considered in Sect. 6.11. Similarly the equations are equally applicable if we use the averaged disturbing functions $\langle \mathcal{R} \rangle$ and $\langle \mathcal{R}' \rangle$.

We have already seen in Sect. 2.9 that the variations in the orbital elements can be expressed in terms of the radial, tangential, and orthogonal forces acting on an orbiting object. However, Lagrange's equations allow us to derive similar variations but based on the Fourier series expansion of the disturbing function discussed in this chapter. As such they provide the basis for most of the perturbation calculations that follow.

6.9 Classification of Arguments in the Disturbing Function

We can now approach the subject of the physical significance of the expansion of the disturbing function. So far we have expressed the perturbing potential as a series involving an infinite number of permissible combinations of angles. But which angles are important in any given problem? In other words, which of the infinite terms in the expansion are important and which can be ignored? To a large extent the answers to these questions depend on the semi-major axis of the perturbed orbit. We can classify all arguments by considering the frequencies or periods associated with the cosine arguments in the expansion.

Each cosine argument contains a linear combination of the angles λ', λ, ϖ', ϖ, Ω', and Ω. We know that in the unperturbed problem the mean longitudes, λ' and λ, increase linearly at rates n' and n respectively. In contrast, all the other angles are constant in the unperturbed problem. Therefore, when we consider the perturbed system λ' and λ are rapidly varying quantities, whereas all the other angles undergo slow variations. Therefore, any valid arguments that do not involve mean longitudes are slowly varying. These give rise to *secular* terms, from the Latin verb *saeculum* meaning century, or long period. This does not imply that all other arguments are of short period. Consider a general argument of the form $\varphi = j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega$ with

$$\lambda' \approx n't + \epsilon' \quad \text{and} \quad \lambda \approx nt + \epsilon \tag{6.157}$$

(see Eq. (6.144)). Therefore $j_1\lambda' + j_2\lambda \approx (j_1n' + j_2n)t +$ constant and so, if the semi-major axes are such that

$$j_1n' + j_2n \approx 0, \tag{6.158}$$

then this argument also has a period longer than either orbital period. Equation (6.158) is satisfied when there is a commensurability between the two mean motions or orbital periods (see Sect. 1.7). We classify such arguments as giving rise to *resonant* terms in the expansion. If we consider the semi-major axes, the equivalent condition is

$$a \approx (|j_1|/|j_2|)^{\frac{2}{3}}\, a'. \tag{6.159}$$

Because of the dependence on semi-major axis, resonant terms are localised. Whereas a particular combination of angles may be slowly varying at one semi-major axis of the perturbed body, the same combination would be varying rapidly at another. In contrast the secular terms can be considered as global.

Any argument that is neither secular nor resonant is considered to give rise to a *short-period* term. In practise the application of the averaging principle mentioned in Sect. 6.7 allows us to ignore the infinite number of short-period terms in the expansion and accept that the dynamics is dominated by the appropriate secular and resonant terms.

Below we provide predictions of motion under secular and resonant terms in the context of the elliptical restricted three-body problem with small inclination, and we compare the answers with the results of numerical integrations. Here we assume that the mass m is negligible and that the orbit of m' is a fixed ellipse in the reference plane. Our starting point is a set of the lowest order form of Lagrange's equations for $\dot{a}$, $\dot{e}$, $\dot{\varpi}$, and $\dot{\Omega}$ derived from inspection of Eqs. (6.145), (6.146), (6.149), and (6.148). The equations of motion are

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\frac{\partial\langle\mathcal{R}\rangle}{\partial\lambda}, \tag{6.160}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{1}{na^2e}\frac{\partial\langle\mathcal{R}\rangle}{\partial\varpi}, \tag{6.161}$$

$$\frac{\mathrm{d}\varpi}{\mathrm{d}t} = +\frac{1}{na^2e}\frac{\partial\langle\mathcal{R}\rangle}{\partial e}, \tag{6.162}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = +\frac{1}{na^2\sin I}\frac{\partial\langle\mathcal{R}\rangle}{\partial I}, \tag{6.163}$$

where $\langle\mathcal{R}\rangle$ is the averaged part of the disturbing function for an external perturber.

6.9.1 Secular Terms

Secular terms arise from those arguments that do not contain the mean longitudes. Inspection of the direct part of the second-order expansion in Eq. (6.107) shows that secular terms are obtained by setting $j = 0$ in those cosine arguments containing $j\lambda' - j\lambda$. This gives

$$\langle\mathcal{R}_{\mathrm{D}}\rangle = C_0 + C_1(e^2 + e'^2) + C_2 s^2 + C_3 e e' \cos(\varpi' - \varpi), \tag{6.164}$$

where

$$C_0 = \frac{1}{2} b_{\frac{1}{2}}^{(0)}(\alpha), \tag{6.165}$$

$$C_1 = \frac{1}{8}\left[2\alpha D + \alpha^2 D^2\right] b_{\frac{1}{2}}^{(0)}(\alpha), \tag{6.166}$$

$$C_2 = -\frac{1}{2}\alpha b_{\frac{3}{2}}^{(1)}(\alpha), \tag{6.167}$$

$$C_3 = \frac{1}{4}\left[2 - 2\alpha D - \alpha^2 D^2\right] b_{\frac{1}{2}}^{(1)}(\alpha). \tag{6.168}$$

Note that there are no ss' or s'^2 terms in $\langle \mathcal{R}_\mathrm{D} \rangle$ because we are taking $s' = 0$ and that C_0 is a function of α only. Furthermore, inspection of the terms in $\mathcal{R}_\mathrm{E}$ (Eq. (6.110)) shows that all the arguments contain at least one mean longitude and so there are no secular contributions from the indirect part of the disturbing function. Hence the low-order version of Lagrange's equations becomes

$$\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)_\mathrm{sec} = 0, \tag{6.169}$$

$$\left(\frac{\mathrm{d}e}{\mathrm{d}t}\right)_\mathrm{sec} = n\alpha(m'/m_\mathrm{c})C_3 e' \sin(\varpi - \varpi'), \tag{6.170}$$

$$\left(\frac{\mathrm{d}\varpi}{\mathrm{d}t}\right)_\mathrm{sec} = n\alpha(m'/m_\mathrm{c})\left[2C_1 + C_3(e'/e)\cos(\varpi - \varpi')\right], \tag{6.171}$$

$$\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)_\mathrm{sec} = n\alpha(m'/m_\mathrm{c})(C_2/2), \tag{6.172}$$

where we have used the fact that $\mu' = \mathcal{G}m' \approx n^2a^3(m'/m_\mathrm{c})$, where m_c is the mass of the central object. If we assume that $e \gg e'$ then the approximate solutions to these equations are

$$a = a_0, \tag{6.173}$$

$$e = e_0 - \frac{n\alpha}{\dot{\varpi}}(m'/m_\mathrm{c})C_3 e' \left[\cos\varpi_0 - \cos\varpi\right], \tag{6.174}$$

$$\varpi = \varpi_0 + n\alpha(m'/m_\mathrm{c})2C_1 t, \tag{6.175}$$

$$\Omega = \Omega_0 + n\alpha(m'/m_\mathrm{c})(C_2/2)t, \tag{6.176}$$

where the subscript 0 denotes the initial ($t = 0$) value of a quantity, and we have taken $\varpi' = 0$. These solutions predict that there is no secular change in a, that e varies sinusoidally with an amplitude of

$$(\Delta e)_\mathrm{sec} = \left|(n\alpha/\dot{\varpi})(m'/m_\mathrm{c})C_3 e'\right|, \tag{6.177}$$

and that ϖ and Ω will either increase or decrease linearly with time depending on the signs of C_1 and C_2.

Figures 6.3a–d show the results of a numerical integration of the full equations of motion of the elliptical restricted three-body problem with $a' = 1$, $e' = 0.048$, $\varpi' = 0$, $I' = 0$, and $m'/m_\mathrm{c} = 1/1047.355$ with starting conditions $a_0 = 0.192$, $e_0 = 0.1$, $\varpi_0 = 130^\circ$, $\Omega_0 = 200^\circ$, $\lambda_0 = 300^\circ$, and $\lambda' = 0^\circ$. Substitution of $\alpha = a/a' = 0.192$ in Eqs. (6.166)–(6.168) gives $C_1 = 0.0148335$, $C_2 = -0.0593339$, and $C_3 = -0.00708688$; note that $2C_1 = -C_2/2$. Since the mass ratio is that of the Jupiter–Sun ratio, the integration was designed to mimic the motion of an asteroid perturbed by Jupiter, and so the time units in the plots are given as Jupiter periods. However, the semi-major axis was deliberately chosen to be far away from Jupiter in order to avoid proximity to strong resonances. In these circumstances the secular perturbations alone should provide a good approximation to the motion.

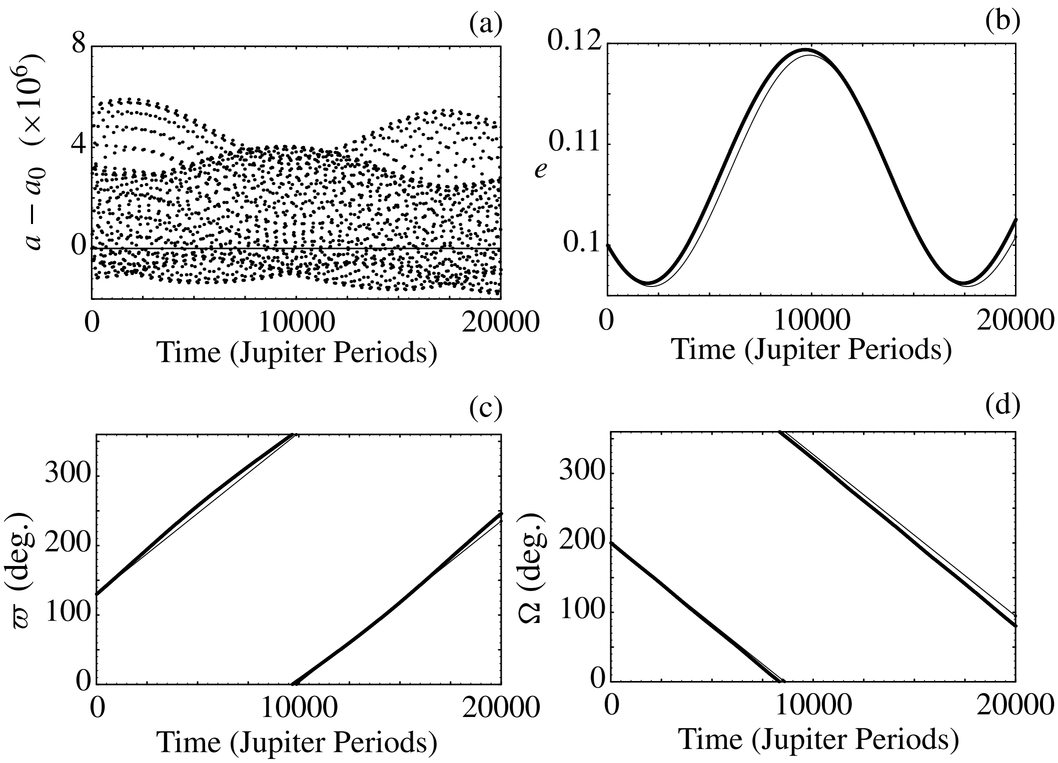

Fig. 6.3. A comparison of the results of a full numerical integration (thick line) with predictions from analytical theory (thin line) for the variation of (a) semi-major axis, (b) eccentricity, (c) longitude of perihelion, and (d) longitude of ascending node for a test particle undergoing predominantly secular perturbations from Jupiter.

The results show that the agreement is excellent over the 20,000 Jupiter periods of the integration. There are variations in a but these are extremely small; note that the scale in Fig. 6.3a is enlarged. The fact that the semi-major axis is almost constant justifies the evaluation of the Laplace coefficients for a fixed value of α. The eccentricity does vary as predicted, and while ϖ increases linearly with time (since $C_1 > 0$), Ω is decreasing linearly at the same rate (since $2C_1 = -C_2/2$; cf. Eqs. (6.175) and (6.176)). Prograde motion of the pericentre (or node) is called *precession* and retrograde motion is called *regression.* The behaviour of ϖ and Ω is a natural consequence of the secular terms in the disturbing function.

Because of the infinite number of short-period terms in the disturbing function, which we have neglected, there should be differences between the results of a full integration and the predictions of our analytical theory. We can see this already in the Fig. 6.3a, where there are small, but detectable short-period changes in the semi-major axis from the constant value predicted by theory. Figure 6.4 shows the difference between the "observed" eccentricity (i.e., the one determined by the numerical integration) and the calculated value from theory, as a function of time for the first 1,000 Jupiter periods of the integration. Here again we can see the effect of the short-period terms inherently included in any full integration.

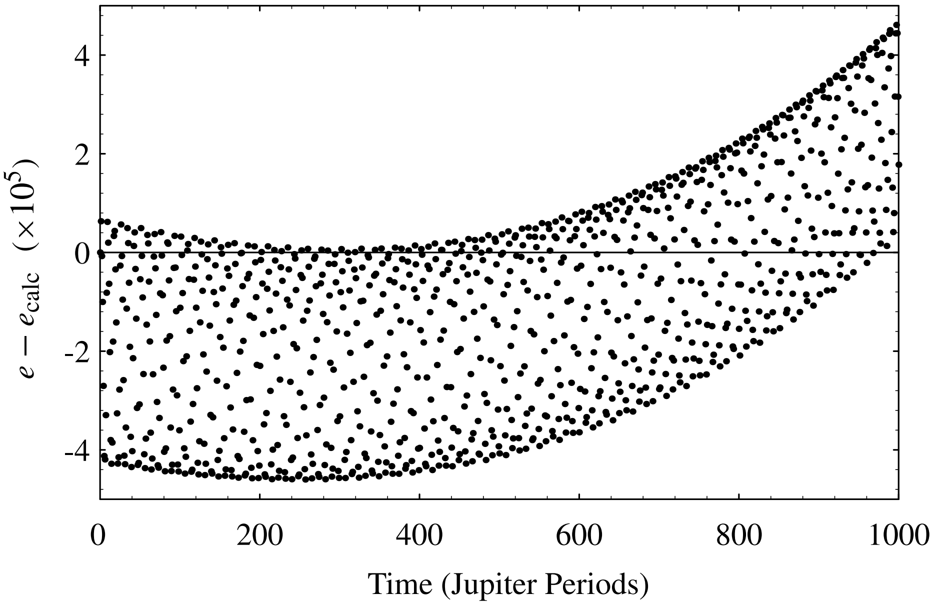

Fig. 6.4. Differences between the observed and calculated values of the test particle's eccentricity as a function of time. The data are sampled every Jupiter period and show the short-period variations in e.

6.9.2 Resonant Terms

Now suppose, for example, that we want to study an asteroid's motion at 3.27 AU, under the perturbing effect of Jupiter. Since Jupiter's semi-major axis is 5.20 AU we have, using Kepler's third law, that the ratio of their periods is $(3.27/5.20)^{3/2} \approx 0.499$. Hence, we have the relation $2n' \approx n$ and we would expect resonant terms to be important. Therefore, in the vicinity of the 2:1 resonance, as well as the secular terms discussed above, we also need to consider those terms in the expansion of the disturbing function that contain $2\lambda' - \lambda$ (i.e., the resonant terms for this location).

Inspection of Eq. (6.107) shows that in a second-order expansion there are two terms in $\langle \mathcal{R}_\mathrm{D} \rangle / a'$ that have a cosine argument containing $2\lambda' - \lambda$ for specific values of j. The relevant direct part of the averaged disturbing function is

$$\begin{aligned} \langle \mathcal{R}_\mathrm{D} \rangle = C_0 + C_1(e^2 + e'^2) + C_2(s^2 + s'^2) + C_3 ee' \cos(\varpi - \varpi') \\ + C_4 e \cos(2\lambda' - \lambda - \varpi) + C_5 e' \cos(2\lambda' - \lambda - \varpi'), \end{aligned} \quad (6.178)$$

where the additional constants C_4 and C_5 are given by

$$C_4 = \frac{1}{2}\,[-4 - \alpha D]\, b_{\frac{1}{2}}^{(2)}(\alpha), \quad (6.179)$$

$$C_5 = \frac{1}{2}\,[3 + \alpha D]\, b_{\frac{1}{2}}^{(1)}(\alpha). \quad (6.180)$$

The second of these two resonant arguments makes no contribution to $\dot{e}$, $\dot{\varpi}$, and $\dot{\Omega}$ but does contribute a term to $\dot{a}$. Inspection of Eq. (6.110) shows that there is also a $-2\alpha e'$ contribution to the same argument from the indirect part.

7
Secular Perturbations

Past and to come seem best, things present worst.

William Shakespeare, *Henry IV, (2), I, iii*

7.1 Introduction

In the last chapter we saw how the disturbing function can be expanded in an infinite series where the individual terms can be classified as secular, resonant, or short period, according to the given physical problem. We have already stated in Sect. 3 that the N-body problem (for $N \geq 3$) is nonintegrable. However, in this chapter we will show how, with suitable approximations, it is possible to find an analytical solution to a particular form of the N-body problem that can be applied to the motion of solar system bodies. We can do this by considering the effects of the purely secular terms in the disturbing function for a system of N masses orbiting a central body. The resulting theory can be applied to satellites orbiting a planet, or planets orbiting the Sun, and then used to study the motion of small objects orbiting in either of these systems. This is the subject of *secular perturbation theory.*

7.2 Secular Perturbations for Two Planets

Consider the motion of two planets of mass m_1 and m_2 moving under their mutual gravitational effects and the attraction of a point-mass central body of mass m_c where $m_1 \ll m_c$ and $m_2 \ll m_c$. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the disturbing functions describing the perturbations on the orbit of the masses m_1 and m_2 respectively, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are functions of the standard *osculating* orbital elements of both bodies. Osculating elements, from the Latin verb *osculare* meaning "to kiss", are instantaneous elements derived from the values of the position and velocity of an object assuming an unperturbed keplerian orbit. The perturbations on the orbital elements are given by Lagrange's equations, Eqs. (6.145)–(6.150).

In the absence of any mean motion commensurabilities between two masses, the secular perturbations arising from the gravitational perturbations between m_1, m_2, and m_c are obtained by isolating the terms in the disturbing function that are independent of the mean longitudes. We can also exclude any terms that depend only on the semi-major axis since, from Eq. (6.145), these will not make any contribution to secular evolution. To second order in the eccentricities and inclinations (and first order in the masses), the only terms in the expansion of the disturbing function that do not contain the mean longitudes are, from Appendix B, the terms 4D0.1, 4D0.2, and 4D0.3 with $j = 0$. Hence the general, averaged, secular, direct part of the disturbing function is

$$\begin{aligned}\mathcal{R}_{\mathrm{D}}^{(\mathrm{sec})} = {} & \frac{1}{8}\left[2\alpha_{12}D + \alpha_{12}^2 D^2\right] b_{\frac{1}{2}}^{(0)}\,(e_1^2 + e_2^2) - \frac{1}{2}\alpha_{12} b_{3/2}^{(1)} (s_1^2 + s_2^2) \\ & + \frac{1}{4}\left[2 - 2\alpha_{12}D - \alpha_{12}^2 D^2\right] b_{\frac{1}{2}}^{(1)}\, e_1 e_2 \cos(\varpi_1 - \varpi_2) \\ & + \alpha_{12} b_{3/2}^{(1)} s_1 s_2 \cos(\Omega_1 - \Omega_2),\end{aligned} \tag{7.1}$$

where the subscripts 1 and 2 refer to the inner and outer body respectively and $\alpha_{12} = a_1/a_2$ where $a_1 < a_2$. There is no indirect part. In fact, as can be seen from Appendix B, all the indirect terms involve at least one mean longitude and hence will never contribute purely secular terms (see Brouwer & Clemence 1961).

When calculating $\mathcal{R}_1$ and $\mathcal{R}_2$ from $\mathcal{R}_{\mathrm{D}}^{(\mathrm{sec})}$ we have to take account of the fact that $\mathcal{R}_1$ arises from an external perturbation by m_2 whereas $\mathcal{R}_2$ comes from an internal perturbation by m_1. Hence, from Eqs. (6.134) and (6.135), $\mathcal{R}_1$ and $\mathcal{R}_2$ can be written as

$$\mathcal{R}_1 = \frac{\mathcal{G}m_2}{a_2}\mathcal{R}_{\mathrm{D}}^{(\mathrm{sec})} = \frac{\mathcal{G}m_2}{a_1}\alpha_{12}\mathcal{R}_{\mathrm{D}}^{(\mathrm{sec})} \tag{7.2}$$

and

$$\mathcal{R}_2 = \frac{\mathcal{G}m_1}{a_1}\alpha_{12}\mathcal{R}_{\mathrm{D}}^{(\mathrm{sec})} = \frac{\mathcal{G}m_1}{a_2}\mathcal{R}_{\mathrm{D}}^{(\mathrm{sec})}. \tag{7.3}$$

Using the following relationships between the Laplace coefficients and their derivatives,

$$2\alpha\frac{\mathrm{d}b_{1/2}^{(0)}}{\mathrm{d}\alpha} + \alpha^2\frac{\mathrm{d}^2 b_{1/2}^{(0)}}{\mathrm{d}\alpha^2} = \alpha b_{3/2}^{(1)}, \tag{7.4}$$

$$2b_{1/2}^{(1)} - 2\alpha\frac{\mathrm{d}b_{1/2}^{(1)}}{\mathrm{d}\alpha} - \alpha^2\frac{\mathrm{d}^2 b_{1/2}^{(1)}}{\mathrm{d}\alpha^2} = -\alpha b_{3/2}^{(2)}, \tag{7.5}$$

and the approximations $\mathcal{G}m_\mathrm{c} \approx n_1^2 a_1^3 \approx n_2^2 a_2^3$, we can write

$$\mathcal{R}_1 = n_1^2 a_1^2 \frac{m_2}{m_\mathrm{c} + m_1}\left[\frac{1}{8}\alpha_{12}^2 b_{3/2}^{(1)} e_1^2 - \frac{1}{8}\alpha_{12}^2 b_{3/2}^{(1)} I_1^2 \right.$$
$$- \frac{1}{4}\alpha_{12}^2 b_{3/2}^{(2)} e_1 e_2 \cos(\varpi_1 - \varpi_2)$$
$$\left. + \frac{1}{4}\alpha_{12}^2 b_{3/2}^{(1)} I_1 I_2 \cos(\Omega_1 - \Omega_2)\right] \tag{7.6}$$

and

$$\mathcal{R}_2 = n_2^2 a_2^2 \frac{m_1}{m_\mathrm{c} + m_2}\left[\frac{1}{8}\alpha_{12} b_{3/2}^{(1)} e_2^2 - \frac{1}{8}\alpha_{12} b_{3/2}^{(1)} I_2^2 \right.$$
$$- \frac{1}{4}\alpha_{12} b_{3/2}^{(2)} e_1 e_2 \cos(\varpi_1 - \varpi_2)$$
$$\left. + \frac{1}{4}\alpha_{12} b_{3/2}^{(1)} I_1 I_2 \cos(\Omega_1 - \Omega_2)\right], \tag{7.7}$$

where we have assumed that I_1 and I_2 are small enough so that the approximations $s_1 = \sin\frac{1}{2}I_1 \approx \frac{1}{2}I_1$ and $s_2 = \sin\frac{1}{2}I_2 \approx \frac{1}{2}I_2$ are valid.

The equations for $\mathcal{R}_1$ and $\mathcal{R}_2$ given in Eqs. (7.6) and (7.7) can be combined to give

$$\mathcal{R}_j = n_j a_j^2\left[\frac{1}{2}A_{jj}e_j^2 + A_{jk}e_1 e_2 \cos(\varpi_1 - \varpi_2)\right.$$
$$\left.\frac{1}{2}B_{jj}I_j^2 + B_{jk}I_1 I_2 \cos(\Omega_1 - \Omega_2)\right], \tag{7.8}$$

where $j = 1, 2$; $k = 2, 1$ $(j \neq k)$; and

$$A_{jj} = +n_j \frac{1}{4}\frac{m_k}{m_\mathrm{c} + m_j}\alpha_{12}\bar{\alpha}_{12} b_{3/2}^{(1)}(\alpha_{12}), \tag{7.9}$$

$$A_{jk} = -n_j \frac{1}{4}\frac{m_k}{m_\mathrm{c} + m_j}\alpha_{12}\bar{\alpha}_{12} b_{3/2}^{(2)}(\alpha_{12}), \tag{7.10}$$

$$B_{jj} = -n_j \frac{1}{4}\frac{m_k}{m_\mathrm{c} + m_j}\alpha_{12}\bar{\alpha}_{12} b_{3/2}^{(1)}(\alpha_{12}), \tag{7.11}$$

$$B_{jk} = +n_j \frac{1}{4}\frac{m_k}{m_\mathrm{c} + m_j}\alpha_{12}\bar{\alpha}_{12} b_{3/2}^{(1)}(\alpha_{12}), \tag{7.12}$$

where $\bar{\alpha}_{12} = \alpha_{12}$ if $j = 1$ (an external perturber) and $\bar{\alpha}_{12} = 1$ if $j = 2$ (an internal perturber). From the definition of the Laplace coefficients given in Sect. 6.4 we have

$$b_{3/2}^{(1)}(\alpha) = \frac{1}{\pi}\int_0^{2\pi}\frac{\cos\psi\,\mathrm{d}\psi}{(1 - 2\alpha\cos\psi + \alpha^2)^{\frac{3}{2}}}, \tag{7.13}$$

$$b_{3/2}^{(2)}(\alpha) = \frac{1}{\pi}\int_0^{2\pi}\frac{\cos 2\psi\,\mathrm{d}\psi}{(1 - 2\alpha\cos\psi + \alpha^2)^{\frac{3}{2}}}. \tag{7.14}$$

Note that in this case $B_{11} = -B_{12}$ and $B_{21} = -B_{22}$. However, the situation is different when we have to take account of terms due to the oblateness of the central body (see Sect. 7.7). All these quantities are frequencies that can be thought of as the constant elements of two matrices **A** and **B** given by

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}. \tag{7.15}$$

Note that the elements of these matrices are only functions of the masses and the (fixed) semi-major axes of the two bodies and that the rows (or columns) of the matrix **B** are not linearly independent.

Taking the lowest order terms in e and I in Eqs. (6.146), (6.148), (6.149), and (6.150) we can easily derive an approximate form of Lagrange's equations for the time variation of the original orbital elements:

$$\dot{e}_j = -\frac{1}{n_j a_j^2 e_j}\frac{\partial \mathcal{R}_j}{\partial \varpi_j}, \qquad \dot{\varpi}_j = +\frac{1}{n_j a_j^2 e_j}\frac{\partial \mathcal{R}_j}{\partial e_j}, \tag{7.16}$$

$$\dot{I}_j = -\frac{1}{n_j a_j^2 I_j}\frac{\partial \mathcal{R}_j}{\partial \Omega_j}, \qquad \dot{\Omega}_j = +\frac{1}{n_j a_j^2 I_j}\frac{\partial \mathcal{R}_j}{\partial I_j}. \tag{7.17}$$

Given the form of the equations above, it is convenient to define the vertical and horizontal components of eccentricity and inclination "vectors" by:

$$h_j = e_j \sin \varpi_j, \qquad k_j = e_j \cos \varpi_j \tag{7.18}$$

and

$$p_j = I_j \sin \Omega_j, \qquad q_j = I_j \cos \Omega_j. \tag{7.19}$$

These variables have the advantage that they avoid the singularities inherent in Eqs. (7.16) and (7.17) for low e and I. The general secular part of the disturbing function can now be written

$$\mathcal{R}_j = n_j a_j^2 \left[\frac{1}{2} A_{jj}(h_j^2 + k_j^2) + A_{jk}(h_j h_k + k_j k_k) \right. \\ \left. + \frac{1}{2} B_{jj}(p_j^2 + q_j^2) + B_{jk}(p_j p_k + q_j q_k) \right]. \tag{7.20}$$

Note that when k is used as a subscript it is always equal to either 1 or 2, denoting the interior or exterior body; this should not be confused with the use of k as the horizontal component of the eccentricity vector.

Since each of the h_j, k_j, p_j, and q_j is a function of two variables we can write

$$\frac{\mathrm{d}h_j}{\mathrm{d}t} = \frac{\partial h_j}{\partial e_j}\frac{\mathrm{d}e_j}{\mathrm{d}t} + \frac{\partial h_j}{\partial \varpi_j}\frac{\mathrm{d}\varpi_j}{\mathrm{d}t}, \qquad \frac{\mathrm{d}k_j}{\mathrm{d}t} = \frac{\partial k_j}{\partial e_j}\frac{\mathrm{d}e_j}{\mathrm{d}t} + \frac{\partial k_j}{\partial \varpi_j}\frac{\mathrm{d}\varpi_j}{\mathrm{d}t}, \tag{7.21}$$

$$\frac{\mathrm{d}p_j}{\mathrm{d}t} = \frac{\partial p_j}{\partial I_j}\frac{\mathrm{d}I_j}{\mathrm{d}t} + \frac{\partial p_j}{\partial \Omega_j}\frac{\mathrm{d}\Omega_j}{\mathrm{d}t}, \qquad \frac{\mathrm{d}q_j}{\mathrm{d}t} = \frac{\partial q_j}{\partial I_j}\frac{\mathrm{d}I_j}{\mathrm{d}t} + \frac{\partial q_j}{\partial \Omega_j}\frac{\mathrm{d}\Omega_j}{\mathrm{d}t}, \tag{7.22}$$

where, from the definitions given above, the partial derivatives are given by

$$\frac{\partial h_j}{\partial e_j} = \frac{h_j}{e_j}, \qquad \frac{\partial k_j}{\partial e_j} = \frac{k_j}{e_j}, \qquad \frac{\partial h_j}{\partial \varpi_j} = +k_j, \qquad \frac{\partial k_j}{\partial \varpi_j} = -h_j \tag{7.23}$$

and

$$\frac{\partial p_j}{\partial I_j} = \frac{p_j}{I_j}, \qquad \frac{\partial q_j}{\partial I_j} = \frac{q_j}{I_j}, \qquad \frac{\partial p_j}{\partial \Omega_j} = +q_j, \qquad \frac{\partial q_j}{\partial \Omega_j} = -p_j. \tag{7.24}$$

After some calculation it can be shown that the perturbation equations can be written as

$$\dot{h}_j = +\frac{1}{n_j a_j^2}\frac{\partial \mathcal{R}_j}{\partial k_j}, \qquad \dot{k}_j = -\frac{1}{n_j a_j^2}\frac{\partial \mathcal{R}_j}{\partial h_j}, \tag{7.25}$$

$$\dot{p}_j = +\frac{1}{n_j a_j^2}\frac{\partial \mathcal{R}_j}{\partial q_j}, \qquad \dot{q}_j = -\frac{1}{n_j a_j^2}\frac{\partial \mathcal{R}_j}{\partial p_j}, \tag{7.26}$$

where $\mathcal{R}_j$ is as given in Eq. (7.20).

The full equations for the variation of h_j, k_j, p_j, and q_j $(j = 1, 2)$ then become

$$\begin{aligned}
\dot{h}_1 &= +A_{11}k_1 + A_{12}k_2, & \dot{k}_1 &= -A_{11}h_1 - A_{12}h_2,\\
\dot{h}_2 &= +A_{21}k_1 + A_{22}k_2, & \dot{k}_2 &= -A_{21}h_1 - A_{22}h_2,\\
\dot{p}_1 &= +B_{11}q_1 + B_{12}q_2, & \dot{q}_1 &= -B_{11}p_1 - B_{12}p_2,\\
\dot{p}_2 &= +B_{21}q_1 + B_{22}q_2, & \dot{q}_2 &= -B_{21}p_1 - B_{22}p_2.
\end{aligned} \tag{7.27}$$

Thus, to lowest order, the equations for the time variation of $\{h_j, k_j\}$ are decoupled from those of $\{p_j, q_j\}$. Furthermore, these are linear differential equations with constant coefficients, and hence the problem of secular perturbations reduces to two sets of eigenvalue problems. The solutions are given by

$$h_j = \sum_{i=1}^{2} e_{ji}\sin(g_i t + \beta_i), \qquad k_j = \sum_{i=1}^{2} e_{ji}\cos(g_i t + \beta_i), \tag{7.28}$$

$$p_j = \sum_{i=1}^{2} I_{ji}\sin(f_i t + \gamma_i), \qquad q_j = \sum_{i=1}^{2} I_{ji}\cos(f_i t + \gamma_i), \tag{7.29}$$

where the frequencies g_i $(i = 1, 2)$ are the eigenvalues of the matrix $\mathbf{A}$, with e_{ji} the components of the two corresponding eigenvectors, and f_i $(i = 1, 2)$ are the eigenvalues of the matrix $\mathbf{B}$, with I_{ji} the components of the corresponding eigenvectors. The phases β_i and γ_i, as well as the amplitudes of the eigenvectors, are determined by the initial conditions. This would correspond to making observations of the osculating eccentricities and inclinations at some time. The solution described by Eqs. (7.28) and (7.29) is the classical *Laplace–Lagrange secular solution* of the secular problem.

With the introduction of the solution to the eigenvalue problem it is easy to confuse the quantities associated with the two bodies and those associated with the two eigenmodes of the system. In our notation the subscript j always denotes the planet number while the subscript i always denotes the mode number.

It is interesting to note that in our case the characteristic equation for $\mathbf{B}$ is

$$\begin{vmatrix} B_{11} - f & B_{12} \\ B_{21} & B_{22} - f \end{vmatrix} = 0, \tag{7.30}$$

which reduces to

$$f\,[f - (B_{11} + B_{22})] = 0 \tag{7.31}$$

since $B_{11}B_{22} - B_{12}B_{21} = 0$ from the definitions given in Eqs. (7.11) and (7.12). Thus one of the roots of the characteristic equation is $f_1 = 0$ and there is a degeneracy in the problem. This highlights a subtle difference between the $\{h, k\}$ and the $\{p, q\}$ solutions. Whereas an eccentric orbit introduces an asymmetry and a reference line into the problem, a spherical or point-mass central body has no natural reference plane. Physically it is only meaningful to talk about a mutual inclination and hence the choice of a reference plane is arbitrary. For example, it is customary to refer satellite orbits to the equatorial plane of the planet (i.e., the plane perpendicular to its spin vector). However, as we shall see, the introduction of a nonspherical planet adds terms to the diagonal elements of $\mathbf{B}$ and removes the degeneracy problem.

Another point concerning our solution is that it is independent of the mean longitudes because these have been deliberately excluded from the averaged part of the disturbing function. Therefore, although we are able to predict the variations in the eccentricities, inclinations, pericentres, and nodes of the two bodies, we have no information about their positions in space.

The solution given in Eqs. (7.28) and (7.29) implies that the resulting motion of all the masses is stable *for all time*. However, it is important to remember the assumptions under which this result was derived: (i) no mean motion commensurabilities, (ii) $\mathbf{r}_1 < \mathbf{r}_2$, and (iii) the es and Is are small enough that a second-order expansion of the disturbing function is sufficient to describe the motion. But the amplitudes of the eccentricity eigenvectors, for example, could be large enough for the orbits to intersect, violating conditions (ii) and (iii). As we shall see there may be situations where no mean motion commensurabilities exist, but where "small divisor" terms are still important. We have derived a theory that is correct only to the first order in the masses and so it is important to realise that there could be significant contributions from a second-order theory.

7.3 Jupiter and Saturn

We will now apply the theory given above to the case of Jupiter (mass m_1) and Saturn (mass m_2) orbiting the Sun (mass m_c). In 1983 the system had the following parameters:

$$
\begin{aligned}
m_1/m_c &= 9.54786 \times 10^{-4}, & m_2/m_c &= 2.85837 \times 10^{-4},\\
a_1 &= 5.202545 \text{ AU}, & a_2 &= 9.554841 \text{ AU},\\
n_1 &= 30.3374^\circ \text{y}^{-1}, & n_2 &= 12.1890^\circ \text{y}^{-1},\\
e_1 &= 0.0474622, & e_2 &= 0.0575481,\\
\varpi_1 &= 13.983865^\circ, & \varpi_2 &= 88.719425^\circ,\\
I_1 &= 1.30667^\circ, & I_2 &= 2.48795^\circ,\\
\Omega_1 &= 100.0381^\circ, & \Omega_2 &= 113.1334^\circ.
\end{aligned}
\tag{7.32}
$$

Since $\alpha = a_1/a_2 = 0.544493$, we can use the definition of Laplace coefficients given in Eqs. (7.13) and (7.14) to get

$$
b_{3/2}^{(1)} = 3.17296, \qquad b_{3/2}^{(2)} = 2.07110. \tag{7.33}
$$

Using the definitions of the matrix elements given in Eqs. (7.9)–(7.12) we have

$$
\mathbf{A} = \begin{pmatrix} +0.00203738 & -0.00132987 \\ -0.00328007 & +0.00502513 \end{pmatrix} {}^\circ \text{y}^{-1} \tag{7.34}
$$

and

$$
\mathbf{B} = \begin{pmatrix} -0.00203738 & +0.00203738 \\ +0.00502513 & -0.00502513 \end{pmatrix} {}^\circ \text{y}^{-1}. \tag{7.35}
$$

We can now find the eigenvalues of **A** and **B** by solving the respective characteristic equations:

$$
\begin{vmatrix} A_{11} - g & A_{12} \\ A_{21} & A_{22} - g \end{vmatrix} = g^2 - (A_{11} + A_{22})\, g + (A_{11}A_{22} - A_{21}A_{12}) = 0 \tag{7.36}
$$

and

$$
\begin{vmatrix} B_{11} - f & B_{12} \\ B_{21} & B_{22} - f \end{vmatrix} = f^2 - (B_{11} + B_{22})\, f + (B_{11}B_{22} - B_{21}B_{12}) = 0. \tag{7.37}
$$

The solutions of the resulting quadratic equations are

$$
g_1 = 9.63435 \times 10^{-4} \,{}^\circ\text{y}^{-1}, \qquad g_2 = 6.09908 \times 10^{-3} \,{}^\circ\text{y}^{-1} \tag{7.38}
$$

and

$$
f_1 = 0, \qquad f_2 = -7.06251 \times 10^{-3} \,{}^\circ\text{y}^{-1}. \tag{7.39}
$$

The eigenvectors of **A** and **B** are the four vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{y}_1$, and $\mathbf{y}_2$ that satisfy the equations

$$
\mathbf{A}\mathbf{x}_i = g_i \mathbf{x}_i \qquad \text{and} \qquad \mathbf{B}\mathbf{y}_i = f_i \mathbf{y}_i \qquad (i = 1, 2). \tag{7.40}
$$

However, it is clear from these definitions that if $\mathbf{x}_i$ is an eigenvector of the matrix **A** then so is $c\mathbf{x}_i$, where c is a constant. Therefore each eigenvector is only determined up to some arbitrary scaling constant. If we let $\bar{e}_{ji}$ and $\bar{I}_{ji}$ denote the components of these unscaled eigenvectors and let S_i and T_i denote the scaling constant (or magnitude) of each eigenvector, then

$$S_i\bar{e}_{ji} = e_{ji} \qquad \text{and} \qquad T_i\bar{I}_{ji} = I_{ji} \qquad (i = 1, 2). \tag{7.41}$$

The values of $\bar{e}_{ji}$ and $\bar{I}_{ji}$ are obtained by solving four sets of two simultaneous linear equations in two unknowns. The four resulting (unscaled) eigenvectors are

$$\begin{pmatrix} \bar{e}_{11} \\ \bar{e}_{21} \end{pmatrix} = \begin{pmatrix} -0.777991 \\ -0.628275 \end{pmatrix}, \qquad \begin{pmatrix} \bar{e}_{12} \\ \bar{e}_{22} \end{pmatrix} = \begin{pmatrix} 0.332842 \\ -1.01657 \end{pmatrix},$$
$$\begin{pmatrix} \bar{I}_{11} \\ \bar{I}_{21} \end{pmatrix} = \begin{pmatrix} 0.707107 \\ 0.707107 \end{pmatrix}, \qquad \begin{pmatrix} \bar{I}_{12} \\ \bar{I}_{22} \end{pmatrix} = \begin{pmatrix} -0.40797 \\ 1.00624 \end{pmatrix}. \tag{7.42}$$

The scaling factors S_i and T_i are determined from the boundary conditions. At time $t = 0$ we have

$$h_1 = 0.0114692, \quad h_2 = 0.0575337, \quad k_1 = 0.0460556, \quad k_2 = 0.00128611 \tag{7.43}$$

and

$$p_1 = 0.0224566, \quad p_2 = 0.0399314, \quad q_1 = -0.00397510, \quad q_2 = -0.0170597, \tag{7.44}$$

where we have converted the inclinations from degrees to radians. Substituting $t = 0$ in our general solution given in Eqs. (7.28) and (7.29) we get

$$h_j = S_1\bar{e}_{j1}\sin\beta_1 + S_2\bar{e}_{j2}\sin\beta_2, \qquad k_j = S_1\bar{e}_{j1}\cos\beta_1 + S_2\bar{e}_{j2}\cos\beta_2 \tag{7.45}$$

and

$$p_j = T_1\bar{I}_{j1}\sin\gamma_1 + T_2\bar{I}_{j2}\sin\gamma_2, \qquad q_j = T_1\bar{I}_{j1}\cos\gamma_1 + T_2\bar{I}_{j2}\cos\gamma_2, \tag{7.46}$$

where the subscript j $(= 1, 2)$ denotes the planet (Jupiter or Saturn). These can be considered as four sets of two simultaneous linear equations in the eight unknowns $S_i \sin\beta_i$, $S_i \cos\beta_i$, $T_i \sin\gamma_i$, and $T_i \cos\gamma_i$ with $(i = 1, 2)$. In our case the solutions are

$$\begin{pmatrix} S_1\sin\beta_1 \\ S_2\sin\beta_2 \end{pmatrix} = \begin{pmatrix} -0.0308089 \\ -0.375549 \end{pmatrix}, \qquad \begin{pmatrix} S_1\cos\beta_1 \\ S_2\cos\beta_2 \end{pmatrix} = \begin{pmatrix} -0.0472469 \\ 0.027935 \end{pmatrix},$$
$$\begin{pmatrix} T_1\sin\gamma_1 \\ T_2\sin\gamma_2 \end{pmatrix} = \begin{pmatrix} 0.0388876 \\ 0.0123566 \end{pmatrix}, \qquad \begin{pmatrix} T_1\cos\gamma_1 \\ T_2\cos\gamma_2 \end{pmatrix} = \begin{pmatrix} -0.0109598 \\ -0.00925221 \end{pmatrix}. \tag{7.47}$$

These give

$$\beta_1 = -146.892^\circ, \quad \beta_2 = -53.3565,^\circ \quad \gamma_1 = 105.74^\circ \quad \gamma_2 = 126.825^\circ \tag{7.48}$$

and

$$S_1 = 0.0564044, \quad S_2 = 0.0468053, \quad T_1 = 0.0404025, \quad T_2 = 0.0154366. \tag{7.49}$$

The resulting, scaled eigenvectors are

$$\begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix} = \begin{pmatrix} -0.0438821 \\ -0.0354375 \end{pmatrix}, \quad \begin{pmatrix} e_{12} \\ e_{22} \end{pmatrix} = \begin{pmatrix} 0.0155788 \\ -0.047581 \end{pmatrix},$$
$$\begin{pmatrix} I_{11} \\ I_{21} \end{pmatrix} = \begin{pmatrix} 0.0285689 \\ 0.0285689 \end{pmatrix}, \quad \begin{pmatrix} I_{12} \\ I_{22} \end{pmatrix} = \begin{pmatrix} -0.00629766 \\ 0.015533 \end{pmatrix}, \qquad (7.50)$$

where the I_{ji} are expressed in radians.

We have now determined all the constants in Eqs. (7.28) and (7.29). Therefore we can obtain h, k, p, and q for Jupiter and Saturn at any time t. The solution is of the form

$$\begin{aligned} h_j &= e_{j1}\sin(g_1 t + \beta_1) + e_{j2}\sin(g_2 t + \beta_2), \\ k_j &= e_{j1}\cos(g_1 t + \beta_1) + e_{j2}\cos(g_2 t + \beta_2), \\ p_j &= I_{j1}\sin(f_1 t + \gamma_1) + I_{j2}\sin(f_2 t + \gamma_2), \\ q_j &= I_{j1}\cos(f_1 t + \gamma_1) + I_{j2}\cos(f_2 t + \gamma_2), \end{aligned} \qquad (7.51)$$

where $j = 1$ for Jupiter and $j = 2$ for Saturn. From these solutions we can derive the orbital elements of the two planets at any time t. For example, the relation $e_j(t) = (h_j^2 + k_j^2)^{1/2}$ is used to calculate the eccentricity of planet j. Using our results we obtain

$$\begin{aligned} e_1(t) &= \sqrt{0.00217 - 0.00137\cos(93.5^\circ + 0.00514\,t)}, \\ e_2(t) &= \sqrt{0.00352 + 0.00337\cos(93.5^\circ + 0.00514\,t)}, \end{aligned} \qquad (7.52)$$

where the phases are in degrees and the frequencies in degrees per year. This implies a fixed periodicity of $\sim$ 70,100 y in the variation of the eccentricity of each planet. Figure 7.1a shows the evolution of the eccentricities of the two planets over a time span of 200,000 y derived from our secular solution; the periodicity in the variation is clear. The different signs in the magnitude of the cosine imply that a maximum in Jupiter's eccentricity coincides with a minimum in Saturn's eccentricity and vice versa.

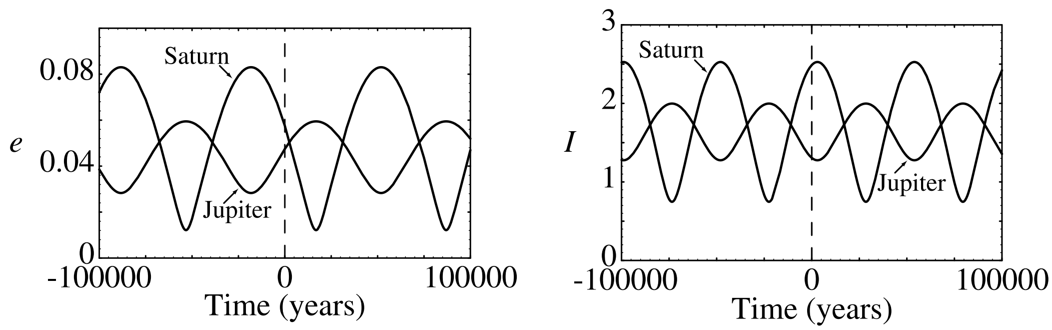

Fig. 7.1. The (a) eccentricities and (b) inclinations of Jupiter and Saturn derived from a secular perturbation theory calculated over a time span of 200,000 y centred on 1983.

Similarly the relation $I_j = (p_j^2 + q_j^2)^{1/2}$ is used to calculate the inclination of planet j at any time t. Our results (in radians) give

$$\begin{aligned} I_1(t) &= \sqrt{0.000856 - 0.00360\cos(21.1^\circ - 0.00706\,t)}, \\ I_2(t) &= \sqrt{0.00106 + 0.000888\cos(21.1^\circ - 0.00706\,t)}. \end{aligned} \tag{7.53}$$

In this case the associated period of the secular variation in each planet is $\sim$51,000 y; since $f_1 = 0$, this period is just $360^\circ/f_2$. The variation for each planet is shown in Fig. 7.1b.

The secular solution that we have derived for Jupiter and Saturn is only an approximation to the actual variations in their orbital elements. In reality the perturbations from the planets Uranus and Neptune exert considerable influence on their orbits. A further complication is that the orbits of Jupiter and Saturn are close to a 5:2 commensurability. This introduces additional perturbations on timescales that are shorter than those associated with the secular variation.

7.4 Free and Forced Elements

We have shown that under certain conditions we can construct a secular solution to the motion of two orbiting bodies moving under their mutual gravitational effects; at any time we can obtain the eccentricities, longitudes of pericentre, inclinations, and longitudes of ascending node of both bodies. We can make use of this solution to study the motion of an additional body, of negligible mass, moving under the influence of the central body and perturbed by the other two bodies.

Following the example given in Sect. 7.2 for the secular theory for two bodies, the disturbing function $\mathcal{R}$ for a test particle with orbital elements a, n, e, I, ϖ, and Ω is given by

$$\mathcal{R} = na^2\left[\frac{1}{2}Ae^2 + \frac{1}{2}BI^2 + \sum_{j=1}^{2} A_j e e_j \cos(\varpi - \varpi_j) + \sum_{j=1}^{2} B_j I I_j \cos(\Omega - \Omega_j)\right], \tag{7.54}$$

where

$$A = +n\frac{1}{4}\sum_{j=1}^{2}\frac{m_j}{m_c}\alpha_j\bar{\alpha}_j b_{3/2}^{(1)}(\alpha_j), \tag{7.55}$$

$$A_j = -n\frac{1}{4}\frac{m_j}{m_c}\alpha_j\bar{\alpha}_j b_{3/2}^{(2)}(\alpha_j), \tag{7.56}$$

$$B = -n\frac{1}{4}\sum_{j=1}^{2}\frac{m_j}{m_c}\alpha_j\bar{\alpha}_j b_{3/2}^{(1)}(\alpha_j), \tag{7.57}$$

$$B_j = +n\frac{1}{4}\frac{m_j}{m_c}\alpha_j\bar{\alpha}_j b_{3/2}^{(1)}(\alpha_j) \tag{7.58}$$