1a:

Proof that Gaussian distribution is normalized.

$$\varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{1}{2\sigma^2}x^2)$$

To prove that the above expression is normalized, we have to show that:

$$\int_{-\infty}^{+\infty} exp(-\frac{1}{2\sigma^2}x^2)dx = 2\sqrt{2\pi\sigma^2}$$

Let:

$$I = \int_{-\infty}^{+\infty} exp(-\frac{1}{2\sigma^2}x^2)dx I^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} exp(-\frac{1}{2\sigma^2}x^2 - \frac{1}{2\sigma^2}y^2)dxdy(1)$$

Make the transformation from Cartesian coordinates (x, y) to polar coordinates (r, θ), which is defined by:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$dxdy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} drd\theta$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} drd\theta = rdrd\theta$$

Equation (1) can be rewritten as:

$$I^2 = \int_0^{2\pi}\int_0^{\infty} exp(-\frac{r^2}{2\sigma^2})rdrd\theta$$

$$= 2\pi\int_0^{\infty} exp(-\frac{r^2}{2\sigma^2})rdr$$

Used the change of variables $u = r^2$:

$$I^2 = 2\pi\int_0^{\infty} exp(-\frac{u}{2\sigma^2})\frac{1}{2}du$$

$$= \pi[exp(-\frac{u}{2\sigma^2})(-2\sigma^2)]_0^{\infty}$$

$$= 2\pi\sigma^2$$

$$\Longrightarrow I = \sqrt{2\pi\sigma^2}$$

We have: $N(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{(x-\mu)^2}{2\sigma^2})$

Make the tranformation $y = x - \mu$ so that:

$$\int_{-\infty}^{\infty} N(x|\mu, \sigma^2)dx = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} exp(-\frac{y^2}{2\sigma^2})dy$$

$$= \frac{I}{\sqrt{s\pi\sigma^2}} = 1$$

1b:
Proof that expectation of Gaussian distribution is mu (mean).
From the definition of the Gaussian distribution, X has probability density function:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

From the definition of the expected value of a continuous random variable:

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

So:

$$E(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x exp(-\frac{(x-\mu)^2}{2\sigma^2}) dx$$

Substituting $t = \frac{x-\mu}{\sqrt{2\sigma^2}}$, $\sqrt{2\sigma^2} t = dx$

$$E(X) = \frac{\sqrt{2\sigma^2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (\sqrt{2\sigma^2} t + \mu) exp(-t^2) dt$$

$$= \frac{1}{\sqrt{\pi}} (\sqrt{2\sigma^2} \int_{-\infty}^{\infty} t exp(-t^2) dt + \mu \int_{-\infty}^{\infty} exp(-t^2) dt)$$

Apply Fundamental Theorem of Calculus, Gaussian Integral:

$$= \frac{1}{\sqrt{\pi}} (\sqrt{2\sigma^2} [\frac{-1}{2} exp(-t^2)]_{-\infty}^{\infty} + \mu\sqrt{x})$$

Exponential Tends to Zero and Infinity

$$= \frac{\mu\sqrt{x}}{\sqrt{x}}$$

$$= \mu$$

1c:
Proof Variance of Gaussian distribution is $sigma^2$ (variance):

$$var(x) = E((x-\mu)^2)$$

$$E((x-\mu)^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu)^2 exp(-\frac{(x-\mu)^2}{2\sigma^2}) dx$$

Let: $y = x - \mu$, $dy = dx$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} y^2 exp(-\frac{y^2}{2\sigma^2}) dy(3)$$

Let:

$$\begin{array}{ll} u = v & du = dv \\ dv = y * exp(\frac{-y^2}{2\sigma^2})dy & v = -\sigma^2 exp(\frac{-y^2}{2\sigma^2}) \end{array}$$

$$(3) = \frac{1}{\sqrt{2\pi\sigma^2}}([-\sigma^2 exp(\frac{-y^2}{2\sigma^2})]_{-\infty}^{\infty} + \sigma^2 \int_{-\infty}^{\infty} exp(\frac{-y^2}{2\sigma^2}dy))$$

$$= 0 + \sigma^2 * 1 = \sigma^2$$