CARDINALITY OF SETS

CARDINALITY

- Georg Cantor (1845–1918) established the field of set theory and discovered that infinite sets can have different sizes.
- *Cardinality* is the general term for the size of a set, whether finite or infinite.
- Georg Cantor work was controversial in the beginning but quickly became a foundation of modern mathematics.

DEFINITION FOR FINITE SETS

- The *cardinality* of a finite set is simply the number of elements in the set.
- For instance the cardinality of {a,b,c} is 3
- The cardinality of the empty set is 0.
- **The set A has cardinality n (n>0) if there is a one-to-one correspondence between the set {1,2,3,…,n} and the set A.**
- We define the one-to-one f:{1,2,3}→{a,b,c} by f(1)=a, f(2)=b, and f(3)=c, thereby proving {a,b,c} has cardinality 3.

DEFINITION OF COUNTABLE

- A set that is either finite or has the same cardinality as the set of positive integers (one-to-one map f: N→a, where N is the set of positive integers) is called *countable.*
- A set that is not countable is called *uncountable.*
- When an infinite set *S* is countable, we denote the cardinality of *s* by ℵ0 (where ℵ is aleph, the first letter of the hebrew alphabet). We write |*S*| = ℵ0 and say that *S* has cardinality "aleph null."
- EXAMPLE 1 Show that the set of odd positive integers is a countable set.

EXAMPLES

- Show that the set of all integers is countable.
- We can list all integers in a sequence by starting with 0 and alternating between positive and negative integers: 0, 1, −1, 2, −2, ... alternatively, we could find a one-to-one correspondence between the set of positive integers and the set of all integers.

The function $f(n) = n/2$ when n is even and $f(n) = -(n-1)/2$ when n is odd is such a function. Consequently, the set of all integers is countable.

SHOW THAT THE SET OF POSITIVE RATIONAL NUMBERS IS COUNTABLE.

Every positive rational number is the quotient p/q of two positive integers. We can arrange the positive rational numbers by listing those with denominator $q = 1$ in the first row, those with denominator $q = 2$ in the second row, and so on.

The key to listing the rational numbers in a sequence is to first list the positive rational numbers p/q with $p + q = 2$, followed by those with $p + q = 3$, followed by those with $p + q = 4$, and so on. Whenever we encounter a number p/q that is already listed, we do not list it again. The initial terms in the list of positive rational numbers we have constructed are 1, 1/2, 2, 3, 1/3, 1/4, 2/3, 3/2, 4, 5, and so on. All positive rational numbers are listed once, as the reader can verify, we have shown that the set of positive rational numbers is countable.

	1	2	3	4	5	6	7	8	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	...
2	$\frac{2}{1}$	$\cancel{\frac{2}{2}}$	$\frac{2}{3}$	$\cancel{\frac{2}{4}}$	$\frac{2}{5}$	$\cancel{\frac{2}{6}}$	$\frac{2}{7}$	$\frac{2}{8}$	...
3	$\frac{3}{1}$	$\frac{3}{2}$	$\cancel{\frac{3}{3}}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	...
4	$\frac{4}{1}$	$\cancel{\frac{4}{2}}$	$\frac{4}{3}$	$\cancel{\frac{4}{4}}$	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$	...
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	...
6	$\frac{6}{1}$	$\cancel{\frac{6}{2}}$	$\cancel{\frac{6}{3}}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	$\frac{6}{7}$	$\frac{6}{8}$	...
7	$\frac{7}{1}$	$\frac{7}{2}$	$\frac{7}{3}$	$\frac{7}{4}$	$\frac{7}{5}$	$\frac{7}{6}$	$\frac{7}{7}$	$\frac{7}{8}$	...
8	$\frac{8}{1}$	$\frac{8}{2}$	$\frac{8}{3}$	$\frac{8}{4}$	$\frac{8}{5}$	$\frac{8}{6}$	$\frac{8}{7}$	$\frac{8}{8}$	...
⋮	⋮								

SCHRÖDER-BERNSTEIN THEOREM if A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there are one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B.

- Show that the $|(0, 1)| = |(0, 1]|$.

Solution: It is not at all obvious how to find a one-to-one correspondence between $(0, 1)$ and $(0, 1]$ to show that $|(0, 1)| = |(0, 1]|$. Fortunately, we can use the Schröder-Bernstein theorem instead.

Finding a one-one function from $(0, 1)$ to $(0, 1]$ is simple. Because $(0, 1) \subset (0, 1]$, $f(x) = x$ is a one-one function from $(0, 1)$ to $(0, 1]$.

Finding a one-to-one function from $(0, 1]$ to $(0, 1)$ is also not difficult. The function $g(x) = x/2$ is clearly one-to-one and maps $(0, 1]$ to $(0, 1/2] \subset (0, 1)$.

As we have found one-one functions from $(0, 1)$ to $(0, 1]$ and from $(0, 1]$ to $(0, 1)$, the Schröder-Bernstein theorem tells us that $|(0, 1)| = |(0, 1]|$.

Show that the set of real numbers is an uncountable set.

(Cantor's diagonal argument-that there are infinite sets which cannot be put into one-to-one correspondence with the infinite set of natural numbers.)

- To show that the set of real numbers is uncountable, we suppose that the set of real numbers is countable and arrive at a contradiction. Then, the subset of all real numbers that fall between 0 and 1 would also be countable (because any subset of a countable set is also countable; see exercise 16). Under this assumption, the real numbers between 0 and 1 can be listed in some order, say, $r1, r2, r3, \ldots$. let the decimal representation of these real numbers be

$r1 = 0.d11d12d13d14 \ldots$

$r2 = 0.d21d22d23d24 \ldots$

$r3 = 0.d31d32d33d34 \ldots$

$r4 = 0.d41d42d43d44 \ldots$

Where $d_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. (For example, if $r_1 = 0.23794102 \ldots$, we have $d_{11} = 2$, $d_{12} = 3$, $d_{13} = 7$, and so on.)

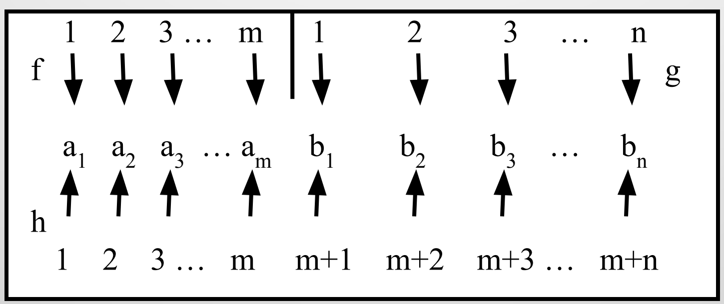
Then, form a new real number with decimal expansion $r = 0.d_1d_2d_3d_4 \ldots$, where the decimal digits are determined by the following rule: $d_i = 1$ if $d_{ii} \neq 1$ and $d_i = 0$ if $d_{ii} = 1$.

Therefore, the real number r is not equal to any of $r_1, r_2, \ldots$ because the decimal expansion of r differs from the decimal expansion of r_i in the ith place to the right of the decimal point, for each i.

Because there is a real number r between 0 and 1 that is not in the list, the assumption that all the real numbers between 0 and 1 could be listed must be false. Therefore, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable. Any set with an uncountable subset is uncountable. Hence, the set of real numbers is uncountable.

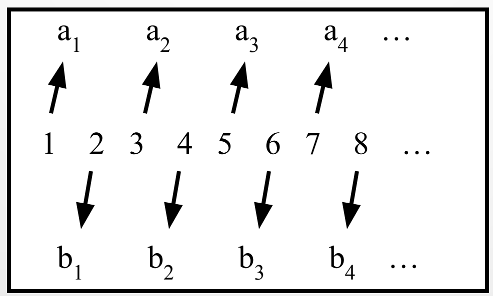
ELEMENTARY THEOREMS

The cardinality of the disjoint union of finite sets is the sum of the cardinalities. That is, suppose $|A|=m$ and $|B|=n$ for nonnegative integers m and n and disjoint sets A and B. Then $|A \cup B|=m+n$.



ELEMENTARY THEOREMS

The disjoint union of countably infinite sets is countably infinite. That is, if A and B are countably infinite and disjoint, then $A \cup B$ is countably infinite.



Theorem-The power set of a set is always of greater cardinality than the set itself.

Proof: There exists an injection but no bijection from S to its power set, P(S) Finding an injection is trivial, as can be seen by considering the function from S to P(S) which maps an element $a \in S$ to the singleton set $\{a\}$.

We show that no onto function exists from an arbitrary set S to its power set, P(S). To

Let f be onto function from S to P(S).

Lets construct a subset C of S, i.e. an element of P(S), which is not in the range of f as follows:

If $a \in f(a)$, then don't put it in C; however, if $a \notin f(a)$, put it in C. Symbolically, $C = \{a : a \in S \text{ and } a \notin f(a)\}$.

For example, if S={1,2,3,4}, then perhaps f(1}={1,3}, f(2}={1,3,4}, f(3)={}, f(4)={2,4}. Therefore C= {2,3}.

Clearly C differs from each element in the range of f (with respect to at least one element). Since f is arbitrary, we conclude there can be no function from S onto € ℘(U). Thus, every set has cardinality smaller than its power set.

The union of countable sets is countable

Suppose that A and B are both countable sets. Without loss of generality, we can assume that A and B are disjoint. (If they are not, we can replace B by B − A, because A ∩ (B − A) = ∅ and A ∪ (B − A) = A ∪ B.) Furthermore, without loss of generality, if one of the two sets is and cases. countably infinite and other finite, we can assume that B is the one that is finite.

There are three cases to consider: (i) A and B are both finite, (ii) A is infinite and B is finite, and (iii) A and B are both countably infinite.

Case (i): Note that when A and B are finite, A ∪ B is also finite, and therefore, countable.

Case (ii): Because A is countably infinite, its elements can be listed in an infinite sequence a1, a2, a3, . . ., an, . . . and because B is finite, its terms can be listed as b1, b2, . . ., bm for some positive integer m. We can list the elements of A ∪ B as b1, b2, . . ., bm, a1, a2, a3, . . ., an, ... This means that A ∪ B is countably infinite.

Case (iii): Because both A and B are countably infinite, we can list their elements as a1, a2, a3, . . ., an, . . . and b1, b2, b3, . . ., bn, . . ., respectively. By alternating terms of these two sequences we can list the elements of A ∪ B in the infinite sequence a1, b1, a2, b2, a3, b3, . . ., an, bn, ... This means A ∪ B must be countably infinite.

MORE ADVANCED THEOREMS

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- The disjoint union of countable sets is countable .
- Subsets of countable sets are countable.
- The Cartesian product of two countable sets is countable.

- The standard symbol for the cardinality of set of Positive Integers is 0א. This is the smallest infinite cardinal. The next are 2 א1 ,א etc.
- Thus the beginning of the list of cardinal numbers looks like 0,1,2,3,…, 2א 0 ,א1 , א….That is, it is an infinite list followed by more infinite lists.
- It is standard to denote the cardinality of $\mathbb{R}$ by C (for continuum). We know that C is greater than 0 א, but it is unknown whether c=1א. This proposition (that c=1א) is called the continuum hypothesis.