Basic counting techniques permutation & Combinations chapter III

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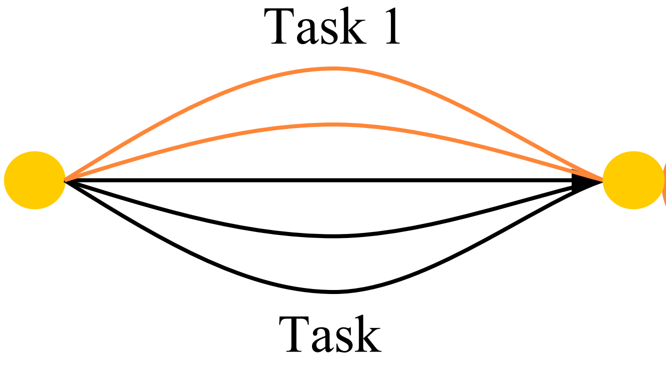
Sum Rule

- Let us consider two tasks:
  - *m* is the number of ways to do **task 1**
  - *n* is the number of ways to do **task 2**
  - Tasks are <u>independent</u> of each other, i.e.,
    - Performing **task 1** does not accomplish **task 2** and vice versa.

- *<u>Sum rule</u>*: the number of ways that "**either** task 1 **or** task 2 can be done, but **not both**", is $\boldsymbol{m + n}$.

- Generalizes to multiple tasks ...



GENERALIZED SUM RULE:

- If we have tasks $T_1, T_2, \ldots, T_m$ that can be done in $n_1, n_2, \ldots, n_m$ ways, respectively, and no two of these tasks can be done at the same time, then there are $n_1 + n_2 + \ldots + n_m$ ways to do one of these tasks.

SUM RULE EXAMPLE

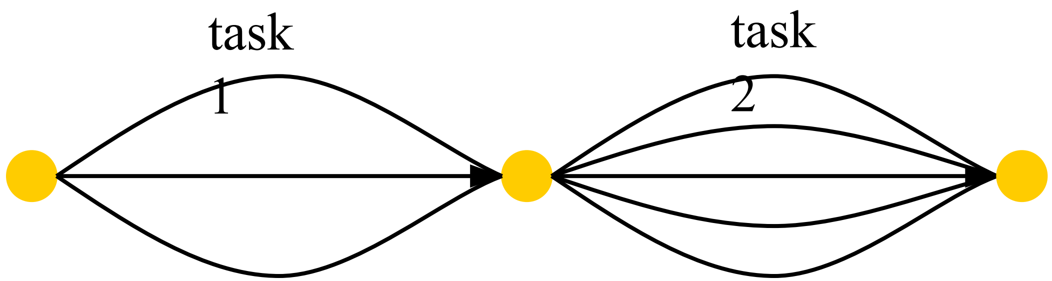
- How many strings of 4 decimal digits, have exactly three digits that are 9s?

- The string can have:
  - The non-9 as the first digit
  - OR the non-9 as the second digit
  - OR the non-9 as the third digit
  - OR the non-9 as the fourth digit
  - Thus, we use the sum rule
- For each of those cases, there are 9 possibilities for the non-9 digit (any number other than 9)
- Thus, the answer is 9+9+9+9 = 36

Product Rule

- Let us consider two tasks:
  - *m* is the number of ways to do **task 1**
  - *n* is the number of ways to do **task 2**
  - Tasks are <u>independent</u> of each other, i.e.,
    - Performing task 1does not accomplish task 2 and vice versa.
- <u>*Product rule*</u>: the number of ways that "**both** tasks 1 and 2 can be done" in *mn*.

- Generalizes to multiple tasks ...



Generalized product rule:

- If we have a procedure consisting of sequential tasks T_1, T_2, …, T_m that can be done in $n_1, n_2, \ldots, n_m$ ways, respectively, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_m$ ways to carry out the procedure.

Product rule example

- There are 18 math majors and 325 CS majors
- How many ways are there to pick one math major and one CS major?

- Total is 18 * 325 = 5850

Product rule example

- How many strings of 4 decimal digits, do not contain the same digit twice?
  - We want to chose a digit, then another that is not the same, then another…
    - First digit: 10 possibilities
    - Second digit: 9 possibilities (all but first digit)
    - Third digit: 8 possibilities
    - Fourth digit: 7 possibilities
  - Total = 10*9*8*7 = 5040

How many strings of 4 decimal digits, end with an even digit?

- First three digits have 10 possibilities
- Last digit has 5 possibilities
- Total = 10*10*10*5 = 5000

- Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters.
- Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected does not matter.

PERMUTATION

- Any arrangement of a set of n objects in a given order is called a *permutation* of the object (taken all at a time).
- Any arrangement of any $r \leq n$ of these objects in a given order is called an "r-permutation" or "a permutation of the n objects taken r at a time." Consider, for example, the set of letters A, B, C, D. Then:

(i) *BDCA*, *DCBA*, and *ACDB* are permutations of the four letters (taken all at a time).

(ii) *BAD*, *ACB*, *DBC* are permutations of the four letters taken three at a time.

(iii) *AD*, *BC*, *CA* are permutations of the four letters taken two at a time.

PERMUTATION

- The number of permutations of n objects taken r at a time will be denoted by $P(n, r)$ (other texts may use nPr, Pn,r , or $(n)r$)$=n!/(n-r)!$.
- There are n! permutations of n objects (taken all at a time).

Permutations with Repetitions

- If set A which contains n elements consists of n_1 elements of the first kind, n_2 elements of the second kind,..., and n_k elements of k-th kind ($n=n_1+n_2+\ldots+n_k$), the number of permutations with repetition is given by:

$$\frac{n!}{n_1!.n_2!....n_k!}$$

- Find the number n of distinct permutations that can be formed from all the letters of each word:

(a) *THOSE*; (*b*) *UNUSUAL*; (*c*) *SOCIOLOGICAL*.

- Find the number m of ways that 7 people can arrange themselves:

  (a) In a row of chairs; (b) Around a circular table.

(a) Here $m = P(7, 7) = 7!$ ways.

(b) One person can sit at any place at the table. The other 6 people can arrange themselves in 6! ways around the table; that is $m = 6!$.

This is an example of a *circular permutation*. In general, n objects can be arranged in a circle in $(n - 1)!$ ways.

- A class contains 8 students. Find the number n of samples of size 3:

(a)With replacement; (b)Without replacement.

Solution:

(a) Each student in the ordered sample can be chosen in 8 ways; hence, there are $n = 8 \cdot 8 \cdot 8 = 512$ samples

(b) The first student in the sample can be chosen in 8 ways, the second in 7 ways, and the last in 6 ways. Thus, there are $n = 8 \cdot 7 \cdot 6 = 336$ samples

- Find the number of automobile license plates where: (a) Each plate contains 2 different letters followed by 3 different digits. (b) The first digit cannot be 0.

Solution:

(a) 26.25.10.9.8

(b) 26.25.9.9.8

- How many 5-digits telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67, for example 67125 etc., and no digit appears more than once?

Solution:

I place after 67 can be filled in 8 ways.

II place after 67 can be filled in 7 ways.

- II place after 67 can be filled in 6 ways.

$\therefore$ No of telephone numbers that can be constructed.

$\Rightarrow 8 \times 7 \times 6$

$\Rightarrow 336$

- How many bit strings of length 8 either start with 1-bit or ends with two bits 00?

Solution:

Starting with the 1 bit =2.2.2.2.2.2.2= 2^7= 128

Ending with the 00 bit =2.2.2.2.2.2= 2^6= 64

Starting with 1 and end with 00 =2.2.2.2.2=32

Total 128+64-32=160

- In how many ways can 3 boys and 3 girls sit in a row?

Solution: There are $6! = 720$ ways 6 people can sit in a row.

- In how many ways can 3 boys and 3 girls sit in a row if the boys and girls are each to sit together?

Solution: (We can have either BBBGGG or GGGBBB, so disregarding the order within boys and within girls, there are 2 possibilities. There are $3!$ ways to order the boys, and $3!$ ways to order the girls, so all together there are $(2)(3!)(3!) = 72$ ways to do this.

- In how many ways if only the boys must sit together?

Solution: If the boys are regards as a unit, then we can first order the objects {B, G1, G2, G3}, where B is the unit of the three boys, and G1, G2, G3 are the girls. There are 4! ways to do this. Then the boys can be ordered in 3! ways. Thus the answer is 4!3! = 144.

- In how many ways if no two people of the same sex are allowed to sit together?

Solution: We can have either BGBGBG or GBGBGB. For each such arrangement, there are 3! ways to arrange the boys, and 3! ways to arrange the girls, so the answer is (2)(3!)(3!) = 72.

COMBINATIONS

- Let S be a set with n elements. A combination of these n elements taken r at a time is any selection of r of the elements where **order does not count**. Such a selection is called an r-combination; it is simply a subset of S with r elements.

The number of such combinations will be denoted by C(n, r) (other texts may use ${}^{n}C_{r}$).

$$C(n, r) = {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

- Combinations A class contains 10 students with 6 men and 4 women. Find the number n of ways to: (a) Select a 4-member committee from the students. (b) Select a 4-member committee with 2 men and 2 women. (c) Elect a president, vice president, and treasurer

Solution:

(a) C(10, 4)

(b) C(6, 2) C(4, 2)

(c) 10.9.8

A woman has 11 close friends. Find the number of ways she can invite 5 of them to dinner where:

(a) There are no restrictions.

(b) Two of the friends are married to each other and will not attend separately.

(c) Two of the friends are not speaking with each other and will not attend together.

Solution:

(b) C(9,3)+C(9,5)

© Case 1: When 1 do not attend the party = 1. C(9,4)

Case 2: When 2 do not attend the party = 1. C(9,4)

Case 3: When both of them do not attend the party =C(9,5)

$\therefore\therefore$ Total no: of ways = 2. C(9,4)+ C(9,5)

- A class contains 8 men and 6 women and there is one married couple in the class. Find the number m of ways a teacher can select a committee of 4 from the class where the husband or wife but not both can be on the committee.

Solution :

$m = C(12, 4) + 2C(12, 3) = 935.$

- A box has 6 blue socks and 4 white socks. Find the number of ways two socks can be drawn from the box where: (a) There are no restrictions. (b) They are different colors. (c) They are the same color.

Solution: $C(10,2)$, $C(6,1)C(4,1)$, $C(6,2)+C(4,2)$

- A women student is to answer 10 out of 13 questions. Find the number of her choices where she must answer: (a) the first two questions; (b) the first or second question but not both; (c) exactly 3 out of the first 5 questions; (d) at least 3 of the first 5 questions.

Solution:

(a) C(11,8)

(b) C(2,1)C(11,9)

(c) C(5,3)C(8,7)

(d) C(5,3)(8,7)+C(5,4)(8,6)+C(5,5)(8,5)