# Chapter I.1

# The electromagnetic field

## I.1.1 The electromagnetic four-tensor

The electromagnetic four tensor is, in terms of the electric/magnetic field

$$F^{\mu\nu} = \begin{bmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{bmatrix} \tag{I.1.1}$$

Some constants in atomic units are

$$\hbar = 1, \quad e = -1, \quad c = 137.036, \quad m = 1 \tag{I.1.2}$$

## I.1.2 The plane wave

### I.1.2.1 The field parameters

The field parameters are

1. $\omega$ = frequency. Typical values are $\omega = 0.05$ au
2. $n = (1, \boldsymbol{n})$ the propagation direction four-vector. Typically, $\boldsymbol{n} = \boldsymbol{e}_z$, i.e. the field propagates along $Oz$
3. $k = \frac{\omega}{c} n$
4. $s$ the polarization four vector, $s \cdot n = 0$, $s^* \cdot s = -1$. Typically $s = (0, \boldsymbol{s})$, $\quad \boldsymbol{s} \cdot \boldsymbol{s}^* = 1, \quad \boldsymbol{s} \cdot \boldsymbol{n} = 0$. The choice for propagation direction along $Oz$ is $s = (\zeta_x, \zeta_y, 0)$ with $|\zeta_x|^2 + |\zeta_y|^2 = 1$
5. the dimensionless intensity parameter $\xi = \frac{|e| A_0}{mc}$.

### I.1.2.2 The electric field

$$\boldsymbol{E}(x) = E_0 \Re\mathfrak{e}\{ \boldsymbol{s} e^{-i\phi - \frac{(\phi - \tau_0)^2}{\tau^2}} \}, \quad \phi \equiv k \cdot x = \omega t - \frac{\omega}{c} \boldsymbol{n} \cdot \boldsymbol{x} \tag{I.1.3}$$

$\frac{\tau}{2\pi}$ is the length of the pulse measured in optical periods $T = \frac{2\pi}{\omega}$

The Cartesian components of the electric field are

$$E_x(\boldsymbol{r},t) = E_0\,\Re\mathfrak{e}\left\{\zeta_x f(\phi)\right\}, \quad f(\phi) = e^{-i\phi-\frac{(\phi-\tau_0)^2}{\tau^2}}, \quad \phi = \omega t - kz \tag{I.1.4}$$

$$E_y(\boldsymbol{r},t) = E_0\,\Re\mathfrak{e}\left\{\zeta_y f(\phi)\right\} \tag{I.1.5}$$

$$E_z(\boldsymbol{r},t) = 0 \tag{I.1.6}$$

### I.1.2.3 The magnetic field

The magnetic field is

$$\boldsymbol{B}(x) = \frac{1}{c}\boldsymbol{n}\times\boldsymbol{E}(x) \tag{I.1.7}$$

The Cartesian components of the magnetic field are

$$B_x(\boldsymbol{r},t) = \frac{E_0}{c}\,\Re\mathfrak{e}\left\{-\zeta_y f(\phi)\right\}, \quad f(\phi) = e^{-i\phi-\frac{(\phi-\tau_{})^2}{\tau^2}}, \quad \phi = \omega t - kz \tag{I.1.8}$$

$$B_y(\boldsymbol{r},t) = \frac{E_0}{c}\,\Re\mathfrak{e}\left\{\zeta_x f(\phi)\right\} \tag{I.1.9}$$

$$B_z(\boldsymbol{r},t) = 0 \tag{I.1.10}$$

## I.1.3 The Laguerre Gauss modes

### I.1.3.1 The field parameters

1. the dimensionless intensity parameter $a_0$; the amplitude of the electric field is $E_0 = \omega\xi\,\frac{mc}{|e|}$
2. propagation direction only along $Oz$ axis. $\boldsymbol{n} = \boldsymbol{e}_z$
3. $\omega$ = frequency; $T = \frac{2\pi}{\omega}$, $k = \frac{\omega}{c}$, $\lambda = cT$
4. $w_0$ = the beam waist
5. $z_R = \frac{kw_0^2}{2}$ = the Rayleigh length
6. $w(z) = w_0\sqrt{1+\frac{z^2}{z_R^2}}$
7. $\psi_G(z) = \arctan\frac{z}{z_R}$
8. $\epsilon = \pm 1$ helicity sign
9. $\zeta_x$, $\zeta_y$ polarization parameters $|\zeta_x|^2 + |\zeta_y|^2 = 1$

### I.1.3.2 The scalar solution

Define the scalar solution

$$u_{pm}(\boldsymbol{r}) = N_{pm}\frac{w_0}{w(z)}\left(\frac{\sqrt{2}}{w(z)}\right)^m e^{-i(2p+m+1)\psi_G(z)}e^{-\frac{k}{2}\frac{x^2+y^2}{z_R+iz}}\,{}_1F_1(-p,m+1,\frac{2(x^2+y^2)}{w(z)^2})(x+i\epsilon y)^m \tag{I.1.11}$$

The normalization constant is

$$N_{pm} = \frac{\sqrt{2}}{m!}\sqrt{\frac{(p+m)!}{p!}} \tag{I.1.12}$$

The derivatives

$$\begin{aligned}\frac{du_{pm}(\boldsymbol{r})}{dx} =& N_{pm}\frac{w_0}{w(z)}\left(\frac{\sqrt{2}}{w(z)}\right)^m e^{-i(2p+m+1)\psi_G(z)}e^{-\frac{k}{2}\frac{x^2+y^2}{z_R+iz}}\\ &\times\left[{}_1F_1(-p,m+1,\frac{2(x^2+y^2)}{w(z)^2})\left(m(x+i\epsilon y)^{m-1}-(x+i\epsilon y)^m\frac{kx}{z_R+iz}\right)\right.\\ &\left.-\frac{p}{m+1}\frac{4x}{w(z)^2}\,{}_1F_1(-p+1,m+2,\frac{2(x^2+y^2)}{w(z)^2})\right]\end{aligned} \tag{I.1.13}$$

$$\begin{aligned}\frac{du_{pm}(\boldsymbol{r})}{dy} =& N_{pm}\frac{w_0}{w(z)}\left(\frac{\sqrt{2}}{w(z)}\right)^m e^{-i(2p+m+1)\psi_G(z)}e^{-\frac{k}{2}\frac{x^2+y^2}{z_R+iz}}\\ &\times\left[{}_1F_1(-p,m+1,\frac{2(x^2+y^2)}{w(z)^2})\left(im\epsilon(x+i\epsilon y)^{m-1}-(x+i\epsilon y)^m\frac{ky}{z_R+iz}\right)\right.\\ &\left.-\frac{p}{m+1}\frac{4y}{w(z)^2}\,{}_1F_1(-p+1,m+2,\frac{2(x^2+y^2)}{w(z)^2})\right]\end{aligned} \tag{I.1.14}$$

### I.1.3.3 The electric field

The Cartesian components of the electric field are

$$E_x(\boldsymbol{r},t) = E_0\,\mathfrak{Re}\left\{\zeta_x u_{pm}(\boldsymbol{r})f(\phi)\right\},\quad f(\phi)=e^{-i\phi-\frac{(\phi-\tau_0)^2}{\tau^2}},\quad \phi=\omega t-kz \tag{I.1.15}$$

$$E_y(\boldsymbol{r},t) = E_0\,\mathfrak{Re}\left\{\zeta_y u_{pm}(\boldsymbol{r})f(\phi)\right\} \tag{I.1.16}$$

$$E_z(\boldsymbol{r},t) = E_0\,\mathfrak{Re}\left\{-\frac{1}{ik}\left[\zeta_x\frac{\partial u_{pm}(\boldsymbol{r})}{\partial x}+\zeta_y\frac{\partial u_{pm}(\boldsymbol{r})}{\partial y}\right]f(\phi)\right\} \tag{I.1.17}$$

### I.1.3.4 The magnetic field

The Cartesian components of the magnetic field are

$$B_x(\boldsymbol{r},t) = \frac{E_0}{c}\,\mathfrak{Re}\left\{-\zeta_y u_{pm}(\boldsymbol{r})f(\phi)\right\},\quad f(\phi)=e^{-i\phi-\frac{(\phi-\tau)^2}{\tau^2}},\quad \phi=\omega t-kz \tag{I.1.18}$$

$$B_y(\boldsymbol{r},t) = \frac{E_0}{c}\,\mathfrak{Re}\left\{\zeta_x u_{pm}(\boldsymbol{r})f(\phi)\right\} \tag{I.1.19}$$

$$B_z(\boldsymbol{r},t) = \frac{E_0}{c}\,\mathfrak{Re}\left\{-\frac{1}{ik}\left[-\zeta_y\frac{\partial u_{pm}(\boldsymbol{r})}{\partial x}+\zeta_x\frac{\partial u_{pm}(\boldsymbol{r})}{\partial y}\right]f(\phi)\right\} \tag{I.1.20}$$

# Chapter I.2

# The classical motion of a charged particle in the electromagnetic field

## I.2.1 The relation between covariant and non-covariant velocity/acceleration

### I.2.1.1 The covariant kinematics

The independent variable is the proper time of the particle $\tau$

1. the trajectory

$$r(\tau) \equiv (ct(\tau), x(\tau), y(\tau), z(\tau)) \tag{I.2.1}$$

2. the 4-velocity

$$u(\tau) = \frac{dr(\tau)}{d\tau}, \quad u \cdot u = c^2 \tag{I.2.2}$$

3. the 4-acceleration

$$\rho(\tau) = \frac{du(\tau)}{d\tau}, \quad \rho \cdot u = 0 \tag{I.2.3}$$

### I.2.1.2 The non-covariant kinematics

The independent variable is the laboratory time $t$. The relation between the proper time $\tau$ and the laboratory time $t$ is

$$dt = \gamma d\tau, \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{u^0}{c}, \quad \frac{d}{dt} = \frac{1}{\gamma}\frac{d}{dt} \tag{I.2.4}$$

1. the trajectory

$$\boldsymbol{r}(t) \equiv (x(t), y(t), z(t)) \tag{I.2.5}$$

2. the velocity

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}(t)}{dt}, \quad \boldsymbol{v}(t) = \frac{\boldsymbol{u}}{\gamma} = c\frac{\boldsymbol{u}}{u^0}, \quad \boldsymbol{\beta} = \frac{\boldsymbol{v}}{c} = \frac{\boldsymbol{u}}{u^0} \tag{I.2.6}$$

3. the acceleration

$$\boldsymbol{a}(t) = \frac{d\boldsymbol{v}(t)}{dt} = \frac{d}{dt}\frac{\boldsymbol{u}}{\gamma} = \frac{1}{\gamma}\frac{d\boldsymbol{u}}{dt} - \boldsymbol{u}\frac{1}{\gamma^2}\frac{d\gamma}{dt} = \frac{1}{\gamma^2}\frac{d\boldsymbol{u}}{d\tau} - \boldsymbol{u}\frac{1}{\gamma^3}\frac{d\gamma}{d\tau} = \frac{1}{\gamma^2}\boldsymbol{\rho} - \boldsymbol{u}\frac{1}{\gamma^3}\frac{d\gamma}{d\tau} \tag{I.2.7}$$

From

$$\gamma = \frac{1}{c}\sqrt{c^2 + \boldsymbol{u}^2} \quad \longrightarrow \quad \frac{d\gamma}{d\tau} = \frac{1}{c\sqrt{c^2+\boldsymbol{u}^2}}\boldsymbol{u}\cdot\frac{d\boldsymbol{u}}{d\tau} = \frac{1}{c^2\gamma}\boldsymbol{u}\cdot\boldsymbol{\rho} \tag{I.2.8}$$

and the acceleration is

$$\boldsymbol{a}(t) = \frac{1}{\gamma^2}\boldsymbol{\rho} - \frac{1}{c^2\gamma^4}(\boldsymbol{u}\cdot\boldsymbol{\rho})\boldsymbol{u} = \frac{1}{\gamma^2}\left(\boldsymbol{\rho} - \frac{1}{c^2}(\boldsymbol{v}\cdot\boldsymbol{\rho})\boldsymbol{v}\right) = \frac{1}{\gamma^2}\left(\boldsymbol{\rho} - (\boldsymbol{\beta}\cdot\boldsymbol{\rho})\boldsymbol{\beta}\right) \tag{I.2.9}$$

$$\dot{\boldsymbol{\beta}}(t) \equiv \frac{d\boldsymbol{\beta}(t)}{dt} = \frac{1}{\gamma^2 c}\left(\boldsymbol{\rho} - (\boldsymbol{\beta}\cdot\boldsymbol{\rho})\boldsymbol{\beta}\right) \tag{I.2.10}$$

## I.2.2 The covariant equations of motion

The equation of motion of a charged particle in the field is

$$\frac{dx^\mu}{d\tau} = u^\mu, \quad \frac{du^\mu}{d\tau} = \frac{e}{m}F^{\mu\nu}u_\nu \tag{I.2.11}$$

where $\tau$ is the proper time.

## I.2.3 The non-covariant equations

The derivatives of the coordinates are

$$\frac{d\boldsymbol{x}}{dt} = \frac{1}{\gamma}d\boldsymbol{x}d\tau = c\frac{\boldsymbol{u}}{u^0} = \boldsymbol{v} = c\boldsymbol{\beta} \tag{I.2.12}$$

Using the explicit form of $F$ we obtain

$$m\frac{d}{dt}(\gamma\boldsymbol{v}) = e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) \tag{I.2.13}$$

i.e.

$$m\frac{d\gamma}{dt}\boldsymbol{v} + m\gamma\frac{d\boldsymbol{v}}{dt} = e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) \tag{I.2.14}$$

We multiply the entire equation by $\boldsymbol{v}$ and we get

$$m\frac{d\gamma}{dt}\boldsymbol{v}^2 + m\gamma\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt} = e\boldsymbol{E}\cdot\boldsymbol{v}, \quad m\frac{d\gamma}{dt}\boldsymbol{v}^2 + m\gamma\frac{1}{2}\frac{d\boldsymbol{v}^2}{dt} = e\boldsymbol{E}\cdot\boldsymbol{v}, \tag{I.2.15}$$

Using the definition of $\gamma$

$$\gamma = \frac{1}{\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}}, \quad \longrightarrow \quad \frac{d\gamma}{dt} = \frac{1}{2}\frac{\gamma^3}{c^2}\frac{d\boldsymbol{v}^2}{dt} \quad \longrightarrow \quad \frac{d\boldsymbol{v}^2}{dt} = \frac{2c^2}{\gamma^3}\frac{d\gamma}{dt}, \quad \boldsymbol{v}^2 = c^2\frac{\gamma^2-1}{\gamma^2} \tag{I.2.16}$$

Finally, we have

$$m\frac{d\gamma}{dt}c^2\frac{\gamma^2-1}{\gamma^2} + m\gamma\frac{1}{2}\frac{2c^2}{\gamma^3}\frac{d\gamma}{dt} = e\boldsymbol{E}\cdot\boldsymbol{v}, \quad mc^2\frac{d\gamma}{dt} = e\boldsymbol{E}\cdot\boldsymbol{v} \tag{I.2.17}$$

and the second equation of motion becomes

$$\frac{e\boldsymbol{E}\cdot\boldsymbol{v}}{c^2}\boldsymbol{v} + m\gamma\frac{d\boldsymbol{v}}{dt} = e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) \tag{I.2.18}$$

i.e.

$$m\gamma\frac{d\boldsymbol{v}}{dt} = e\left(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} - \frac{1}{c^2}(\boldsymbol{E}\cdot\boldsymbol{v})\boldsymbol{v}\right) \tag{I.2.19}$$

The system of 6 equations that has to be solved numerically is

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v} \tag{I.2.20}$$

$$\frac{d\boldsymbol{v}}{dt} = \frac{e}{m\gamma}\left(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} - \frac{1}{c^2}(\boldsymbol{E}\cdot\boldsymbol{v})\boldsymbol{v}\right), \quad \gamma = \frac{1}{\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}} \tag{I.2.21}$$

# Chapter I.3

# The ponderomotive force theory

## I.3.1 Equations of motion

Consider the relativistic equations of motion for a particle with a time and coordinate dependent mass $\mathcal{M}(\boldsymbol{x}, t)$ The covariant Lagrange function will be

$$L_{(\tau)} = -\mathcal{M}(x)c\sqrt{\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau}} = -\mathcal{M}(x)c\sqrt{\dot{x}_\mu \dot{x}^\mu} \tag{I.3.1}$$

The equations of motion are obtained from

$$\frac{\partial L_{(\tau)}}{\partial \dot{x}_\mu} = -\mathcal{M}(x)c\frac{\dot{x}^\mu}{\sqrt{\dot{x}_\mu \dot{x}^\mu}} \quad \frac{\partial L_{(\tau)}}{\partial x_\mu} = -\frac{\partial \mathcal{M}(x)}{\partial x_\mu}c\sqrt{\dot{x}_\mu \dot{x}^\mu}, \tag{I.3.2}$$

$$\frac{d}{d\tau}\left(-\mathcal{M}(x)c\frac{\dot{x}^\mu}{\sqrt{\dot{x}_\mu \dot{x}^\mu}}\right) = -\frac{\partial \mathcal{M}(x)}{\partial x_\mu}c\sqrt{\dot{x}_\mu \dot{x}^\mu} \tag{I.3.3}$$

If we assume that

$$\dot{x}_\mu \dot{x}^\mu = c^2 \tag{I.3.4}$$

as it should be then the equations of motion are

$$\frac{d}{d\tau}\left(\mathcal{M}(x)\dot{x}^\mu\right) = c^2\frac{\partial \mathcal{M}(x)}{\partial x_\mu} \tag{I.3.5}$$

The expression of the four acceleration $\frac{du}{d\tau}$ will be

$$\mathcal{M}(x)\frac{d\dot{x}^\mu}{d\tau} = c^2\frac{\partial \mathcal{M}(x)}{\partial x_\mu} - \dot{x}^\mu\frac{d\mathcal{M}(x)}{d\tau} = c^2\frac{\partial \mathcal{M}(x)}{\partial x_\mu} - \dot{x}^\mu \dot{x}^\nu\frac{\partial \mathcal{M}(x)}{\partial x_\nu} = \left(c^2 g^{\mu\nu} - \dot{x}^\mu \dot{x}^\nu\right)\frac{\partial \mathcal{M}(x)}{\partial x^\nu} \tag{I.3.6}$$

The covariant form of the equations of motion

$$\boxed{\frac{dx^\mu}{d\tau} = u^\mu, \quad \frac{du^\mu}{d\tau} = \frac{1}{\mathcal{M}(x)}\left(c^2 g^{\mu\nu} - \dot{x}^\mu \dot{x}^\nu\right)\frac{\partial \mathcal{M}(x)}{\partial x^\nu}} \tag{I.3.7}$$

To check the correctness of the above formula we shoud verify that

$$u \cdot \frac{du}{d\tau} = 0 \tag{I.3.8}$$

In our case,

$$u_\mu \frac{du^\mu}{d\tau} = \frac{1}{\mathcal{M}(x)} \left( c^2 g^{\mu\nu} - u^\mu u^\nu \right) u_\mu \frac{\partial \mathcal{M}(x)}{\partial x^\nu} = 0 \tag{I.3.9}$$

The above formulas are valid if $u^\mu \approx 0$, or $u^\mu \ll c$. If $u$ is not small, the formulas are still valid, but in a reference frame in which the dressed electron is at rest, and we need to perform a Lorentz boost to find the acceleration in the laboratory frame.

## I.3.2 The dressed mass

In our case we define the variable mass in terms of a 4-vector potential

$$A(x) = \left( \frac{\Phi(x)}{c}, \boldsymbol{A}(x) \right) \tag{I.3.10}$$

as

$$\mathcal{M}(x) = m \sqrt{1 - \frac{e^2 \langle A^2 \rangle (x)}{(mc)^2}} \tag{I.3.11}$$

where $m$ is the (constant) mass of the isolated particle and $\langle A^2 \rangle(x)$ is the time average

$$\langle A^2 \rangle (x) = \frac{1}{T} \int_{-T/2}^{T/2} d\tau A^2(t + \tau, \boldsymbol{x}) \tag{I.3.12}$$

In a Coulomb gauge, the four vector potential reduces to

$$A(x) = (0, \boldsymbol{A}(x)), \quad A^2(x) = -\boldsymbol{A}^2(x) \tag{I.3.13}$$

and the dressed mass becomes

$$\mathcal{M}(x) = m \sqrt{1 + \frac{e^2 \langle \boldsymbol{A}^2 \rangle (x)}{(mc)^2}}, \quad \langle \boldsymbol{A}^2 \rangle (x) = \frac{1}{T} \int_{-T/2}^{T/2} d\tau \boldsymbol{A}^2(t + \tau, \boldsymbol{x}) = -\frac{1}{T} \int_{-T/2}^{T/2} d\tau A^2(t + \tau, \boldsymbol{x}) \tag{I.3.14}$$

Define the dimensionless quantity

$$a(x) = \frac{e^2}{(mc)^2} \frac{1}{T} \int_{-T/2}^{T/2} d\tau \boldsymbol{A}^2(t + \tau, \boldsymbol{x}) = -\frac{e^2}{(mc)^2} \frac{1}{T} \int_{-T/2}^{T/2} d\tau A^2(t + \tau, \boldsymbol{x}) \tag{I.3.15}$$

The derivatives are

$$\frac{\partial a(x)}{\partial x_\mu} = \frac{2e^2}{(mc)^2} \frac{1}{T} \int_{-T/2}^{T/2} d\tau \boldsymbol{A}(t + \tau, \boldsymbol{x}) \cdot \partial^\mu \boldsymbol{A}(t + \tau, \boldsymbol{x}) = -\frac{2e^2}{(mc)^2} \frac{1}{T} \int_{-T/2}^{T/2} d\tau A(t + \tau, \boldsymbol{x}) \cdot \partial^\mu A(t + \tau, \boldsymbol{x}) \tag{I.3.16}$$

We can define the 4-vector

$$v = (1, 0, 0, 0), \quad \text{and} \quad c\tau v = (c\tau, 0, 0, 0) \tag{I.3.17}$$

and define $\boldsymbol{A}$ as a function of the four-vector $x \equiv (ct, \boldsymbol{x})$. Then in covariant notations we get

$$\boxed{a(x) = -\frac{e^2}{(mc)^2}\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} d\chi A^2(x+\chi v) = -\frac{e^2}{(mc)^2}\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} d\chi A_\alpha(x+\chi v)A^\alpha(x+\chi v)} \tag{I.3.18}$$

$$\boxed{\partial^\mu a(x) = -\frac{2e^2}{(mc)^2}\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} d\chi A(x+\chi v)\cdot\partial^\mu A(x+\chi v) = -\frac{2e^2}{(mc)^2}\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} d\chi A_\alpha(x+\chi v)\cdot\partial^\mu A^\alpha(x+\chi v)} \tag{I.3.19}$$

$$\boxed{\mathcal{M}(x) = m\sqrt{1+a(x)}} \tag{I.3.20}$$

$$\boxed{\partial^\mu \mathcal{M}(x) = m\partial^\mu\sqrt{1+a(x)} = \frac{m}{2}\partial^\mu a(x)\frac{1}{\sqrt{1+a(x)}}} \tag{I.3.21}$$

# Appendix A

# Elements of relativistic dynamics

## A.1 The relativistic Lagrangian theory for a particle in a potential force

For a particle of mass $m$, moving in a potential field

$$V(\boldsymbol{x}) \tag{A.1}$$

the Lagrange (non-covariant) function is

$$L_{(t)} = -mc\sqrt{c^2 - \boldsymbol{v}^2(t)} - V(\boldsymbol{x}) \tag{A.2}$$

where

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{x}(t)}{dt} \tag{A.3}$$

is the non-covariant velocity. The action integral is

$$S = \int_{t_1}^{t_2} dt L_{(t)} \tag{A.4}$$

and the extremum condition for the action becomes

$$\begin{aligned}\delta S =& \delta \int_{t_1}^{t_2} dt L_{(t)}(\boldsymbol{x}, \boldsymbol{v}) = \int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial \boldsymbol{x}} \cdot \delta \boldsymbol{x} + \frac{\partial L}{\partial \boldsymbol{v}} \cdot \delta \boldsymbol{v} \right) = \int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial \boldsymbol{x}} \cdot \delta \boldsymbol{x} + \frac{\partial L}{\partial \boldsymbol{v}} \cdot \delta \frac{d\boldsymbol{x}}{dt} \right) = \int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial \boldsymbol{x}} \cdot \delta \boldsymbol{x} + \frac{\partial L}{\partial \boldsymbol{v}} \cdot \frac{d\delta \boldsymbol{x}}{dt} \right) = \\ &= \int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial \boldsymbol{x}} \cdot \delta \boldsymbol{x} + \frac{d}{dt} \left( \frac{\partial L}{\partial \boldsymbol{v}} \cdot \delta \boldsymbol{x} \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial \boldsymbol{v}} \right) \cdot \delta \boldsymbol{x} \right) = \int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial \boldsymbol{x}} - \frac{d}{dt} \frac{\partial L}{\partial \boldsymbol{v}} \right) \cdot \delta \boldsymbol{x}\end{aligned} \tag{A.5}$$

The Lagrange equations become

$$\frac{\partial L_{(t)}}{\partial \boldsymbol{x}} - \frac{d}{dt} \frac{\partial L_{(t)}}{\partial \boldsymbol{v}} = \boldsymbol{0} \tag{A.6}$$

i.e.

$$-\frac{\partial V}{\partial \boldsymbol{x}} = \frac{d}{dt} \frac{mc\boldsymbol{v}}{\sqrt{c^2 - \boldsymbol{v}^2}} \tag{A.7}$$

The relativistic momentum is

$$\boldsymbol{p} = \frac{\partial L_{(t)}}{\partial \boldsymbol{v}} = \frac{mc\boldsymbol{v}}{\sqrt{c^2 - \boldsymbol{v}^2}} = \frac{m\boldsymbol{v}}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}} = m\gamma\boldsymbol{v} \tag{A.8}$$

and the three dimensional force acting on the particle is

$$\boldsymbol{F} = -\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \tag{A.9}$$

and we obtained the equation of motion in the non-covariant form

$$\boldsymbol{F} = -\frac{\partial V}{\partial \boldsymbol{x}} = \frac{d\boldsymbol{p}}{dt}, \quad p = m\gamma\boldsymbol{v} \tag{A.10}$$

$$\boxed{\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}, \quad \frac{d}{dt}(m\gamma\boldsymbol{v}) = \boldsymbol{F} = -\frac{\partial V}{\partial \boldsymbol{x}}} \tag{A.11}$$

To get the covariant version of the equation of motion we define the proper time $\tau$, with

$$dt = \gamma d\tau, \quad d\tau = \frac{dt}{\gamma}, \quad \frac{d}{d\tau} = \gamma\frac{d}{dt} \tag{A.12}$$

and

$$x = (ct, \boldsymbol{x}), \quad u = \frac{dx}{d\tau} = (\gamma c, \gamma\boldsymbol{v}), \quad p = mu = (\gamma mc, \gamma\boldsymbol{v}) = (mc\gamma, \boldsymbol{p}) \tag{A.13}$$

The four vector force is

$$K = \frac{dp}{d\tau} = \left(mc\frac{d\gamma}{d\tau}, \frac{d\boldsymbol{p}}{d\tau}\right) \tag{A.14}$$

The derivative of $\gamma$ is calculated directly

$$\frac{d\gamma}{dt} = \frac{d}{dt}\frac{1}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}} = \frac{1}{c^2}\frac{\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt}}{(1 - \frac{\boldsymbol{v}^2}{c^2})^{3/2}} \tag{A.15}$$

and

$$\frac{d\gamma}{dt} = \frac{\gamma^3}{c^2}\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt}, \quad mc\frac{d\gamma}{d\tau} = \gamma^4\frac{m}{c}\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt} \tag{A.16}$$

We want to prove that the four-force $K$ is orthogonal on the four velocity $u$, i.e.

$$mc\frac{d\gamma}{d\tau}c\gamma = \frac{d\boldsymbol{p}}{d\tau}\cdot\gamma\boldsymbol{v} = m\frac{d\gamma\boldsymbol{v}}{d\tau}\cdot\gamma\boldsymbol{v} \tag{A.17}$$

A direct proof is obvious, i.e.

$$m\frac{1}{2}\frac{d}{d\tau}(c^2\gamma^2) - m\frac{1}{2}\frac{d}{d\tau}(\gamma\boldsymbol{v})^2 = \frac{m}{2}\frac{d}{d\tau}\left(\gamma^2(c^2 - \boldsymbol{v}^2)\right) = \frac{mc^2}{2}\frac{d}{d\tau}1 = 0 \tag{A.18}$$

A more involved proof would be

$$mc\frac{d\gamma}{d\tau} = \frac{1}{c\gamma}\frac{d\boldsymbol{p}}{d\tau}\cdot\gamma\boldsymbol{v} = \frac{\gamma}{c}\frac{d\boldsymbol{p}}{dt}\cdot\boldsymbol{v} = m\frac{\gamma}{c}\frac{d\gamma\boldsymbol{v}}{dt}\cdot\boldsymbol{v} = m\frac{\gamma}{c}\left(\frac{d\gamma}{dt}\boldsymbol{v}^2 + \gamma\frac{d\boldsymbol{v}}{dt}\cdot\boldsymbol{v}\right) \tag{A.19}$$

or

$$mc\gamma\frac{d\gamma}{dt} = m\frac{\gamma}{c}\left(\frac{d\gamma}{dt}\boldsymbol{v}^2 + \gamma\frac{d\boldsymbol{v}}{dt}\cdot\boldsymbol{v}\right), \quad \frac{d\gamma}{dt} = \frac{1}{c^2}\left(\frac{d\gamma}{dt}\boldsymbol{v}^2 + \gamma\frac{d\boldsymbol{v}}{dt}\cdot\boldsymbol{v}\right), \quad \frac{1}{\gamma^2}\frac{d\gamma}{dt} = \frac{1}{c^2}\gamma\frac{d\boldsymbol{v}}{dt}\cdot\boldsymbol{v} \tag{A.20}$$

which is equivalent to the result (A.16).

The covariant version of the equation of motion is

$$x = (ct, \boldsymbol{x}), \quad \frac{dx}{d\tau} = u, \quad \frac{d(mu)}{d\tau} = K, \quad K = \left(\gamma\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{F}, \gamma\boldsymbol{F}\right) = \left(\frac{\boldsymbol{u}}{c}\cdot\boldsymbol{F}, \frac{u^0}{c}\boldsymbol{F}\right), \quad \boldsymbol{F} = -\frac{\partial V}{\partial\boldsymbol{x}} \tag{A.21}$$

or

$$\boxed{x = (ct, \boldsymbol{x}), \quad \frac{dx}{d\tau} = \frac{p}{m}, \quad \frac{dp}{d\tau} = K, \quad K = \left(\frac{\boldsymbol{p}}{mc}\cdot\boldsymbol{F}, \frac{p^0}{mc}\boldsymbol{F}\right), \quad \boldsymbol{F} = -\frac{\partial V}{\partial\boldsymbol{x}}} \tag{A.22}$$

The direct derivation of the above equations can be obtained directly from the covariant Lagrange function. The action integral can e written as

$$S = \int dt L_{(t)} = \int d\tau\gamma L_{(t)} = \int d\tau L_{(\tau)}, \quad L_{(\tau)} = \gamma L_{(t)} = -mc\gamma\sqrt{c^2 - \boldsymbol{v}^2} - \gamma V(\boldsymbol{x}) \tag{A.23}$$

i.e.

$$L_{(\tau)} = -mc\sqrt{\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau}} - \frac{1}{c}\frac{dx^0}{d\tau}V(\boldsymbol{x}) \tag{A.24}$$

The covariant equations of motion are

$$\frac{d}{d\tau}\frac{\partial L_{(\tau)}}{\partial\dot{x}_\mu} = \frac{\partial L_{(\tau)}}{\partial x_\mu}, \quad \dot{x}_\mu = \frac{dx_\mu}{d\tau} \tag{A.25}$$

In our case

$$\frac{\partial L_{(\tau)}}{\partial\dot{x}_i} = -m\dot{x}^i, \quad \frac{\partial L_{(\tau)}}{\partial x_i} = -\frac{1}{c}\dot{x}^0\frac{\partial V}{\partial x_i} = \frac{1}{c}\dot{x}^0\frac{\partial V(\boldsymbol{x})}{\partial x^i} = -\gamma F^i = -K^i \tag{A.26}$$

and the spatial components of the equations of motions become

$$\frac{d}{d\tau}(m\dot{x}^i) = K^i \tag{A.27}$$

for the temporal components we get

$$\frac{\partial L_{(\tau)}}{\partial\dot{x}_0} = -mc\dot{x}^0 - \frac{1}{c}V(\boldsymbol{x}), \quad \frac{d}{d\tau}\frac{\partial L_{(\tau)}}{\partial\dot{x}_0} = -mc\frac{d\dot{x}^0}{d\tau} - \frac{1}{c}\frac{\partial V}{\partial x^i}\dot{x}^i = -mc\frac{d\dot{x}^0}{d\tau} + \frac{1}{c}\boldsymbol{F}\cdot\boldsymbol{u} \tag{A.28}$$

i.e.

$$mc\frac{du^0}{d\tau} = \frac{1}{c}\boldsymbol{F}\cdot\boldsymbol{u} \tag{A.29}$$

with the notation

$$u = \dot{x}, \quad p = mu \tag{A.30}$$

we get

$$\boxed{\dot{x} = u = \frac{p}{m}, \quad \dot{p} = K, \quad K = \left(\frac{1}{cm}\boldsymbol{F}\cdot\boldsymbol{p}, \frac{p^0}{mc}\boldsymbol{F}\right), \quad F^i = -\frac{\partial V(\boldsymbol{x})}{\partial x^i}} \tag{A.31}$$

## A.2 The relativistic Lagrangian theory for a particle in an electromagnetic field

In that case the potential is

$$V(x,\dot{x}) = q(\frac{\Phi}{c}c - \boldsymbol{A}\cdot\boldsymbol{v}) \tag{A.32}$$

and the covariant Lagrange function

$$L_{(\tau)} = -mc\sqrt{\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau}} - \frac{1}{c}\frac{dx^0}{d\tau}V(x,\dot{x}) = -mc\sqrt{\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau}} - qA_\mu(x)\cdot\frac{dx_\mu}{d\tau} \tag{A.33}$$

where $A$ is the four potential

$$A(x) = \left(\frac{\Phi(x)}{c}, \boldsymbol{A}(x)\right) \tag{A.34}$$

Then the covariant Lagrange equations become

$$\frac{\partial L_{(\tau)}}{\partial\left(\frac{du^\mu}{d\tau}\right)} = -m\frac{dx_\mu}{d\tau} - qA_\mu(x), \quad \frac{d}{d\tau}\frac{\partial L_{(\tau)}}{\partial\left(\frac{du^\mu}{d\tau}\right)} = -m\frac{d}{d\tau}\frac{dx_\mu}{d\tau} - q\partial_\nu A_\mu(x)\frac{dx_\nu}{d\tau} \tag{A.35}$$

$$\frac{\partial L_{(\tau)}}{\partial x^\mu} = -q\partial_\mu A_\nu \frac{dx^\nu}{d\tau} \tag{A.36}$$

$$\frac{d}{d\tau}(m\frac{dx_\mu}{d\tau}) = q\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)\frac{dx^\nu}{d\tau} \tag{A.37}$$

or

$$\boxed{\frac{dx^\mu}{d\tau} = \frac{p^\mu}{m}} \tag{A.38}$$

$$\boxed{\frac{dp^\mu}{d\tau} = \frac{q}{m}F^{\mu\nu}(x)p_\mu, \quad F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)} \tag{A.39}$$