# Chapter 5 Schwarzschild spacetime

## Introduction

The previous chapter introduced Einstein's field equations of general relativity. These equations assert the direct proportionality of the geometric Einstein tensor $[G_{\mu\nu}]$ that represents the gravitational 'field', and the energy–momentum tensor $[T_{\mu\nu}]$ that represents the 'sources' of the gravitational field. However, at a deeper level, once the Einstein tensor has been expanded in terms of the Ricci tensor $[R_{\mu\nu}]$, the Ricci tensor expressed in terms of components of the Riemann tensor $[R^{\rho}{}_{\sigma\mu\nu}]$, and the Riemann tensor related to the connection coefficients and hence to components of the metric tensor $[g_{\mu\nu}]$, it is seen that the Einstein field equations are actually a set of complicated non-linear differential equations that relate the metric coefficients $g_{\mu\nu}$ of some region of spacetime to quantities that describe the density and flow of energy and momentum in that region. Solving the Einstein field equations for some specified region (if that can be done) provides all the information needed to determine the four-dimensional line element $(\mathrm{d}s)^2$ in that region along with all the other geometric properties that follow from it. This includes the set of time-like and null geodesic pathways through an event that represent the possible world-lines of massive and massless particles present at that event.

In four-dimensional spacetime the Einstein field equations can have non-trivial solutions even in regions where there are no sources, i.e. in regions of spacetime that are devoid of matter and radiation (in this chapter we shall ignore dark energy). In the absence of sources $[T_{\mu\nu}] = 0$, and the field equations require that the Ricci tensor must vanish, but the relationship between the Ricci and Riemann tensors is such that the vanishing of the Ricci tensor does not necessarily imply that the Riemann tensor should be zero. If the Riemann tensor is not zero, then the spacetime must be curved and the metric tensor $[g_{\mu\nu}]$ that satisfies the Einstein field equations must differ from the 'trivial' Minkowski metric $[\eta_{\mu\nu}]$ that describes a flat spacetime. In this sense the Einstein field equations can describe gravitational fields in empty space, just as Maxwell's equations can describe non-trivial electric and magnetic fields in a vacuum. As we noted in the previous chapter, the solutions that arise when $[T_{\mu\nu}] = 0$ are called *vacuum solutions*.

This chapter is mainly concerned with one of these vacuum solutions — the *Schwarzschild solution*, the first and arguably the most important non-trivial solution of the Einstein field equations. We shall start by simply writing down the Schwarzschild solution so that you can see what a solution looks like and how it is conventionally presented. Next we shall outline how this particular solution can be obtained and then go on to examine its properties and some of its consequences for observations regarding intervals in space and time. These investigations of a particularly simple curved spacetime can be seen as the analogues of those that we carried out in Chapter 1 when investigating time dilation and length contraction in the flat spacetime described by the Minkowski metric of special relativity.

In Section 5.4 we shall use the metric provided by the Schwarzschild solution to determine geodesic pathways in a region described by that solution. This will enable us to study the motion of massive and massless particles in such a region and thus discuss the behaviour of massive bodies and light pulses that move under the influence of gravity alone.

In case all of this sounds like a purely mathematical exploration of some particular solution of the Einstein field equations, it's worth pointing out that many years after its discovery the Schwarzschild solution was recognized as describing the most basic type of *black hole*. The study of the Schwarzschild solution is therefore the natural precursor and preparation for the study of black holes, which have done much to revolutionize thinking in astrophysics. Black holes will be the subject of the next chapter.

## 5.1 The metric of Schwarzschild spacetime

The Schwarzschild solution takes its name from the German astrophysicist Karl Schwarzschild (Figure 5.1) who published the relevant results in 1916, shortly after Einstein completed his theory of general relativity. Schwarzschild had been a university professor and Director of the Potsdam Observatory outside Berlin but joined the German army at the outbreak of the First World War and was serving on the Eastern front when he made his discovery. He posted his results to Einstein, who was surprised that such a simple solution could be found.

**Figure 5.1** Karl Schwarzschild (1873–1916) discovered the first exact solution of the Einstein field equations. He served as an artillery officer in the First World War, but contracted a serious skin disease and was invalided out of the army. He died in May 1916, not long after completing the work for which he is mainly remembered.

### 5.1.1 The Schwarzschild metric

The 'exterior' Schwarzschild solution discussed here describes the spacetime geometry in the empty region surrounding a non-rotating, spherically symmetric body of mass $M$. (You might like to think of that body as a simplified model of a star.) The presentation of the Schwarzschild solution, like that of any solution of the Einstein field equations, involves specifying, as explicit functions of the spacetime coordinates $x^0, x^1, x^2, x^3$, the sixteen components of the metric tensor $[g_{\mu\nu}]$ that correspond to the energy–momentum tensor $[T_{\mu\nu}]$ in the region of interest. In the case of the Schwarzschild solution, the relevant energy–momentum tensor is $[T_{\mu\nu}] = 0$ since we are dealing with the empty region *outside* the mass distribution. Nonetheless, the symmetry of the region involved suggests that it would be wise to use a system of spherical coordinates originating at the centre of the massive body, and it also seems likely that the solution will involve the mass $M$ in some way. We shall have more to say about the significance of $M$ and the precise meaning of the coordinates later; for the moment we shall simply refer to the coordinates as **Schwarzschild coordinates** and denote them by $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$.

Due to the symmetry of the metric tensor, only ten of its sixteen components $g_{\mu\nu}$ are independent. Moreover, in the particular case of the Schwarzschild solution, thanks to the spherical symmetry, the lack of time-dependence and the judicious choice of coordinates, only four of the components turn out to be non-zero, and none of them depends on $x^0$. In fact, the solution can be represented by the diagonal matrix

$$[g_{\mu\nu}] = \begin{pmatrix} 1 - \frac{2GM}{c^2 r} & 0 & 0 & 0 \\ 0 & -\dfrac{1}{1 - \frac{2GM}{c^2 r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}. \qquad (5.1)$$

Though clear, this is a rather cumbersome way of presenting the metric, so it is actually more common to see the non-zero components presented as the metric coefficients in the four-dimensional line element of the spacetime region being described. This is usually written as follows.

**The Schwarzschild metric**

$$(\mathrm{d}s)^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 (\mathrm{d}t)^2 - \frac{(\mathrm{d}r)^2}{1 - \frac{2GM}{c^2 r}} - r^2 (\mathrm{d}\theta)^2 - r^2 \sin^2\theta\, (\mathrm{d}\phi)^2. \qquad (5.2)$$

Although the terminology that we have been using leads us to refer to this expression as a line element, what it really tells us is the functional form of the non-zero components of the metric tensor. Because of this it is often referred to as the **Schwarzschild metric**. You should also be aware that built into it is the choice that we made regarding the use of an $x^0$ coordinate to represent time (some authors prefer $x^4$) and some other decisions regarding signs and symbols. The upshot of all this is that although we have adopted a range of common conventions, you should not be surprised to find that other authors may make different decisions and will therefore write the Schwarzschild solution in a related but different form.

### 5.1.2 Derivation of the Schwarzschild metric

In empty space $T_{\mu\nu} = 0$, so the Einstein field equations become

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\, R = 0. \qquad (5.3)$$

These equations are known as the **vacuum field equations**. Multiplying them by $g^{\mu\nu}$ and contracting over the indices $\mu$ and $\nu$ gives

$$\sum_{\mu,\nu} g^{\mu\nu} \left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\, R\right) = 0, \qquad (5.4)$$

that is,

$$\sum_{\nu} \left(R^{\nu}{}_{\nu} - \tfrac{1}{2} \delta^{\nu}{}_{\nu}\, R\right) = 0. \qquad (5.5)$$

Summing $R^{\nu}{}_{\nu}$ over all values of $\nu$ gives the curvature scalar $R$, while summing $\delta^{\nu}{}_{\nu}$ over all possible values of $\nu$ gives $\delta^0{}_0 + \delta^1{}_1 + \delta^2{}_2 + \delta^3{}_3 = 4$. Substituting these results into Equation 5.5, we get

$$R - \tfrac{1}{2} 4R = 0,$$

showing that $R = 0$ in this case and hence (from the vacuum field equations) that $R_{\mu\nu} = 0$ for all values of $\mu$ and $\nu$. Thus the Ricci tensor and the curvature scalar must both vanish for a vacuum solution, but remember, this is not sufficient to make spacetime flat.

It would be straightforward (though time-consuming) to show that the Schwarzschild metric written down earlier does indeed lead to a vanishing Ricci tensor and therefore *is* a solution of the vacuum field equations. However, that is not the aim of this section. Rather, our approach here is to write down the most general metric that exhibits the symmetries expected of the Schwarzschild solution and then use the additional requirement that the metric satisfies the vacuum field equations to lead us to a specific metric that will turn out to be the Schwarzschild solution. This is closer to the approach actually followed by Schwarzschild.

Note that you are not expected to remember all the steps in this derivation, but you should be able to follow them and they should provide helpful examples of many of the tensor quantities that were introduced earlier. The derivation omits a lot of detailed algebra, simply quoting results in its place. If you really want to get a feel for relativity, you might like to fill in some of the missing steps, but don't try this if you are short of time!

Since the Schwarzschild solution describes the geometry of the empty spacetime region surrounding a spherically symmetric body, it is natural to use a system of spherical coordinates centred on the middle of that spherically symmetric body (see Figure 5.2). In addition we shall assume the following.

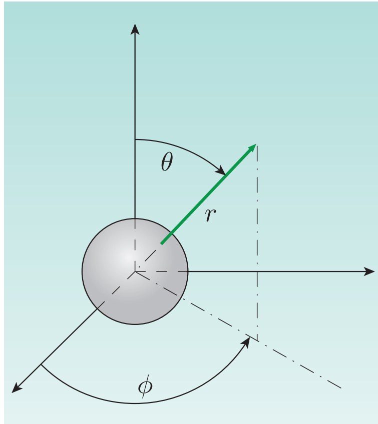


**Figure 5.2** The spatial part of the Schwarzschild coordinate system, with origin at the centre of a spherically symmetric body.

1. The spacetime far from the spherically symmetric body is flat. This is described by saying that the metric is **asymptotically flat** and is consistent with the idea that gravitational effects become weaker as the distance from their source increases.
2. The metric coefficients do not depend on time. This is described by saying that the metric is **stationary** and is consistent with the idea that nothing is moving from place to place.
3. The line element is unchanged if $t$ is replaced by $-t$. This is described by saying that the metric is **static** and is consistent with the idea that nothing is rotating.

We shall say more about these assumptions and about the definition and meaning of the Schwarzschild coordinates later. For the moment we shall simply use them.

The most general spacetime line element that meets all of the listed requirements may be written as

$$\begin{aligned}(\mathrm{d}s)^2 &= \sum_{\mu,\nu} g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \\ &= \mathrm{e}^{2A}(c\,\mathrm{d}t)^2 - \mathrm{e}^{2B}(\mathrm{d}r)^2 - r^2(\mathrm{d}\theta)^2 - r^2\sin^2\theta(\mathrm{d}\phi)^2, \qquad (5.6)\end{aligned}$$

where $A$ and $B$ are functions of the radial coordinate $r$ alone. You may wonder why we choose to include exponential functions of the form $\mathrm{e}^{2A}$ and $\mathrm{e}^{2B}$ rather than simply using functions such as $f(r)$ and $g(r)$. The reason is that the use of exponentials ensures that the signs of the metric components will be preserved in the desired $(+,-,-,-)$ pattern. The absence of terms proportional to $\mathrm{d}x^i\,\mathrm{d}t$ (where $i = 1, 2$ or 3) reflects the static property of the spacetime, while the absence of $\mathrm{d}x^i\,\mathrm{d}x^j$ terms reflects the spherical symmetry.

Our aim now is to determine the precise form of the functions $A(r)$ and $B(r)$ using the fact that the metric must satisfy the vacuum field equations. The first step in this process is the determination of the connection coefficients that correspond to the metric given in Equation 5.6. This involves applying the general formula

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2}\sum_{\rho} g^{\sigma\rho}\left\{\frac{\partial g_{\rho\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\rho}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right\}$$

to the case where $g_{00} = \mathrm{e}^{2A}$, $g_{11} = -\mathrm{e}^{2B}$, $g_{22} = -r^2$ and $g_{33} = -r^2\sin^2\theta$. Because the metric is represented by a diagonal matrix in this case, each contravariant component $g^{\mu\nu}$ is simply the reciprocal of the corresponding covariant component $g_{\mu\nu}$, so $g^{00} = \mathrm{e}^{-2A}$, $g^{11} = -\mathrm{e}^{-2B}$, $g^{22} = -1/r^2$ and $g^{33} = -1/r^2\sin^2\theta$. Substituting these values into the expression for $\Gamma^{\sigma}{}_{\mu\nu}$ shows that only nine of the forty independent connection coefficients for this metric are non-zero. Using a prime to indicate differentiation with respect to $r$, so that $A' = \mathrm{d}A(r)/\mathrm{d}r$ and $B' = \mathrm{d}B(r)/\mathrm{d}r$, these nine independent non-zero connection coefficients can be written as

$$\begin{aligned}
\Gamma^{0}{}_{01} &= A' \; (= \Gamma^{0}{}_{10}),\\
\Gamma^{1}{}_{00} &= A'\mathrm{e}^{2(A-B)},\\
\Gamma^{1}{}_{11} &= B',\\
\Gamma^{1}{}_{22} &= -r\mathrm{e}^{-2B},\\
\Gamma^{1}{}_{33} &= -\mathrm{e}^{-2B}r\sin^2\theta,\\
\Gamma^{2}{}_{12} &= \frac{1}{r} \; (= \Gamma^{2}{}_{21}),\\
\Gamma^{2}{}_{33} &= -\sin\theta\cos\theta,\\
\Gamma^{3}{}_{13} &= \frac{1}{r} \; (= \Gamma^{3}{}_{31}),\\
\Gamma^{3}{}_{23} &= \cot\theta \; (= \Gamma^{3}{}_{32}).
\end{aligned}$$

These non-zero connection coefficients can be used to determine the non-zero components of the Riemann curvature tensor using the general formula

$$R^{\rho}{}_{\sigma\mu\nu} = \frac{\partial \Gamma^{\rho}{}_{\sigma\nu}}{\partial x^{\mu}} - \frac{\partial \Gamma^{\rho}{}_{\sigma\mu}}{\partial x^{\nu}} + \sum_{\lambda}\Gamma^{\lambda}{}_{\sigma\nu}\,\Gamma^{\rho}{}_{\lambda\mu} - \sum_{\lambda}\Gamma^{\lambda}{}_{\sigma\mu}\,\Gamma^{\rho}{}_{\lambda\nu}.$$

Again, there are many symmetries so not all the non-zero curvature tensor components are independent, though these are the six that are:

$$\begin{aligned}
R^{0}{}_{101} &= A'B' - A'' - \left(A'\right)^2,\\
R^{0}{}_{202} &= -r\mathrm{e}^{-2B}A',\\
R^{0}{}_{303} &= -r\mathrm{e}^{-2B}A'\sin^2\theta,\\
R^{1}{}_{212} &= r\mathrm{e}^{-2B}B',\\
R^{1}{}_{313} &= r\mathrm{e}^{-2B}B'\sin^2\theta,\\
R^{2}{}_{323} &= \left(1 - \mathrm{e}^{-2B}\right)\sin^2\theta,
\end{aligned}$$

where the double prime indicates the second derivative with respect to $r$. Contraction of the Riemann tensor gives the Ricci tensor with components

$$R_{\mu\nu} = \sum_{\lambda} R^{\lambda}{}_{\mu\nu\lambda},$$

and reveals (after much algebra) that only the four diagonal components of the Ricci tensor are not identically zero:

$$\begin{aligned}
R_{00} &= -\mathrm{e}^{2(A-B)}\left(A'' + (A')^2 - A'B' + \frac{2A'}{r}\right),\\
R_{11} &= A'' + (A')^2 - A'B' - \frac{2B'}{r},\\
R_{22} &= \mathrm{e}^{-2B}\left(1 + r\left(A' - B'\right)\right) - 1,\\
R_{33} &= \sin^2\theta\left(\mathrm{e}^{2B}\left[1 + r\left(A' - B'\right)\right] - 1\right).
\end{aligned}$$

Now, we already know that for a vacuum solution all four of these components must be equal to zero. Nonetheless, for the sake of completeness, we shall use the expressions that we have obtained to calculate the curvature scalar

$$R = \sum_{\mu,\nu} g^{\mu\nu} R_{\mu\nu},$$

which in this case becomes

$$R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}$$

and yields

$$R = -2\mathrm{e}^{-2B}\left(A'' + (A')^2 - A'B' + \frac{2}{r}\left(A' - B'\right) + \frac{1}{r^2}\right) + \frac{2}{r^2}.$$

When evaluated, this too must vanish for a vacuum solution.

Combining the results for the curvature scalar and the components of the Ricci tensor, we can determine the Einstein tensor components given by

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R,$$

the only ones that are not identically zero in this case being

$$\begin{aligned}
G_{00} &= -\frac{2\mathrm{e}^{2(A-B)}}{r}B' + \frac{\mathrm{e}^{2(A-B)}}{r^2} - \frac{\mathrm{e}^{2A}}{r^2},\\
G_{11} &= -\frac{2A'}{r} + \frac{\mathrm{e}^{2B}}{r^2} - \frac{1}{r^2},\\
G_{22} &= -r^2\mathrm{e}^{-2B}\left(A'' + (A')^2 + \frac{A' - B'}{r} - A'B'\right),\\
G_{33} &= -r^2\mathrm{e}^{-2B}\sin^2\theta\left(A'' + (A')^2 + \frac{A' - B'}{r} - A'B'\right).
\end{aligned}$$

Now, the vacuum field equations demand that even these Einstein tensor components should each be zero in the space outside the spherically symmetric body. One consequence of this is that $\mathrm{e}^{-2A}G_{00} + \mathrm{e}^{-2B}G_{11} = 0$, but this implies that

$$\frac{2\mathrm{e}^{-2B}}{r}\left(A' + B'\right) = 0,$$

implying that $A' + B' = 0$, which can be integrated to give $A(r) + B(r) = C$, where $C$ is a constant. This constant can be set to zero without loss of generality, since any other choice can be represented by a rescaling of the $r$-coordinate, which still has an arbitrary scale at this stage. (This is one of the points that we shall return to later.) Making use of this freedom to set $C = 0$, we see that $A(r) = -B(r)$, and the equation $G_{00} = 0$ can be rewritten as

$$\frac{1}{r^2}\frac{\mathrm{d}\big(r[1-\mathrm{e}^{-2B}]\big)}{\mathrm{d}r} = 0,$$

which, after ignoring $1/r^2$, can also be integrated, to yield $\mathrm{e}^{-2B} = 1 - R_\mathrm{S}/r$, where the integration constant, $R_\mathrm{S}$, has the units of distance. The constant $R_\mathrm{S}$ is called the **Schwarzschild radius**.

Since $\mathrm{e}^{2A} = \mathrm{e}^{-2B}$, we can now identify the explicit form that must be taken by the two exponential functions in the line element of Equation 5.6 if the corresponding metric is to satisfy the vacuum field equations. Explicitly,

$$\mathrm{e}^{2A} = 1 - \frac{R_\mathrm{S}}{r}, \quad \mathrm{e}^{2B} = \frac{1}{1-\frac{R_\mathrm{S}}{r}}.$$

This shows that the line element of the Schwarzschild solution can be written as

$$(\mathrm{d}s)^2 = \left(1 - \frac{R_\mathrm{S}}{r}\right)c^2(\mathrm{d}t)^2 - \frac{1}{1-\frac{R_\mathrm{S}}{r}}(\mathrm{d}r)^2 - r^2\left((\mathrm{d}\theta)^2 + \sin^2\theta\,(\mathrm{d}\phi)^2\right). \tag{5.7}$$

The final step in our modern derivation is to use the principle of consistency and the Newtonian limit to relate the Schwarzschild radius to the mass $M$ of the spherically symmetric body centred on the origin. We saw in Section 4.3.3 that for weak fields, in the Newtonian limit $g_{00} = 1 + h_{00} = 1 + 2\Phi/c^2$, where $\Phi$ is the Newtonian gravitational potential (i.e. the potential energy per unit mass). In the case of a spherically symmetric body of mass $M$ centred on the origin, the Newtonian gravitational potential outside the body, at a distance $r$ from the origin, is $\Phi = -GM/r$. It follows that in the Newtonian limit $g_{00} = 1 - 2GM/rc^2$, and comparing this with the metric coefficient that occupies the position of $g_{00}$ in Equation 5.7, we see that the two will agree provided that we assign the Schwarzschild radius the value

$$R_\mathrm{S} = 2GM/c^2. \tag{5.8}$$

We can now represent the metric tensor of the Schwarzschild solution in the diagonal matrix form

$$[g_{\mu\nu}] = \begin{pmatrix} 1-\frac{2GM}{c^2r} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{2GM}{c^2r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{Eqn 5.1}$$

or in its more common form as the line element

$$(\mathrm{d}s)^2 = \left(1 - \frac{2GM}{c^2r}\right)c^2(\mathrm{d}t)^2 - \frac{(\mathrm{d}r)^2}{1-\frac{2GM}{c^2r}} - r^2(\mathrm{d}\theta)^2 - r^2\sin^2\theta\,(\mathrm{d}\phi)^2, \tag{Eqn 5.2}$$

which relates incremental changes in the spacetime interval $\mathrm{d}s$ to incremental changes in intervals of Schwarzschild coordinate time $t$ and the Schwarzschild spatial coordinates $r, \theta, \phi$ between neighbouring events.

There are shortcuts that could have been taken in this section; for instance, we could have used the condition that the components of the Ricci tensor must vanish in the case of a vacuum solution rather than working out the Einstein tensor components and applying the full field equations. The approach we have taken has the advantage of showing you explicit examples of each of the major tensor quantities. Now that we know what they look like, we can investigate their meaning and significance in this particular case.

**Exercise 5.1** Confirm the value for $G_{00}$ given above. ■

## 5.2 Properties of Schwarzschild spacetime

Several properties of the Schwarzschild metric were mentioned early in the previous section, where they were used to determine the general line element given in Equation 5.6. One of the most basic was spherical symmetry. We shall start by considering that property in more detail.

### 5.2.1 Spherical symmetry

At any particular value of $t$, call it $T$, fixing the value of $r$ to have some particular value $R$ ensures that $\mathrm{d}t = 0$ and $\mathrm{d}r = 0$, and reduces the Schwarzschild line element to

$$(\mathrm{d}s)^2 = -R^2(\mathrm{d}\theta)^2 - R^2\sin^2\theta\,(\mathrm{d}\phi)^2, \tag{5.9}$$

which describes the two-dimensional geometry on the surface of a sphere of radius $R$. Now, from a physical point of view, no point on this spherical surface is any more 'special' than any other point. The fact that no value of $\phi$ is picked out is clear from the fact that $\phi$ does not appear in any of the metric coefficients. However, the same is not true of $\theta$ — that does appear in the metric coefficient that multiplies $(\mathrm{d}\phi)^2$. This makes it seem that there might be something special about certain values of $\theta$ even though we have already said that there can't be. The reason why $\theta$ is picked out in this way has nothing to do with the gravitation of a spherically symmetric body; it is entirely due to the way in which we define spherical coordinates. When we use such coordinates we have to choose some radial direction to be the 'north polar axis'. That direction is assigned the special coordinate value $\theta = 0$ even though in the case of a non-rotating spherically symmetric body there is nothing physically 'special' about the direction chosen to play that role. Any other direction from the origin could just as easily have been chosen as the north polar axis.

This illustrates an important point in general relativity that we shall come back to later. Locations that appear to be 'special' in metrics and line elements may be physically special in some way, or they may only appear to be special because of some particular feature of the coordinate system being used. It is always important to distinguish between real physical effects and non-physical effects produced by the coordinate system alone. The need for this distinction is clear, but as you will soon see it is not always easy to tell whether a particular feature is the result of coordinates or gravitation.

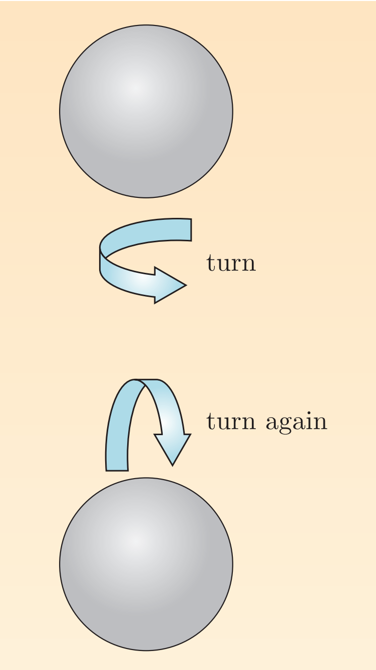


**Figure 5.3** A sphere (spherical shell) exhibits spherical symmetry; the sphere is invariant under arbitrary rotations about the origin.

The Schwarzschild solution is *spherically symmetric*: at any given value of $t$, all points with the same value of $r$ are physically equivalent. The spacetime has the same symmetries as a sphere (by which mathematicians mean it has the symmetries of the surface of a ball), so it is said to be 'invariant under rotations about the origin' (see Figure 5.3).

Of course, this does not mean that points with *different* values of $r$ are physically equivalent. Indeed, we have already seen that in the Newtonian limit, points at different values of $r$ will correspond to different values of the gravitational potential. Also, one of the main outcomes of the derivation was that the metric coefficients in the Schwarzschild line element contain terms of the form $1 - 2GM/c^2r$ that are functions of $r$.

**Exercise 5.2** Suppose that the Schwarzschild coordinate system $ct, r, \theta, \phi$ used to describe the spacetime outside a non-rotating spherically symmetric body is replaced by a different system that uses the coordinates $ct, r, \theta, \phi'$, where $\phi' = \phi + \phi_0$.

(a) Show that the Schwarzschild metric is form-invariant when the new coordinates are substituted for the old ones.

(b) Give a physical justification for the mathematical fact stated in part (a). ■

## 5.2.2 Asymptotic flatness

In the Schwarzschild line element, the factor $1 - 2GM/c^2r$ appears in the metric coefficients of the $c^2(\mathrm{d}t)^2$ term and the $(\mathrm{d}r)^2$ term. The factor is independent of direction and approaches 1 as $r$ becomes large. The meaning of 'large' in this context depends on the value of $M$; what is meant is that $r$ is sufficiently large to make the term $2GM/c^2r$ very much smaller than 1. Where that condition is satisfied, $1 - 2GM/c^2r \to 1$ and the Schwarzschild line element

$$(\mathrm{d}s)^2 = \left(1 - \frac{2GM}{c^2r}\right)c^2(\mathrm{d}t)^2 - \frac{(\mathrm{d}r)^2}{1 - \frac{2GM}{c^2r}} - r^2(\mathrm{d}\theta)^2 - r^2\sin^2\theta\,(\mathrm{d}\phi)^2 \qquad \text{(Eqn 5.2)}$$

takes the form of the Minkowski line element

$$(\mathrm{d}s)^2 = c^2(\mathrm{d}t)^2 - (\mathrm{d}r)^2 - r^2(\mathrm{d}\theta)^2 - r^2\sin^2\theta\,(\mathrm{d}\phi)^2 \qquad (5.10)$$

that describes the flat spacetime of special relativity in spherical coordinates. This is the form that we should expect the Schwarzschild line element to take 'far' from the origin where gravitational effects due to the mass of the spherically symmetric body will be negligible.

Remembering that this 'flatness' only applies at sufficiently large values of $r$, we say that the Schwarzschild metric has the property of *asymptotic flatness*.

## 5.2.3 Time-independence

Two other properties of the Schwarzschild metric that were briefly mentioned earlier related to its time-independence. The first of these is the property of being *stationary*, implying that none of the metric coefficients depends on $t$. So, if $t_1$ and $t_2$ represent the time coordinates of neighbouring events, then $\mathrm{d}t = t_2 - t_1 = (t_2 + t_0) - (t_1 + t_0) = t'_2 - t'_1 = \mathrm{d}t'$, and the metric is invariant under a coordinate transformation of the form $t \to t' = t + t_0$, where $t_0$ is a constant. This specific aspect of time-independence is described as 'invariance under translation in time' and is another symmetry of the solution.

The second feature relating to time-independence introduced earlier was the property of being *static*. This concerns invariance under transformations that reverse time, such as $t \to -t$. The fact that the Schwarzschild metric is stationary ensures that time reversal will have no effect on any of the metric coefficients since they do not depend on $t$ at all. However, in order that the metric should be static, it is also important that the line element should not contain any terms of the form $\mathrm{d}r\,\mathrm{d}t$, $\mathrm{d}\theta\,\mathrm{d}t$ or $\mathrm{d}\phi\,\mathrm{d}t$. Such terms are often referred to as 'cross terms' or 'mixed terms' and are typical of situations involving rotation.

The Schwarzschild metric is both stationary and static.

### 5.2.4 Singularity

A striking feature of the Schwarzschild metric is its odd behaviour as $r$ approaches the Schwarzschild radius $R_\mathrm{S} = 2GM/c^2$. As $r \to R_\mathrm{S}$, the factor $1 - R_\mathrm{S}/r$ causes the metric coefficient $g_{00} \to 0$ while the factor $(1 - R_\mathrm{S}/r)^{-1}$ causes $g_{11} \to \infty$. The unlimited growth of the latter factor is described by saying that there is a **singularity** in the Schwarzschild metric. This particular singularity is in fact a consequence of the coordinates that we are using to describe the Schwarzschild solution. That is, it is a **coordinate singularity**, not a physically meaningful **gravitational singularity**. As a coordinate singularity it can be removed by an appropriate transformation of coordinates in a way that would not be possible for a true gravitational singularity. Nonetheless it is a feature of the solution as described by Schwarzschild coordinates and an indicator of the significance of $R_\mathrm{S}$.

When considering this coordinate singularity it is important to remember that the exterior Schwarzschild solution that we are discussing describes the spacetime *outside* a spherically symmetric body of mass $M$. It is therefore interesting to ask if $R_\mathrm{S}$ is likely to be larger or smaller than the radius of such a body. If $R_\mathrm{S}$ is smaller than the body's radius, the coordinate singularity will be outside the domain in which the Schwarzschild solution is applicable, and the solution itself will be non-singular throughout the region that it actually describes.

For a body with the mass of the Sun (about $2.0 \times 10^{30}$ kg), the Schwarzschild radius is $3.0$ km. This compares with a solar radius of about $0.7$ *million* km. So in the case of a normal star-like body, the Schwarzschild radius is deep inside the body. Of course, not all bodies of astronomical interest are 'normal' or 'star-like'. As you will see later, the Schwarzschild radius is of great importance in the study of black holes. A body can become a black hole if its surface shrinks within its Schwarzschild radius.

A final point to note is that the Schwarzschild metric also has a singularity at $r = 0$. This is a gravitational singularity, marked by the unlimited growth of invariants related to the curvature, and cannot be removed by any change of coordinates. This singularity is of little relevance to the exterior solution that we have been discussing in this section, but it will be significant when we come to discuss black holes.

### 5.2.5 Generality

According to the Schwarzschild solution, the spacetime geometry outside a static spherically symmetric body is characterized by a single quantity $M$, which represents the total mass of that distribution.

In 1923 the American mathematician George Birkhoff proved that even if the source of gravitation is not static (and therefore not necessarily stationary), and as long as its effect is **isotropic** (i.e. the same in all directions), the vacuum solution of the Einstein field equations in the region exterior to the source is still stationary and is still the Schwarzschild solution.

This result is known as **Birkhoff's theorem**. One of its implications is that a spherically symmetric body that is expanding or contracting in a purely radial way, or even one that is pulsating radially, cannot produce any gravitational signs of that radial motion beyond the spherical region that contains the material of the body itself. So, if a fixed mass $M$ were contained within a sphere of radius $r_1$, then the Schwarzschild metric would apply throughout the region $r > r_1$, but if the mass distribution were to shrink in an isotropic way to a smaller radius $r_2$, then the spacetime would be unaffected in the region $r > r_1$ but now the Schwarzschild metric would apply throughout the larger region $r > r_2$.

This is a surprising result. It indicates the special nature of vacuum solutions as well as the generality of the Schwarzschild solution. As you will see later when we discuss gravitational radiation, it also indicates that sources that only pulsate radially cannot produce gravitational waves.

To summarize, we have the following.

**Properties of the Schwarzschild solution**

The Schwarzschild metric is a static (and therefore stationary), spherically symmetric solution of the Einstein field equations in the empty region exterior to any distribution of energy and momentum characterized by mass $M$ that produces purely isotropic effects in that region. The solution is asymptotically flat, approaching the Minkowski metric in spherical coordinates for sufficiently large values of $r$. The solution has a coordinate singularity at the Schwarzschild radius $r = R_S = 2GM/c^2$ and a gravitational singularity at $r = 0$, though neither of these singularities is within the region described by the solution for normal 'star-like' bodies.

## 5.3 Coordinates and measurements in Schwarzschild spacetime

We now need to deal with an issue that has been present since we first introduced the Schwarzschild coordinates $ct, r, \theta, \phi$ near the start of this chapter. The issue concerns the relationship between coordinate values and physically meaningful intervals of time and distance.

When confronted by a system of coordinates that includes a $t$-coordinate and an $r$-coordinate, it is tempting to assume that the $t$ must represent time and the $r$ radial distance from the origin. However, such an assumption is always dangerous and often wrong.

The simple fact is that in general relativity, coordinates are essentially arbitrary systems of markers chosen to distinguish one event from another. This gives us great freedom in how we define coordinates, a freedom that we exploited in the derivation of the Schwarzschild metric. The relationship between the coordinate differences separating events and the corresponding intervals of time or distance that would be measured by a specified observer must be worked out using the metric of the spacetime. It cannot be assumed that the 'physical' times and distances that would be measured by clocks or measuring sticks are directly specified by the coordinates. This situation is described by saying that:

In general relativity, coordinates do not have immediate metrical significance.

Einstein found this a perplexing feature of general relativity. In his own account of how the general theory developed after 1908 he says:

> Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not easy to free oneself from the idea that coordinates must have an immediate metrical meaning.
>
> Quoted in Schilpp, P. A. (ed.) (1969) *Albert Einstein — Philosopher Scientist*, 3rd edn, Illinois, Open Court.

Intervals of time and distance must be measured by an observer who must make use of a frame of reference, so we start with a discussion of the observers and frames that will be relevant to our discussion of Schwarzschild spacetime.

### 5.3.1 Frames and observers

We saw in the discussion of special relativity that the phenomena of time dilation and length contraction made it important to be clear about who was performing measurements of time and distance, and to be especially careful when relating time and distance measurements made by different inertial observers. In special relativity, inertial fames are 'global', in principle stretching out to infinity. We needed to be clear about the frame that an observer was using but we emphasized the distinction between 'seeing' and 'observing', and stressed that observers were concerned with the latter, which made their location irrelevant for most purposes.

In general relativity, the situation is very different. There is no 'special' class of frames, and the frames that are used are generally 'local' so an observer's location is important. In this chapter we shall be particularly concerned with observations made in three 'local' frames: the frame used by a freely falling observer, a frame that is at rest at some specified location, and the frame of a 'distant' observer located far from the spherically symmetric body at the origin of Schwarzschild coordinates. The frame of the freely falling observer is locally inertial; gravity has effectively been 'turned off' and special relativity applies locally. The observer at a fixed location will need to take steps to avoid falling freely; they might need to locate themselves in a rocket, for example. For such an observer special relativity will work locally but only if the observer supposes that every body is subject to a 'gravitational force' that is proportional to the mass of the body. This is really a 'fictitious force', introduced to account for the fact that the observer's frame is not freely falling and is therefore not really locally inertial. To this extent the observer maintaining a fixed position is in a similar situation to a passenger in a bus turning a corner who 'feels' the effect of a (fictitious) centrifugal force. The distant observer will be in a region of spacetime that is effectively flat, so special relativity will again apply locally and there will not be any local effects of gravitation to take into account. Such an observer can remain at rest without needing the support of a rocket and can even be regarded as falling freely while remaining at rest!

### 5.3.2 Proper time and gravitational time dilation

Consider two events involving the emission of light, that happen in the Schwarzschild spacetime surrounding a static spherically symmetric body. Suppose that the two emission events are described by the Schwarzschild coordinates $(t_{\rm em}, r_{\rm em}, \theta_{\rm em}, \phi_{\rm em})$ and $(t_{\rm em} + {\rm d}t_{\rm em}, r_{\rm em}, \theta_{\rm em}, \phi_{\rm em})$, so they are separated by a difference in coordinate time ${\rm d}t_{\rm em}$, while their other coordinate separations are all zero: ${\rm d}r_{\rm em} = {\rm d}\theta_{\rm em} = {\rm d}\phi_{\rm em} = 0$.

According to the Schwarzschild metric, the infinitesimal spacetime separation of these events is given by

$$({\rm d}s_{\rm em})^2 = \left(1 - \frac{2GM}{c^2 r_{\rm em}}\right) c^2 ({\rm d}t_{\rm em})^2, \tag{5.11}$$

and the **proper time** between the events, as would be measured by a clock at rest at the location of the events, is ${\rm d}\tau_{\rm em} = {\rm d}s_{\rm em}/c$, so

$${\rm d}\tau_{\rm em} = {\rm d}s_{\rm em}/c = \left(1 - \frac{2GM}{c^2 r_{\rm em}}\right)^{1/2} {\rm d}t_{\rm em}. \tag{5.12}$$

Note that the proper time separating the events, according to a stationary clock at the location of the events, is less than the coordinate time separating the events.

Now consider what will be seen by an observer at rest at some other location with the same angular coordinates $\theta$ and $\phi$ but a different value of the radial coordinate $r = r_{\rm ob}$. As will be shown in Chapter 6, such an observer will find that the coordinate time separating the signals from the two events when they arrive at $r = r_{\rm ob}$ will be the same as the coordinate time between the emission of those signals. We can indicate this by writing ${\rm d}t_{\rm ob} = {\rm d}t_{\rm em}$. All the other coordinate differences ${\rm d}r$, ${\rm d}\theta$ and ${\rm d}\phi$ will still be zero. It follows that the spacetime separation between the observations of the two signals at $r = r_{\rm ob}$ will be

$$({\rm d}s_{\rm ob})^2 = \left(1 - \frac{2GM}{c^2 r_{\rm ob}}\right)^{1/2} {\rm d}t_{\rm ob} = \left(1 - \frac{2GM}{c^2 r_{\rm ob}}\right) c^2 ({\rm d}t_{\rm em})^2, \tag{5.13}$$

and the proper time between the observations of the two signals will be

$${\rm d}\tau_{\rm ob} = {\rm d}s_{\rm ob}/c = \left(1 - \frac{2GM}{c^2 r_{\rm ob}}\right)^{1/2} {\rm d}t_{\rm em}. \tag{5.14}$$

There are two important consequences that follow from these relationships.

First, for a distant observer fixed at a sufficiently large value of $r$, effectively at $r_{\text{ob}} = \infty$, it follows from Equation 5.14 that

$$\mathrm{d}\tau_\infty = \mathrm{d}t_{\text{em}}. \tag{5.15}$$

Integrating both sides of this equation shows that even for two emission events separated by a finite coordinate time difference $\Delta t_{\text{em}}$, that difference will still equal $\Delta\tau_\infty$, the difference in the proper time between observations of those events made by a stationary observer at infinity. This establishes that the Schwarzschild coordinate time separating two events at a fixed location can actually be determined by measuring the proper time between observations of those two events using a stationary clock at infinity. This gives us a way, in principle at least, of assigning Schwarzschild coordinate times to events.

- Should we be worried by the fact that this argument involves an observer at infinity? Does that invalidate the process?

- No. All it means is that the observer should be far enough away to be in the asymptotically flat region of spacetime where $2GM/c^2 r_{\text{ob}}$ is negligible compared with 1.

Second, it follows from Equation 5.15 and the relation between $\mathrm{d}\tau_{\text{em}}$ and $\mathrm{d}t_{\text{em}}$ in Equation 5.12 that

$$\mathrm{d}\tau_\infty = \frac{\mathrm{d}\tau_{\text{em}}}{\left(1 - \frac{2GM}{c^2 r_{\text{em}}}\right)^{1/2}}. \tag{5.16}$$

This shows that the proper time between the observation of the two light signals at infinity, $\mathrm{d}\tau_\infty$, is greater than the proper time between their emission as measured at the site of the emission, $\mathrm{d}\tau_{\text{em}}$.

If we suppose that the two events that we have been discussing represent the beginning and the end of a single tick of a clock fixed at $r = r_{\text{em}}$, then our second result shows that the duration of that tick as seen by a distant observer will be increased by a factor $1/\left(1 - \frac{2GM}{c^2 r_{\text{em}}}\right)^{1/2}$. This shows that the distant observer will find that the clock at $r = r_{\text{em}}$ is running slow.

- If the stationary clock emitting the light signals was moved closer to the surface of the spherically symmetric body, how would the observations of its rate of ticking by a distant fixed observer be affected?

- The distant observer would find that the clock ticked even more slowly. Moving the clock closer to the surface reduces the value of $r_{\text{em}}$, which has the effect of increasing the factor $1/\left(1 - \frac{2GM}{c^2 r_{\text{em}}}\right)^{1/2}$.

This effect, the slowing of the rate of ticking of a clock in a gravitational field, as seen by a distant observer, is sometimes referred to as **gravitational time dilation**. Note, however, that there is a significant difference between this effect and the time dilation in special relativity that we studied in Chapter 1. In that earlier case we were careful to ignore the effects of signal travel time and only considered the time intervals between the events themselves as measured by different inertial observers, irrespective of the observer's location. In the general

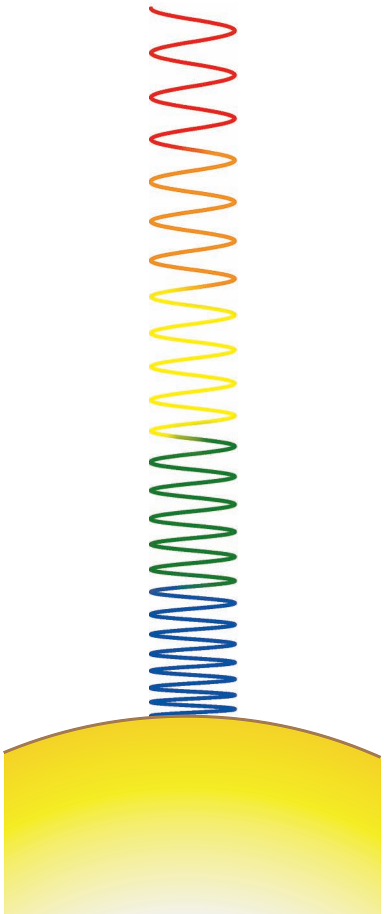

**Figure 5.4** A schematic representation of the redshift of radiation as it escapes from a massive body.

relativistic case there is no relative motion; both the clock and the distant observer are at rest, and we are very deliberately considering the proper time between the arrival of light signals at that distant observer's location. The distant observer is still making observations, but the observations are of local events — the arrival of the light signals, not their emission.

The general relativistic effect can be given another interpretation. Suppose that the two 'emission' events represent the emission of successive peaks of an electromagnetic wave (a light wave), so that $\mathrm{d}\tau_{\mathrm{em}}$ represents the period of that wave at its point of emission. Then $\mathrm{d}\tau_{\mathrm{ob}}$ will represent the period of that same radiation as measured by a distant observer. The periods will still be related by

$$\mathrm{d}\tau_\infty = \frac{\mathrm{d}\tau_{\mathrm{em}}}{\left(1 - \frac{2GM}{c^2 r_{\mathrm{em}}}\right)^{1/2}}, \qquad \text{(Eqn 5.16)}$$

but now we can say that the reciprocal of the period represents the frequency of the radiation, so the frequency observed by the distant observer will be

$$f_\infty = f_{\mathrm{em}} \left(1 - \frac{2GM}{c^2 r_{\mathrm{em}}}\right)^{1/2}. \qquad (5.17)$$

This shows that the observed (proper) frequency is less than the emitted (proper) frequency. It follows that light rising through a gravitational field will be redshifted. This phenomenon is known as **gravitational redshift** (see Figure 5.4). You saw in Section 4.1.1 that a local version of this phenomenon was already predicted as a consequence of the principle of equivalence. Now, with the aid of the Einstein field equations and the Schwarzschild metric, you can see the full effect, not limited to a local frame, but relating quantities that might be measured in two widely separated local frames. This is an effect that might be measured by an astronomer, and we shall discuss such measurements in Chapter 7.

**Exercise 5.3** Treating the Sun as a non-rotating, spherically symmetric body, and regarding the surrounding space as well described by the Schwarzschild metric, at what value of the Schwarzschild coordinate $r$ do intervals of proper time $\mathrm{d}\tau$ and coordinate time $\mathrm{d}t$ differ by no more than 1 part in $10^8$? ■

To summarize, we have the following.

### Proper time and gravitational time dilation

The Schwarzschild coordinate time separating two events at a fixed location is equal to the proper time between sightings of those two events by a distant stationary observer.

The rate of ticking of a stationary clock at Schwarzschild coordinate distance $r$ will be seen to be slowed by a factor of $\left(1 - \frac{2GM}{c^2 r_{\mathrm{em}}}\right)^{-1/2}$ as measured by a distant stationary observer. This same effect will lead to a gravitational redshift — seen as a reduction in frequency by a factor $\left(1 - \frac{2GM}{c^2 r_{\mathrm{em}}}\right)^{1/2}$ — of the radiation from a stationary source as measured by a distant stationary observer.

### 5.3.3 Proper distance

Just as we related differences in Schwarzschild coordinate time to intervals of proper time that might be measured by clocks, so we must relate differences in Schwarzschild coordinate position to proper distances that might be measured using measuring sticks. Consider two events that happen in Schwarzschild spacetime at the same coordinate time but at infinitesimally separated positions, so that their spacetime separation is given by the negative quantity

$$(\mathrm{d}s)^2 = -\frac{(\mathrm{d}r)^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2(\mathrm{d}\theta)^2 - r^2\sin^2\theta\,(\mathrm{d}\phi)^2. \tag{5.18}$$

The **proper distance** between those two events will be given by $\mathrm{d}\sigma = \sqrt{-(\mathrm{d}s)^2}$.

We saw earlier, when discussing the spherical symmetry of the Schwarzschild solution (see Subsection 5.2.1), that the events occurring at fixed values of $t$ and $r$ form a spherical shell described by the familiar metric of such a shell. To this extent the Schwarzschild spacetime can be regarded as consisting of a set of nested spheres surrounding the spherically symmetric body. The proper distance between neighbouring points on the sphere of coordinate radius $r$ is given by

$$\mathrm{d}\sigma = r^2(\mathrm{d}\theta)^2 + r^2\sin^2\theta\,(\mathrm{d}\phi)^2. \tag{5.19}$$

There is nothing unusual about the geometry of any of these spherical surfaces; the sphere of coordinate radius $r$ has proper circumference $2\pi r$ and proper area $4\pi r^2$. In principle either of these quantities could be measured using ordinary measuring rods. This provides a method, in principle at least, of determining the Schwarzschild radial coordinate $r$ of any event: use measuring sticks to measure the proper circumference $C$ of a circle centred on the origin that passes through the location of the event, then divide that circumference by $2\pi$ to find the coordinate radius $r = C/2\pi$.

What is unusual is that the radial coordinate $r$ does not provide a direct measure of the proper radius of such a sphere, and differences in the radial coordinate $r$ do not indicate the proper distance between different spherical shells. Consider two events that occur at the same coordinate time and with the same angular coordinates $\theta$ and $\phi$ but at different radial coordinates $r$ and $r + \mathrm{d}r$. The proper distance between those events will be

$$\mathrm{d}\sigma = \frac{\mathrm{d}r}{\left(1 - \frac{2GM}{c^2 r}\right)^{1/2}}. \tag{5.20}$$

This equation shows that $\mathrm{d}\sigma$ is generally greater than $\mathrm{d}r$, provided that $r$ is greater than the Schwarzschild radius. The differences will be particularly large close to the Schwarzschild radius (see Figure 5.5 overleaf). This result may be integrated to determine the proper radial distance between any two events on the same radial coordinate line.

Stretching a point, so to speak, the relation between coordinate distance and proper distance can be inverted to show that the coordinate distance is contracted relative to the proper distance. This could be described as ‘gravitational length contraction’, but the comparison with the length contraction of special relativity is very weak since $\mathrm{d}r$ is not really a ‘physical’ distance at all.

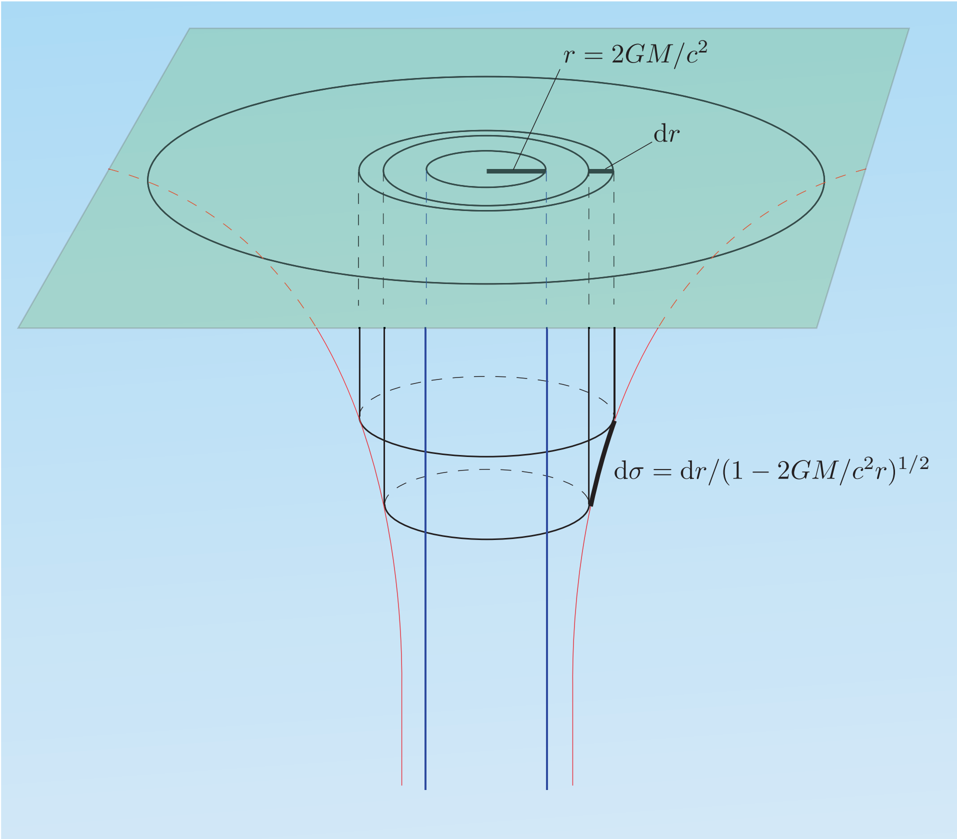


**Figure 5.5** A schematic representation of the relation between the Schwarzschild radial coordinate and the proper distance for events close to the Schwarzschild radius $r = R_\mathrm{S} = 2GM/c^2$.

**Exercise 5.4** Confirm that the proper distance around a circle (proper circumference) in the $\theta = \pi/2$ plane centred at $r = 0$ is $C = 2\pi r$, according to the Schwarzschild geometry.

**Proper distance**

The Schwarzschild metric describes the spacetime around a static, spherically symmetric body as a set of nested spheres. The coordinate radius $r$ of any one of those spheres can be determined by dividing its proper circumference by $2\pi$.

Two events occurring at the same coordinate time and separated only by a radial coordinate distance $\mathrm{d}r$ will be separated by a proper radial distance

$$\mathrm{d}\sigma = \frac{\mathrm{d}r}{\left(1 - \frac{2GM}{c^2 r}\right)^{1/2}}. \qquad \text{(Eqn 5.20)}$$

## 5.4 Geodesic motion in Schwarzschild spacetime

According to the geodesic principle discussed in Chapter 4, the time-like and null geodesics of a spacetime represent the possible world-lines of massive and massless particles moving under the influence of gravity alone. Remember, a world-line is a pathway through spacetime, not just a trajectory through space. So once we know the world-line of a freely falling particle — i.e. once we know the specific geodesic that it moves along — we know everything about that particular particle's motion. In this section we examine some aspects of geodesic motion in the Schwarzschild spacetime around a static spherically symmetric body. We shall be particularly interested in motions relevant to astrophysics, so we shall be mainly concerned with orbital motion.

### 5.4.1 The geodesic equations

As you saw in Chapters 3 and 4, the geodesics of a spacetime are usually presented as parameterized curves, represented by four coordinate functions $x^\mu(\lambda)$, where $\lambda$ is an affine parameter that varies along the geodesic. The choice of parameter is not completely arbitrary. In the case of a massive particle moving along a time-like geodesic, the affine parameter is usually taken to be the proper time $\tau$ that would be measured by a clock falling with the particle. It is also possible to use any linearly related parameter such as $a\tau + b$, where $a$ and $b$ are constants, though this would be unusual. These choices are not possible for a null geodesic since $\mathrm{d}\tau = \mathrm{d}s/c = 0$ for each of its elements, so some other affine parameter must be adopted. In either case the parameter is chosen to be an affine parameter since this ensures that the coordinate functions will satisfy geodesic equations of the relatively simple form

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \sum_{\nu,\rho} \Gamma^\mu_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} = 0,$$

where the $\Gamma^\mu_{\nu\rho}$ are the connection coefficients that follow directly from the spacetime metric.

The general form of the non-zero connection coefficients was given in Section 5.1.2 at the start of the derivation of the Schwarzschild metric. Now that we know the explicit form of the Schwarzschild radius and the functions $A(r)$ and $B(r)$, we can write down the explicit form of all the non-zero connection coefficients:

$$\begin{aligned}
\Gamma^0{}_{01} &= \frac{GM}{r^2c^2\left(1 - \frac{2GM}{c^2r}\right)} \; (= \Gamma^0{}_{10}),\\
\Gamma^1{}_{00} &= \frac{GM\left(1 - \frac{2GM}{c^2r}\right)}{r^2c^2},\\
\Gamma^1{}_{11} &= -\frac{GM}{r^2c^2\left(1 - \frac{2GM}{c^2r}\right)},\\
\Gamma^1{}_{22} &= -r\left(1 - \frac{2GM}{c^2r}\right),\\
\Gamma^1{}_{33} &= -r\left(1 - \frac{2GM}{c^2r}\right)\sin^2\theta,\\
\Gamma^2{}_{12} &= \frac{1}{r} \; (= \Gamma^2{}_{21}),\\
\Gamma^2{}_{33} &= -\sin\theta\cos\theta,\\
\Gamma^3{}_{13} &= \frac{1}{r} \; (= \Gamma^3{}_{31}),\\
\Gamma^3{}_{23} &= \cot\theta \; (= \Gamma^3{}_{32}).
\end{aligned}$$

Using these connection coefficients, the geodesic equations provide the following four differential equations that must be satisfied by the four coordinate functions $x^0 = t(\lambda)$, $x^1 = r(\lambda)$, $x^2 = \theta(\lambda)$, $x^3 = \phi(\lambda)$ that describe any affinely parameterized geodesic in Schwarzschild spacetime:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} + \frac{2GM}{c^2 r^2 \left(1 - \frac{2GM}{c^2 r}\right)} \frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}t}{\mathrm{d}\lambda} = 0, \tag{5.21}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2} + \frac{GM}{r^2}\left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 - \frac{GM}{c^2 r^2 \left(1 - \frac{2GM}{c^2 r}\right)}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2$$
$$- r\left(1 - \frac{2GM}{c^2 r}\right)\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2 + \sin^2\theta \left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\right] = 0, \tag{5.22}$$

$$\frac{\mathrm{d}^2 \theta}{\mathrm{d}\lambda^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 = 0, \tag{5.23}$$

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}\lambda^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = 0. \tag{5.24}$$

Given the initial location of a particle in Schwarzschild spacetime and the initial values of the four components of its tangent vector $t^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$, these four coupled, second-order, ordinary differential equations can be solved (numerically if not analytically) to determine the unique world-line of the particle. If the particle is massless, the magnitude of the initial tangent vector will be zero, showing the particle to be travelling at the speed of light, and the relevant world-line will turn out to be a null geodesic. For a particle with mass, the world-line will be a time-like geodesic.

As far as motion under gravity is concerned, the geodesic equations are the general relativistic analogues of Newton's second law of motion. Both sets of equations may be expressed as differential equations, and their solution allows initial data to be used to predict subsequent motion. However, as you can see, the geodesic equations look formidable and can be very difficult to solve. Because of their difficulty we shall not attempt a direct solution in this case. There are simplifying techniques that can be used based on the Lagrangian approach introduced when we first derived the geodesic equations in Chapter 3, but those methods are beyond the level of this book. Instead, we shall take a lesson from Newtonian mechanics, where problems involving motion are often simplified by making use of constants of the motion such as energy and angular momentum.

**Exercise 5.5** Confirm the form of the first of the four geodesic equations given above. ■

### 5.4.2 Constants of the motion in Schwarzschild spacetime

To start, we recall that when geodesics were first introduced we described them as parameterized curves defined by $x^\mu(\lambda)$ with the particular property that the tangent vector $\mathrm{d}x^\mu/\mathrm{d}\lambda$ at any point remained parallel to itself under parallel transport. (This was a property that they shared with straight lines in a flat space.) Choosing the parameter $\lambda$ to be an affine parameter ensures that as the tangent vector is transported along the geodesic, it not only remains self-parallel but also has a constant magnitude (more properly called a *norm* in this context). The square of that norm at every point on the geodesic is given by

$$n^2 = \sum_{\mu,\nu} g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = \text{constant}, \tag{5.25}$$

and will be zero in the case of a null geodesic.

If we regard the geodesic as the world-line of a massive particle and choose to use the proper time $\tau$ (as measured by a clock falling with the particle) as the parameter $\lambda$, then the tangent vector components $\mathrm{d}x^\mu/\mathrm{d}\lambda$ become $\mathrm{d}x^\mu/\mathrm{d}\tau$ and are seen to be the components of the particle's four-velocity $[U^\mu]$. Now, for the four-velocity of a massive particle,

$$\sum_{\mu,\nu} g_{\mu\nu}\, U^\mu\, U^\nu = c^2. \tag{5.26}$$

So in this case the constant $n^2$ in Equation 5.25 will be given by $n^2 = c^2$, and we can use our explicit knowledge of the Schwarzschild metric coefficients $g_{\mu\nu}$ to expand Equation 5.25 as

$$\begin{aligned} c^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 \\ - r^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 - r^2 \sin^2\theta \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2. \end{aligned} \tag{5.27}$$

This still looks complicated, but apart from $n^2 = c^2$ there are four other **constants of the motion** that can help to simplify Equation 5.27. There are many ways of deducing these four conserved quantities, most of them drawing on the symmetry of the Schwarzschild solution. There are deep connections between symmetries and conservation laws throughout physics, so it is not surprising that the many symmetries of the Schwarzschild solution should give rise to conserved quantities in this case. In particular, we noted earlier that the static nature of the Schwarzschild solution indicates a symmetry associated with invariance under translation in time. This kind of symmetry is generally associated with the conservation of energy. Similarly, the solution's invariance under rotations about the origin indicates spherical symmetry, and is associated with the conservation of angular momentum.

In the specific context of a freely falling body of non-zero mass $m$, moving along a time-like geodesic in Schwarzschild spacetime, the conserved quantity that plays the role of total energy (actually the energy per unit mass energy) is

$$\frac{E}{mc^2} = \left(1 - \frac{2GM}{c^2 r}\right) \frac{\mathrm{d}t}{\mathrm{d}\tau}. \tag{5.28}$$

When dealing with the analogue of angular momentum, which is a vector, there are three conserved scalar quantities. These are most conveniently regarded as the magnitude of the angular momentum per unit mass, $J/m$, and two angles that determine the direction of the angular momentum vector. In practice, rather than dealing with whatever direction the angular momentum actually has, it is usually easier to transform the coordinates so that the angular momentum points along the polar axis, with the consequence that the motion is confined to the plane in which

$\theta = \pi/2$ and consequently $\mathrm{d}\theta/\mathrm{d}t = 0$. So, without any real loss of generality, two of the three constants of the motion associated with angular momentum are represented by the single condition

$$\theta = \pi/2, \tag{5.29}$$

while the third turns out to be

$$\frac{J}{m} = r^2 \sin^2\theta \, \frac{\mathrm{d}\phi}{\mathrm{d}\tau}. \tag{5.30}$$

Take care to note that the quantities $E/mc^2$ and $J/m$ are specific to the Schwarzschild metric; they do not represent general definitions that can automatically be applied to other cases. If we now use Equations 5.28, 5.29 and 5.30 to simplify Equation 5.27, we see that

$$c^2 = \frac{E^2}{m^2c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1} - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{J^2}{m^2r^2}. \tag{5.31}$$

Rearranging this gives

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{J^2}{m^2r^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{2GM}{r} = c^2\left[\left(\frac{E}{mc^2}\right)^2 - 1\right]. \tag{5.32}$$

This equation, which already incorporates the general relativistic analogues of energy conservation and angular momentum conservation, describes the changes in the radial position coordinate with proper time for a freely falling particle of non-zero mass moving in the equatorial plane $\theta = \pi/2$. The phrase 'freely falling' can give the impression that the particle is plummeting radially inwards towards the central body. That is a possible form of freely falling motion, but not the only one. All 'freely falling' really means is that the motion is determined by gravity alone. In this sense the Moon is (very nearly) freely falling around the Earth and the Earth is (very nearly) freely falling around the Sun. So Equation 5.32 holds the key to describing orbital motion about the central massive body in Schwarzschild spacetime, and that is how we shall use it in the next subsection. Before doing that, however, let's see how Equation 5.32 together with the definitions contained in Equations 5.28 and 5.30 *can* be used to solve a problem involving purely radial motion.

### Worked Example 5.1

Show that in Schwarzschild spacetime, the motion of a test particle in radial free fall (i.e. directly towards $r = 0$) satisfies the relation

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = -\frac{GM}{r^2}.$$

### Solution

To determine the equation of motion for a freely falling body travelling along a radial geodesic, we can use Equation 5.32, together with the supplementary Equations 5.28 and 5.30 that define $E$ and $J$. In the case of purely radial motion $\phi$ is constant, so $\mathrm{d}\phi/\mathrm{d}\tau = 0$, so Equation 5.30 shows that $J = 0$. Equation 5.32 therefore reduces to

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = c^2\left[\left(\frac{E}{mc^2}\right)^2 - 1\right] + \frac{2GM}{r}.$$

Differentiating with respect to $\tau$ gives

$$2\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} = -\frac{2GM}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau},$$

and dividing through by $\mathrm{d}r/\mathrm{d}\tau$ gives

$$\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} = -\frac{GM}{r^2},$$

as required.

The result that has just been derived in this worked example looks very much like the corresponding Newtonian result for free fall under the gravitational pull of a spherically symmetric mass in Euclidean space. Note, however, the several differences between the general relativistic result and its Newtonian counterpart. In the first place, talking about free fall under gravity is fine in general relativity, but talking of the 'pull' of gravity or gravitational 'attraction' would be quite wrong since there is no gravitational 'force' in general relativity, and even the term gravitational 'field' only retains a meaning when interpreted in terms of the metric coefficients, which can vary from place to place. Similarly, the Newtonian result directly relates the second derivative of the radial distance with respect to time to the inverse square of the radial distance, but in the general relativistic result the second derivative is with respect to proper time $\tau$, and $r$ is the coordinate distance, not the 'physical' proper distance. In the Newtonian limit, when $\mathrm{d}r/\mathrm{d}\tau \ll c$ and the particle is sufficiently far from the spherical mass for the field to be weak, these differences vanish, and the general relativistic result does reduce to the Newtonian result. This shows how Einstein's theory of motion under gravity encompasses Newton's theory and reduces to it under appropriate conditions. Nonetheless, away from the Newtonian limit, especially when close to the Schwarzschild radius, the differences are real and significant.

To summarize, we have the following.

### Freely falling motion in Schwarzschild spacetime

The motion of a particle of mass $m$ falling freely in the $\theta = \pi/2$ plane of a Schwarzschild spacetime is described by the **radial motion equation**

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{J^2}{m^2r^2}\left(1 - \frac{2GM}{c^2r}\right) - \frac{2GM}{r} = c^2\left[\left(\frac{E}{mc^2}\right)^2 - 1\right], \quad \text{(Eqn 5.32)}$$

where $\tau$ is the proper time as would be measured by a clock falling with the particle, and the constants of the motion, $E/mc^2$ and $J/m$, the Schwarzschild analogues of energy per unit mass energy and angular momentum magnitude per unit mass, are determined by

$$\frac{E}{mc^2} = \left(1 - \frac{2GM}{c^2r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}, \quad \text{(Eqn 5.28)}$$

$$\frac{J}{m} = r^2\sin^2\theta\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}. \quad \text{(Eqn 5.30)}$$

### 5.4.3 Orbital motion in Schwarzschild spacetime

The shape of an orbit in the $\theta = \pi/2$ plane of Schwarzschild spacetime is described by expressing $r$ as a function of $\phi$. In the previous subsection we developed a differential equation relating $r$ to $\tau$; we now need to convert that into a tractable relation between $r$ and $\phi$, and then investigate its solution. We start by noting that

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\phi}, \tag{5.33}$$

and then use the fact that $J/m = r^2\,\mathrm{d}\phi/\mathrm{d}\tau$, in the plane $\theta = \pi/2$, to eliminate $\mathrm{d}\phi/\mathrm{d}\tau$, giving

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{J}{r^2 m}\frac{\mathrm{d}r}{\mathrm{d}\phi}. \tag{5.34}$$

Substituting this result into Equation 5.32 gives

$$\begin{aligned}&\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + r^2\left(1 - \frac{2GM}{c^2 r}\right) - m^2 r^3 \frac{2GM}{J^2}\\ &= \left(\frac{r^2 mc}{J}\right)^2\left[\left(\frac{E}{mc^2}\right)^2 - 1\right].\end{aligned} \tag{5.35}$$

Now we apply a standard ‘trick’ of orbital analysis by introducing the reciprocal variable $u = 1/r$, and rewrite this equation as

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + u^2 = \left(\frac{mc}{J}\right)^2\left[\left(\frac{E}{mc^2}\right)^2 - 1\right] + \frac{2GMum^2}{J^2} + \frac{2GMu^3}{c^2}.$$

Differentiating with respect to $\phi$ and dividing the resulting equation by $\mathrm{d}u/\mathrm{d}\phi$ gives the **orbital shape equation** that we need.

**Orbital shape equation**

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{GMm^2}{J^2} + \frac{3GMu^2}{c^2}. \tag{5.36}$$

It is informative to compare this result with the analogous result from Newtonian mechanics for orbits around a massive spherically symmetric body. In the Newtonian case the result is

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{GMm^2}{J^2}. \tag{5.37}$$

This is the same as the Schwarzschild expression, apart from the absence of the final relativistic term $3GMu^2/c^2$. That additional term will vanish in the limit as $u$ approaches zero, showing that as long as $r$ is sufficiently large, the Newtonian orbits will be recovered from the relativistic orbit equation, as they should be. Of course, for ‘small’ values of $r$ (meaning close to $2GM/c^2$), the value of $u$ will be large and the additional term will not be negligible. There will then be significant differences between the Newtonian and relativistic behaviours.

Additional insight into the behaviour of orbits comes from a study of energy, so it is useful here to rewrite the radial motion equation (Equation 5.32) that we developed in the previous subsection in a way that emphasizes the role of energy:

$$\frac{c^2}{2}\left[\left(\frac{E}{mc^2}\right)^2 - 1\right] = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{J^2}{2m^2r^2}\left(1 - \frac{2GM}{c^2r}\right) - \frac{GM}{r}. \quad (5.38)$$

The quantity on the left is not an energy, but for a particle of given mass it is determined by the orbital energy. The expression on the right consists of a 'kinetic' term (proportional to $(\mathrm{d}r/\mathrm{d}\tau)^2$) added to a sum of terms that depend only on $r$ for given values of $J$ and $m$. This is sufficient to earn the sum of those $r$ dependent terms the name 'effective potential' and the symbol $V_{\text{eff}}$. Thus we can write

$$\frac{c^2}{2}\left[\left(\frac{E}{mc^2}\right)^2 - 1\right] = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\text{eff}}, \quad (5.39)$$

where

$$V_{\text{eff}} = \frac{J^2}{2m^2r^2}\left(1 - \frac{2GM}{c^2r}\right) - \frac{GM}{r}. \quad (5.40)$$

Now, a very similar equation arises in Newtonian orbital analysis, where the constant orbital energy $E^{\text{Newton}}$ is given by

$$\frac{E^{\text{Newton}}}{m} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + V_{\text{eff}}^{\text{Newton}}, \quad (5.41)$$

with

$$V_{\text{eff}}^{\text{Newton}} = \frac{J^2}{2m^2r^2} - \frac{GM}{r}. \quad (5.42)$$

The Newtonian and Schwarzschild effective potentials for a positive value of $J$ are shown in Figure 5.6. In the Newtonian case the angular momentum magnitude $J$ is the source of an infinite 'effective potential barrier' that prevents particles with non-zero angular momentum magnitude from reaching $r = 0$. In the Schwarzschild case the behaviour at small values of $r$ is quite different. Indeed, for sufficiently small values of $J$ there is no barrier at all.

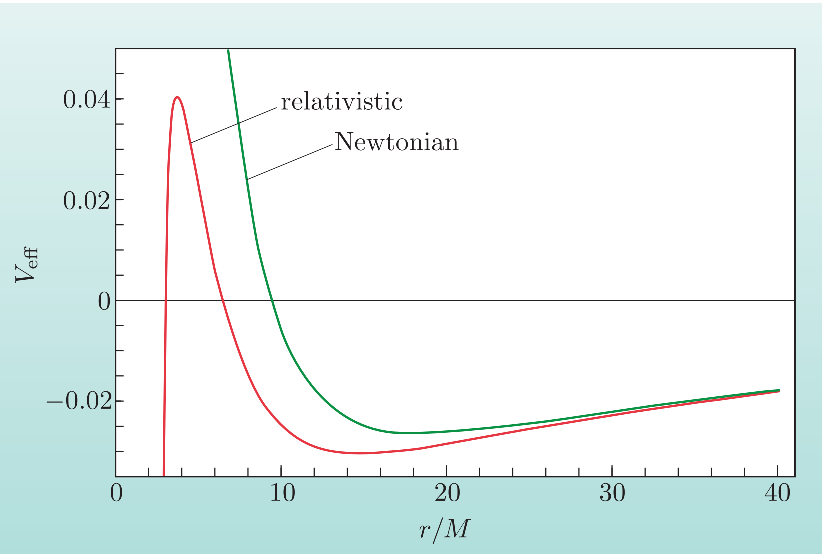


**Figure 5.6** Effective potentials for orbital motion with fixed angular momentum magnitude $J$ in Newtonian gravity and general relativity.

The difference between the Newtonian and Schwarzschild effective potentials comes from the extra term $-GMJ^2/m^2c^2r^3$ in the Schwarzschild case. One of its effects is to cause the orbits of particles to rotate in the $\theta = \pi/2$ plane. This effect is negligible at large values of $r$ but significant for small values, preventing elliptical orbits from closing and causing them to follow the kind of rosette pattern shown in Figure 5.7. This is another effect with astronomically observable consequences to which we shall return in Chapter 7.

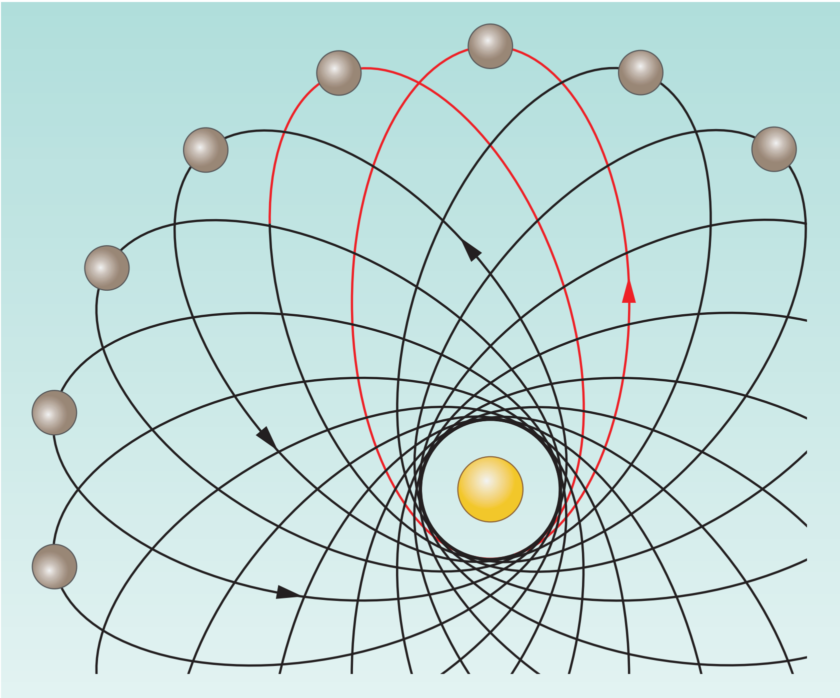

**Figure 5.7** The rosette orbit created by rotating a nearly elliptical orbit in its own plane. Part of the path is coloured to clarify the motion.

**Exercise 5.6** Both Newtonian and Schwarzschild orbital dynamics allow stable circular orbits to exist at large values of $r$, but in the Schwarzschild case there is a lower limit to the radius of a stable circular orbit that corresponds to $J/m = 2\sqrt{3}GM/c$.

(a) What is the (coordinate) radius of that orbit?

(b) What is the corresponding value of the parameter $E$? ■

## Summary of Chapter 5

1. The Schwarzschild metric tensor is

$$[g_{\mu\nu}] = \begin{pmatrix} 1 - \frac{2GM}{c^2 r} & 0 & 0 & 0 \\ 0 & -\dfrac{1}{1 - \frac{2GM}{c^2 r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}, \quad \text{(Eqn 5.1)}$$