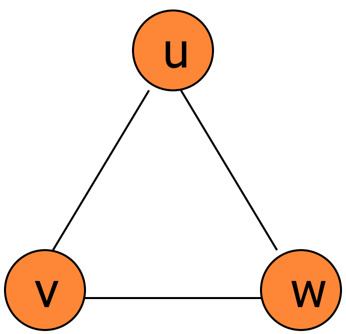
G - {u, w



G - {{u, w}, {

REMOVING VERTEX OF A GRAPH

- If v is a vertex in G, then $G - v$ is the subgraph of G obtained by deleting v from G and deleting all edges in G which contain v.



G

G - u

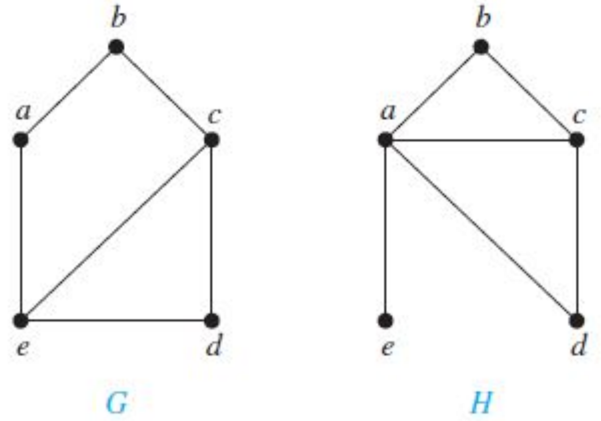
Isomorphic Graphs

- The simple graphs $G1 = (V1,E1)$ and $G2 = (V2,E2)$ are *isomorphic* if there exists a one to-one and onto function f from $V1$ to $V2$ with the property that a and b are adjacent in $G1$ if and only if $f(a)$ and $f(b)$ are adjacent in $G2$, for all a and b in $V1$.
- Such a function f is called an *isomorphism*.
- Two simple graphs that are not isomorphic are called *nonisomorphic*.

Determining whether Two Simple Graphs are Isomorphic

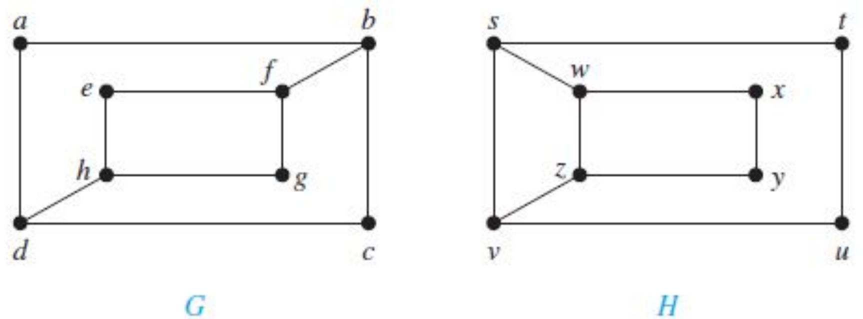
1. **Isomorphic simple graphs must have the same number of vertices**, as there is a one-to-one correspondence between the sets of vertices of the graphs.
2. **A vertex v of degree d in G must correspond to a vertex f (v) of degree d in H**, as a vertex w in G is adjacent to v if and only if f (v) and f (w) are adjacent in H.

Example : Show that the below graphs are not isomorphic.



Solution: Both G and H have five vertices and six edges. However, H has a vertex of degree one, namely, e, whereas G has no vertices of degree one. It follows that G and H are not isomorphic.

Example : Determine whether the below graphs are isomorphic.



Solution: The graphs G and H both have eight vertices and 10 edges. They also both have four vertices of degree two and four of degree three. Because these invariants all agree, it is still conceivable that these graphs are isomorphic.

However, G and H are not isomorphic. To see this, note that because $\deg(a) = 2$ in G, a must correspond to either t , u, x, or y in H, because these are the vertices of degree two in H. However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G.

Applications

- Graph isomorphisms, arise in applications of graph theory to chemistry and to the design of electronic circuits, and other areas including bioinformatics and computer vision.

- Chemists use multigraphs, known as molecular graphs, to model chemical compounds. In these graphs, vertices represent atoms and edges represent chemical bonds between these atoms.
- When a potentially new chemical compound is synthesized, a database of molecular graphs is checked to see whether the molecular graph of the compound is the same as one already known.
- Graph isomorphism can also be used to determine whether a chip from one vendor includes intellectual property from a different vendor.

Graphs Representation

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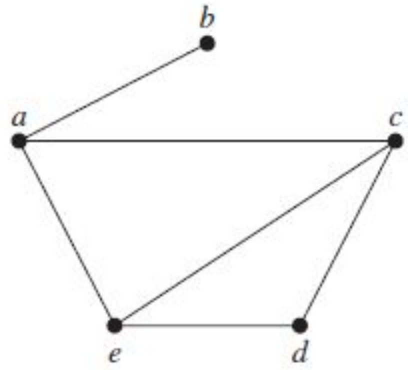
- **Adjacency Matrix :** Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $v1, v2, \ldots,$ vn. The a**djacency matrix A** (or **A**G) of G, with respect to this listing of the vertices, is the n x n zero–one matrix with 1 as its (i, j)th entry when vi and vj are adjacent, and 0 as its (i, j)th entry when they are not adjacent. In other words, if its adjacency matrix is $\mathbf{A} = [aij]$, then

$$a_{ij} = \begin{cases} 1 & if\ \{v_i, v_j\} is\ an\ edge\ of\ G \\ 0 & otherwise \end{cases}$$

Example : Find the **adjacency** matrix of the following graph.



Vertex	Adjacent Vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

- The adjacency matrix of a simple graph is symmetric, that is, $a_{ij} = a_{ji}$, because both of these entries are 1 when v_i and v_j are adjacent, and both are 0 otherwise.
- Adjacency matrices can also be used to represent undirected graphs with loops and with multiple edges.
- A loop at the vertex v_i is represented by a 1 at the (i, i)th position of the adjacency matrix.
- When multiple edges connecting the same pair of vertices v_i and v_j, the (i, j)th entry of this matrix equals the number of edges that are associated to $\{v_i , v_j \}$.

Example : Find the **adjacency** matrix of the following graph.

Initial Vertex	*Terminal Vertices with multiplicity*
a	b(3), d(2)
b	a(3), c(1), d(1)
c	c(1), b(1), d(2)
d	a(2),b(1), c(2)

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

- The matrix for a directed graph $G = (V, E)$ has a 1 in its (i, j)th position if there is an edge from vi to vj , where $v1, v2, \ldots, vn$ is an arbitrary listing of the vertices of the directed graph.
- In other words, if $\mathbf{A} = [aij]$ is the adjacency matrix for the directed graph with respect to this listing of the vertices, then

$$a_{ij} = \begin{cases} 1 & if\ \{v_i, v_j\} is\ an\ edge\ of\ G \\ 0 & otherwise \end{cases}$$

- The adjacency matrix for a directed graph does not have to be symmetric, because there may not be an edge from vj to vi when there is an edge from vi to vj

Example : Find the **adjacency** matrix of the following graph.

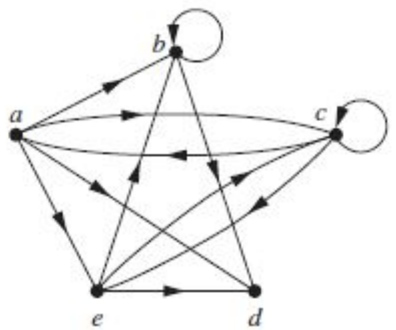


FIGURE 2 A Directed Graph.

TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

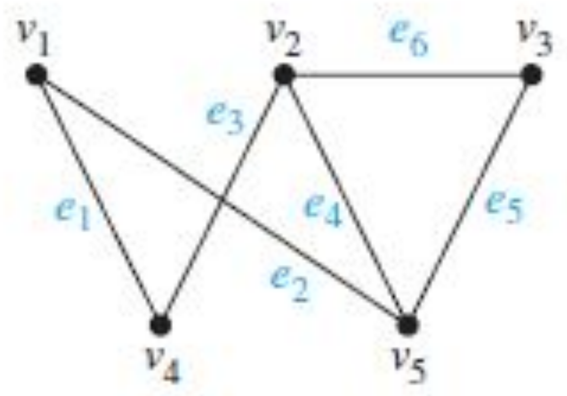
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Incidence Matrices

- Let $G = (V, E)$ be an undirected graph. Suppose that $v1, v2, \ldots, vn$ are the vertices and $e1, e2, \ldots, em$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [mij]$,

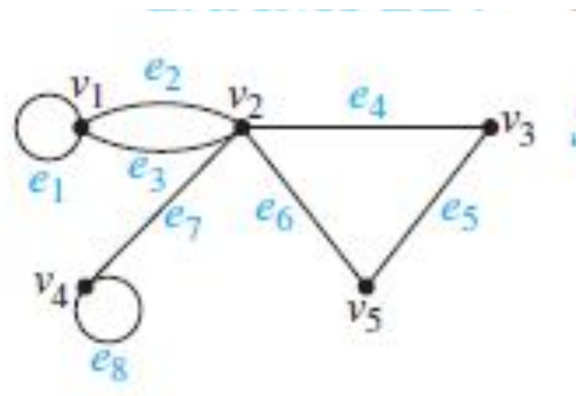
$$m_{ij} = \begin{cases} 1 & if \text{edge } ej \text{ is incident with } vi \\ 0 & otherwise \end{cases}$$

Example : Find the **incidence** matrix of the following graph.



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Example : Find the **incidence** matrix of the following graph.



$$
\begin{array}{c}
\\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}.
\end{array}
$$

APPLICATIONS

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- Show that f is an isomorphism, we can show that the adjacency matrix of G is the same as the adjacency matrix of H, when rows and columns are labeled to correspond to the images under f of the vertices in G that are the labels of these rows and columns in the adjacency matrix of G.