4
Lambert's Problem

4.1 Introduction

A fundamental problem in astrodynamics is the transfer of a spacecraft from one point in space to another. An example application is spacecraft targeting, in which the final point (the "target") is a planet or space station moving in a known orbit. In this situation, one might want the spacecraft to either *intercept* the target (match position only) or *rendezvous* with the target (match both position and velocity).

The initial point for an orbital rendezvous or interception is typically the location of the spacecraft in its orbit at the initial time. However, in other applications, such as ascent trajectories from the surface of the moon, the initial point can be at rest on the surface. Common to all orbit transfer applications is the determination of two-body orbits that connect specified initial and final points.

4.2 Transfer Orbits Between Specified Points

As shown in Fig. 4.1, consider points P_1 and P_2 described by radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to the focus F at the center of attraction. The end points P_1 and P_2 are separated by the transfer angle θ and the chord c. The triangle FP_1P_2 is sometimes referred to as the *space triangle* for the transfer.

First, let us investigate the possible transfer orbits between the specified endpoints P_1 and P_2. For the case of elliptic transfer orbits, this can be accomplished using the simple geometric property shown in Fig. 4.2. A similar analysis can be done for parabolic and hyperbolic transfer orbits, but is not presented here.

The geometric property used is that the sum of the distances from any point on the ellipse to the focus and the vacant focus is a constant having value $2a$. This is the familiar property by which one can draw an ellipse by anchoring a piece of string at two points with thumbtacks, draw the string taut with the point of a pencil, and trace out an ellipse. In this mechanical device the thumbtacks locate the focus and vacant focus, and the string length is $2a$. Thus in Fig. 4.2

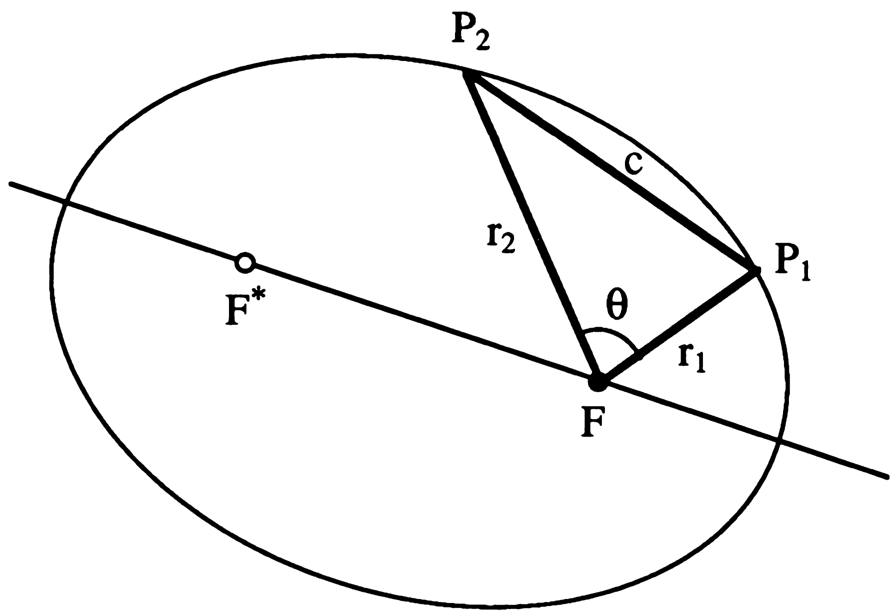

Fig. 4.1 Transfer Orbit Geometry

$$P_1F + P_1F^* = 2a \tag{4.1}$$

and

$$P_2F + P_2F^* = 2a$$

or

$$P_1F^* = 2a - r_1 \tag{4.2}$$

and

$$P_2F^* = 2a - r_2$$

For the remainder of the discussion, let us assume that $r_2 \geq r_1$, which implies no loss of generality, since the transfer orbit can be traversed in either direction. Because gravity is a conservative (nondissipative) force, one can determine the orbit that solves the boundary value problem in the reverse direction (starting at the final point P_2 and ending at the initial point P_1) by simply letting time run backward on the original orbit from P_2 to P_1. This represents a valid forward time solution with the original velocity vector replaced by its negative.

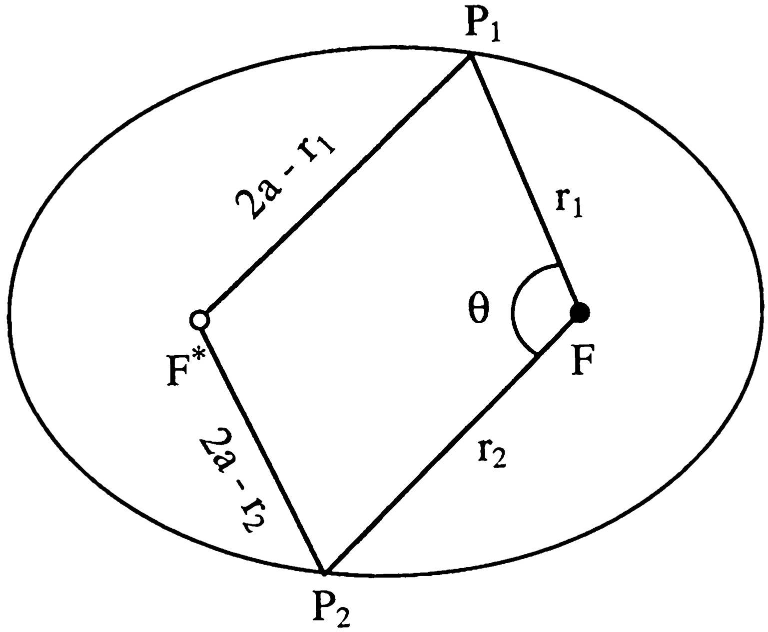

Fig. 4.2 A Geometric Property of Ellipses

For a given space triangle and specified value of semimajor axis a, Fig. 4.3 shows that a vacant focus is located at the intersection of two circles centered at P_1 and P_2 having respective radii $2a - r_1$ and $2a - r_2$ [Eq. (4.2)].

As shown in Fig. 4.3, the circular arcs for a given value of $a = a_k$ intersect at two points labeled F_k^* and $\tilde{F}_k^*$ that are equidistant from the chord c. This means that for the value of a depicted, there are two elliptic transfer orbits between P_1 and P_2. As we will see, these two transfer orbits for the same value of a have different eccentricities and transfer times, but they have the same total energy.

From Fig. 4.3 it is evident that the distance FF^* is less than the distance $F\tilde{F}^*$. Because the distance from the focus of an ellipse to the vacant focus is $2ae$ (Sec. 1.5), this implies that the ellipse with vacant focus at F^* has the smaller eccentricity: $e < \tilde{e}$. Figure 4.4 shows the two elliptic transfer orbits for the case $r_2 = 1.524\ r_1$ (earth to Mars) with $\theta = 107\ °$ and a specified semimajor axis value of $a = 1.36\ r_1$. The numerical values of the two eccentricities are $e = 0.26$ and $\tilde{e} = 0.68$ for this case.

Returning to Fig. 4.3 two other aspects of the problem are evident from the geometry. First, as the value of a is varied, the vacant foci describe a locus formed by the intersections of the circles of varying radii centered at P_1 and P_2. This *locus of the focus* has the property that at any point on it the *difference* in the distances to the fixed points P_1 and P_2 is a constant, equal to $r_2 - r_1$. This implies that the locus itself (the solid line in Fig. 4.3) is a *hyperbola* with foci at P_1 and P_2 !

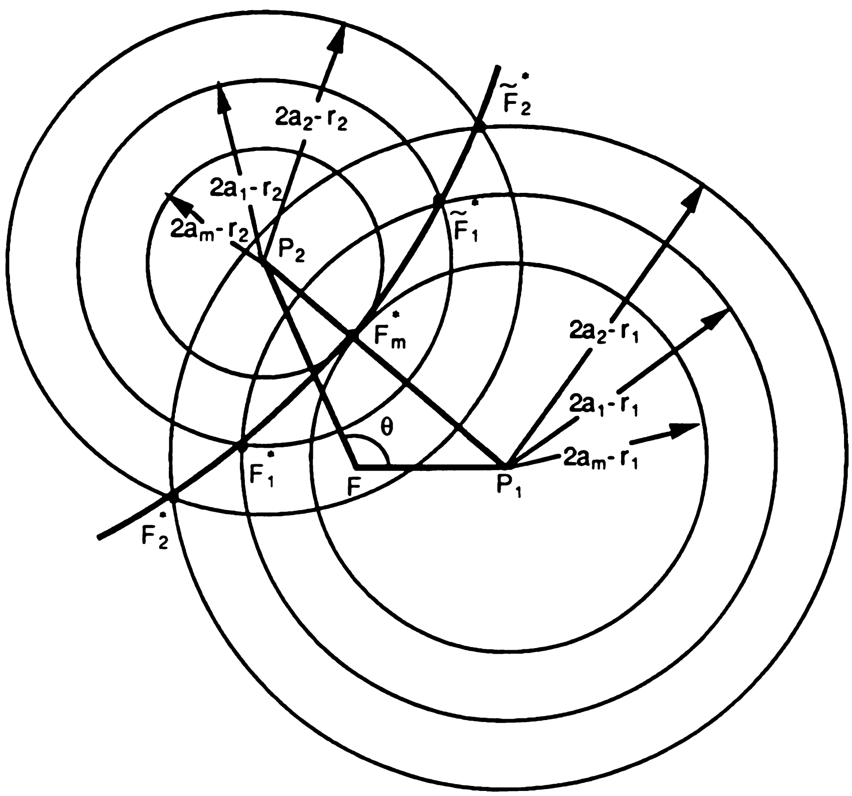

Fig. 4.3 Vacant Focus Locations

The second aspect of the problem that is evident from Fig. 4.3 is that as the value of a is decreased from the values shown, the two vacant foci approach the point $F_m{}^*$ on the chord between P_1 and P_2. For all values of a less than this value there is *no intersection* of the circles centered at P_1 and P_2, which implies that no elliptic transfer connecting P_1 and P_2 exists for values of a less than a certain minimum value. This minimum value is denoted by a_m and its value is easily calculated from the geometry of the point $F_m{}^*$ in Fig. 4.3:

$$(2\,a_m - r_2) + (2\,a_m - r_1) = c \tag{4.3}$$

or

$$a_m = s/2 \tag{4.4}$$

where

$$s \equiv (r_1 + r_2 + c)/2 \tag{4.5}$$

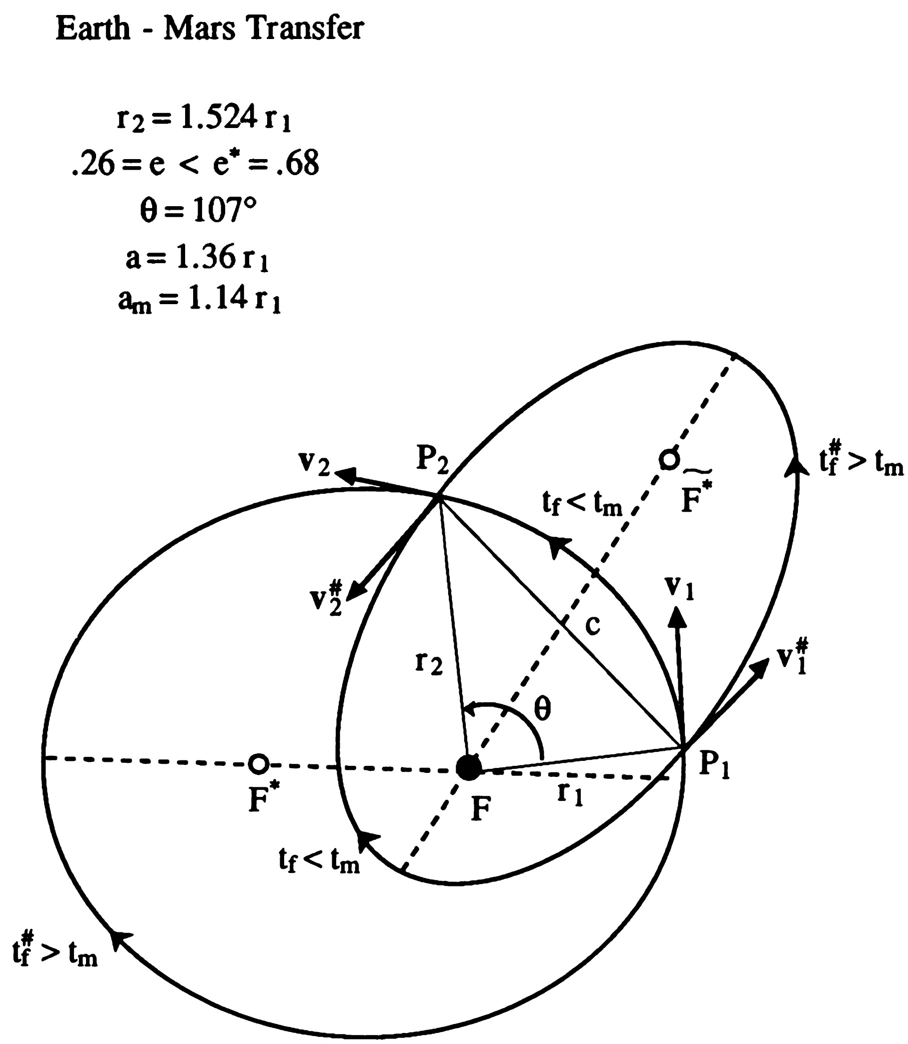

Fig. 4.4 Two Elliptic Transfer Orbits with the Same Value of a

is the *semiperimeter* of the space triangle FP_1P_2.

In addition to describing the value of a_m geometrically, one can also interpret it dynamically by recalling that the value of a for a conic orbit is a measure of its total energy (Sec. 2.4). Thus the ellipse having semimajor axis a_m is the *minimum-energy ellipse* that connects the specified endpoints P_1 and P_2. Orbits having a value of a less than a_m simply do not have enough energy at point P_1 to reach point P_2. Some days are like that.

One other interesting geometric property of the elliptic transfer orbits between P_1 and P_2 concerns the eccentricities of these orbits. As will be shown, the locus of the eccentricity vectors is a straight line that is normal to the chord, as shown by Battin, Fill, and Shepperd in [4.1].

To demonstrate this, one uses the basic polar equation for a conic section

$$r = \frac{p}{1 + e\cos f} \tag{4.6}$$

to write

$$\mathbf{e} \cdot \mathbf{r}_1 = p - r_1 \; ; \; \mathbf{e} \cdot \mathbf{r}_2 = p - r_2 \tag{4.7}$$

Subtracting and dividing by the chord c yields

$$-\mathbf{e} \cdot (\mathbf{r}_2 - \mathbf{r}_1)/c = (r_2 - r_1)/c \tag{4.8}$$

Because $(\mathbf{r}_2 - \mathbf{r}_1)/c$ is a unit vector along the chord directed from P_1 to P_2, Eq. (4.8) implies that the eccentricity vectors for all the transfer orbits have a constant projection along the chord direction. This, in turn, implies that the locus of the eccentricity vectors is a straight line normal to the chord as shown in Fig. 4.5.

Also evident from Fig. 4.5 is the fact that there is a transfer ellipse of *minimum eccentricity* e_s whose value is simply [4.3]

$$e_s = \frac{r_2 - r_1}{c} \tag{4.9}$$

which is, interestingly, the reciprocal of the eccentricity of the hyperbolic locus of the vacant focus. This minimum eccentricity ellipse is also termed the *fundamental ellipse* because the point P_1 has the same relationship to the occupied focus F as P_2 does to the vacant focus $F,^*$ due to the fact that the major axis of the ellipse is parallel to the chord.

4.3 Lambert's Theorem

A primary concern in orbit transfer is the transfer time, defined as the time required to travel from point P_1 to point P_2. In the spacecraft targeting example mentioned earlier, the spacecraft is at point P_1 in its orbit at a time t_1 and the target will be at point P_2 in its orbit at a later time t_2. The transfer time is then $t_2 - t_1$ and the crucial issue is the determination of the transfer orbit that connects the specified endpoints in the given transfer time. Because of the work of Johann Heinrich Lambert (1728–1779), this is often called *Lambert's problem*.

The theorem which bears his name is due to his conjecture in 1761, based on geometric reasoning, that the time required to traverse an elliptic arc between specified endpoints *depends only on the semimajor axis of the ellipse, and on two geometric properties of the space triangle, namely the chord length and the sum of the radii from the focus to points P_1 and P_2*: