STAT302: Time Series Analysis

Chapter 3. Time Series Regression

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Outline

Linear Models with Time Series

Harmonic Regression

Splines for Nonlinear Trend

Model Selection and Forecasting

Decompositon of time series

- Our general strategy is to decompose Y_t by non-stationary parts and stationary part (Wold decomposition, Doob-Meier decomposition).
- For example,

$$Y_t = T_t + S_t + R_t$$

  - $T_t =$ trend;
  - $S_t =$ seasonality with period d in the sense that $S_t = S_{t+d}$;
  - $R_t =$ weakly stationary errors

Decompositon of time series

- Thus, before estimating mean and covariance of R_t, we will first model/remove trend and seasonality.
- Four major methods are
  1. Regression
  2. Decomposition
  3. Smoothing (local regression)
  4. Differencing

Multiple linear regression

We may consider a time series regression:

$$Y_t = \beta_0 + \beta_1 X_{1,t} + \beta_2 X_{2,t} + \cdots + \beta_k X_{k,t} + \varepsilon_t.$$

- Y_t is the "response" variable
- Each $X_{j,t}$ is numerical and is called a "predictor". They are usually assumed to be known for all past and future times.
- The coefficients $\beta_1, \ldots, \beta_k$ measure the effect of each predictor after taking account of the effect of all other predictors in the model.
- That is, the coefficients measure the **marginal effects**.
- ε_t is a white noise error term.

The graph of the population data, which contains no apparent periodic component, suggests trying a model of the form

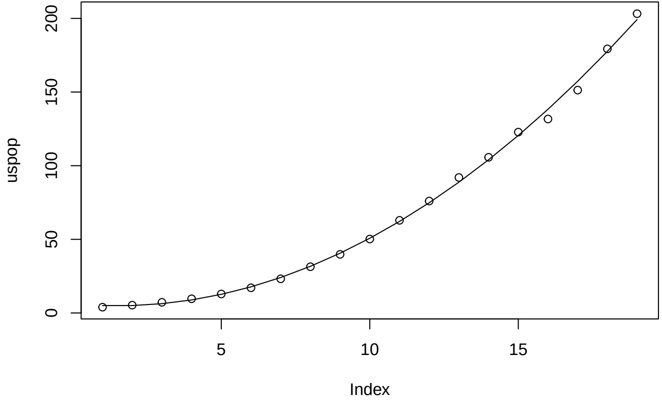
$$Y_t = T_t + R_t$$

with a 2nd-order polynormial regression

$$T_t = a_0 + a_1 t + a_2 t^2.$$

uspop data

```
uspop=as.numeric(uspop)
time=1:length(uspop)
fit=lm(uspop~time+I(time^2))
plot(uspop)
lines(predict(fit)~time)
```



Linear regression in matrix formulation

- Let $\boldsymbol{Y} = (Y_1, \ldots, Y_T)'$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_T)'$, and

$$\boldsymbol{X} = \begin{bmatrix} 1 & X_{1,1} & X_{2,1} & \ldots & X_{k,1} \\ 1 & X_{1,2} & X_{2,2} & \ldots & X_{k,2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & X_{1,T} & X_{2,T} & \ldots & X_{k,T} \end{bmatrix}.$$

- Then, the linear regression takes the form:

$$\boldsymbol{Y} = \boldsymbol{X}\beta + \boldsymbol{\varepsilon},$$

where $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_k)'$ is the regression coefficient parameter.

Least squares estimation (LSE)

- Ordinary least squares (OLS) estimation finds the coefficient β by minimizing the error sum of squares (SSE):

$$Q(\beta) = (\boldsymbol{Y} - \boldsymbol{X}\beta)'(\boldsymbol{Y} - \boldsymbol{X}\beta)$$

- Differentiating it wrt $\boldsymbol{\beta}$ gives the normal equation:

$$\boldsymbol{X}'(\boldsymbol{Y} - \boldsymbol{X}\beta) = 0,$$

which results in the least-squares estimator (LSE):

$$\hat{\beta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$$

- The variance can be estimated by

$$\hat{\sigma}^2 = \frac{1}{n-k-1}(\boldsymbol{Y} - \boldsymbol{X}\hat{\beta})'(\boldsymbol{Y} - \boldsymbol{X}\hat{\beta})$$

Maximum likelihood estimator (MLE)

- If the errors are iid and normally distributed, then

$$\boldsymbol{Y} \sim N(\boldsymbol{X}\beta, \sigma_\varepsilon^2 \boldsymbol{I}).$$

- The likelihood function is

$$L(\beta) = \frac{1}{\sigma^n (2\pi)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}(\boldsymbol{Y} - \boldsymbol{X}\beta)'(\boldsymbol{Y} - \boldsymbol{X}\beta)\right)$$

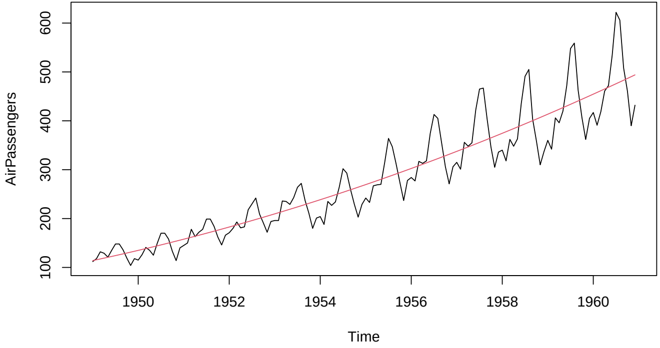
which is maximized when $Q(\beta)$ is minimized.
- So, **MLE = OLS** under the normality assumption.
- Moreover, $\hat{\beta}$ is asymptotically normally distributed in the sense:

$$\sqrt{n}(\hat{\beta} - \beta) \approx N(0, \sigma_\varepsilon^2 C), \quad C = (\boldsymbol{X}'\boldsymbol{X})^{-1}$$

AirPassengers data

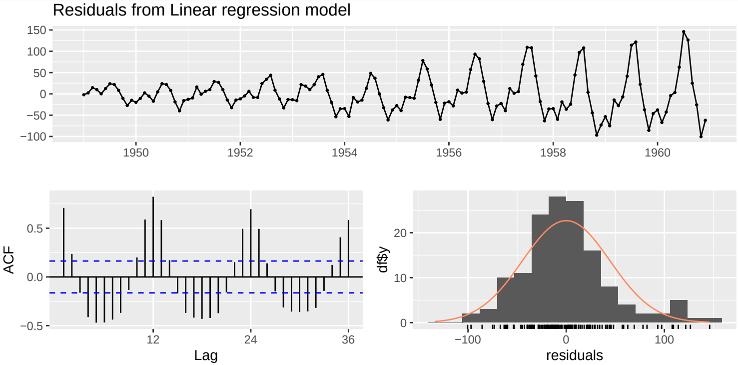
A naive regression approach, however, cannot afford to explain oscillations by seasonal effects and heterogeneity of variance.

```
library(forecast)
time=time(AirPassengers)
fit=tslm(AirPassengers~time+I(time^2))
ts.plot(AirPassengers)
lines(fitted(fit),col=2)
```



AirPassengers data

```r
library(fpp2)
checkresiduals(fit)
```

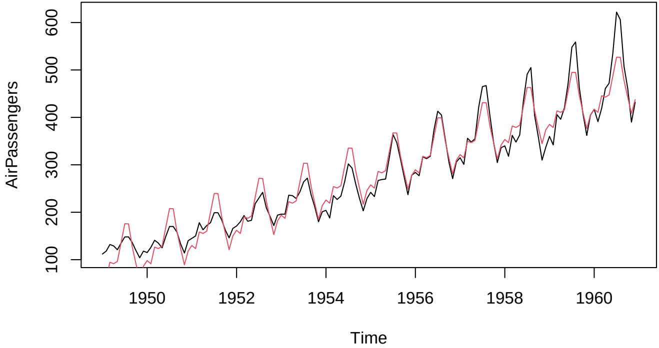



```
##
##  Breusch-Godfrey test for serial correlation of order up to 24
##
## data:  Residuals from Linear regression model
## LM test = 137.86, df = 24, p-value < 2.2e-16
```

AirPassengers data

When `month` is added in the model, the fit becomes slightly better.

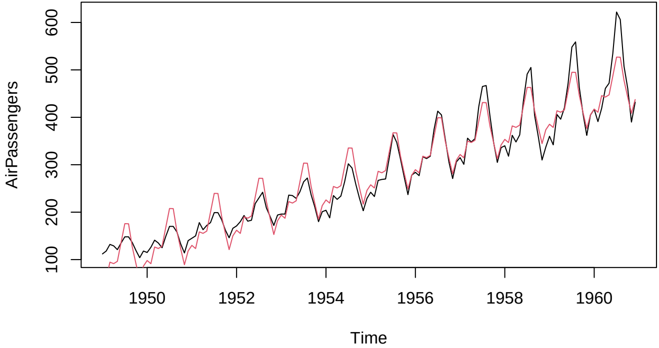
```
library(tidyverse)
month = AirPassengers %>% cycle %>% as.factor
fit=tslm(AirPassengers~time+month)
ts.plot(AirPassengers)
lines(fitted(fit),col=2)
```



AirPassengers data

- TS linear regression can be implemented by calling the `tslm` function in the `forecast` library.
- Here, `trend` is a time variable and `season` is a dummy variable for seasonal effect.

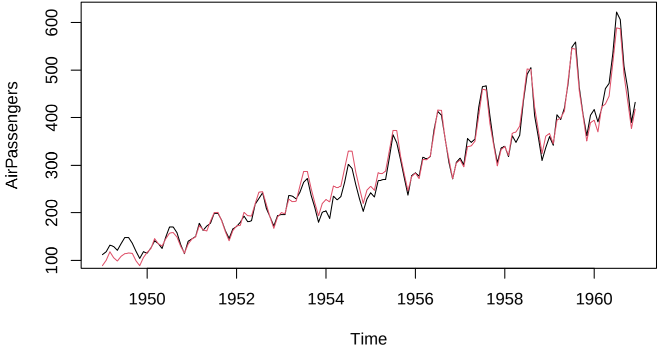
```
library(forecast)
fit=tslm(AirPassengers ~ trend+season)
ts.plot(AirPassengers)
lines(fitted(fit),col=2)
```



AirPassengers data

- Inclusion of the interaction term seems to improve the fit very much. Notice that additive model vs. multiplicative model.

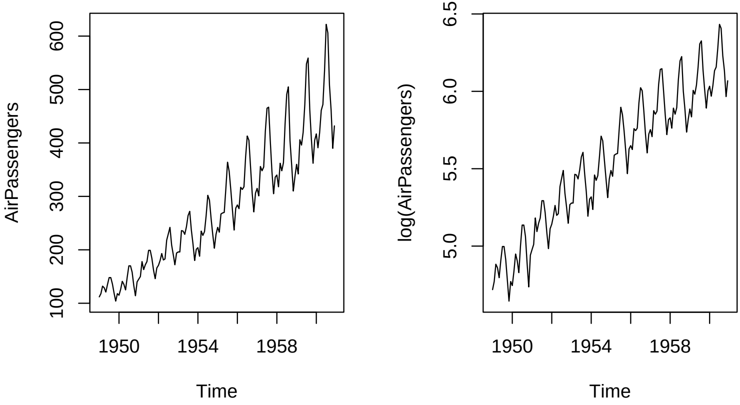
```
fit=tslm(AirPassengers ~ trend*season)
ts.plot(AirPassengers)
lines(fitted(fit),col=2)
```



Variance stabilization

- Sometimes, it is very helpful to take some transformation of the time series variable for variace stabilization.

```
par(mfrow=c(1,2))
ts.plot(AirPassengers)
ts.plot(log(AirPassengers))
```



Power transformation

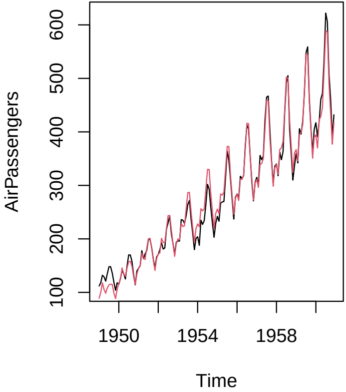
- **Box-Cox transformation** is a family of functions applied to create a monotonic transformation of data using power functions.

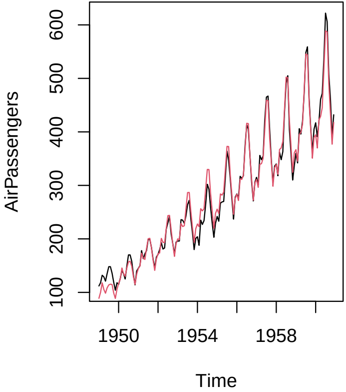
$$y^{(\lambda)} = \begin{cases} \frac{y^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \log(y) & \text{if } \lambda = 0 \end{cases}$$

- It is a data transformation technique used to stabilize variance, make the data more normal distribution-like, improve the validity of measures of association.
- You might take a log-transformation by setting `lambda=0`:

Power transformation

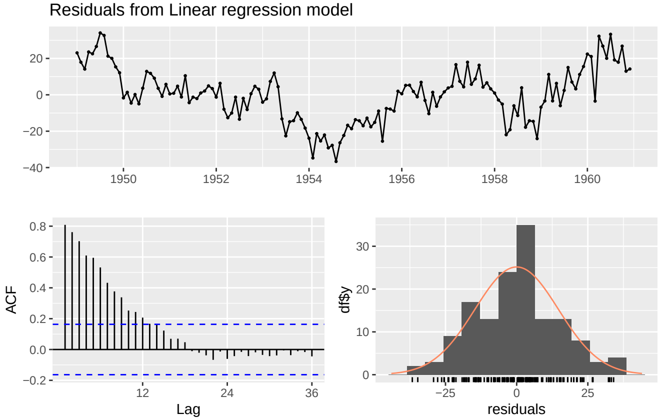
```
fit1=tslm(AirPassengers ~ trend*season)
lambda = BoxCox.lambda(AirPassengers)
fit2=tslm(AirPassengers ~ trend*season, lamabda = lambda)
par(mfrow=c(1,2))
ts.plot(AirPassengers)
lines(fitted(fit1),col=2)
ts.plot(AirPassengers)
lines(fitted(fit2),col=2)
```





Power transformation

```r
checkresiduals(fit2)
```




```
##
##  Breusch-Godfrey test for serial correlation of order up to 27
##
## data:  Residuals from Linear regression model
## LM test = 113.25, df = 27, p-value = 1.558e-12
```

US consumption data

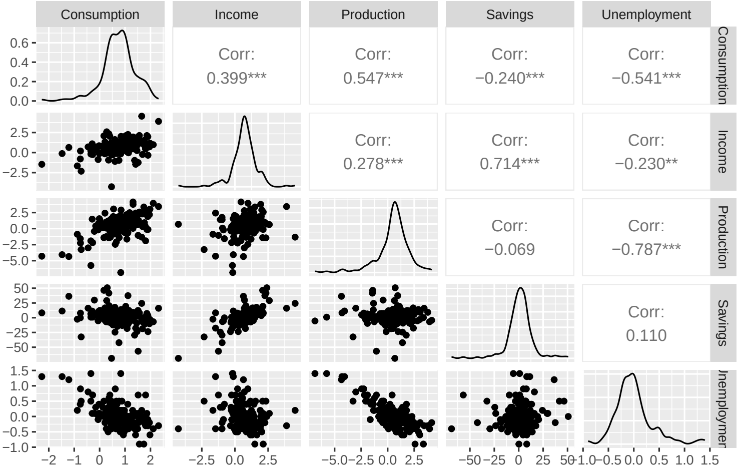
```r
library(fpp2)
autoplot(uschange[,c("Consumption","Income")]) +
  ylab("% change") + xlab("Year")
```

US consumption data

```
tslm(Consumption ~ Income, data=uschange) %>% summary
##
## Call:
## tslm(formula = Consumption ~ Income, data = uschange)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.40845 -0.31816  0.02558  0.29978  1.45157
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.54510    0.05569   9.789  < 2e-16 ***
## Income       0.28060    0.04744   5.915 1.58e-08 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6026 on 185 degrees of freedom
## Multiple R-squared:  0.159,  Adjusted R-squared:  0.1545
## F-statistic: 34.98 on 1 and 185 DF,  p-value: 1.577e-08
```

US consumption data

```r
uschange %>% as.data.frame %>% GGally::ggpairs()
```

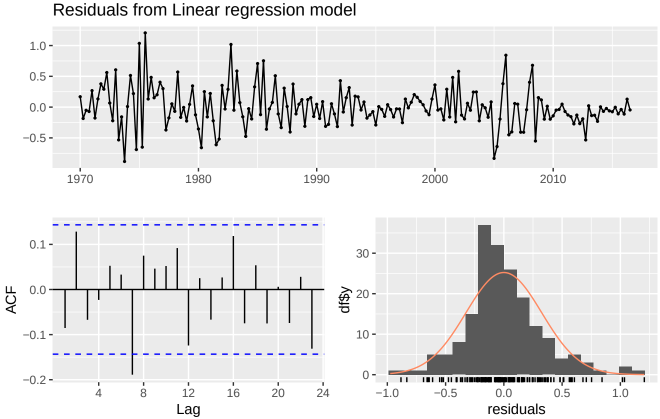


US consumption data

```r
fit.consMR <- tslm(
  Consumption ~ Income + Production + Unemployment + Savings, data=uschange)
summary(fit.consMR)
##
## Call:
## tslm(formula = Consumption ~ Income + Production + Unemployment +
##     Savings, data = uschange)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.88296 -0.17638 -0.03679  0.15251  1.20553
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)   0.26729    0.03721   7.184 1.68e-11 ***
## Income        0.71449    0.04219  16.934  < 2e-16 ***
## Production    0.04589    0.02588   1.773   0.0778 .
## Unemployment -0.20477    0.10550  -1.941   0.0538 .
## Savings      -0.04527    0.00278 -16.287  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.3286 on 182 degrees of freedom
## Multiple R-squared:  0.754,  Adjusted R-squared:  0.7486
## F-statistic: 139.5 on 4 and 182 DF,  p-value: < 2.2e-16
```

US consumption data

```r
checkresiduals(fit.consMR)
```




```
##
##  Breusch-Godfrey test for serial correlation of order up to 8
##
## data:  Residuals from Linear regression model
## LM test = 14.874, df = 8, p-value = 0.06163
```

Residual diagnostics

- For forecasting purposes, we require the following assumptions:
  - ε_t are uncorrelated and zero mean
  - ε_t are uncorrelated with each $x_{j,t}$.
- It is **useful** to also have $\varepsilon_t \sim N(0, \sigma^2)$ when producing prediction intervals or doing statistical tests.
- Useful for spotting outliers and whether the linear model was appropriate.
  - Scatterplot of residuals ε_t against each predictor $x_{j,t}$.
  - Scatterplot residuals against the fitted values $\hat{y}_t$
  - Expect to see scatterplots resembling a horizontal band with no values too far from the band and no patterns such as curvature or increasing spread.

Residual diagnostics

- If a plot of the residuals vs any predictor in the model shows a pattern, then the relationship is nonlinear.
- If a plot of the residuals vs any predictor **not** in the model shows a pattern, then the predictor should be added to the model.
- If a plot of the residuals vs fitted values shows a pattern, then there is heteroscedasticity in the errors. (Could try a transformation.)

Durbin-Watson statistic

- The **Durbin–Watson statistic** is a test statistic used to detect the presence of autocorrelation at lag 1 in the residuals (prediction errors) from a regression analysis.
- It tests $H_0 : \phi = 0$ in the AR(1) model: $e_t = \phi e_{t-1} + \nu_t$.
- If e_t is the residual, the Durbin-Watson test statistic is

$$d = \frac{\sum_{t=2}^{n}(e_t - e_{t-1})^2}{\sum_{t=1}^{n} e_t^2},$$

  where n is the number of observations.
- For large n, d is approximately equal to $2(1-\hat{\rho})$, where $\hat{\rho}$ is the sample autocorrelation of the residuals, $d = 2$ therefore indicates no autocorrelation.
- The value of d always lies between 0 and 4.

Durbin-Watson statistic

To test for positive autocorrelation at significance α, the test statistic d is compared to lower and upper critical values.

- If $d < d_{L,\alpha}$, there is statistical evidence that the error terms are positively autocorrelated.
- If $d > d_{U,\alpha}$, there is no statistical evidence that the error terms are positively autocorrelated.
- If $d_{L,\alpha} < d < d_{U,\alpha}$, the test is inconclusive.

Positive serial correlation is serial correlation in which a positive error for one observation increases the chances of a positive error for another observation.

Durbin-Watson statistic

To test for negative autocorrelation at significance α, the test statistic $(4 - d)$ is compared to lower and upper critical values:

- If $(4 - d) < d_{L,\alpha}$, there is statistical evidence that the error terms are negatively autocorrelated.
- If $(4 - d) > d_{U,\alpha}$, there is no statistical evidence that the error terms are negatively autocorrelated.
- If $d_{L,\alpha} < (4 - d) < d_{U,\alpha}$, the test is inconclusive.

Negative serial correlation implies that a positive error for one observation increases the chance of a negative error for another observation.

Ljung–Box and Breusch-Godfrey tests

- The Ljung–Box or Breusch-Godfrey test is a type of statistical test of whether any of a group of autocorrelations of a time series are different from zero.
- If R^2 statistic is calculated, then

$$(n-p)R^2 \sim \chi^2_p,$$

when there is no serial correlation up to lag p, and T, length of series.

- Breusch-Godfrey test better than Ljung-Box for regression models.

```
checkresiduals(fit.consMR, plot=FALSE)
##
##  Breusch-Godfrey test for serial correlation of order up to 8
##
## data:  Residuals from Linear regression model
## LM test = 14.874, df = 8, p-value = 0.06163
```

Outline

Harmonic regression

- Joseph Fourier (1768-1830) showed that

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \ldots\}$$

forms a basis for $L^2(-\pi, \pi]$, hence f in $L^2(-\pi, \pi]$ can be represented as

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

Harmonic regression

- Based on this theory, we will consider a finite order approximation of s_t:

$$s_t = a_0 + \sum_{j=1}^{k} \left(a_j \cos\left(\lambda_j t\right) + b_j \sin\left(\lambda_j t\right)\right)$$

where $a_0, a_1, ..., a_k$ and $b_1, ..., b_k$ are unknown parameters and $\lambda_1, ..., \lambda_k$ are fixed frequencies, each being some integer multiple of $2\pi/d$.

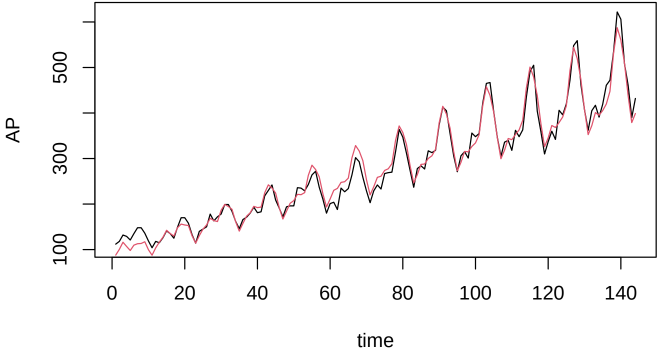
Harmonic regression

- Once k, the number of basis, and corresponding λ_j is selected, we can simply apply OLS to get estimates of coefficients.
- We will assume that k is known. Otherwise, as in the regression, you can apply variable selection to choose k. In practice $k = 1, ..., 4$.
- How to choose λ_j ?
  1. Set $f_1 = [n/d]$. This is a number of cycles that s_t repeated in the data. Take $f_j = jf_1$.
  2. $\lambda_j = f_j(2\pi/n)$
- For example if $n = 72$ and $d = 12$,

$$f_1 = [72/12] = 6, \quad \lambda_j = j \times 6 \times 2\pi/72$$

AirPassengers data

```r
L=12
AP = as.numeric(AirPassengers); time = 1:length(AP)
sin1=sin(2*pi*time/L); sin2=sin(2*pi*time/L*2); sin4=sin(2*pi*time/L*4)
cos1=cos(2*pi*time/L); cos2=cos(2*pi*time/L*2); cos4=cos(2*pi*time/L*4)
fit=lm(AP~time*(sin1+cos1+sin2+cos2+sin4+cos4))
plot(AP~time, type="l")
lines(time,predict(fit),col=2)
```




```r
fit=tslm(AirPassengers ~ trend * fourier(AirPassengers, K=2))
```

Outline

Linear Models with Time Series

Harmonic Regression

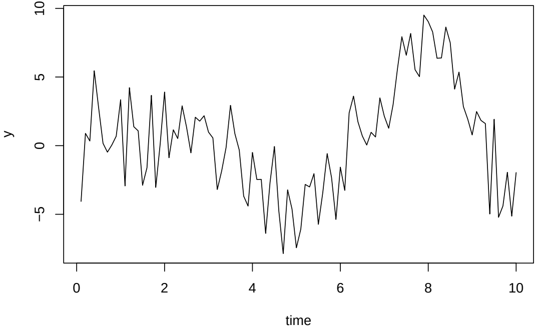
Splines for Nonlinear Trend

Model Selection and Forecasting

Simple spline examples

- Consider the following times series data with non-linear trend.

```
time = c(1:100)/10
y = time * sin(time) + rnorm(100,sd=2)
plot(y ~ time, type="l")
```

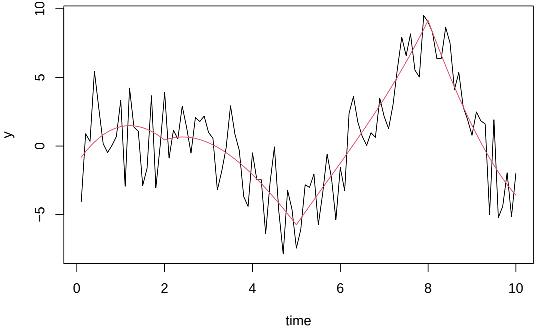


- Clearly, a linear regression is not a good choice to take off the non-linear trend.
- We may use a spline regression with (2, 5, 8) as knot points.

Simple spline examples

- We may use a linear spline regression with (2, 5, 8) as knot points.

```
x1 = time; x2 = time^2; x3 = time^3
z1 = pmax(time, 2); z2 = pmax(time, 5); z3 = pmax(time, 8)
fit = lm(y ~ x1+x2+x3+z1+z2+z3)
plot(y ~ time, type="l")
lines(time,predict(fit),col=2)
```



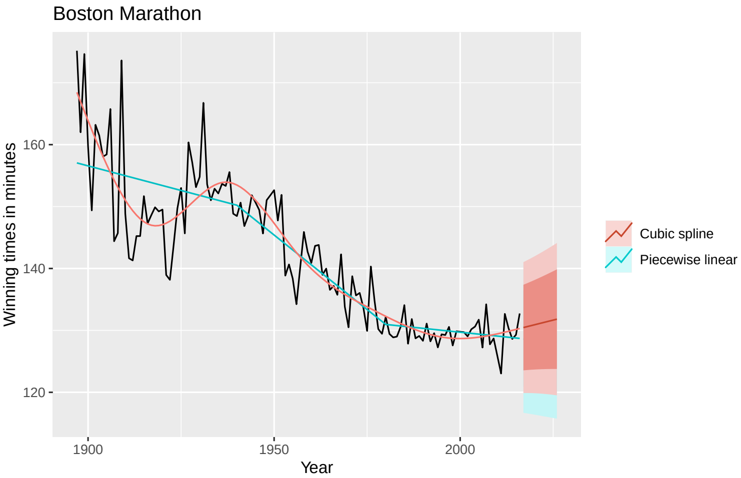
Interpolating splines for non-linear trend

- A spline is a continuous function $f(x)$ interpolating all points (κ_j, y_j) for $j = 1, \ldots, K$ and consisting of polynomials between each consecutive pair of 'knots' κ_j and κ_{j+1}.
- Parameters constrained so that $f(x)$ is continuous.
- Further constraints imposed to give continuous derivatives.
- For example, we can use a natural spline as follows:
  - Let $\kappa_1 < \kappa_2 < \cdots < \kappa_K$ be **knots** in interval (a, b).
  - Let $x_1 = x$, $x_j = (x - \kappa_{j-1})_+$ for $j = 2, \ldots, K+1$.
  - Then the regression is piecewise linear with bends at the knots.
  - Let $x_1 = x$, $x_2 = x^2$, $x_3 = x^3$, $x_j = (x - \kappa_{j-3})_+^3$ for $j = 4, \ldots, K+3$.
  - Then the regression is piecewise cubic, but smooth at the knots.

Boston marathon winning times

```
library(splines)
t <- time(marathon)
fit.splines <- lm(marathon ~ ns(t, df=6))
summary(fit.splines)
##
## Call:
## lm(formula = marathon ~ ns(t, df = 6))
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -13.0028  -2.5722   0.0122   2.1242  21.5681
##
## Coefficients:
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)      168.447      2.086  80.743  < 2e-16 ***
## ns(t, df = 6)1    -6.948      2.688  -2.584    0.011 *
## ns(t, df = 6)2   -28.856      3.416  -8.448 1.16e-13 ***
## ns(t, df = 6)3   -35.081      3.045 -11.522  < 2e-16 ***
## ns(t, df = 6)4   -32.563      2.652 -12.279  < 2e-16 ***
## ns(t, df = 6)5   -64.847      5.322 -12.184  < 2e-16 ***
## ns(t, df = 6)6   -21.002      2.403  -8.741 2.46e-14 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
```

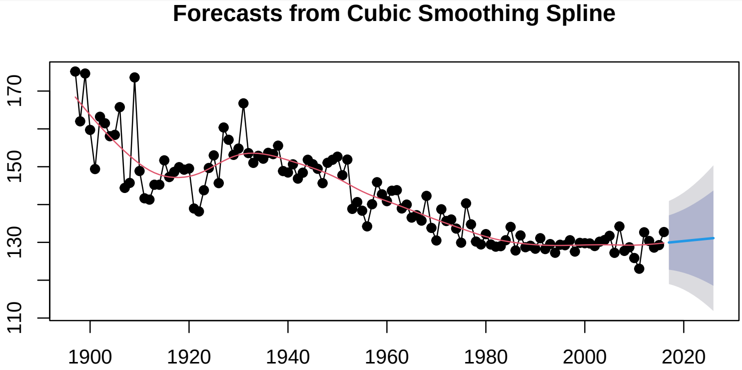
Boston marathon winning times



Spline forecasting with `splinef`

A slightly different type of spline is provided by `splinef`

```
fc = splinef(marathon)
plot(fc)
```



Spline forecasting with `splinef`

- Cubic **smoothing** splines (rather than cubic regression splines).
- Still piecewise cubic, but with many more knots (one at each observation).
- Coefficients constrained to prevent the curve becoming too "wiggly".
- Degrees of freedom selected automatically.
- Equivalent to ARIMA(0,2,2) and Holt's method.

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Comparing regression models

Computer output for regression will always give the R^2 value. This is a useful summary of the model.

- It is equal to the square of the correlation between y and $\hat{y}$.
- It is often called the "coefficient of determination".
- It can also be calculated as follows:

$$R^2 = \frac{\sum(\hat{y}_t - \bar{y})^2}{\sum(y_t - \bar{y})^2}$$

- It is the proportion of variance accounted for (explained) by the predictors.

Comparing regression models

However,

- R^2 does not allow for "degrees of freedom".
- Adding *any* variable tends to increase the value of R^2, even if that variable is irrelevant.

To overcome this problem, we can use *adjusted* R^2:

$$\bar{R}^2 = 1 - (1 - R^2)\frac{n-1}{n-k-1}$$

where $k =$ no. predictors and $n =$ no. observations.

Cross-validation (CV)

Cross-validation for regression

(Assuming future predictors are known)

- Select one observation for test set, and use *remaining* observations in training set. Compute error on test observation.
- Repeat using each possible observation as the test set.
- Compute accuracy measure over all errors.

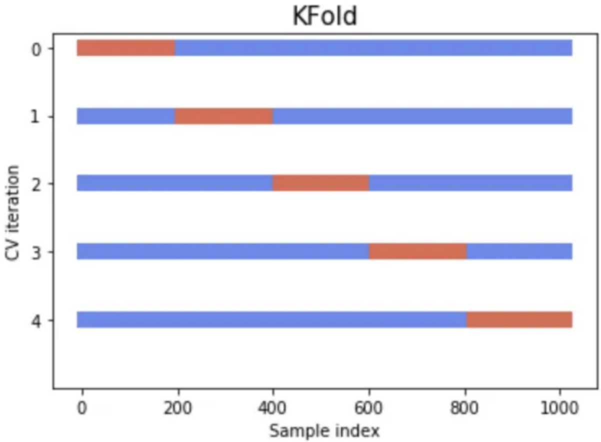
Cross-validation (CV)

Cross-validation:

1. Split randomly data in train and test set.
2. Focus on train set and split it again randomly in chunks (called folds).
3. Let's say you got 5 folds; train on 4 of them and test on the 5th.
4. Repeat step three 5 times to get 5 accuracy measures on 5 different and separate folds.
5. Compute the average of the 5 accuracies which is the final reliable number telling us how the model is performing.

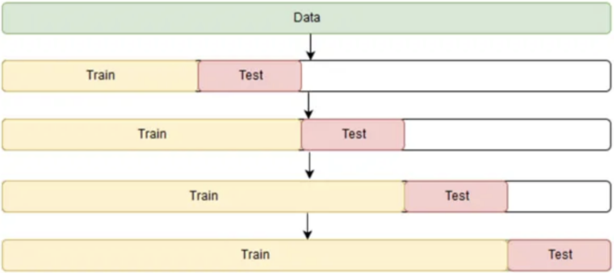
The best model is the one with minimum CV.

Conventional CV



Time series CV

- In the case of time series, the cross-validation is not trivial.
- We may use cross-validation on a time-rolling basis.



Akaike's Information Criterion (AIC)

$$\text{AIC} = -2\log(L) + 2(k+2)$$

where L is the likelihood and k is the number of predictors in the model.

- This is a *penalized likelihood* approach.
- *Minimizing* the AIC gives the best model for prediction.
- AIC penalizes terms more heavily than $\bar{R}^2$.
- Minimizing the AIC is asymptotically equivalent to minimizing MSE via leave-one-out cross-validation.

Corrected AIC (AICc)

For small values of n, the AIC tends to select too many predictors, and so a bias-corrected version of the AIC has been developed.

$$\text{AIC}_{\text{C}} = \text{AIC} + \frac{2(k+2)(k+3)}{n-k-3}$$

As with the AIC, the AIC_{C} should be minimized.

Bayesian Information Criterion (BIC)

$$\text{BIC} = -2\log(L) + (k+2)\log(n)$$

where L is the likelihood and k is the number of predictors in the model.

- BIC penalizes terms more heavily than AIC
- Also called SBIC and SC.
- Minimizing BIC is asymptotically equivalent to leave-v-out cross-validation when $v = n[1 - 1/(\log(n) - 1)]$.

Choosing informative regression variables

Best subsets regression

- Fit all possible regression models using one or more of the predictors.
- Choose the best model based on one of the measures of predictive ability (CV, AIC, AICc).

Warning!

- If there are a large number of predictors, this is not possible.
- For example, 44 predictors leads to 18 trillion possible models!

Performance metrics

We may also consider the mean square error (MSE), root-mean-square error (RMSE), and mean absolute percentage error (MAPE), to evaluate the model's performance:

$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}$$

$$\text{MAPE}(\%) = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{y_i - \hat{y}_i}{y_i}\right| \times 100$$

Model selection

```
tslm(Consumption ~ Income + Production + Unemployment + Savings,
  data=uschange) %>% CV()
##          CV          AIC         AICc          BIC        AdjR2
##   0.1163477 -409.2980298 -408.8313631 -389.9113781    0.7485856
tslm(Consumption ~ Income + Production + Unemployment,
  data=uschange) %>% CV()
##          CV          AIC         AICc          BIC        AdjR2
##   0.2776928 -243.1635677 -242.8320760 -227.0080246    0.3855438
tslm(Consumption ~ Income + Production + Savings,
  data=uschange) %>% CV()
##          CV          AIC         AICc          BIC        AdjR2
##   0.1178681 -407.4669279 -407.1354362 -391.3113848    0.7447840
tslm(Consumption ~ Income + Unemployment + Savings,
  data=uschange) %>% CV()
##          CV          AIC         AICc          BIC        AdjR2
##   0.1160223 -408.0941325 -407.7626408 -391.9385894    0.7456386
tslm(Consumption ~ Production + Unemployment + Savings,
  data=uschange) %>% CV()
##          CV          AIC         AICc          BIC        AdjR2
##   0.2927095 -234.3734580 -234.0419663 -218.2179149    0.3559711
```

Building a predictive regression model

- Assumes possible scenarios for the predictor variables
- Prediction intervals for scenario based forecasts do not include the uncertainty associated with the future values of the predictor variables.
- If getting forecasts of predictors is difficult, you can use lagged predictors instead.

$$Y_t = \beta_0 + \beta_1 X_{1,t-h} + \cdots + \beta_k X_{k,t-h} + \varepsilon_t$$

- A different model for each forecast horizon h.

Regression forecasting

- Optimal forecasts:

$$\hat{y}^* = \mathrm{E}(y^*|\boldsymbol{Y}, \boldsymbol{X}, \boldsymbol{x}^*) = \boldsymbol{x}^*\hat{\beta} = \boldsymbol{x}^*(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$$

  where $\boldsymbol{x}^*$ is a row vector containing the values of the predictors for the forecasts (in the same format as $\boldsymbol{X}$).
- Forecast variance:

$$\mathrm{Var}(y^*|\boldsymbol{X}, \boldsymbol{x}^*) = \sigma^2\left[1 + \boldsymbol{x}^*(\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{x}^*)'\right]$$

  - This ignores any errors in $\boldsymbol{x}^*$.
  - 95% prediction intervals assuming normal errors:

$$\hat{y}^* \pm 1.96\sqrt{\mathrm{Var}(y^*|\boldsymbol{X}, \boldsymbol{x}^*)}.$$

Regression forecasting

- Fitted values:

$$\hat{\boldsymbol{Y}} = \boldsymbol{X}\hat{\beta} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} = \boldsymbol{H}\boldsymbol{Y}$$

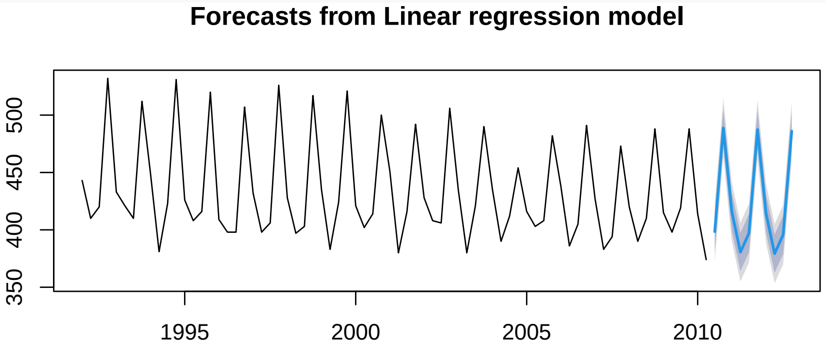
  where $\boldsymbol{H} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ is the "hat matrix"
- **Leave-one-out residuals**
- Let $h_1, \ldots, h_n$ be the diagonal values of $\boldsymbol{H}$, then the cross-validation statistic is

$$\text{CV} = \frac{1}{n}\sum_{t=1}^{n}[e_t/(1-h_t)]^2,$$

  where e_t is the residual obtained from fitting the model to all n observations.

Beer production data

```r
beer2 = window(ausbeer, start=1992)
fit.beer = tslm(beer2 ~ trend + season)
fcast = forecast(fit.beer)
plot(fcast)
```

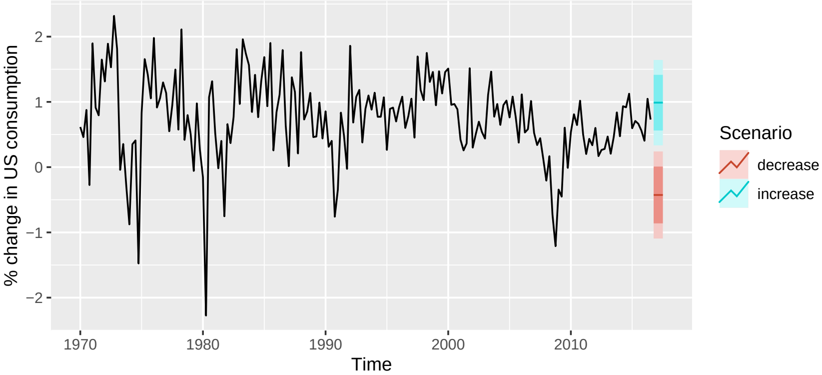


US consumption data

```r
fit.consBest <- tslm(
  Consumption ~ Income + Savings + Unemployment,
  data = uschange)
h <- 4
newdata <- data.frame(
    Income = c(1, 1, 1, 1),
    Savings = c(0.5, 0.5, 0.5, 0.5),
    Unemployment = c(0, 0, 0, 0))
fcast.up <- forecast(fit.consBest, newdata = newdata)
newdata <- data.frame(
    Income = rep(-1, h),
    Savings = rep(-0.5, h),
    Unemployment = rep(0, h))
fcast.down <- forecast(fit.consBest, newdata = newdata)
```

US consumption data

```r
autoplot(uschange[, 1]) +
  ylab("% change in US consumption") +
  autolayer(fcast.up, PI = TRUE, series = "increase") +
  autolayer(fcast.down, PI = TRUE, series = "decrease") +
  guides(colour = guide_legend(title = "Scenario"))
```



Correlation is not causation

- When X is useful for predicting Y, it is not necessarily causing Y.
- e.g., predict number of drownings Y using number of ice-creams sold X.
- Correlations are useful for forecasting, even when there is no causality.
- Better models usually involve causal relationships (e.g., temperature X and people Z to predict drownings Y).

Multicollinearity

In regression analysis, multicollinearity occurs when:

- Two predictors are highly correlated (i.e., the correlation between them is close to ± 1).
- A linear combination of some of the predictors is highly correlated with another predictor.
- A linear combination of one subset of predictors is highly correlated with a linear combination of another subset of predictors.

Multicollinearity

If multicollinearity exists,

- the numerical estimates of coefficients may be wrong (worse in Excel than in a statistics package)
- don't rely on the p-values to determine significance.
- there is no problem with model *predictions* provided the predictors used for forecasting are within the range used for fitting.
- omitting variables can help.
- combining variables can help.

Outliers and influential observations

Things to watch for

- *Outliers*: observations that produce large residuals.
- *Influential observations*: removing them would markedly change the coefficients. (Often outliers in the X variable).
- *Lurking variable*: a predictor not included in the regression but which has an important effect on the response.
- Points should not normally be removed without a good explanation of why they are different.