



BASIC COUNTING TECHNIQUES PERMUTATION & COMBINATIONS

CHAPTER III

Dr Honey Sharma

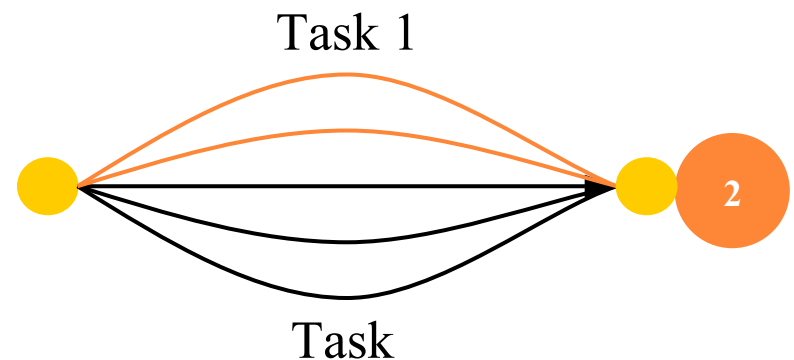
GGI, Ludhiana

SUM RULE

- Let us consider two tasks:
 - m is the number of ways to do **task 1**
 - n is the number of ways to do **task 2**
 - Tasks are independent of each other, i.e.,
 - Performing **task 1** does not accomplish **task 2** and vice versa.

- Sum rule: the number of ways that “**either** task 1 **or** task 2 can be done, but **not both**”, is $m + n$.

- Generalizes to multiple tasks ...



GENERALIZED SUM RULE:

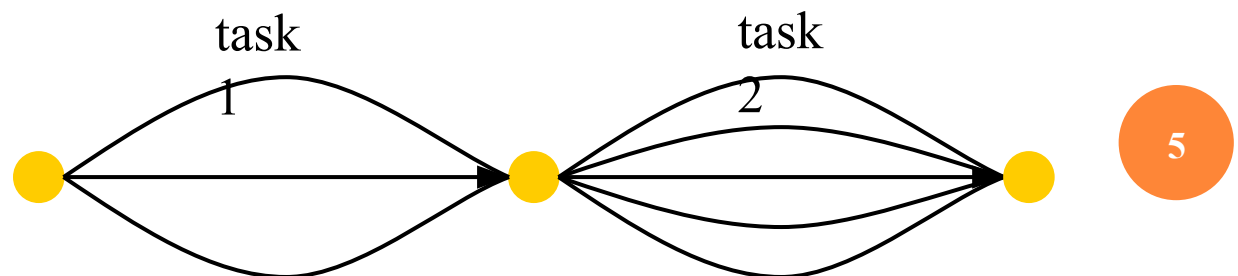
- If we have tasks T_1, T_2, \dots, T_m that can be done in n_1, n_2, \dots, n_m ways, respectively, and no two of these tasks can be done at the same time, then there are $n_1 + n_2 + \dots + n_m$ ways to do one of these tasks.

SUM RULE EXAMPLE

- How many strings of 4 decimal digits, have exactly three digits that are 9s?
 - The string can have:
 - The **non-9** as the first digit
 - OR the non-9 as the second digit
 - OR the non-9 as the third digit
 - OR the non-9 as the fourth digit
 - Thus, we use the sum rule
 - For each of those cases, there are 9 possibilities for the non-9 digit (any number other than 9)
 - Thus, the answer is $9+9+9+9 = 36$

PRODUCT RULE

- Let us consider two tasks:
 - m is the number of ways to do **task 1**
 - n is the number of ways to do **task 2**
 - Tasks are independent of each other, i.e.,
 - Performing task 1 does not accomplish task 2 and vice versa.
- Product rule: the number of ways that “**both** tasks 1 and 2 can be done” in mn .
- Generalizes to multiple tasks ...



GENERALIZED PRODUCT RULE:

- If we have a procedure consisting of sequential tasks T_1, T_2, \dots, T_m that can be done in n_1, n_2, \dots, n_m ways, respectively, then there are $n_1 \cdot n_2 \cdot \dots \cdot n_m$ ways to carry out the procedure.

PRODUCT RULE EXAMPLE

- There are 18 math majors and 325 CS majors
 - How many ways are there to pick one math major and one CS major?
- Total is $18 * 325 = 5850$

PRODUCT RULE EXAMPLE

□ How many strings of 4 decimal digits, do not contain the same digit twice?

- We want to choose a digit, then another that is not the same, then another...
 - First digit: 10 possibilities
 - Second digit: 9 possibilities (all but first digit)
 - Third digit: 8 possibilities
 - Fourth digit: 7 possibilities
- Total = $10 \times 9 \times 8 \times 7 = 5040$

How many strings of 4 decimal digits, end with an even digit?

- First three digits have 10 possibilities
- Last digit has 5 possibilities
- Total = $10 \times 10 \times 10 \times 5 = 5000$

- Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters.
- Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected does not matter.

PERMUTATION

- Any arrangement of a set of n objects in a given order is called a *permutation* of the object (taken all at a time).
- Any arrangement of any $r \leq n$ of these objects in a given order is called an “ r -permutation” or “a permutation of the n objects taken r at a time.” Consider, for example, the set of letters A, B, C, D . Then:
 - (i) $BDCA$, $DCBA$, and $ACDB$ are permutations of the four letters (taken all at a time).
 - (ii) BAD , ACB , DBC are permutations of the four letters taken three at a time.
 - (iii) AD , BC , CA are permutations of the four letters taken two at a time.

PERMUTATION

- The number of permutations of n objects taken r at a time will be denoted by $P(n, r)$ (other texts may use nPr , $P_{n,r}$, or $(n)r = n!/(n-r)!$).
- There are $n!$ permutations of n objects (taken all at a time).

PERMUTATIONS WITH REPETITIONS

- ❏ If set A which contains n elements consists of n_1 elements of the first kind, n_2 elements of the second kind,..., and n_k elements of k-th kind ($n=n_1+n_2+\dots+n_k$), the number of permutations with repetition is given by:

$$\frac{n!}{n_1!.n_2!....n_k!}$$

- Find the number n of distinct permutations that can be formed from all the letters of each word:

(a) *THOSE*; (b) *UNUSUAL*; (c) *SOCIOLOGICAL*.

□ Find the number m of ways that 7 people can arrange themselves:

(a) In a row of chairs; (b) Around a circular table.

(a) Here $m = P(7, 7) = 7!$ ways.

(b) One person can sit at any place at the table. The other 6 people can arrange themselves in $6!$ ways around the table; that is $m = 6!$.

This is an example of a *circular permutation*. In general, n objects can be arranged in a circle in $(n - 1)!$ ways.

□ A class contains 8 students. Find the number n of samples of size 3:

(a) With replacement; (b) Without replacement.

Solution:

- (a) Each student in the ordered sample can be chosen in 8 ways; hence, there are $n = 8 \cdot 8 \cdot 8 = 512$ samples
- (b) The first student in the sample can be chosen in 8 ways, the second in 7 ways, and the last in 6 ways. Thus, there are $n = 8 \cdot 7 \cdot 6 = 336$ samples

- Find the number of automobile license plates where: (a) Each plate contains 2 different letters followed by 3 different digits. (b) The first digit cannot be 0.

Solution:

- (a) $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8$
(b) $26 \cdot 25 \cdot 9 \cdot 9 \cdot 8$

- How many 5-digits telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67, for example 67125 etc., and no digit appears more than once?

Solution:

I place after 67 can be filled in 8 ways.

II place after 67 can be filled in 7 ways.

- II place after 67 can be filled in 6 ways.

∴ No of telephone numbers that can be constructed.

$$\Rightarrow 8 \times 7 \times 6$$

$$\Rightarrow 336$$

- How many bit strings of length 8 either start with 1-bit or ends with two bits 00?

Solution:

Starting with the 1 bit = $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$

Ending with the 00 bit = $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64$

Starting with 1 and end with 00 = $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$

Total $128 + 64 - 32 = 160$

- In how many ways can 3 boys and 3 girls sit in a row?

Solution: There are $6! = 720$ ways 6 people can sit in a row.

- In how many ways can 3 boys and 3 girls sit in a row if the boys and girls are each to sit together?

Solution: (We can have either BBBGGG or GGGBBB, so disregarding the order within boys and within girls, there are 2 possibilities. There are $3!$ ways to order the boys, and $3!$ ways to order the girls, so all together there are $(2)(3!)(3!) = 72$ ways to do this.

- In how many ways if only the boys must sit together?

Solution: If the boys are regarded as a unit, then we can first order the objects $\{B, G1, G2, G3\}$, where B is the unit of the three boys, and G1, G2, G3 are the girls. There are $4!$ ways to do this. Then the boys can be ordered in $3!$ ways. Thus the answer is $4!3! = 144$.

- In how many ways if no two people of the same sex are allowed to sit together?

Solution: We can have either BGBGBG or GBGBGB. For each such arrangement, there are $3!$ ways to arrange the boys, and $3!$ ways to arrange the girls, so the answer is $(2)(3!)(3!) = 72$.

COMBINATIONS

Let S be a set with n elements. A combination of these n elements taken r at a time is any selection of r of the elements where **order does not count**. Such a selection is called an r -combination; it is simply a subset of S with r elements.

The number of such combinations will be denoted by $C(n, r)$ (other texts may use nC_r).

$$C(n, r) = {}^nC_r = \frac{n!}{r!(n-r)!}$$

- Combinations A class contains 10 students with 6 men and 4 women. Find the number n of ways to: (a) Select a 4-member committee from the students. (b) Select a 4-member committee with 2 men and 2 women. (c) Elect a president, vice president, and treasurer

Solution:

- (a) $C(10, 4)$
(b) $C(6, 2) C(4, 2)$
(c) 10.9.8

A woman has 11 close friends. Find the number of ways she can invite 5 of them to dinner where:

- (a) There are no restrictions.
- (b) Two of the friends are married to each other and will not attend separately.
- (c) Two of the friends are not speaking with each other and will not attend together.

Solution:

(b) $C(9,3)+C(9,5)$

© Case 1: When 1 do not attend the party = 1. $C(9,4)$

Case 2: When 2 do not attend the party = 1. $C(9,4)$

Case 3: When both of them do not attend the party = $C(9,5)$

$\therefore \therefore$ Total no: of ways = 2. $C(9,4)+ C(9,5)$

- A class contains 8 men and 6 women and there is one married couple in the class. Find the number m of ways a teacher can select a committee of 4 from the class where the husband or wife but not both can be on the committee.

Solution :

$$m = C(12, 4) + 2C(12, 3) = 935.$$

- A box has 6 blue socks and 4 white socks. Find the number of ways two socks can be drawn from the box where: (a) There are no restrictions. (b) They are different colors. (c) They are the same color.

Solution: $C(10, 2)$, $C(6, 1)C(4, 1)$, $C(6, 2) + C(4, 2)$

- A women student is to answer 10 out of 13 questions. Find the number of her choices where she must answer:
- (a) the first two questions; (b) the first or second question but not both; (c) exactly 3 out of the first 5 questions; (d) at least 3 of the first 5 questions.

Solution:

- (a) $C(11,8)$
- (b) $C(2,1)C(11,9)$
- (c) $C(5,3)C(8,7)$
- (d) $C(5,3)C(8,7)+C(5,4)C(8,6)+C(5,5)C(8,5)$