Propositional logic
Unit III

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1.1 Propositional Logic

Introduction

- A **proposition** is a declarative sentence (a sentence that declares a fact) that is either true or false, but not both.
- Are the following sentences propositions?
  - Toronto is the capital of Canada. (Yes)
  - Read this carefully. (No)
  - 1+2=3 (Yes)
  - x+1=2 (No)
  - What time is it? (No)

1.1 Propositional Logic

- Propositional Logic – the area of logic that deals with propositions
- Propositional Variables – variables that represent propositions: p, q, r, s
  - E.g. Proposition p – “Today is Friday.”
- Truth values – T, F

1.1 Propositional Logic

DEFINITION 1

Let p be a proposition. The negation of p, denoted by $\neg p$, is the statement "It is not the case that p."

The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$ is the opposite of the truth value of p.

- Examples
  - Find the negation of the proposition "Today is Friday." and express this in simple English.

    Solution: The negation is "It is not the case that *today is Friday*." In simple English, "Today is not Friday." or "It is not Friday today."
  - Find the negation of the proposition "At least 10 inches of rain fell today in Miami." and express this in simple English.

    Solution: The negation is "It is not the case that *at least 10 inches of rain fell today in Miami*." In simple English, "Less than 10 inches of rain fell today in Miami."

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- Note: Always assume fixed times, fixed places, and particular people unless otherwise noted.
- Truth table:

The Truth Table for the Negation of a Proposition.	
p	$\neg p$
T	F
F	T

- Logical operators are used to form new propositions from two or more existing propositions. The logical operators are also called connectives.

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DEFINITION 2

Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition "p and q". The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

- Examples
  - Find the conjunction of the propositions p and q where p is the proposition "Today is Friday." and q is the proposition "It is raining today.", and the truth value of the conjunction.

    Solution: The conjunction is the proposition "Today is Friday and it is raining today." The proposition is true on rainy Fridays.

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DEFINITION 3

Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is the proposition "p or q". The conjunction $p \vee q$ is false when both p and q are false and is true otherwise.

- Note:

  inclusive or : The disjunction is true when at least one of the two propositions is true.

  - E.g. "Students who have taken calculus or computer science can take this class." – those who take one or both classes.

  exclusive or : The disjunction is true only when one of the proposition is true.

  - E.g. "Students who have taken calculus or computer science, but not both, can take this class." – only those who take one of them.

- Definition 3 uses *inclusive or*.

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DEFINITION 4

Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The Truth Table for the Exclusive *Or* (*XOR*) of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

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Conditional Statements

DEFINITION 5

Let p and q be propositions. The *conditional statement* $p \rightarrow q$, is the proposition "if p, then q." The conditional statement is false when **p is true and q is false**, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

- A conditional statement is also called an implication.
- Example: "If I am elected, then I will lower taxes." $p \rightarrow q$

implication:	p	q	$p \rightarrow q$
elected, lower taxes.	T	T	T
not elected, lower taxes.	F	T	T
not elected, not lower taxes.	F	F	T
elected, not lower taxes.	T	F	F

1.1 Propositional Logic

- Example:
  - Let p be the statement “Maria learns discrete mathematics.” and q the statement “Maria will find a good job.” Express the statement $p \longrightarrow q$ as a statement in English.

  Solution: Any of the following -

  “If Maria learns discrete mathematics, then she will find a good job.

  “Maria will find a good job when she learns discrete mathematics.”

  “For Maria to get a good job, it is sufficient for her to learn discrete mathematics.”

  “Maria will find a good job unless she does not learn discrete mathematics.”

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- Other conditional statements:
  - Converse of $p \longrightarrow q$: $q \longrightarrow p$
  - Contrapositive of $p \longrightarrow q$: $\neg q \longrightarrow \neg p$
  - Inverse of $p \longrightarrow q$: $\neg p \longrightarrow \neg q$

- What are the contrapositive, the converse, and the inverse of the conditional statement “The home team wins whenever it is raining?”

Solution: Because “q whenever p” is one of the ways to express the conditional statement

$p \to q$, the original statement can be rewritten as

“If it is raining, then the home team wins.”

Consequently, the contrapositive of this conditional statement is

“If the home team does not win, then it is not raining.”

The converse is

“If the home team wins, then it is raining.”

The inverse is

“If it is not raining, then the home team does not win.”

Only the contrapositive is equivalent to the original statement.

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DEFINITION 6

Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

- $p \leftrightarrow q$ has the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$
- "*if and only if*" can be expressed by "*iff*"
- Example:
  - Let p be the statement "You can take the flight" and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement

    "You can take the flight if and only if you buy a ticket."

    Implication:

    If you buy a ticket you can take the flight.

    If you don't buy a ticket you cannot take the flight.

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The Truth Table for the Biconditional $p \leftrightarrow q$.		
p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

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Truth Tables of Compound Propositions

- We can use connectives to build up complicated compound propositions involving any number of propositional variables, then use truth tables to determine the truth value of these compound propositions.
- Example: Construct the truth table of the compound proposition

$$(p \vee \neg q) \longrightarrow (p \wedge q).$$

The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

1.1 Propositional Logic

Precedence of Logical Operators

- We can use parentheses to specify the order in which logical operators in a compound proposition are to be applied.
- To reduce the number of parentheses, the precedence order is defined for logical operators.

Precedence of Logical Operators.	
Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

E.g. $\neg p \wedge q = (\neg p) \wedge q$

$p \wedge q \vee r = (p \wedge q) \vee r$

$p \vee q \wedge r = p \vee (q \wedge r)$

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Translating English Sentences

- English (and every other human language) is often ambiguous. Translating sentences into compound statements removes the ambiguity.
- Example: How can this English sentence be translated into a logical expression?

  "You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Solution: Let q, r, and s represent "You can ride the roller coaster," "You are under 4 feet tall," and "You are older than 16 years old." The sentence can be translated into:

$$(r \wedge \neg s) \rightarrow \neg q.$$

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- Example: How can this English sentence be translated into a logical expression?

  "You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Solution: Let a, c, and f represent "You can access the Internet from campus," "You are a computer science major," and "You are a freshman." The sentence can be translated into:

$$(c \lor \neg f) \rightarrow a$$

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Logic and Bit Operations

- Computers represent information using bits.
- A **bit** is a symbol with two possible values, 0 and 1.
- By convention, 1 represents T (true) and 0 represents F (false).
- A variable is called a Boolean variable if its value is either true or false.
- Bit operation – replace true by 1 and false by 0 in logical operations.

Table for the Bit Operators *OR*, *AND*, and *XOR*.

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

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DEFINITION 7

A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

- Example: Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit string 01 1011 0110 and 11 0001 1101.

Solution:

01 1011 0110

11 0001 1101

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11 1011 1111 bitwise *OR*

01 0001 0100 bitwise *AND*

10 1010 1011 bitwise *XOR*

1.2 Propositional Equivalences

Introduction

DEFINITION

- A compound proposition that is always true, no matter what the truth values of the propositions that occurs in it, is called a ***tautology***.
- A compound proposition that is always false is called a ***contradiction***.
- A compound proposition that is neither a tautology or a contradiction is called a ***contingency***.

Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

1.2 Propositional Equivalences

Logical Equivalences

DEFINITION 2

The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- Compound propositions that have the same truth values in all possible cases are called logically equivalent.
- Example: Show that $\neg p \vee q$ and $p \rightarrow q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Algebra of propositions

- **Idempotent laws:**

  (1a) $p \lor p \equiv p$

  (1b) $p \land p \equiv p$

- **Associative laws:**

  (2a) $(p \lor q) \lor r \equiv p \lor (q \lor r)$

  (2b) $(p \land q) \land r \equiv p \land (q \land r)$

- **Commutative laws:**

  (3a) $p \lor q \equiv q \lor p$

  (3b) $p \land q \equiv q \land p$

- **Distributive laws:**

  (4a) $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

  (4b) $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$

Algebra of propositions

- **Identity laws:**

  (5a) $p \lor F \equiv p$ (5b) $p \land T \equiv p$

  (6a) $p \lor T \equiv T$ (6b) $p \land F \equiv F$
- **Involution law: (7) $\neg\neg p \equiv p$**
- **Complement laws:**

  (8a) $p \lor \neg p \equiv T$ (8b) $p \land \neg p \equiv T$

  (9a) $\neg T \equiv F$ (9b) $\neg F \equiv T$
- **DeMorgan's laws:**

  (10a) $\neg(p \lor q) \equiv \neg p \land \neg q$

  (10b) $\neg(p \land q) \equiv \neg p \lor \neg q$

Constructing New Logical Equivalences

- Example: Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

  Solution:

  $$\begin{aligned} \neg(p \to q) &\equiv \neg(\neg p \vee q) \\ &\equiv \neg(\neg p) \wedge \neg q \\ &\equiv p \wedge \neg q \end{aligned}$$

- *Example: Show that $(p \wedge q) \to (p \vee q)$* is a tautology.

  Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to T.

  $$\begin{aligned} (p \wedge q) \to (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) \\ &\equiv (\neg p \vee \neg q) \vee (p \vee q) \\ &\equiv (\neg p \vee p) \vee (\neg q \vee q) \\ &\equiv \mathrm{T} \vee \mathrm{T} \\ &\equiv \mathrm{T} \end{aligned}$$

- *Note: The above examples can also be done using truth tables*.

Arguments

- An *argument* is an assertion that a given set of propositions $P1, P2, \ldots, Pn$, called *premises*, yields (has a consequence) another proposition Q, called the *conclusion*. Such an argument is denoted by

  $P1, P2, \ldots, Pn \vdash Q$

The notion of a “logical argument” or “valid argument” is formalized as follows:

- **Definition** An argument $P1, P2, \ldots, Pn \vdash Q$ is said to be *valid* if Q is true whenever all the premises $P1, P2, \ldots, Pn$ are true.
- An argument which is not valid is called *fallacy*.

- **EXAMPLE**

(a) The following argument is valid: $p, p \rightarrow q \vdash q$ (***Law of Detachment***)

(b) The following argument is a fallacy: $p \rightarrow q, q \vdash p$

Now the propositions $P1, P2, \ldots, Pn$ are true simultaneously if and only if the proposition $P1 \wedge P2 \wedge \ldots Pn$ is true. Thus the argument $P1, P2, \ldots, Pn \vdash Q$ is valid if and only if Q is true whenever $P1 \wedge P2 \wedge \ldots \wedge Pn$ is true or, equivalently, if the proposition $(P1 \wedge P2 \wedge \ldots \wedge Pn) \rightarrow Q$ is a tautology. We state this result formally.

Theorem: The argument $P1, P2, \ldots, Pn \vdash Q$ is valid if and only if the proposition $(P1 \wedge P2 \ldots \wedge Pn) \rightarrow Q$ is a tautology.

EXAMPLE A fundamental principle of logical reasoning states: "If *p* implies *q* and *q* implies *r*, then *p* implies *r*"

- **EXAMPLE** Consider the following argument:

  $S1$: If a man is a bachelor, he is unhappy.

  $S2$: If a man is unhappy, he dies young.

  S : Bachelors die young

Here the statement S below the line denotes the conclusion of the argument, and the statements $S1$ and $S2$ above the line denote the premises. We claim that the argument $S1, S2 \vdash S$ is valid.

For the argument is of the form $p \to q, q \to r \vdash p \to r$ where p is “He is a bachelor,” q is “He is unhappy” and r is “He dies young;”

Predicates

Statements involving variables, such as

"$x > 3$," "$x = y + 3$," "$x + y = z$,"

and

"computer *x* is under attack by an intruder,"

and

"computer *x* is functioning properly,"

are often found in mathematical assertions, in computer programs, and in system specifications.

These statements are neither true nor false when the values of the variables are not specified. In

this section, we will discuss the ways that propositions can be produced from such statements.

- The statement “*x* is greater than 3” has two parts. The first part, the variable *x*, is the subject of the statement. The second part—the **predicate**, “is greater than 3”—refers to a property that the subject of the statement can have.
- We can denote the statement “*x* is greater than 3” by *P(x)*, where *P* denotes the predicate “is greater than 3” and *x* is the variable. The statement *P(x)* is also said to be the value of the **propositional function** *P* at *x*.
- Once a value has been assigned to the variable *x*, the statement *P(x)* becomes a proposition and has a truth value. We obtain the statement P(4) by setting x = 4 in the statement “x > 3.” Hence, P(4), which is the statement “4 > 3,” is true. However, P(2), which is the statement “2 > 3,” is false.
- Let Q(x, y) denote the statement “x = y + 3.” What are the truth values of the propositions Q(1, 2) and Q(3, 0)?

Quantifiers

When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value.

However, there is another important way, called **quantification**, to create a proposition from a propositional function.

Quantification expresses the extent to which a predicate is true over a range of elements. In English, the words *all*, *some*, *many*, *none*, and *few* are used in quantifications.

Quantifiers

When the variables in a propositional function are assigned We will focus on two types of quantification here:

universal quantification, which tells us that a predicate is true for every element under consideration, and existential quantification, which tells us that there is one or more element under consideration for which the predicate is true.

The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

The *universal quantification* of *P(x)* is the statement "*P(x)* for all values of *x* in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of *P(x)*. Here $\forall$ is called the **universal quantifier.** We read $\forall x\, P(x)$ as "for all *x* *P(x)*" or "for every *x* *P(x)*."

An element for which *P(x)* is false is called a **counterexample** of $\forall x\, P(x)$.

When all the elements in the domain can be listed—say, x1, x2, . . ., xn—it follows that the

universal quantification $\forall x P(x)$ is the same as the conjunction $P(x1) \land P(x2) \land \cdots \land P(xn)$.

because this conjunction is true if and only if P(x1), P(x2), . . . , P (xn) are all true.

The existential quantification of P(x) is the proposition “There exists an element x in the domain such that P(x).”

We use the notation $\exists xP(x)$ for the existential quantification of P(x). Here $\exists$ is called the **existential quantifier**.

Besides the phrase “there exists,” we can also express existential quantification in many otherways, such as by using the words “for some,” “for at least one,” or “there is.”

When all elements in the domain can be listed—say, $x1, x2, \ldots,$ xn—the existential quantification $\exists xP(x)$ is the same as the disjunction

$P(x1) \lor P(x2) \lor \cdots \lor P(xn),$

because this disjunction is true if and only if at least one of $P(x1),$ $P(x2), \ldots, P(xn)$ is true.

The quantifiers $\forall$ and $\exists$ have **higher precedence** than all logical operators from propositional calculus. For example, $\forall xP(x) \lor Q(x)$ is the disjunction of $\forall xP(x)$ and Q(x). In other words, it means $(\forall xP(x)) \lor Q(x)$ rather than $\forall x(P(x) \lor Q(x))$.

Statement	When True?	When False?
$\forall x P(x)$	P(x) is true for every x.	There is an x for which P(x) is false.
$\exists x P(x)$	There is an x for which P(x) is true.	P(x) is false for every x.

Example: What is the truth value of $\forall x(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

Solution: The universal quantification $\forall x(x^2 \geq x)$, where the domain consists of all real numbers, is false.

For example, $(1/2)^2 < 1/2$. Note that $x^2 \geq x$ if and only if $x^2 - x = x(x - 1) \geq 0$.

Consequently, $x^2 \geq x$ if and only if $x \leq 0$ or $x \geq 1$. It follows that $\forall x(x^2 \geq x)$ is false if the domain consists of all real numbers (because the inequality is false for all real numbers x with $0 < x < 1$).

However, if the domain consists of the integers, $\forall x(x2 \geq x)$ is true, because there are no integers x with $0 < x < 1$.

Logical Equivalences Involving Quantifiers

Definition : Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.

Example: Show that $\forall x(P(x) \land Q(x))$ and $\forall xP(x) \land \forall xQ(x)$ are logically equivalent.

Solution: We can show that $\forall x(P(x) \land Q(x))$ and $\forall xP(x) \land \forall xQ(x)$ are logically equivalent by doing two things. First, we show that if $\forall x(P(x) \land Q(x))$ is true, then $\forall xP(x) \land \forall xQ(x)$ is true. Second, we show that if $\forall xP(x) \land \forall xQ(x)$ is true, then $\forall x(P(x) \land Q(x))$ is true.

So, suppose that $\forall x(P(x) \land Q(x))$ is true. This means that if a is in the domain, then $P(a) \land Q(a)$ is true. Hence, $P(a)$ is true and $Q(a)$ is true. Because $P(a)$ is true and $Q(a)$ is true for every element in the domain, we can conclude that $\forall xP(x)$ and $\forall xQ(x)$ are both true. This means that $\forall xP(x) \land \forall xQ(x)$ is true.

Next, suppose that $\forall xP(x) \land \forall xQ(x)$ is true. It follows that $\forall xP(x)$ is true and $\forall xQ(x)$ is true. Hence, if a is in the domain, then $P(a)$ is true and $Q(a)$ is true. It follows that for all a, $P(a) \land Q(a)$ is true. It follows that $\forall x(P(x) \land Q(x))$ is true.

Problem: Show that $\forall x P(x) \lor \forall x Q(x)$ and $\forall x(P(x) \lor Q(x))$ are not logically equivalent.

Solution:

Under the following interpretation: domain: set of people in the world $P(x)$ =“x is male”.

$Q(x)$ =“x is female”.

We have: $\forall x(P(x) \lor Q(x))$ (every person is a male or a female) is true;

while $\forall x P(x) \lor \forall x Q(x)$ (every person is a male or every person is a female) is false.

Problem: Show that $\exists x P(x) \wedge \exists x Q(x)$ and $\exists x (P(x) \wedge Q(x))$ are not logically equivalent.

Solution:

Let U = N. Set

P(x): “x is prime” and Q(x): “x is composite” (ie not prime).

Then $\exists x (P(x) \wedge Q(x))$ is false, but $\exists x P(x) \wedge \exists x Q(x)$ is true

Negating Quantified Expressions

Consider the negation of the statement

"Every student in your class has taken a course in calculus."

This statement is a universal quantification, namely, $\forall x P(x)$,

where $P(x)$ is the statement "x has taken a course in calculus" and the domain consists of the students in your class.

The negation of this statement is "It is not the case that every student in your class has taken a course in calculus." This is equivalent to "There is a student in your class who has not taken a course in calculus."

This is simply the existential quantification of the negation of the original propositional function, namely, $\exists x \neg P(x)$.

Problem: Show that $\neg\forall x P(x) \equiv \exists x \neg P(x)$ is logical equivalence.

Solution:

To show that $\neg\forall x P(x)$ and $\exists x P(x)$ are logically equivalent no matter what the propositional function $P(x)$ is and what the domain is, first note that

$\neg\forall x P(x)$ is true if and only if $\forall x P(x)$ is false. Next, $\forall x P(x)$ is false if and only if there is an element x in the domain for which $P(x)$ is false. This holds if and only if there is an element x in the domain for which $\neg P(x)$ is true. Finally, it means $\exists x \neg P(x)$ is true.

Therefore, we can conclude that $\neg\forall x P(x)$ is true if and only if $\exists x \neg P(x)$ is true. It follows that $\neg\forall x P(x)$ and $\exists x \neg P(x)$ are logically equivalent.

TABLE 2 De Morgan's Laws for Quantifiers.			
Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

When the domain has n elements x1, x2, . . . , xn, it follows that $\neg\forall x P(x)$ is the same as $\neg(P(x1) \wedge P(x2) \wedge \cdots \wedge P(xn))$, which is equivalent to $\neg P(x1) \vee \neg P(x2) \vee \cdots \vee \neg P(xn)$ by De Morgan's laws, and this is the same as $\exists x \neg P(x)$.

Similarly, $\neg\exists x P(x)$ is the same as $\neg(P(x1) \vee P(x2) \vee \cdots \vee P(xn))$, which by De Morgan's laws is equivalent to $\neg P(x1) \wedge \neg P(x2) \wedge \cdots \wedge \neg P(xn)$, and this is the same as $\forall x \neg P(x)$

What are the negations of the statements "There is an honest politician" and "All Americans eat cheeseburgers"?

Solution: Let H(x) denote "x is honest." Then the statement "There is an honest politician" is represented by $\exists xH(x)$, where the domain consists of all politicians. The negation of this statement is $\neg\exists xH(x)$, which is equivalent to $\forall x\neg H(x)$. This negation can be expressed as

"Every politician is dishonest."

Let C(x) denote "x eats cheeseburgers." Then the statement "All Americans eat cheeseburgers" is represented by $\forall xC(x)$, where the domain consists of all Americans. The negation of this statement is $\neg\forall xC(x)$, which is equivalent to $\exists x\neg C(x)$. This negation can be expressed in several different ways, including "Some American does not eat cheeseburgers" and "There is an American who does not eat cheeseburgers."

Show that $\neg\forall x(P(x)\rightarrow Q(x))$ and $\exists x(P(x) \land \neg Q(x))$ are logically equivalent.

Solution: By De Morgan's law for universal quantifiers, we know that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(\neg(P(x) \rightarrow Q(x)))$ are logically equivalent. We know that $\neg(P(x) \rightarrow Q(x))$ and $P(x) \land \neg Q(x)$ are logically equivalent for every x.

It follows that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \land \neg Q(x))$ are logically equivalent.

Problem: Consider these statements, of which the first three are premises and the fourth is a valid conclusion.

"All hummingbirds are richly colored."

"No large birds live on honey."

"Birds that do not live on honey are dull in color."

"Hummingbirds are small."

Solution:

Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a hummingbird," "x is large," "x lives on honey," and "x is richly colored," respectively. Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

We can express the statements in the argument as

$$\forall x(P(x) \rightarrow S(x)).$$

$$\neg \exists x(Q(x) \wedge R(x)).$$

$$\forall x(\neg R(x) \rightarrow \neg S(x)).$$

$$\forall x(P(x) \rightarrow \neg Q(x)).$$