

4

Tides, Rotation, and Shape

What fates impose, that men must needs abide;
It boots not to resist both wind and tide.

William Shakespeare, *Henry VI (3), IV, iii*

4.1 Introduction

So far we have considered all objects as being point masses with no physical dimensions. Since this is evidently not the case for real bodies, we must now consider the effects of the application of universal gravitation to the matter that forms the bodies of the solar system. A *tide* is raised on one body by another because of the effect of the gravitational gradient or the variation of the gravitational force across the body. For example, if we consider the tide raised on a planet by an orbiting satellite, the force experienced by the side of the planet facing the satellite is stronger than that experienced by the far side of the planet. Since none of the bodies that make up the solar system is perfectly rigid, there will be a distortion that gives rise to a *tidal bulge*.

The magnitude of the tidal bulge on a body is determined in part by its internal density distribution and thus, in principle, a measurement of the tidal amplitude could lead to a determination of the internal structure. Such measurements are not possible for any of the planets in the solar system other than the Earth. However, the deforming potential associated with planetary rotation acts in a similar way as that due to tides and a measurement of the rotational deformation of a planet can be used to determine its internal density distribution; this knowledge can then be used to estimate the response of the planet to a tidal potential. Satellites in the solar system that are in synchronous rotation are deformed by both rotational and tidal forces and measurements of their triaxial ellipsoidal figures have been used to determine their internal structures.

The response of the satellite to the tide it has raised can result in dynamical evolution of the system. Since friction is always present to some degree, tides are a dissipative phenomenon and the tide raised by a satellite on a planet can lead to orbital evolution of the satellite and a change in the spin rate of the planet. Just as the satellite raises a tide on the planet, so the planet also raises a tide on the satellite. This can be especially important when the satellite's orbit is eccentric. In some cases the effect of tidal dissipation in a satellite can lead to dramatic consequences, such as the runaway tidal heating of the jovian satellite Io.

4.2 The Tidal Bulge

Consider the case of the tides raised on a planet of mass m_p by a satellite of mass m_s . If we represent the objects as point masses, then Newton's law of gravitation gives the mean magnitude of the mutual force, $\langle F \rangle$, as

$$\langle F \rangle = \mathcal{G} \frac{m_p m_s}{r^2}, \quad (4.1)$$

where r is the separation of the centres. If we assume that the bodies move in circular orbits about their common centre of mass (Fig. 4.1), then we have already seen in Sect. 2.7 that the semi-major axes of the orbits are related to the masses by

$$a_s/a_p = m_p/m_s, \quad (4.2)$$

where the bodies have a constant separation $a = a_p + a_s$. The motion of the particle P_1 at the centre of the planet with respect to the centre of mass, C_1 , is a circle

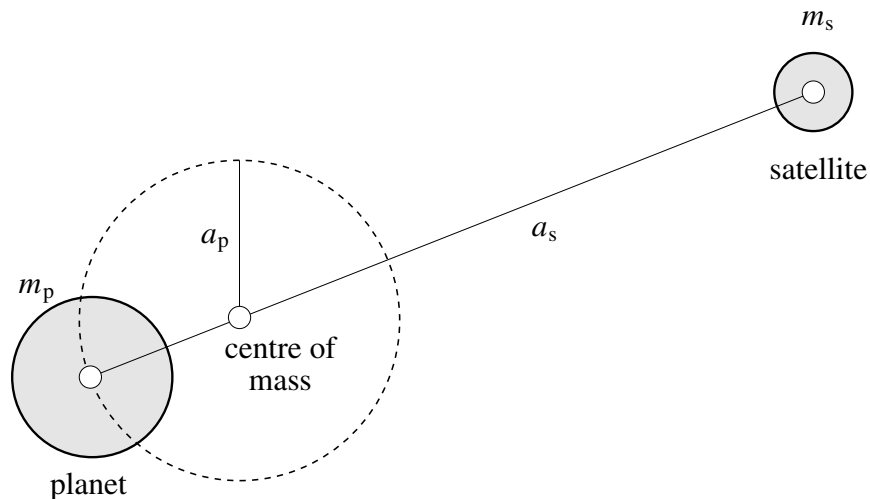


Fig. 4.1. A planet and satellite moving about their common centre of mass in circular orbits. Their semi-major axes with respect to the centre of mass are a_p and a_s , while the semi-major axis of the satellite with respect to the planet is $a = a_p + a_s$.

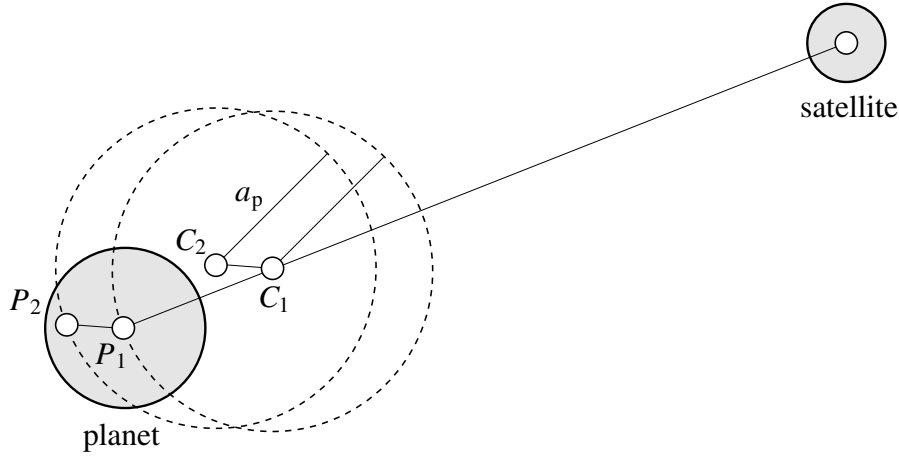


Fig. 4.2. All the particles in the planet move in similar circles of identical radii a_p , but with different centres. The particles P_1 and P_2 are on circles with centres C_1 and C_2 , respectively.

of radius a_p . If we neglect rotation it follows that the motion of any other point, P_2 , in the planet is a circle of the same radius but with a centre, C_2 , displaced from C_1 to the same extent that P_2 is displaced from P_1 (see Fig. 4.2). It follows that all particles that make up the planet are acted on by equal (in magnitude and direction) centrifugal forces but not by equal gravitational forces, \mathbf{F} . The common centrifugal force is equal to the mean gravitational force, $\langle \mathbf{F} \rangle$, that is,

$$\langle \mathbf{F} \rangle = \text{centrifugal force} \neq \mathbf{F}. \quad (4.3)$$

The tide-generating force, $\mathbf{F}_{\text{tidal}}$, that deforms the planet is defined by

$$\mathbf{F}_{\text{tidal}} = \mathbf{F} - \langle \mathbf{F} \rangle. \quad (4.4)$$

Rotational forces can also deform a body (see Sect. 4.7), but if the tidal and rotational deformations are small, they can be calculated separately and added linearly.

We approach the problem of determining the shape of the tidal bulge on the planet by considering the potential V at some point P on the surface of the planet due to the satellite (treated as a point mass). We have

$$V = -G \frac{m_s}{\Delta}, \quad (4.5)$$

where Δ is the distance of the point P from the centre of the satellite. From the cosine rule (see Fig. 4.3) we have

$$\Delta = a \left[1 - 2 \left(\frac{R_p}{a} \right) \cos \psi + \left(\frac{R_p}{a} \right)^2 \right]^{\frac{1}{2}}, \quad (4.6)$$

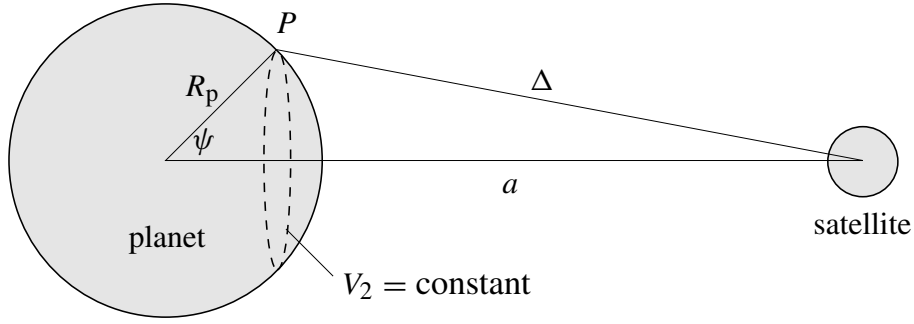


Fig. 4.3. The relationships among the radius of the planet, R_p , the semi-major axis of the satellite, a , and the distance Δ from a point P on the surface of the planet. The dashed line denotes the plane defined by the equipotential surface $V_2 = -\mathcal{G}(m_s/a^2)R_p \cos \psi = \text{constant}$.

where ψ is measured from the line joining the centres of the two bodies. In most cases of interest, $R_p/a \ll 1$. For example, the equatorial radius of the Earth is 6,378 km and the semi-major axis of the Moon's orbit is 384,400 km (see Tables A.4 and A.5). Consequently we can expand Eq. (4.6) binomially to obtain

$$V = -\mathcal{G} \frac{m_s}{a} \left[1 + \left(\frac{R_p}{a} \right) \cos \psi + \left(\frac{R_p}{a} \right)^2 \frac{1}{2} (3 \cos^2 \psi - 1) + \dots \right] \quad (4.7)$$

$$\approx V_1 + V_2 + V_3,$$

where we have neglected the higher order terms in the expansion.

The first term in Eq. (4.7), $V_1 = -\mathcal{G}m_s/a$, is a constant and since $\mathbf{F}/m_p = -\nabla V$ this term produces no force on the planet. The second term in Eq. (4.7), $V_2 = -\mathcal{G}(m_s/a^2)R_p \cos \psi$, gives rise to the force on the particle at the point P needed for motion in a circle. The force arising at any point from a potential is in the direction of the potential gradient and perpendicular to the equipotential surface that passes through that point. In this case, this is the plane perpendicular to the line connecting the centres of the gravitating bodies and the potential gradient is given by

$$-\frac{\partial V_2}{\partial (R_p \cos \psi)} = \mathcal{G} \frac{m_s}{a^2} = \frac{\langle F \rangle}{m_p}. \quad (4.8)$$

The potential at the surface of the planet due to the third term in Eq. (4.7) can be written as

$$V_3(\psi) = -\mathcal{G} \frac{m_s}{a^3} R_p^2 \mathcal{P}_2(\cos \psi), \quad (4.9)$$

where

$$\mathcal{P}_2(\cos \psi) = \frac{1}{2} (3 \cos^2 \psi - 1) \quad (4.10)$$

is the Legendre polynomial of degree 2 in $\cos \psi$ (see Sect. 4.3). Because

Importante en la exposicion
$$\frac{\mathbf{F}_{\text{tidal}}}{m_p} = \frac{\mathbf{F}}{m_p} - \frac{\langle \mathbf{F} \rangle}{m_p} = -\nabla V - \frac{\langle \mathbf{F} \rangle}{m_p} \approx -\nabla V_3(\psi), \quad (4.11)$$

this is the tide raising part of the potential.

We can also write $V_3(\psi)$ as

$$V_3(\psi) = -\zeta g \mathcal{P}_2(\cos \psi), \quad (4.12)$$

where

$$\zeta = \frac{m_s}{m_p} \left(\frac{R_p}{a} \right)^3 R_p \quad (4.13)$$

and

$$g = \frac{\mathcal{G}m_p}{R_p^2} \quad (4.14)$$

is the gravitational acceleration, or surface gravity, of the planet. In this case $\zeta \mathcal{P}_2(\cos \psi)$ is said to be the amplitude of the *equilibrium tide* for any value of ψ on the planet's surface. Note that $\mathcal{P}_2(\cos \psi)$ is a maximum at $\theta = 0$ or π and a minimum at $\theta = \pi/2$ or $3\pi/2$. Given that the Earth rotates on its axis with respect to the stars once every twenty-four hours, this explains why the Moon produces two high tides and two low tides on the Earth approximately every day.

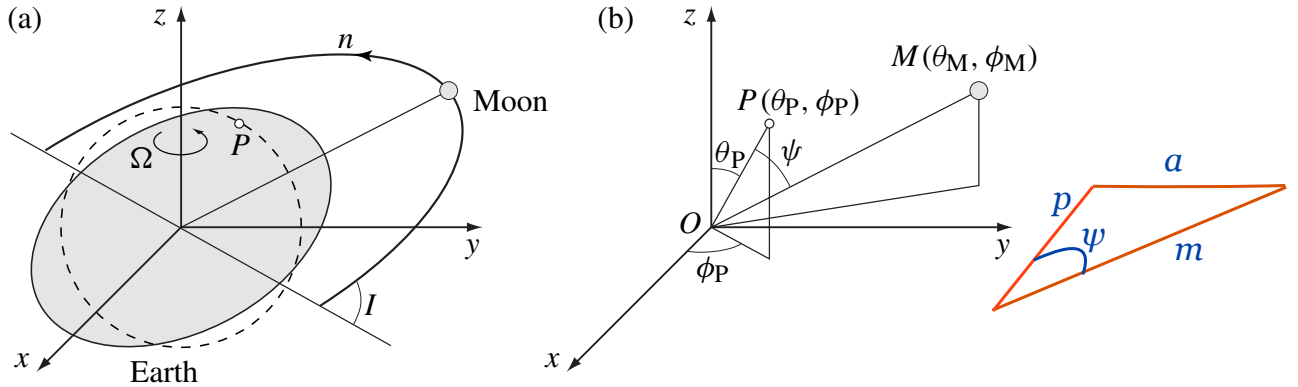


Fig. 4.4. Schematic diagram of the shape of the tidal distortion (solid line) arising from the $V_3(\psi)$ term in the gravitational potential compared with the circular, zero-distortion shape (dashed line). (a) The axis of symmetry of the tidal distortion passes through the centres of Earth and Moon, while the Earth rotates about the z axis with angular speed Ω ; the Moon has a mean motion n and an orbital inclination I with respect to the equatorial plane of the Earth. (b) The colatitudes and longitudes of the Moon (θ_M, ϕ_M) and a point P (θ_P, ϕ_P) on the surface of the Earth. Note that the longitude ϕ_P (and ϕ_M) is measured from a fixed direction in space and not from a direction that rotates with the Earth.

The tidal deformation of the Earth is made more complicated by the facts that (i) both the Sun and the Moon raise significant tides and (ii) in a geocentric frame both of these bodies orbit the Earth in paths that are eccentric and inclined with respect to the Earth's equator. If we neglect the orbital eccentricities, then the Sun and Moon both raise three principal tides on the Earth. In Fig. 4.4a the axis of symmetry of the tidal distortion passes through the centres of the Earth and Moon, the Earth rotates about the z axis with angular speed Ω , and we allow that the Moon has a mean motion n and an inclination I with respect to the Earth's equatorial plane. In Fig. 4.4b we show the colatitudes and longitudes of the Moon (θ_M, ϕ_M) and of a point P (θ_P, ϕ_P) on the surface of the Earth. Note that the longitudes ϕ_P and ϕ_M are measured from a fixed direction in space and not from a direction that rotates with the Earth.

Using the cosine rule we can show that the angle ψ between the position vectors OP and OM is given by

$$\cos \psi = \cos \theta_P \cos \theta_M + \sin \theta_P \sin \theta_M \cos(\phi_P - \phi_M). \quad (4.15)$$

Hence

$$\begin{aligned} \frac{1}{2} (3 \cos^2 \psi - 1) &= \frac{1}{2} (3 \cos^2 \theta_P - 1) \frac{1}{2} (3 \cos^2 \theta_M - 1) \\ &\quad + \frac{3}{4} \sin^2 \theta_P \sin^2 \theta_M \cos 2(\phi_P - \phi_M) \\ &\quad + \frac{3}{4} \sin 2\theta_P \sin 2\theta_M \cos(\phi_P - \phi_M). \end{aligned} \quad (4.16)$$

Look at variations with every parameter

Given that the colatitude θ_P of a fixed point on the Earth is constant, the tidal amplitude at P varies with ϕ_P , θ_M , and ϕ_M . The variation of the first term in Eq. (4.16) is determined by the variation with time of $\cos^2 \theta_M = (1/2)(1 + \cos 2\theta_M)$. Hence, this term varies with frequency $2n$ and gives rise to a *fortnightly tide*. The second term varies with frequency $2(\Omega - n) \approx 2\Omega$ and gives rise to a *semidiurnal tide*, while the third term varies with frequency $(\Omega - n) \approx \Omega$ and gives rise to a *diurnal tide*. Because the latter term contains the factor $\sin 2\theta_M$, the diurnal tide has a strong fortnightly modulation. Other tidal terms are associated with the Moon's orbital eccentricity.

The corresponding solar *semiannual*, *semidiurnal*, and *diurnal tides* have frequencies $2n_{\text{Sun}}$, $2(\Omega - n_{\text{Sun}}) \approx 2\Omega$, and $(\Omega - n_{\text{Sun}}) \approx \Omega$, where n_{Sun} is the solar (or the Earth's) mean motion. It follows from Eq. (4.13) that the ratio of the amplitudes of the corresponding solar and lunar tides is given, in each case, by

$$\frac{m_{\text{Sun}}}{m_{\text{Moon}}} \left(\frac{a_{\text{Moon}}}{a_{\text{Sun}}} \right)^3 \approx 0.46. \quad (4.17)$$

For the tide raised by the Moon on the Earth, $\zeta = 0.36$ m, while for the solar tide $\zeta = 0.16$ m.

4.3 Potential Theory

Before proceeding to the calculation of tidal and rotational deformation it is useful to summarise some results derived from potential theory. The gravitational potential of a homogeneous, spherical body of density γ and radius C can be found by considering the internal and external potentials of a thin spherical shell of radius r , thickness δr , and mass δm (Ramsey 1940). The potential exterior to the shell at some point distant r' from its centre is given by

$$V_{\text{ext}}(r') = -\frac{\mathcal{G}\delta m}{r'} \quad (4.18)$$

and is the same as if all the mass were concentrated at the centre of the shell. Thus the external potential at the surface of a uniform sphere is given by

$$V_{\text{ext}}(C) = -\frac{\mathcal{G} \sum \delta m}{C} = -\frac{4}{3}\pi\gamma\mathcal{G}C^2. \quad (4.19)$$

Because the gravitational force is described by an inverse square law, it follows that the force on a particle interior to the shell is zero. Hence, the gravitational potential interior to the shell must be a constant and can be determined by calculating the potential at any point in the interior. By calculating the potential at the centre of the shell, we obtain

$$V_{\text{int}}(r) = -\frac{\mathcal{G} \sum \delta m}{r} = -4\pi\gamma\mathcal{G}r\delta r. \quad (4.20)$$

It follows that the internal potential of a uniform shell of external radius C and internal radius r is given by

$$V_{\text{int}}(C, r) = -4\pi\gamma\mathcal{G} \int_r^C r \, dr = -2\pi\gamma\mathcal{G} (C^2 - r^2). \quad (4.21)$$

Hence, the interior ($r < C$) and exterior ($r > C$) potentials of a homogeneous spherical body at some point distant r from its centre are given by

$$V_{\text{int}}(r) = -\frac{4}{3}\pi\gamma\mathcal{G}r^2 - 2\pi\gamma\mathcal{G} (C^2 - r^2) = -\frac{2}{3}\pi\gamma\mathcal{G} (3C^2 - r^2), \quad (4.22)$$

$$V_{\text{ext}}(r) = -\frac{4}{3}\pi\gamma\mathcal{G} \frac{C^3}{r}. \quad (4.23)$$

The internal and external potentials of a deformed body can be expressed in terms of spherical harmonic functions. The following brief discussion of the use of these functions in potential theory is based on the fuller accounts given by Ramsey (1940), MacRobert (1967), Bullen (1975), and Blakely (1995).

The gravitational potential V in free space satisfies Laplace's equation:

$$\nabla^2 V = 0. \quad (4.24)$$

A function V is said to be *homogeneous of degree n* if it satisfies *Euler's equation*:

$$x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} = nV. \quad (4.25)$$

Homogeneous functions that also satisfy Laplace's equations are called *spherical solid harmonics*. They have the important property that when transformed into spherical coordinates, they can be factored into three functions, each of which depends on only one of the three variables r , θ , or ϕ . (For a good account of spherical harmonic analysis, see Blakely (1995).)

Using spherical polar coordinates (r, θ, ϕ) , where r is the radial distance from the centre of mass, θ is the colatitude measured from the polar axis, and ϕ is a longitude measured from some arbitrary fixed direction, we can write Laplace's equation as

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial V}{\partial \mu} \right) + \frac{1}{(1 - \mu^2)} \frac{\partial^2 V}{\partial \phi^2} = 0, \quad (4.26)$$

where $\mu = \cos \theta$. Laplace's equation can be solved by substituting the trial solution $V = r^n S_n(\mu, \phi)$, where $S_n(\mu, \phi)$ is independent of r , into Eq. (4.26). This gives

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) = n(n+1) r^n S_n, \quad (4.27)$$

and Eq. (4.26) reduces to

$$\frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial S_n}{\partial \mu} \right) + \frac{1}{1 - \mu^2} \frac{\partial^2 S_n}{\partial \phi^2} + n(n+1) S_n = 0. \quad (4.28)$$

This is called *Legendre's equation* and any function S_n that satisfies this equation is called a *spherical surface harmonic* of degree n . Because $n(n+1)$ remains unchanged when we write $-(n+1)$ for n , the general solution of Laplace's equation can be written as

$$V = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-(n+1)} \right) S_n(\mu, \phi). \quad (4.29)$$

Each of the terms in this equation is called a *solid harmonic* of degree n and $-(n+1)$, respectively (Ramsey 1940).

In the applications discussed in this chapter, where the deformations, either tidal or rotational, have axial symmetry, the solution of Legendre's equation reduces to

$$(1 - \mu^2) \frac{\partial^2 \mathcal{P}_n(\mu)}{\partial \mu^2} - 2\mu \frac{\partial \mathcal{P}_n(\mu)}{\partial \mu} + n(n+1) \mathcal{P}_n(\mu) = 0, \quad (4.30)$$

where Legendre's polynomial, $\mathcal{P}_n(\mu)$, is given by the terminating series

$$\mathcal{P}_n(\mu) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \left[\mu^n - \frac{n(n-1)}{2(2n-1)} \mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} \mu^{n-4} + \cdots \right] \quad (4.31)$$

or by Rodrigues's formula

$$\mathcal{P}_n(\mu) = \frac{1}{2^n n!} \frac{d^n(\mu^2 - 1)^n}{d\mu^n} \quad (4.32)$$

and the colatitude θ is measured from the axis of symmetry of the deformation. Legendre polynomials are *zonal harmonics*; the first five are given by

$$\mathcal{P}_0(\mu) = 1, \quad (4.33)$$

$$\mathcal{P}_1(\mu) = \mu = \cos \theta, \quad (4.34)$$

$$\mathcal{P}_2(\mu) = \frac{1}{2}(3\mu^2 - 1) = \frac{1}{4}(3 \cos 2\theta + 1), \quad (4.35)$$

$$\mathcal{P}_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu) = \frac{1}{8}(5 \cos 3\theta + 3 \cos \theta), \quad (4.36)$$

$$\mathcal{P}_4(\mu) = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3) = \frac{1}{64}(35 \cos 4\theta + 20 \cos 2\theta + 9). \quad (4.37)$$

Surface harmonics are orthogonal functions and double integrals of their products over the surface of the sphere have the following useful properties. The element of area on a unit sphere ($r = 1$) is given by $\sin \theta \, d\theta \, d\phi = -d\mu \, d\phi$. If $Y_m(\mu, \phi)$ and $S_n(\mu, \phi)$ are two surface harmonics of degrees m and n , respectively, and $m \neq n$, then

$$\int_0^{2\pi} \int_{-1}^{+1} Y_m(\mu, \phi) S_n(\mu, \phi) \, d\mu \, d\phi = 0. \quad (4.38)$$

If the two surface harmonics have the same degree, n , and one is a zonal harmonic, $\mathcal{P}_n(\mu)$, then

$$\int_0^{2\pi} \int_{-1}^{+1} S_n(\mu, \phi) \mathcal{P}_n(\mu) \, d\mu \, d\phi = \frac{4\pi}{2n+1} S_n(1), \quad (4.39)$$

where $S_n(1)$ is the value of $S_n(\mu, \phi)$ at the pole of $\mathcal{P}_n(\mu)$.

Consider two points on a unit sphere: a variable point (θ', ϕ') and a fixed point (θ, ϕ) . Let ψ be the angle subtended by these points at the centre of the sphere, and consider the integral

$$\int_0^{2\pi} \int_{-1}^{+1} S_n(\theta', \phi') \mathcal{P}_n(\cos \psi) \, d\mu' \, d\phi'. \quad (4.40)$$

Let the axes of the coordinates of the variable point be changed such that the new axis defining the variable colatitude passes through the fixed point (θ, ϕ) , and let the new angular coordinates of the variable point be (Θ', Φ') , such that $\Theta' = \psi$. Also let $S_n(\theta', \phi')$ become $Y_n(\Theta', \Phi')$. Then, from Eq. (4.39), we obtain

$$\int_0^{2\pi} \int_{-1}^{+1} Y_n(\Theta', \Phi') \mathcal{P}_n(\cos \Theta') \, d(\cos \Theta') \, d\Phi' = \frac{4\pi}{2n+1} Y_n(1). \quad (4.41)$$

However,

$$Y_n(1) = S_n(\theta, \phi). \quad (4.42)$$

It follows from this that

$$\int_0^{2\pi} \int_{-1}^{+1} S_n(\mu', \phi') \mathcal{P}_n(\cos \psi) d\mu' d\phi' = \frac{4\pi}{2n+1} S_n(\mu, \phi), \quad (4.43)$$

where $S_n(\mu, \phi)$ is the same function of (μ, ϕ) that $S_n(\mu', \phi')$ is of (μ', ϕ') (MacRobert 1967).

Now consider the potential at some fixed point P due to a homogeneous, nearly spherical body whose surface is defined by

$$R(\theta') = C [1 + \epsilon_2 \mathcal{P}_2(\cos \theta')], \quad (4.44)$$

where $\epsilon_2 (\ll 1)$ is a constant and C is now the mean radius. The point P can be either internal ($r < C$) or external ($r > C$) and has spherical coordinates (r, μ, ϕ) , where $\mu = \cos \theta$ and the colatitude θ is measured from the axis of symmetry of the tidal bulge (Fig. 4.5). The total gravitational potential at P is the sum of two parts. The part due to the spherical body is given by Eqs. (4.22) and (4.23), while the other, noncentral, part of the potential is due to the thin distribution of matter between the surface of the deformed body and the mean sphere. At some point $P'(r', \mu', \phi')$, the radial thickness of this thin layer of matter is $\epsilon_2 C \mathcal{P}_2(\mu')$ and the element of volume at that point is $\epsilon_2 C^3 \mathcal{P}_2(\mu') d\mu' d\phi'$. The potential at P due to the element of mass is determined by the distance $PP' = \Delta$, where

$$\frac{1}{\Delta} = (C^2 + r^2 - 2Cr \cos \psi)^{-1/2}. \quad (4.45)$$

If $r < C$, then

$$\frac{1}{\Delta} = \frac{1}{C} \left[1 + \left(\frac{r}{C} \right)^2 - 2\mu \left(\frac{r}{C} \right) \right]^{-1/2}, \quad (4.46)$$

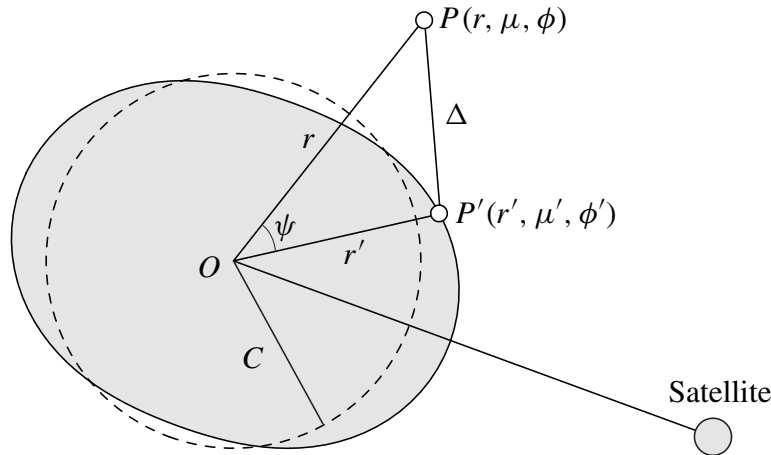


Fig. 4.5. The potential at the point P due to a deformed central planet arises from the spherical body of mean radius C and the distribution of matter associated with the tidal bulge.

and on expanding binomially and ordering the terms in powers of r/C we obtain

$$\begin{aligned} \frac{1}{\Delta} = \frac{1}{C} & \left[1 + \left(\frac{r}{C}\right) \mu + \left(\frac{r}{C}\right)^2 \left(-\frac{1}{2} + \frac{3}{2}\mu^2\right) \right. \\ & \left. + \left(\frac{r}{C}\right)^3 \left(-\frac{3}{2}\mu + \frac{5}{2}\mu^3\right) + \dots \right], \end{aligned} \quad (4.47)$$

and inspection of Eqs. (4.33)–(4.37) shows that Eq. (4.47) can be written as

$$\frac{1}{\Delta} = \frac{1}{C} \sum_{n=0}^{\infty} \left(\frac{r}{C}\right)^n \mathcal{P}_n(\cos \psi) + \mathcal{O}(\epsilon_2). \quad (4.48)$$

Hence, the noncentral contribution to the total potential at P is given by

$$V_{\text{nc,int}} = -\gamma \mathcal{G} C^2 \epsilon_2 \sum_{n=0}^{\infty} \left(\frac{r}{C}\right)^2 \int \int \mathcal{P}_2(\mu') \mathcal{P}_n(\cos \psi) d\mu' d\phi'. \quad (4.49)$$

From Eq. (4.43) we obtain

$$\sum_{n=0}^{\infty} \left(\frac{r}{C}\right)^n \int \int \mathcal{P}_2(\mu') \mathcal{P}_n(\cos \psi) d\mu' d\phi' = \frac{4\pi}{5} \left(\frac{r}{C}\right)^2 \mathcal{P}_2(\cos \theta). \quad (4.50)$$

Hence, the internal, noncentral contribution to the potential at P is given by

$$V_{\text{nc,int}} = -\frac{4\pi}{5} \gamma \mathcal{G} r^2 \epsilon_2 \mathcal{P}_2(\cos \theta), \quad (4.51)$$

and the total internal potential at P is the sum of this term and that given by Eq. (4.22), that is,

$$V_{\text{int}}(r, \theta) = -\frac{4}{3} \pi C^3 \gamma \mathcal{G} \left[\frac{3C^2 - r^2}{2C^3} + \frac{3}{5} \frac{r^2}{C^3} \epsilon_2 \mathcal{P}_2(\cos \theta) \right]. \quad (4.52)$$

Similarly, if $r > C$, then

$$\frac{1}{\Delta} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{C}{r}\right)^n \mathcal{P}_n(\cos \psi) + \mathcal{O}(\epsilon_2) \quad (4.53)$$

and the total external potential is given by

$$V_{\text{ext}}(r, \theta) = -\frac{4}{3} \pi C^3 \gamma \mathcal{G} \left[\frac{1}{r} + \frac{3}{5} \frac{C^2}{r^3} \epsilon_2 \mathcal{P}_2(\cos \theta) \right]. \quad (4.54)$$

4.4 Tidal Deformation

If $h(\psi)$ is the local height of the equipotential surface, then given that the tidal potential at the surface of the planet is $-\zeta g \mathcal{P}_2(\cos \psi)$, the total potential is

$$V_{\text{total}}(r, \psi) = -\frac{\mathcal{G} m_{\text{p}}}{B} + gh(\psi) - \zeta g \mathcal{P}_2(\cos \psi), \quad (4.55)$$

where B is the mean radius of the planet. On an equipotential surface this must be independent of ψ ; hence we must have $h(\psi) = \zeta \mathcal{P}_2(\cos \psi)$, where it is understood that ψ is measured from the axis of symmetry of the tidal bulge. The equilibrium tide defines the shape of a shallow, zero-density ocean covering an inflexible, spherical planet. In practice, of course, no fluid has zero density and no solid is inflexible. We need to find the tidal deformation of a real solid or fluid body.

In order to show the factors involved, we consider the case of a simple, two-component model planet that has a mean radius B , a homogeneous, incompressible, fluid ocean of density σ , and a homogeneous, incompressible, solid core of mean radius A , density ρ , and rigidity μ (Street 1925, Dermott 1979a; see Fig. 4.6). The rigidity μ is a measure of the force needed to deform the elastic body.

The equilibrium tide is the second-order surface harmonic that describes the equipotential surface that would exist close to a completely rigid, ocean-free, spherical planet circled by a single, distant satellite. If the planet is not completely rigid and has an ocean, then the surfaces of the ocean and the equilibrium tide will not coincide. This is the case even if we neglect effects due to the kinetic energy of the ocean currents. To calculate the response of the ocean and core to the gravitational field of the satellite we must take into account the effect of the gravitational field of the tidal bulge itself (self-gravitation) and the effect of the elastic forces within the solid. We must also allow for the elastic yielding of the core under the action of all the forces that act on both the core and the ocean; that is, we must allow for the potentials of the deformed core and the ocean tide

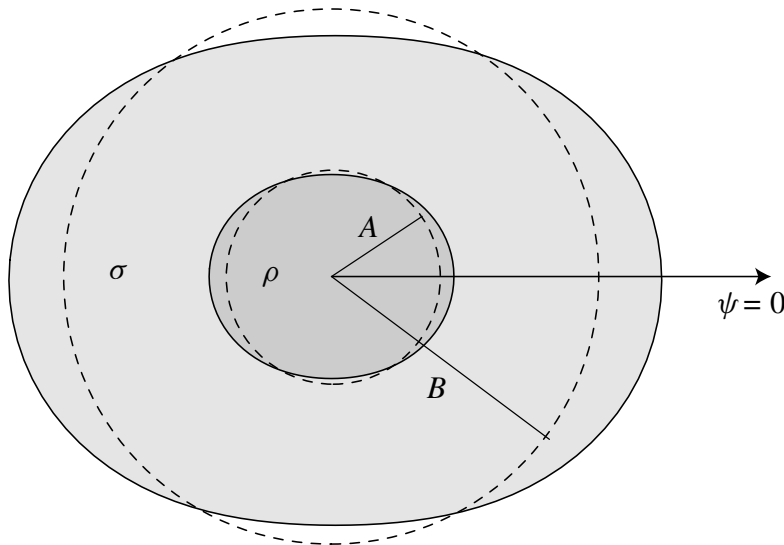


Fig. 4.6. Schematic diagram showing a model planet consisting of a deformed core of mean radius A and density ρ , surrounded by a deformed ocean of mean radius B and density σ . The circles of radii A and B are denoted by dashed lines.

itself, as well as the loading effects of the tides raised in the various parts of the planet (Street 1925).

Since the tide-raising potential is a solid spherical harmonic of the second degree, it follows that the deformation of the planet must be described in terms of the same harmonic function (Love 1911). If this were not the case, then it would not be possible for the surface of the ocean, for example, to be an equipotential surface. We have already seen in Sect. 4.2 that the tidal potential depends only on one angle, ψ , implying axial symmetry about the line joining the two centres. Thus, we can describe the deformed shapes of the core boundary and ocean surface by

$$R_{cb}(\psi) = A [1 + S_2 \mathcal{P}_2(\cos \psi)] \quad (4.56)$$

and

$$R_{os}(\psi) = B [1 + T_2 \mathcal{P}_2(\cos \psi)] \quad (4.57)$$

respectively, where S_2 and T_2 are constants. We will determine S_2 and T_2 from (i) the fact that the surface of the static ocean must be an equipotential and (ii) by considering the equilibrium of all the forces acting on the mean core boundary.

The potential within the ocean, $V_o(r, \psi)$, is the sum of three potentials: (i) the tidal potential due to the satellite,

$$V_3(r, \psi) = -\frac{\mathcal{G}m_s}{a^3} r^2 \mathcal{P}_2(\cos \psi) = -\zeta g \left(\frac{r}{B}\right)^2 \mathcal{P}_2(\cos \psi), \quad (4.58)$$

which is a generalisation of Eqs. (4.9) and (4.12), (ii) $V_{int}(r, \psi)$ due to the ocean, and (iii) $V_{ext}(r, \psi)$ due to the core. Thus, the total potential within the ocean is given by

$$\begin{aligned} V_o(r, \psi) = & -\zeta g \left(\frac{r}{B}\right)^2 \mathcal{P}_2(\cos \psi) \\ & - \frac{4}{3} \pi B^3 \sigma \mathcal{G} \left(\frac{3B^2 - r^2}{2B^3} + \frac{3}{5} \frac{r^2}{B^3} T_2 \mathcal{P}_2(\cos \psi) \right) \\ & - \frac{4}{3} \pi A^3 (\rho - \sigma) \mathcal{G} \left(\frac{1}{r} + \frac{3}{5} \frac{A^2}{r^3} S_2 \mathcal{P}_2(\cos \psi) \right). \end{aligned} \quad (4.59)$$

The contributions of the ocean and the core to the total were derived by first calculating the internal potential due to an all-ocean planet and then adding the external potential due to a solid core with a density, $\rho - \sigma$, equal to the difference of the core and ocean densities.

We calculate the potential at the ocean surface, $V_{os}(r, \psi)$, by substituting the surface equation, $r = B [1 + T_2 \mathcal{P}_2(\cos \psi)]$, into Eq. (4.59) and expanding the resulting expression, neglecting second and higher order terms in ζ/B , S_2 and

T_2 . This gives

$$V_{\text{os}}(r, \psi) = -\zeta g \mathcal{P}_2(\cos \psi) - \frac{4}{3} \pi B^2 \sigma \mathcal{G} \left(\frac{1}{2} - \frac{2}{5} T_2 \mathcal{P}_2(\cos \psi) \right) - \frac{4}{3} \pi \frac{A^3}{B} (\rho - \sigma) \mathcal{G} \left(1 - T_2 \mathcal{P}_2(\cos \psi) + \frac{3}{5} \left(\frac{A}{B} \right)^2 S_2 \mathcal{P}_2(\cos \psi) \right). \quad (4.60)$$

The terms in this equation that do not depend on ψ give rise to compressive forces only; since we have assumed that the core and ocean are incompressible, these have no role in determining the shape of the planet and can be ignored. Given that the ocean surface is an equipotential, the sum of those terms that do depend on ψ must be zero. Hence we must have

1 relación entre S_2 y T_2
$$\frac{\zeta_c}{A} = \left[\frac{2}{5} \frac{\sigma}{\rho} + \left(\frac{A}{B} \right)^3 \left(1 - \frac{\sigma}{\rho} \right) \right] T_2 - \frac{3}{5} \left(\frac{A}{B} \right)^5 \left(1 - \frac{\sigma}{\rho} \right) S_2. \quad (4.61)$$

Here we have introduced

$$\zeta_c = \frac{m_s}{m_c} \left(\frac{A}{a} \right)^3 A, \quad (4.62)$$

where m_c is the mass of the core and

$$g_c = \frac{\mathcal{G} m_c}{A^2} \quad (4.63)$$

is the gravity at the core boundary. The quantity ζ_c is the amplitude of the “equilibrium tide” that would exist at the core boundary if the ocean were removed; it is related to ζ in Eq. (4.13) by

$$\zeta g A^2 = \zeta_c g_c B^2. \quad (4.64)$$

Equation (4.61) gives us our first relation between S_2 and T_2 . A second relation is obtained by considering the equilibrium of all the forces acting on the mean core boundary.

Within the solid body of the core, the deforming potential, $V_c(r, \psi)$ is the sum of the tidal potential due to the satellite, $V_3(r, \psi)$, and the internal potentials due to the ocean and core. Thus,

$$V_c(r, \psi) = -\zeta_c g_c \left(\frac{r}{A} \right)^2 \mathcal{P}_2(\cos \psi) - \frac{4}{3} \pi B^3 \sigma \mathcal{G} \left(\frac{3B^2 - r^2}{2B^3} + \frac{3}{5} \frac{r^2}{B^3} T_2 \mathcal{P}_2(\cos \psi) \right) - \frac{4}{3} \pi A^3 (\rho - \sigma) \mathcal{G} \left(\frac{3A^2 - r^2}{2A^3} + \frac{3}{5} \frac{r^2}{A^3} S_2 \mathcal{P}_2(\cos \psi) \right). \quad (4.65)$$

Consider a sphere within the core, concentric with the planet’s centre. For a fixed value of r , the effective deforming potential is given by those terms in $V_c(r, \psi)$

that only depend on $\mathcal{P}_2(\cos \psi)$. The other terms are compressive and can be ignored. Hence the effective deforming potential can be written as

$$V_c(r, \psi) = -Zr^2\mathcal{P}_2(\cos \psi), \quad (4.66)$$

where

$$Z = \frac{g_c}{A} \left(\frac{\zeta_c}{A} + \frac{3}{5} \frac{\sigma}{\rho} (T_2 - S_2) + \frac{3}{5} S_2 \right). \quad (4.67)$$

Chree (1896a) showed that the yielding of the core under the force resulting from this deforming potential is the same as that which would be produced by an outward normal force per unit area of amount $\rho Z A^2 \mathcal{P}_2(\cos \psi)$ acting at the mean core boundary, $r = A$.

Other pressures also act at this boundary. These arise from loading terms caused by the hydrostatic pressure in the ocean and the tide in the solid core. For example, in the case of a shallow ocean for which $(B - A) \ll B$ and there is no variation of g within the ocean, these pressures arise from (i) the variation with ψ of the ocean depth and (ii) the variation with ψ of the distance of the core boundary from the centre of the planet. These pressures are given by the product of the local gravity, the density, and the height of the tide and can be written as

$$P_o(\psi) = g\sigma B(T_2 - S_2)\mathcal{P}_2(\cos \psi) \quad (4.68)$$

and

$$P_c(\psi) = g_c \rho A S_2 \mathcal{P}_2(\cos \psi). \quad (4.69)$$

For a deep ocean g is not a constant and although the loading term due to the tide in the core remains the same, the angular variation of the hydrostatic pressure on the core is no longer determined by the height of the ocean tide alone. We must also take account of the angular variation of gravity within the ocean.

In the general case of a deep ocean, the pressure on the core boundary due to the ocean is given by

$$P_o(\psi) = \int_{R_{cb}}^{R_{os}} \sigma(r) \frac{\partial V_o(r, \psi)}{\partial r} dr, \quad (4.70)$$

where the integral limits are from the core boundary to the ocean surface. As the ocean in our model is assumed to be incompressible and of uniform density, this reduces to

$$P_o(\psi) = \sigma [V_o(R_{os}, \psi) - V_o(R_{cb}, \psi)]. \quad (4.71)$$

Furthermore, since only the variable part of the potential can contribute to the deforming forces (we are neglecting compression) and since the surface of the ocean is an equipotential surface, it follows that $V_o(R_{os}, \psi)$ is a constant and can be neglected.

We can obtain the potential at the core boundary, $V_{cb}(\psi)$, by substituting the equation for the core boundary, $R_{cb} = A[1 + S_2 \mathcal{P}_2(\cos \psi)]$, into Eq. (4.60) or

Eq. (4.65) and expanding the resulting expression, neglecting second and higher order terms in ζ/B , S_2 , and T_2 . This gives

$$V_{cb}(\psi) = \text{constant} - Ag_c \left(\frac{\zeta_c}{A} + \frac{3}{5} \frac{\sigma}{\rho} (T_2 - S_2) - \frac{2}{5} S_2 \right) \mathcal{P}_2(\cos \psi), \quad (4.72)$$

where the ψ dependence is contained in the second term on the right-hand side of the equation; this is the part that will contribute to the variable part of the pressure at the core boundary, $P_o(\psi)_\psi$. Hence

$$P_o(\psi)_\psi = \sigma Ag_c \left(\frac{\zeta_c}{A} + \frac{3}{5} \frac{\sigma}{\rho} (T_2 - S_2) - \frac{2}{5} S_2 \right) \mathcal{P}_2(\cos \psi). \quad (4.73)$$

The total effective outward normal force per unit area at the mean core boundary is the sum of the elastic forces within the core and the loading terms due to the ocean and the core tides. We write this as $X\mathcal{P}_2(\cos \psi)$, where

$$\begin{aligned} X &= \rho A^2 Z - P_o(\psi)_\psi - \rho g_c A S_2 \\ &= \frac{2}{5} \rho g_c A \left(1 - \frac{\sigma}{\rho} \right) \left(\frac{5}{2} \frac{\zeta_c}{A} - S_2 + \frac{3}{2} \frac{\sigma}{\rho} (T_2 - S_2) \right). \end{aligned} \quad (4.74)$$

Note that $X \rightarrow 0$ as $\sigma \rightarrow \rho$, which must be the case if the ocean is in hydrostatic equilibrium.

Love (1944) showed that the radial displacement of the solid core produced by this deforming pressure is

$$\Delta R(\psi) = \frac{5}{19} \frac{A}{\mu} X \mathcal{P}_2(\cos \psi), \quad (4.75)$$

which must be the same as $AS_2\mathcal{P}_2(\cos \psi)$. This gives us our second relationship between S_2 and T_2 :

$$S_2 = \frac{1}{\tilde{\mu}} \left(1 - \frac{\sigma}{\rho} \right) \left(\frac{5}{2} \frac{\zeta_c}{A} - S_2 + \frac{3}{2} \frac{\sigma}{\rho} (T_2 - S_2) \right), \quad (4.76)$$

where $\tilde{\mu}$, the *effective rigidity* of the solid core, is a dimensionless quantity defined by

$$\tilde{\mu} = \frac{19\mu}{2\rho g_c A}. \quad (4.77)$$

It is a measure of the ratio of the elastic and gravitational forces acting at the core boundary.

If $\tilde{\mu} \ll 1$, then the core responds like a fluid, whereas for $\tilde{\mu} \gg 1$ elastic forces within the core dominate. Note that if $\sigma = \rho$, then $S_2 = 0$ and the elastic core is undeformed. If the planet is ocean free, then $\sigma = 0$ and, from Eq. (4.76), the amplitude of the tide on the isolated core is given by

$$AS_2 = \frac{(5/2)\zeta_c}{1 + \tilde{\mu}}. \quad (4.78)$$

In the general case, we can write

$$AS_2 = F \frac{(5/2)\zeta_c}{1 + \tilde{\mu}} \quad \text{and} \quad BT_2 = H \frac{5}{2}\zeta, \quad (4.79)$$

in which case F is a dimensionless quantity that is a measure of the effect of the ocean on the amplitude of the core tide, and H is a measure of the effect of the internal structure on the external shape of the planet. By eliminating T_2 from Eqs. (4.61) and (4.76), we obtain

$$F = \frac{(1 + \tilde{\mu})(1 - \sigma/\rho)(1 + 3/2\alpha)}{1 + \tilde{\mu} - \sigma/\rho + (3\sigma/2\rho)(1 - \sigma/\rho) - (9/4\alpha)(A/B)^5(1 - \sigma/\rho)^2} \quad (4.80)$$

and

$$H = \frac{2\langle\rho\rangle}{5\rho} \left(\frac{1 + \tilde{\mu} + (3/2)(A/B)^2 F \delta}{(1 + \tilde{\mu})(\delta + 2\sigma/5\rho)} \right), \quad (4.81)$$

where

$$\alpha = 1 + \frac{5}{2} \frac{\rho}{\sigma} \left(\frac{A}{B} \right)^3 \left(1 - \frac{\sigma}{\rho} \right) \quad \text{and} \quad \delta = \left(\frac{A}{B} \right)^3 \left(1 - \frac{\sigma}{\rho} \right) \quad (4.82)$$

and $\langle\rho\rangle$ is the mean density. If the planet is fluid throughout, or if because of thermal creep the solid core has relaxed to hydrostatic equilibrium (this has application to tidal bulges on satellites with spin rates equal to or synchronous with their orbital mean motions), then $\tilde{\mu} = 0$ and the hydrostatic value of H is given by

$$H_h = \frac{2\langle\rho\rangle}{5\sigma} \left(\frac{1 + (3\delta/5\gamma)(A/B)^2}{\delta + 2\sigma/5\rho - (9\delta\sigma/25\gamma\rho)(A/B)^2} \right), \quad (4.83)$$

where

$$\gamma = \frac{2}{5} + \frac{3\sigma}{5\rho} \quad (4.84)$$

(Dermott 1979a).

We can now use these results to determine the tidal amplitudes in the limiting case of a planet with a shallow, uniform ocean. If $A = B$, $\zeta_c = \zeta$, and $\langle\rho\rangle = \rho$ the condition for the ocean surface to be an equipotential surface, Eq. (4.61), reduces to

$$\frac{\zeta}{A} = \frac{2}{5}S_2 + \left(1 - \frac{3}{5} \frac{\sigma}{\rho} \right) (T_2 - S_2). \quad (4.85)$$

Eliminating the terms containing S_2 alone from Eqs. (4.61) and (4.76) we obtain

$$A(T_2 - S_2) = \frac{\zeta \tilde{\mu}}{1 - \sigma/\rho + \tilde{\mu}(1 - 3\sigma/5\rho)} \quad (4.86)$$

for the amplitude of the ocean tide. This agrees with that first given by Chree (1896b). We note that $A(T_2 - S_2) \rightarrow 0$ as $\tilde{\mu} \rightarrow 0$. Furthermore, if $\sigma = \rho$, then

$$A(T_2 - S_2) = \frac{5}{2}\zeta \quad (4.87)$$

and is independent of $\tilde{\mu}$. Thus, when the core and ocean have the same density, the core is undeformed and the amplitude of the ocean tide is a factor 5/2 larger than the “equilibrium” tide given by Eq. (4.13).

We can find the amplitude of the solid body tide in the case of a shallow ocean by substituting the expression for $T_2 - S_2$ from Eq. (4.86) into Eq. (4.61). This gives

$$AS_2 = \frac{5}{2}\zeta \left[\frac{(1 - \sigma/\rho)}{1 - \sigma/\rho + \tilde{\mu}(1 - 3\sigma/5\rho)} \right]. \quad (4.88)$$

For an ocean-free planet, $\sigma = 0$ and

$$AS_2 = \frac{(5/2)\zeta}{1 + \tilde{\mu}}, \quad (4.89)$$

which agrees with the expression first given by Lord Kelvin (Thompson 1863). Kelvin applied this result to observations of the fortnightly tide for which $AS_2 \approx 0.6\zeta$ and hence deduced that the rigidity of the Earth is $\sim 1.2 \times 10^{11} \text{ Nm}^{-2}$ or $\sim 50\%$ greater than the rigidity of uncompressed steel. At the time, this was a surprising result because the interior of the Earth was then thought to be largely molten (Bullen 1975).

Interesante o.O

In the case of the Earth it is perhaps more realistic to consider the planet as being composed of a solid core of density ρ and rigidity μ covered by a shallow ocean of density σ . In which case, the amplitudes of the tides in the ocean and the solid core are given by Eqs. (4.86) and (4.88). However, these amplitudes were calculated by assuming that the ocean is in hydrostatic equilibrium and thus that the tidal currents in the ocean have no role in determining the shape of the ocean surface. For the Earth, this is not a good assumption.

The effects of ocean currents depend on the natural oscillation frequency of the ocean basin and this is determined by the size, shape, and depth of the basin. For heuristic purposes we follow Proudman (1953) and consider a uniform, equatorial canal of depth d bounded by two parallels of latitude. In a reference frame centred on the Earth and rotating with the mean motion of the Moon the tidal bulge is stationary. Hence the speed of the bulge with respect to the surface of the Earth is $U = 2\pi A/T_E \approx 500 \text{ ms}^{-1}$, where T_E is the spin period of the Earth. However, U is not the speed of the tidal current in the ocean. The fluid in the tidal bulge is supplied by a current of speed u flowing uniformly throughout the depth of the ocean (see Fig. 4.7). If, at a given longitude, ζ denotes the height of the surface above the ocean floor, then from the equation of continuity we

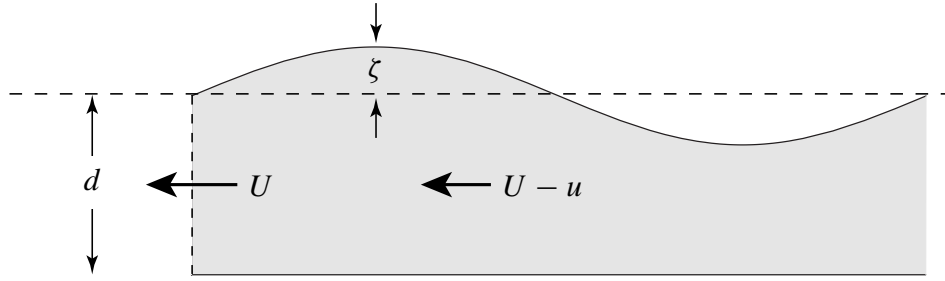


Fig. 4.7. Schematic diagram of a tidal wave in a uniform equatorial canal of depth d bounded by two parallels of latitude. The tidal bulge is stationary in a reference frame centred on the Earth and rotating with the Moon. The speed of the bulge with respect to the solid Earth is U . However, the fluid in the tidal bulge is supplied by a current of speed u , which, at a given longitude, flows uniformly throughout the depth of the ocean.

have

$$(U - u)(d + \zeta) = Ud. \quad (4.90)$$

If $\zeta \ll d$, then

$$u = \frac{\zeta}{d}U. \quad (4.91)$$

The mean ocean depth on the Earth is ~ 4 km, and so $u \sim 0.1 \text{ ms}^{-1}$. From Bernouilli's theorem, which relates work done by the hydrostatic pressure to the kinetic and potential energies of the fluid in streamline flow, we obtain

$$\frac{1}{2}(U - u)^2 + g\zeta + \Psi = \text{constant}, \quad (4.92)$$

where Ψ is the tidal potential. If we assume that at the surface of the canal $\Psi = -g\bar{\zeta}$, then given that U^2 is a constant and $u^2 \ll uU$, we have

$$Uu = g(\zeta - \bar{\zeta}). \quad (4.93)$$

Substituting for u from Eq. (4.91) we obtain

$$\zeta = \frac{\bar{\zeta}}{1 - U^2/gd}. \quad (4.94)$$

For the Earth resonance would occur in an equatorial canal of depth $d_{\text{res}} = U^2/g \approx 22$ km and given that the mean depth of the ocean is less than d_{res} we would expect the tides to be inverted. This reasoning cannot be applied to the Earth as a whole because for near-global oceans we cannot assume that the total tidal potential is decoupled from the tidal currents. However, Eq. (4.94) provides a good indication that in calculating the shapes of the Earth's oceans we must include effects due to the ocean currents and consider the possibility of resonance in the various ocean basins.