# Relation & Function chapter I

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# RELATION

- A binary relation from set A to B is subset of cartesian product of A and B ($A \times B$).

- A relation is a set of ordered pairs. If ordered pair (a, b) belongs to relation R then a is related to b.

- In other words, for a binary relation R we have $R \subseteq A \times B$. We use the notation aRb to denote that $(a, b) \in R$.

# RELATIONS

**Example:** Let A be a set of students and B be a set of Courses. R be the relation describing which students opt which course(s).

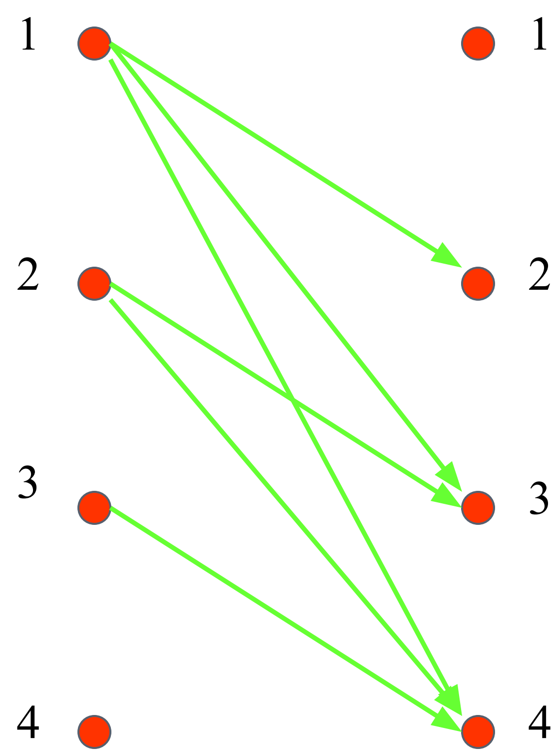
A = {Ram, Sita, Ramesh, Roma},

B = {CS401, CS402, CS403}

C = {(Ram, CS 402), (Sita, CS403), (Roma, CS401), (Roma, cs402)}

# Relations on a Set

$R = \{ (1,2), (1, 3), (1,4), (2,3), (2, 4), (3, 4)\}$




| R | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | | X | X | X |
| 2 | | | X | X |
| 3 | | | | X |
| 4 | | | | |

# Relations on a Set

- **How many different relations can we define on a set A with n elements?**
- A relation on a set A is a subset of $A \times A$.
- There are $n^2$ elements in $A \times A$.
- The number of subsets that we can form out of a set with m elements is $2^m$. Therefore, $2^{n^2}$ subsets can be formed out of $A \times A$.
- We can define $2^{n^2}$ different relations on A.

# PROPERTIES OF RELATIONS

- A relation R on a set A is called **reflexive** if for all $a \in A$, $(a, a) \in R$.
- Are the following relations on {1, 2, 3, 4} reflexive?

$R = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$ No.

$R = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$ Yes.

$R = \{(1, 1), (2, 2), (3, 3)\}$ No.

A relation on a set A is called **irreflexive** if $(a, a) \notin R$ for every element $a \in A$.

# Properties of Relations

- A relation R on a set A is called **symmetric** for all $(a, b) \in R$, $(b, a) \in R$.
- A relation R on a set A is called **antisymmetric** if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$. It can be reflexive but not symmetric.
- A relation R on a set A is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$ for all $a, b \in A$. It can not be reflexive and symmetric.

# Properties of Relations

- Are the following relations on {1, 2, 3, 4} symmetric, antisymmetric, or asymmetric?

| | |
|---|---|
| R = {(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)} | symmetric |
| R = {(1, 1)} | sym. and antisym. |
| R = {(1, 3), (3, 2), (2, 1)} | antisym. and asym. |
| R = {(4, 4), (3, 3), (1, 4)} | antisym. |

# Properties of Relations

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- **Definition:** A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

- Are the following relations on $\{1, 2, 3, 4\}$ transitive?

$R = \{(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)\}$ Yes.

$R = \{(1, 3), (3, 2), (2, 1)\}$ No.

$R = \{(2, 4), (4, 3), (2, 3), (4, 1)\}$ No.

# Equivalence Relations

- **Equivalence relations** are used to relate objects that are similar in some way.
- **Definition:** A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.
- Two elements that are related by an equivalence relation R are called **equivalent**.

# Equivalence Relations

- **Example:** Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?
- **Solution:**
  - R is reflexive, because $l(a) = l(a)$ and therefore aRa for any string a.
  - R is symmetric, because if $l(a) = l(b)$ then $l(b) = l(a)$, so if aRb then bRa.
  - R is transitive, because if $l(a) = l(b)$ and $l(b) = l(c)$, then $l(a) = l(c)$, so aRb and bRc implies aRc.
- R is an equivalence relation.

# Combining Relations

- Relations are sets, and therefore, we can apply the usual **set operations** to them.

- If we have two relations $R_1$ and $R_2$, and both of them are from a set A to a set B, then we can combine them to $R_1 \cup R_2$, $R_1 \cap R_2$, or $R_1 - R_2$.

- In each case, the result will be **another relation from A to B**.

# Combining Relations

- **Definition:** Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite** of S and R ($S \circ R$) is the relation such that

  for $(a, b) \in R$ and $(b, c) \in S$, $(a, c) \in S \circ R$.

# COMBINING RELATIONS

- **Example:** Let D and S be relations on A = {1, 2, 3, 4}.
- D = {(a, b) | b = 5 - a} “b equals (5 – a)”
- S = {(a, b) | a < b} “a is smaller than b”

- D = {(1, 4), (2, 3), (3, 2), (4, 1)}
- S = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}
- $S \circ D$ = {

(2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)}

D maps an element a to the element (5 – a), and afterwards S maps (5 – a) to all elements larger than (5 – a), resulting in $S \circ D = \{(a,b) \mid b > 5 - a\}$ or $S \circ D = \{(a,b) \mid a + b > 5\}$.

# Combining relations

- There are two ways to combine relations $R_1$ and $R_2$
  - Via Boolean operators
  - Via relation "composition"

# Combining relations via Boolean operators

- Consider two relations $R_{\geq}$ and $R_{\leq}$
- We can combine them as follows:
  - $R_{\geq} \cup R_{\leq}$ = all numbers ≥ OR ≤
    - That's all the numbers
  - $R_{\geq} \cap R_{\leq}$ = all numbers ≥ AND ≤
    - That's all numbers equal to
  - $R_{\geq} \oplus R_{\leq}$ = all numbers ≥ or ≤, but not both
    - That's all numbers not equal to
  - $R_{\geq} - R_{\leq}$ = all numbers ≥ that are not also ≤
    - That's all numbers strictly greater than
  - $R_{\leq} - R_{\geq}$ = all numbers ≤ that are not also ≥
    - That's all numbers strictly less than
- Note that it's possible the result is the empty set

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# Combining relations via relational composition

- Let $R$ be a relation from $A$ to $B$, and $S$ be a relation from $B$ to $C$
  - Let $a \in A$, $b \in B$, and $c \in C$
  - Let $(a,b) \in R$, and $(b,c) \in S$
  - Then the composite of R and S consists of the ordered pairs $(a,c)$
    - We denote the relation by $S \circ R$
    - Note that $S$ comes first when writing the composition!

# Combining relations via relational composition

- Let $M$ be the relation "is mother of"
- Let $F$ be the relation "is father of"
- What is $M \circ F$?
  - If $(a,b) \in F$, then $a$ is the father of $b$
  - If $(b,c) \in M$, then $b$ is the mother of $c$
  - Thus, $M \circ F$ denotes the relation "maternal grandfather"
- What is $F \circ M$?
  - If $(a,b) \in M$, then $a$ is the mother of $b$
  - If $(b,c) \in F$, then $b$ is the father of $c$
  - Thus, $F \circ M$ denotes the relation "paternal grandmother"
- What is $M \circ M$?
  - If $(a,b) \in M$, then $a$ is the mother of $b$
  - If $(b,c) \in M$, then $b$ is the mother of $c$
  - Thus, $M \circ M$ denotes the relation "maternal grandmother"
- Note that $M$ and $F$ are not transitive relations!!!

# Combining relations via relational composition

- Given relation $R$
  - $R \circ R$ can be denoted by $R^2$
  - $R^2 \circ R = (R \circ R) \circ R = R^3$
  - Example: $M^3$ is your mother's mother's mother

# REPRESENTING RELATIONS

- We already know different ways of representing relations. We will now take a closer look at two ways of representation: **Zero-one matrices** and **directed graphs**.
- If R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, then R can be represented by the zero-one matrix $M_R = [m_{ij}]$ with
- $m_{ij} = 1$, if $(a_i, b_j) \in R$, and
- $m_{ij} = 0$, if $(a_i, b_j) \notin R$.
- Note that for creating this matrix we first need to list the elements in A and B in a **particular, but arbitrary order**.

# Representing Relations

- **Example:** How can we represent the relation $R = \{(2, 1), (3, 1), (3, 2)\}$ as a zero-one matrix?

- **Solution:** The matrix $M_R$ is given by

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

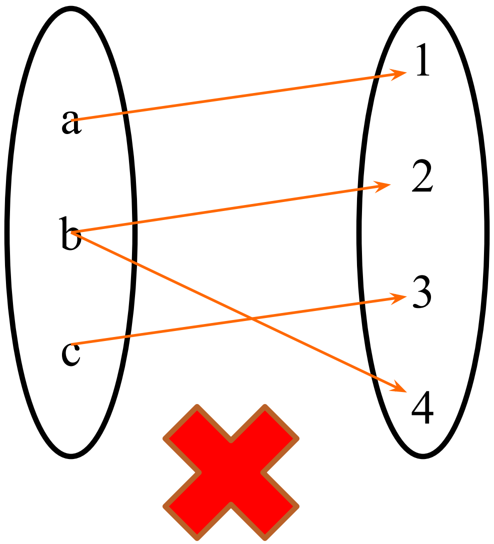
# Functions

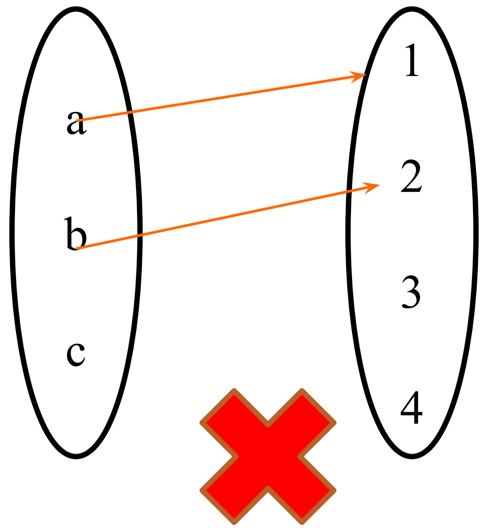
- A binary relation f from A to B is said to be function if for every element a in A, there is a unique element b in B so that (a, b) is in f.

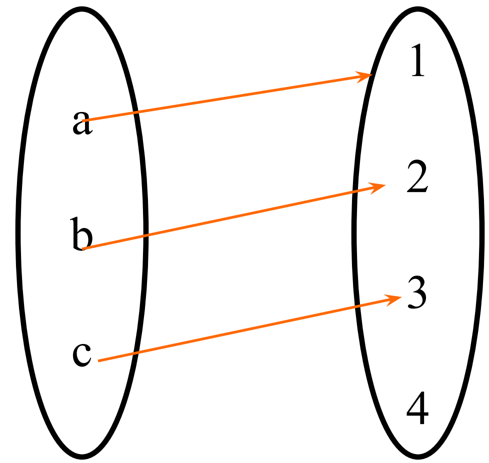
OR

- *A function f from a set A to B is correspondence such that for each element a in A there exists unique element b in B .*

# Function ??????




**Function**

# Functions as Relations

- The **graph** of f is the set of ordered pairs (a, b) such that b = f(a).
- Since the graph of f is a subset of A×B, it is a **relation** from A to B.
- Moreover, for each element **a** of A, there is exactly one ordered pair in the graph that has **a** as its first element.

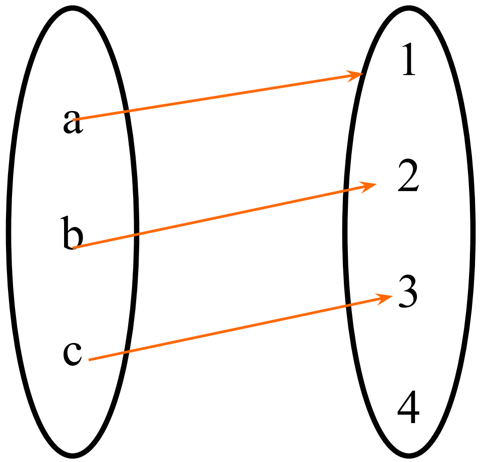
# Functions as Relations

- Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph.

- This is done by assigning to an element $a \in A$ the unique element $b \in B$ such that $(a, b) \in R$.

# Domain and Range

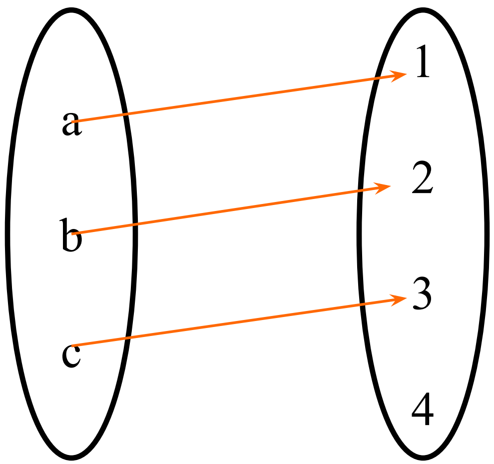
- Let f: A→B be a function then A is domain and B is Co-domain of function. Range of function f is set { y: y=f(x), x in A}.



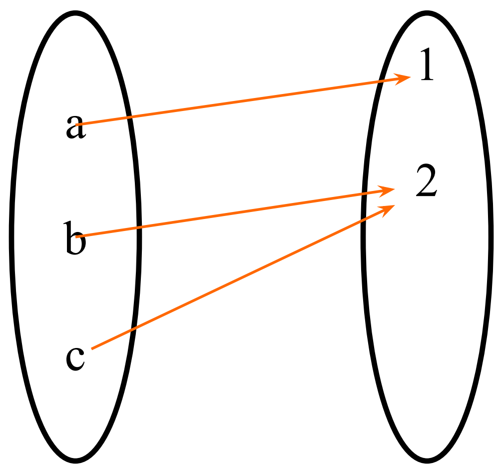

- Domain :{A,B,C}
- Co-domain : {1,2,3,4}
- Range : {1, 2, 3}

# ONE-TO-ONE AND ONTO FUNCTIONS

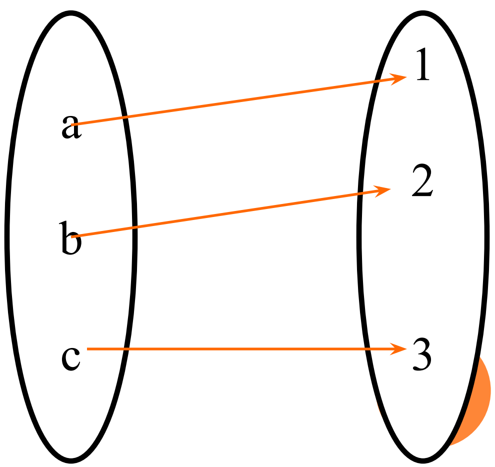
- A function *f is said to be one-to-one, or an injunction, if and only if f (a) = f (b) implies that a = b for all a and b in the domain of f.*
- A function *f from A to B is called onto, or a surjection, if and only if for every element b ∈ B there is an element a ∈ A with f (a) = b.*




one-to-one




onto but not one-to-one

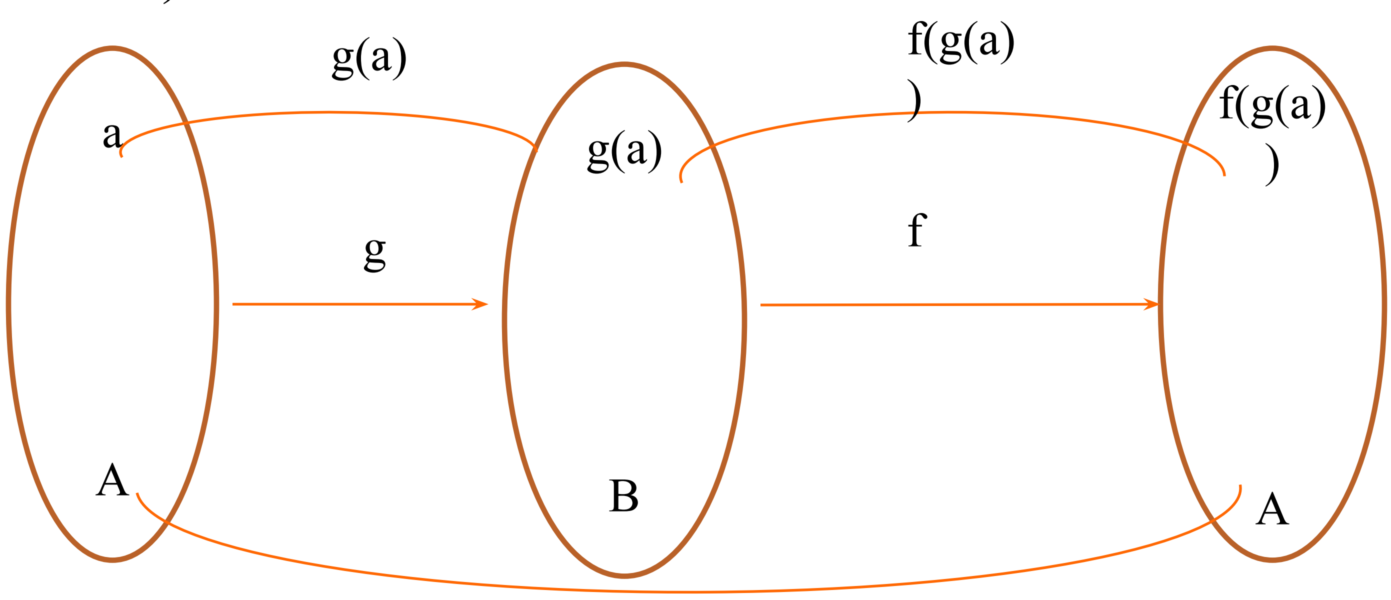



Onto and one-to-one

# Inverse function and Composition

- Let f be a function from A to B. The inverse function of f (denoted by $f^{-1}$) is the function from B to A such that $f^{-1}(b) = a$ whenever $f(a) = b$.
- Inverse of function exists if only if it is bijection (one to one and onto).

- Let g be a function from the set A to the set B and f be a function from the set B to the set C. The composition of the functions f and g, denoted $(f \circ g)$ from A to C as $(f \circ g)(a) = f(g(a))$.

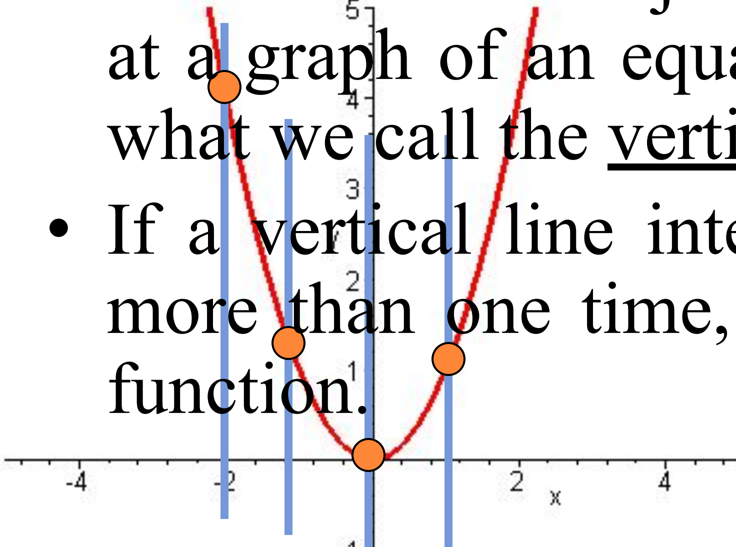



**f ∘ g**

# Graphs and Function

$$y = x^2 \qquad\qquad x = y^2$$

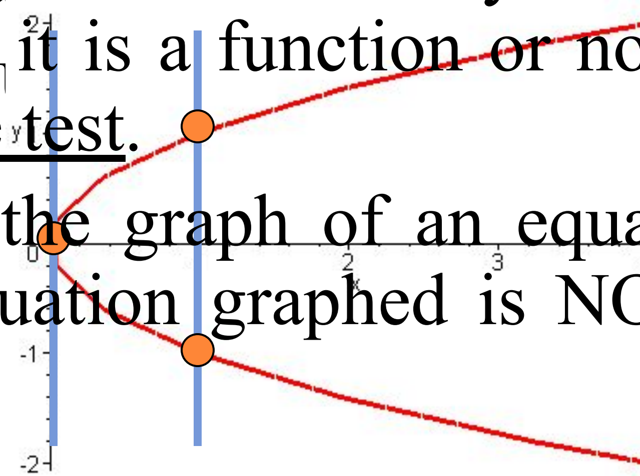
- From what we've just seen, we can check by looking at a graph of an equation if it is a function or not by what we call the vertical line test.
- If a vertical line intersects the graph of an equation more than one time, the equation graphed is NOT a function.




$x$ = -2 $x$ = -1 $x$ = 0 $x$ = 1

**Look at different *x* values and see there is only one *y* value on the graph for it.**

**This IS a function**



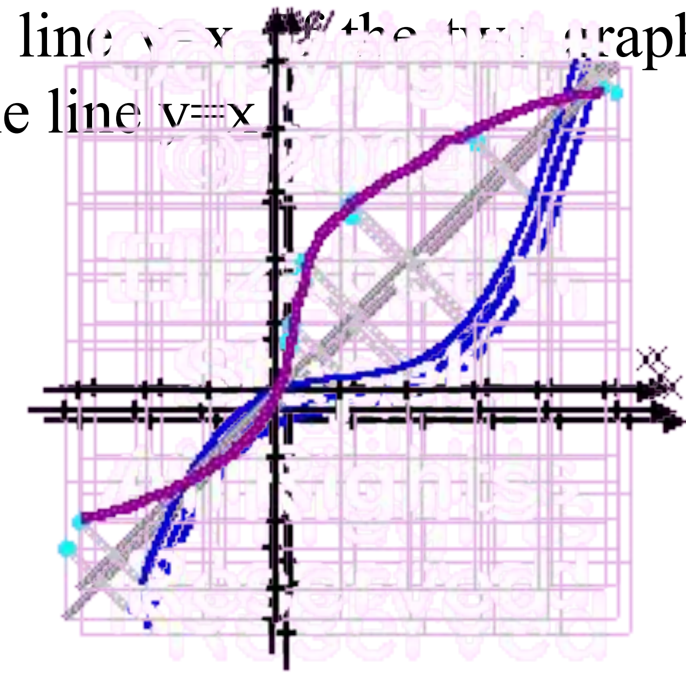

$x = 0$ $x = 1$

**At *x* = 1 there are two *y* values.**

**This then is NOT a function**

# Graphs and Inverse functions

- The graph of y=f(x) and its inverse $y=f^{-1}(x)$ are symmetrical about line y=x. If the two graphs intersect, then intersect on the line y=x.



**Consider the Function**

**y=x**

**Now plot some point symmetric about y=x**

**Now connect the points and we get inverse function**

# Reference Books

- Liu C. L., Mohapatra D. P., Elements of Discrete Mathematics, Tata McGraw Hill.
- Rosen K. N., Discrete Mathematics and its Applications, The McGraw Hill.

The cost of being wrong is less than the cost of doing nothing


THANKS !