MATHEMATICS

SECTION A

January 30, 2024

1 Matrix

1. If A is a square matrix satisfying $A'A = I$, write the value of $\left|A\right|$.
2. Using properties of determinants, show that

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3\,(a+b+c)\,(ab+bc+ca)$$

2 Differentiation

3. If $x\sqrt{1+y}+y\sqrt{1+x}=0$ and $x \neq y$, prove that $\frac{dy}{dx} = -\frac{1}{(x+1)^2}$.
4. If $y = x\left|x\right|$, find $\frac{dy}{dx}$ for $x < 0$.
5. Solve the differential equation:

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$$

6. If $(\cos x)^y = (\sin y)^x$, find $\frac{dy}{dx}$.
7. Find the differential equation representing the family of curves $y = ae^{2x} + 5$, where a is an arbitrary constant.
8. If $x = ae^t\,(\sin t + \cos t)$ and $y = ae^t\,(\sin t - \cos t)$, then prove that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

3 Integration

9. Find:

$$\int \sqrt{3-2x-x^2}dx$$

10. Find:

$$\int \frac{x^2+x+1}{(x+2)\left(x^2+1\right)}dx$$

11. Find:

$$\int \frac{x-5}{(x-3)^3} e^x dx$$

12. Find:

$$\int \frac{2\cos x}{(1-\sin x)\left(2-\cos^2 x\right)} dx$$

4 Vectors

13. A line passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction of the vector $\hat{i} + \hat{j} - 2\hat{k}$. Find the equation of the line in cartesian form.

14. Find the volume of a cuboid whose edges are given by $-3\hat{i} + 7\hat{j} + 5\hat{k}$,$-5\hat{i} + 7\hat{j} - 3\hat{k}$ and $7\hat{i} - 5\hat{j} - 3\hat{k}$.

15. The scalar product of the vector $\overrightarrow{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vectors $\overrightarrow{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\overrightarrow{c} = \hat{\lambda} + 2\hat{j} + 3\hat{k}$ is equal to 1. Find the value of λ and hence find the unit vector along $\overrightarrow{b} + \overrightarrow{c}$.

16. Show that the points $A\left(-2\hat{i} + 3\hat{j} + 5\hat{k}\right)$, $B\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$ and $C\left(7\hat{i} - \hat{k}\right)$ are collinear.

17. Find the direction cosines of a line which makes equal angles with the coordinate axes.

18. If the lines $\frac{x-1}{-3} = \frac{y-2}{2\lambda} = \frac{z-3}{2}$ and $\frac{x-1}{3\lambda} = \frac{y-1}{2} = \frac{z-6}{-5}$ are perpendicular, find the value of λ. Hence find weather the lines are intersecting or not.

19. If $\left|\overrightarrow{a}\right| = 2, \left|\overrightarrow{b}\right| = 7$ and $\overrightarrow{a} \times \overrightarrow{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$, find the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$.

5 Probability

20. A bag contains 5 red and 4 black balls, a second bag contains 3 red and 6 black balls. One of the two bags is selected at random and two balls are drawn at random (without replacement) both of which are found to be red. Find the probability that the balls are drawn from the second bag.

21. Find the probability distribution of X, the number of heads in a simultaneous toss of two coins.

22. There are three coins. One is a two-headed coin, another is a biased coin that comes up heads 75% of the time and the third is an unbiased coin. One of the three coins is chosen at random and tossed. If it shows heads, what is the probability that it is the two-headed coin ?

6 Functions

23. Check whether the relation R defined on the set $A = \{1, 2, 3, 4, 5, 6\}$ as $R = \{(a, b) : b = a + 1\}$ is reflexive, symmetric or transitive.

24. Let $f : N \to Y$ be a function defined as $f(x) = 4x + 3$, where $Y = \{y \in N : y = 4x + 3, \text{ for some } x \in N\}$. Show that f is invertible. Find its inverse.

7 Intersection of coins

25. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $\left(-1, 4\right)$.

26. Find the point on the curve $y^2 = 4x$, which is nearest to the point $\left(2, -8\right)$.

8 Optimization

27. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

28. The volume of a cube is increasing at the rate of $8cm^3/s$. How fast is the surface area increasing when the length of its edge is $12cm$?

9 Algebra

29. Solve for x:

$$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right)$$