

1. Question 1

- (1.1) The Routh array was constructed by first rewriting the characteristic equation of the closed-loop system using the substitution $T_I = \frac{1}{\tau_I}$ to simplify the algebra. The goal was to determine the range of K_C and τ_I values that lead to a stable system based on Routh-Hurwitz criteria.

Question 1

$G_C = K_C + T_I \frac{1}{s} = \frac{K_C s + T_I}{s}$

1.1

$$\frac{Y(s)}{R(s)} = \frac{\frac{2}{s^3 + 4s^2 + 5s + 2} \left(\frac{K_C s + T_I}{s} \right)}{1 + \frac{2}{s^3 + 4s^2 + 5s + 2} \left(\frac{K_C s + T_I}{s} \right)}$$

$$= \frac{\frac{2K_C s + 2T_I}{s^4 + 4s^3 + 5s^2 + 2s}}{1 + \frac{2K_C s + 2T_I}{s^4 + 4s^3 + 5s^2 + 2s}}$$

$$= \frac{2K_C s + 2T_I}{(s^4 + 4s^3 + 5s^2 + 2s) + 2K_C s + 2T_I}$$

$$= \frac{2K_C s + 2T_I}{s^4 + 4s^3 + 5s^2 + (2 + 2K_C)s + 2T_I}$$

$$D(s) = s^4 + 4s^3 + 5s^2 + (2 + 2K_C)s + 2T_I = 0$$

Routh Array

4	1	5	$2T_I$
3	4	$2 + 2K_C$	0
2	b_3	b_1	b_0
1	c_3	c_1	c_0
0	d_3	d_1	d_0

Figure 1: Step 1 of Routh Array derivation.

$$b_3 = -\frac{1}{4} \begin{vmatrix} 1 & 5 \\ 4 & 2+2K_c \end{vmatrix}$$

$$= -\frac{1}{4} [(2+2K_c) - 5(4)]$$

$$= -\frac{1}{2} - \frac{1}{2}K_c + 5$$

$$= 4.5 - \frac{1}{2}K_c$$

$$b_1 = -\frac{1}{4} (-4(2T_I))$$

$$= 2T_I$$

$$b_0 = 0$$

$$c_3 = -\frac{1}{b_3} [4b_1 - (2+2K_c)b_3]$$

$$c_1 = 0$$

$$c_0 = 0$$

$$= -\frac{4b_1}{b_3} + (2+2K_c)$$

$$= \frac{-8T_I}{4.5 - \frac{1}{2}K_c} + 2 + 2K_c$$

$$d_3 = -\frac{1}{c_3} (-b_1 c_3)$$

$$d_1 = 0$$

$$d_0 = 0$$

$$= b_1$$

$$= 2T_I$$

Figure 2: Step 2 of Routh Array derivation.

Conditions

$$b_3: 4.5 - \frac{1}{2}K_c > 0$$

$$4.5 > \frac{1}{2}K_c$$

$$9 > K_c$$

$$d_3: 2T_I > 0 \Rightarrow T_I > 0$$

$$c_3: \frac{-8T_I}{4.5 - \frac{1}{2}K_c} + 2 + 2K_c > 0$$

$$-8T_I + 2(4.5 - \frac{1}{2}K_c) + 2K_c(4.5 - \frac{1}{2}K_c) > 0$$

$$-8T_I + 9 - K_c + 9K_c - K_c^2 > 0$$

$$-K_c^2 + 8K_c + 9 > 8T_I$$

$$-(K_c - 9)(K_c + 1) > 8T_I$$

$$\frac{(9 - K_c)(K_c + 1)}{8} > T_I$$

\therefore Final conditions:

$$\textcircled{1} \quad 9 > K_c$$

$$\textcircled{2} \quad T_I > 0 \Rightarrow \frac{1}{\tau_i} > 0 \Rightarrow \tau_i > 0$$

$$\textcircled{3} \quad \frac{(9 - K_c)(K_c + 1)}{8} > T_I$$

\rightarrow +ve due to $\textcircled{1} \rightarrow \therefore K_c > -1$ to keep this +ve
 \uparrow 0 due to $\textcircled{2}$

$\tau_i > \frac{8}{(9 - K_c)(K_c + 1)}$

$$-1 < K_c < 9$$

$$T_I > 0 \approx \tau_i > 0$$

Figure 3: Final Routh Array with derived inequalities for stability.

- (1.2) Using the inequalities derived in 1.1, the feasible set was plotted to visualize combinations of K_C and τ_I that result in closed-loop stability. The red shaded region indicates infeasible (unstable) parameter combinations.

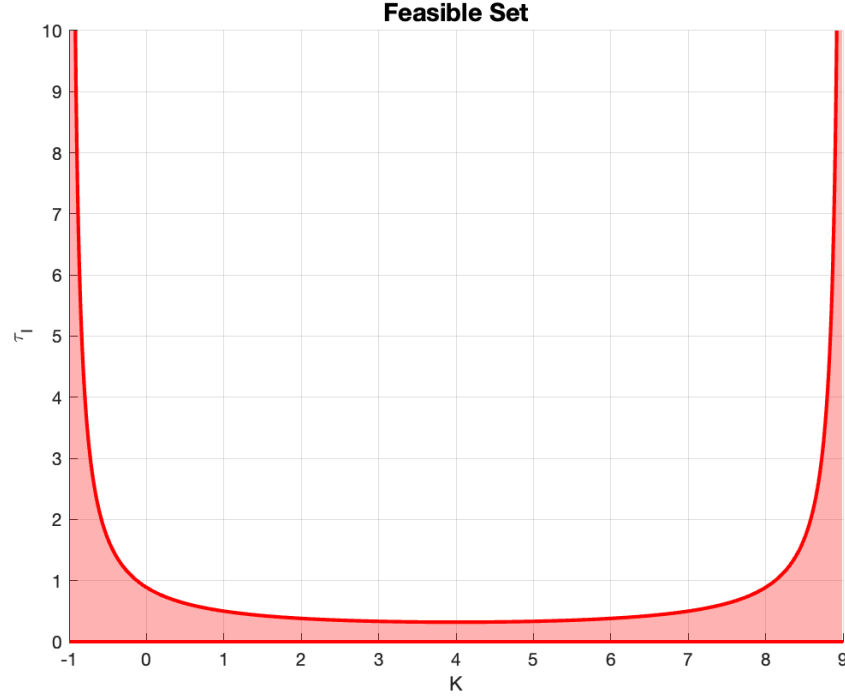


Figure 4: Feasible set for K_C and τ_I from Routh-Hurwitz conditions.

- (1.3) A Simulink model was developed to simulate the closed-loop response of the system. Three scenarios were tested: one inside the feasible region, one outside, and one on the edge.

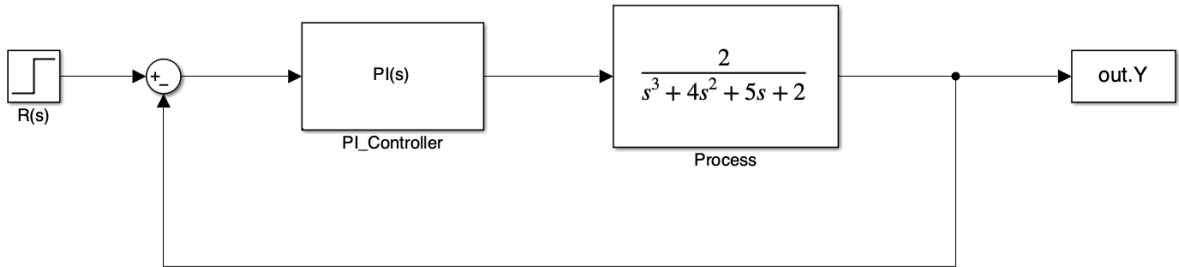


Figure 5: Simulink model of the PI-controlled system.

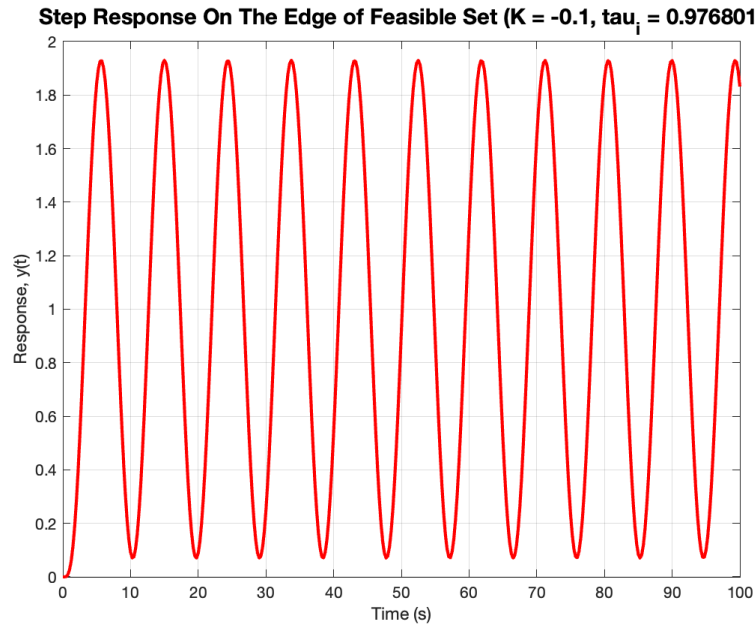


Figure 6: Step response on the boundary of the feasible set. The system oscillates and the amplitude grows very slowly, indicating marginal instability.

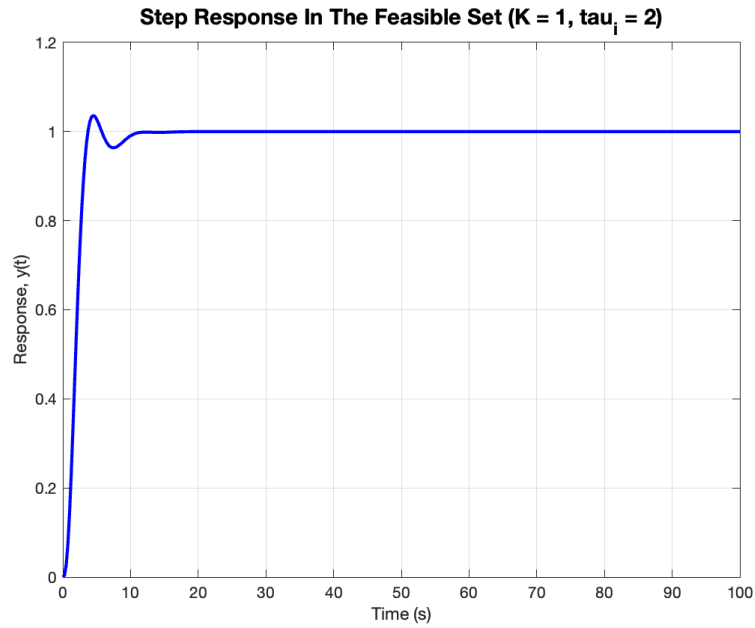


Figure 7: Step response for parameters inside the feasible set. The system is stable and relatively well damped, approaching steady state smoothly.

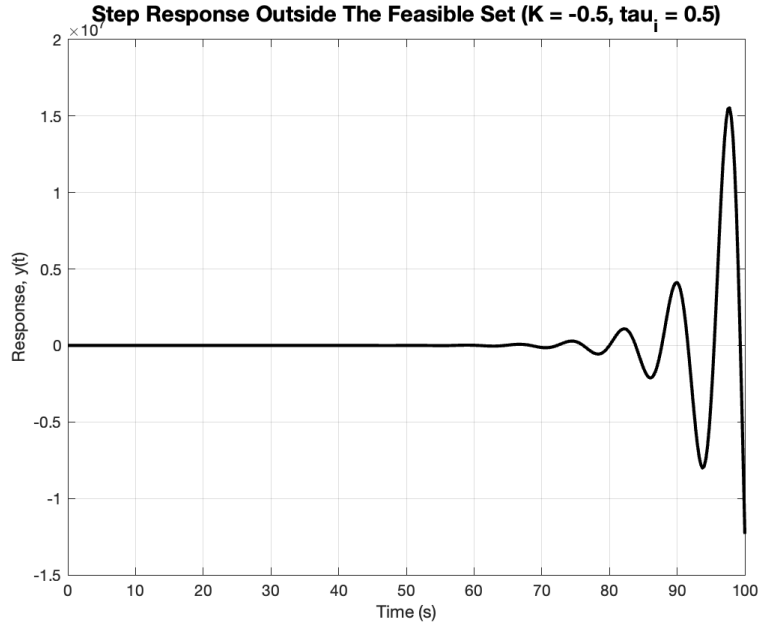


Figure 8: Step response for parameters outside the feasible set. The system exhibits fast-growing oscillations, clearly demonstrating instability.

The edge case exhibits slow-growing oscillations, which indicates marginal instability. Although the system initially appears to behave stably, the oscillations increase in amplitude over time. This is consistent with the fact that the chosen parameters lie directly on the boundary of the feasible set, where the Routh-Hurwitz conditions require strict inequalities ($>$) rather than allowing equality. The second case, which is well within the feasible region, produces a stable and mostly well-damped response. The final case, with parameters outside the feasible set, immediately shows rapidly growing oscillations, confirming instability.

- (1.4) The Routh-Hurwitz array was recomputed with the new transport delay (using first-order Padé approximation) to determine how the delay affects system stability.

1.4

$$\begin{aligned}
 G_c &= K_c + T_I \frac{1}{s} = \frac{K_c s + T_I}{s} & G_H &= e^{-0.5s} \\
 & & &\approx \frac{1 - 0.25s}{1 + 0.25s} \\
 \frac{Y(s)}{R(s)} &= \frac{\frac{2}{s^3 + 4s^2 + 5s + 2} \left(\frac{K_c s + T_I}{s} \right)}{1 + \frac{2}{s^3 + 4s^2 + 5s + 2} \left(\frac{K_c s + T_I}{s} \right) \left(\frac{1 - 0.25s}{1 + 0.25s} \right)} \\
 &= \frac{(2K_c s + 2T_I)(1 + 0.25s)}{(s^4 + 4s^3 + 5s^2 + 2s)(1 + 0.25s) + (2K_c s + 2T_I)(1 - 0.25s)} \\
 &= \frac{2K_c s + 2T_I + 0.5K_c s^2 + 0.5T_I s}{s^4 + 4s^3 + 5s^2 + 2s + 0.25s^5 + s^4 + \frac{5}{4}s^3 + 0.5s^2 + 2K_c s + 2T_I - 0.5K_c s^2 - 0.5T_I s} \\
 &= \frac{0.5K_c s^2 + (2K_c + 0.5T_I)s + 2T_I}{0.25s^5 + 2s^4 + \frac{21}{4}s^3 + (5.5 - 0.5K_c)s^2 + (2 + 2K_c - 0.5T_I)s + 2T_I}
 \end{aligned}$$

$$D(s) = 0.25s^5 + 2s^4 + \frac{21}{4}s^3 + (5.5 - 0.5K_c)s^2 + (2 + 2K_c - 0.5T_I)s + 2T_I = 0$$

Routh Array

5	0.25	$\frac{21}{4}$	$2 + 2K_c - 0.5T_I$
4	2	$5.5 - 0.5K_c$	$2T_I$
3	b_4	b_2	b_0
2	c_4	c_2	c_0
1	d_4	d_2	d_0
0	e_4	e_2	e_0

Figure 9: Step 1 of Routh Array with transport delay.

$$\begin{aligned}
b_4 &= -\frac{1}{2} \left[(0.25)(5.5 - 0.5K_c) - 2\left(\frac{21}{4}\right) \right] & b_2 &= -\frac{1}{2} \left[(0.25)(2T_I) - 2(2 + 2K_c - 0.5T_I) \right] \\
&= -\frac{11}{16} + \frac{1}{16}K_c + \frac{21}{4} & &= -0.25T_I + 2 + 2K_c - 0.5T_I \\
&= \frac{1}{16}K_c + \frac{73}{16} & &= 2K_c - 0.75T_I + 2 \\
& & b_0 &= 0 \\
c_4 &= -\frac{1}{b_4} [2b_2 - (5.5 - 0.5K_c)b_4] & c_2 &= -\frac{1}{b_4} [-2b_4] \\
&= \frac{-2b_2}{b_4} + 5.5 - 0.5K_c & &= 2T_I \\
& & c_0 &= 0 \\
&= \frac{-4K_c + 1.5T_I - 4}{\frac{1}{16}K_c + \frac{73}{16}} + 5.5 - 0.5K_c & & \xrightarrow{\quad} (5.5 - 0.5K_c) \left(\frac{1}{16}K_c + \frac{73}{16} \right) \\
& & & \frac{11}{32}K_c + \frac{803}{32} - \frac{1}{32}K_c^2 - \frac{73}{32}K_c \\
& & & -\frac{1}{32}K_c^2 - \frac{31}{16}K_c + \frac{803}{32} \\
d_4 &= -\frac{1}{c_4} [(b_4)c_2 - b_2(c_4)] & d_2 &= 0 \\
&= b_2 - \frac{2T_I b_4}{c_4} & d_0 &= 0 \\
&= 2K_c - 0.75T_I + 2 - \frac{\left(\frac{1}{16}K_c + \frac{73}{16}\right)(2T_I)(K_c + 73)}{\left(-\frac{1}{2}K_c^2 - 95K_c + \frac{675}{2} + 24T_I\right)} & & c_4 = \frac{-\frac{1}{32}K_c^2 - \frac{95}{16}K_c + \frac{675}{32} + 1.5T_I}{\frac{1}{16}K_c + \frac{73}{16}} \\
& & & = \frac{-\frac{1}{2}K_c^2 - 95K_c + \frac{675}{2} + 24T_I}{K_c + 73} \\
e_4 &= -\frac{1}{d_4} [(c_4)d_2 - c_2(d_4)] & e_2 &= 0 \\
&= c_2 - \frac{c_4 d_2}{d_4} & e_0 &= 0 \\
&= 2T_I
\end{aligned}$$

Figure 10: Step 2 of Routh Array with transport delay.

Conditions

$$e_4 = 2 > 0 \quad \checkmark$$

$$b_4 = \frac{1}{16}K_c + \frac{73}{16} > 0$$

$$\textcircled{1} \quad K_c > -73$$

$$c_4 = \frac{-\frac{1}{2}K_c^2 - 95K_c + \frac{675}{2} + 24T_I}{K_c + 73} > 0$$

$$\textcircled{2} \quad T_I > \frac{\frac{1}{2}K_c^2 + 95K_c - \frac{675}{2}}{24}$$

$$\textcircled{3} \quad d_4 = 2K_c - 0.75T_I + 2 - \frac{(\frac{1}{16}K_c + \frac{73}{16})(2T_I)(K_c + 73)}{(-\frac{1}{2}K_c^2 - 95K_c + \frac{675}{2} + 24T_I)} > 0$$

↳ used matlab +symbolab to solve for T_I

Figure 11: Final Routh Array with delay-based conditions.

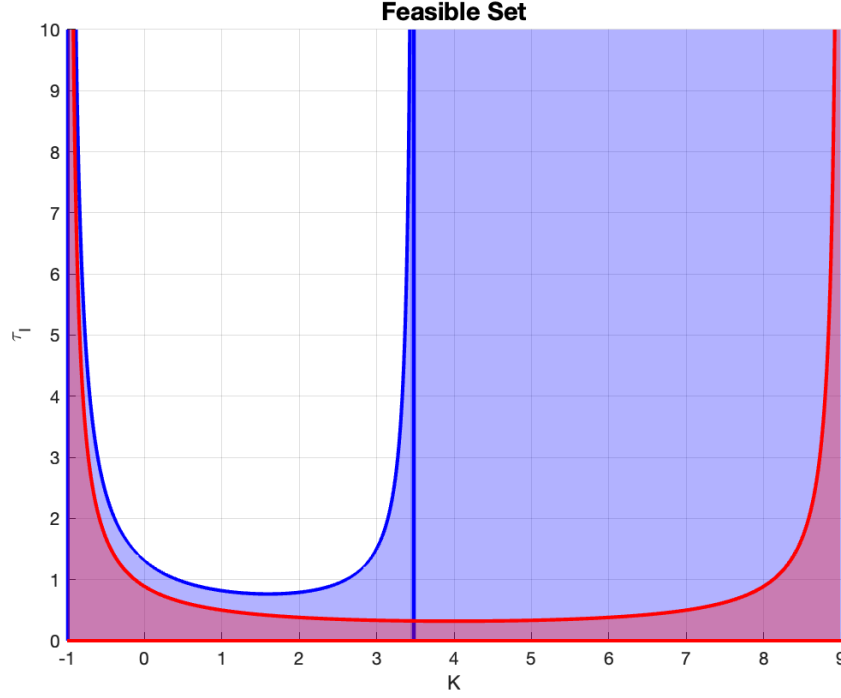


Figure 12: Feasible sets with and without delay. Blue indicates infeasible region of the new delayed system. Red is the same infeasible region from earlier.

As seen in the figure, the introduction of transport delay reduces the size of the feasible region, particularly in areas with small integral time τ_I or large controller gain K_C . This is because transport delay introduces additional phase lag into the system, which decreases the phase margin and brings the system closer to instability. From the perspective of the Bode plot, delay shifts the phase curve to the left, meaning that the phase crosses -180° at a lower frequency. This corresponds to a higher critical gain A_c , which implies a lower allowable K_C for stability. In practical terms, this means that the system must be detuned (K_C must be reduced) when delay is present. If the gain is not reduced accordingly, the aggressive controller action causes excessive oscillations, and the system approaches instability. This explains why many of the previously stable combinations of K_C and τ_I now fall outside the feasible set when delay is introduced.

- (1.5) The system was simulated again using the same controller parameters ($K_C = 2$, $\tau_I = 1$) for both the delay and non-delay versions. These values were chosen by looking at the feasible sets and picking values that lay in both the time delay and non-time delayed feasible sets. The updated Simulink model includes the transport delay block.

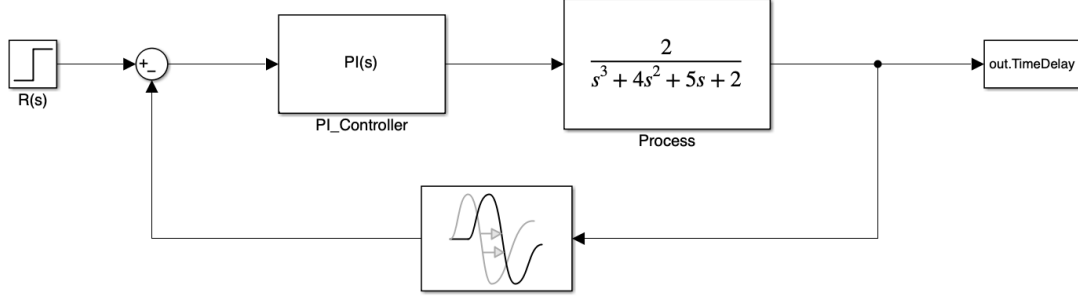


Figure 13: Simulink model with transport delay included.

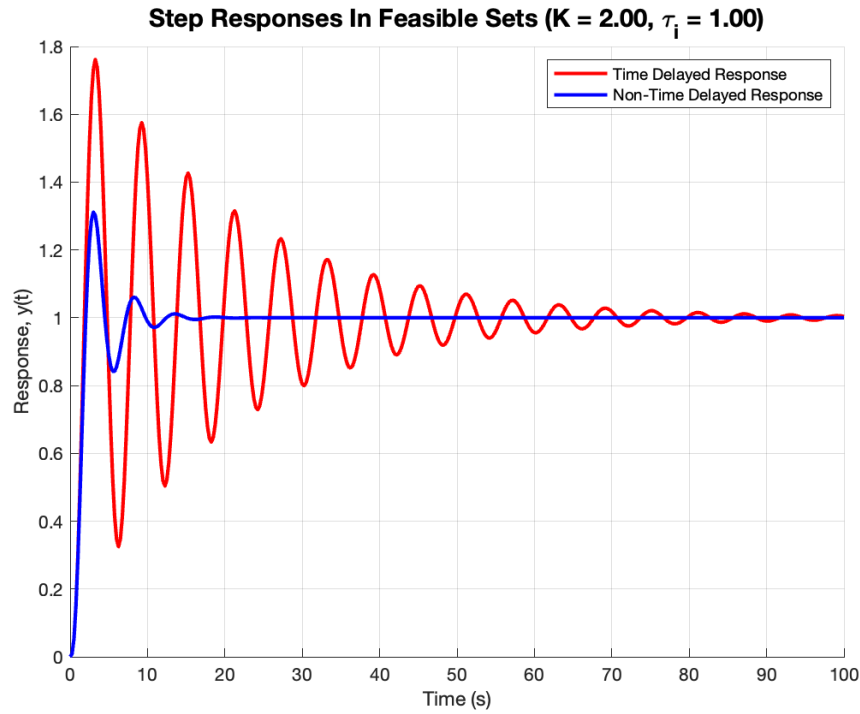


Figure 14: Comparison of system response with and without transport delay.

The system with transport delay exhibits more oscillatory behavior and takes longer to settle compared to the delay-free case, despite using identical PI tuning parameters. This aligns with the analysis in 1.4, where the added phase lag from delay was shown to reduce stability margins and narrow the feasible set. The delay effectively pushes the system closer to instability, and without detuning K_C , even stable configurations become more oscillatory and sluggish in response.

2. Question 2

- (2.1) The approximated first-order system equation for the closed-loop system is given by:

$$\frac{Y(s)}{U(s)} = \frac{-1.2e^{-5s}}{6s + 1} = \frac{Ke^{-\theta s}}{\tau s + 1} \quad (1)$$

We first start with the **Cohen-Coon** heuristic. From class, we are given the following equations:

$$K_C = \frac{1}{K} \cdot \frac{\tau}{\theta} \left(\frac{4}{3} + \frac{\theta}{4\tau} \right) \quad \tau_I = \theta \cdot \frac{32 + \frac{6\theta}{\tau}}{13 + \frac{8\theta}{\tau}} \quad K_D = \theta \cdot \frac{4}{11 + \frac{2\theta}{\tau}}$$

We thus get the answers below:

$$K_c = \frac{1}{(-1.2)} \left(\frac{6}{5} \right) \left(\frac{4}{3} + \frac{5}{4 \cdot 6} \right) = -1.54166$$

$$\tau_I = (5) \cdot \frac{32 + \frac{6(5)}{(6)}}{13 + \frac{8(5)}{(6)}} = 9.40677$$

$$K_D = (5) \cdot \frac{4}{11 + \frac{2(5)}{(6)}} = 1.5789$$

We then move onto the **Ciancone** heuristic. Since this system is for setpoint tracking, we use the correlation graphs as shown in class. To avoid wasting digital ink, the graphs were not attached.

For all graphs, we need to derive the fraction dead time first:

$$\frac{\theta}{\theta + \tau} = \frac{5}{5 + 6} = 0.454545$$

For the graphs $K_c K_p$, $\frac{\tau_I}{\theta + \tau}$, and $\frac{K_D}{\theta + \tau}$, we get the values 0.73, 0.73, and 0.08 respectively. Solving for K_c , τ_I , and K_D gives us:

$$K_C = \frac{0.73}{-1.2} = -6.083 \quad \tau_I = (5 + 6) \cdot 0.73 = 8.03 \quad K_D = (5 + 6) \cdot 0.08 = 0.88$$

Lastly, we calculate the **ITAE** heuristic. We start with the equation:

$$Y = A \left(\frac{\theta}{\tau} \right)^B$$

The values of A and B are given in the table below:

Controller	Mode	A	B
PID	P	0.965	-0.85
	I	0.796	-0.1465
	D	0.308	0.929

Table 1: Controller Parameters A and B for PID Mode

We also note the following equations to calculate the parameters in set-point tracking:

$$Y_P = K K_C \quad Y_I = \frac{\tau}{\tau_I} = A + B \left(\frac{\theta}{\tau} \right) \quad Y_D = \frac{K_D}{\tau}$$

We do some mathematics:

$$Y_P = (0.965) \left(\frac{\theta}{\tau} \right)^{-0.85} = 1.12675 \quad K_C = \frac{1.12675}{-1.2} = -0.938958$$

$$Y_I = (0.796) + (-0.1465) \cdot \left(\frac{\theta}{\tau} \right) = 0.67391 \quad \tau_I = \frac{6}{0.67391} = 8.895$$

$$Y_D = (0.308) \left(\frac{\theta}{\tau} \right)^{0.929} = 0.26001 \quad K_D = 0.26001 \cdot 6 = 1.56006$$

(2.2) The Bode plot is shown below in Figure 15.

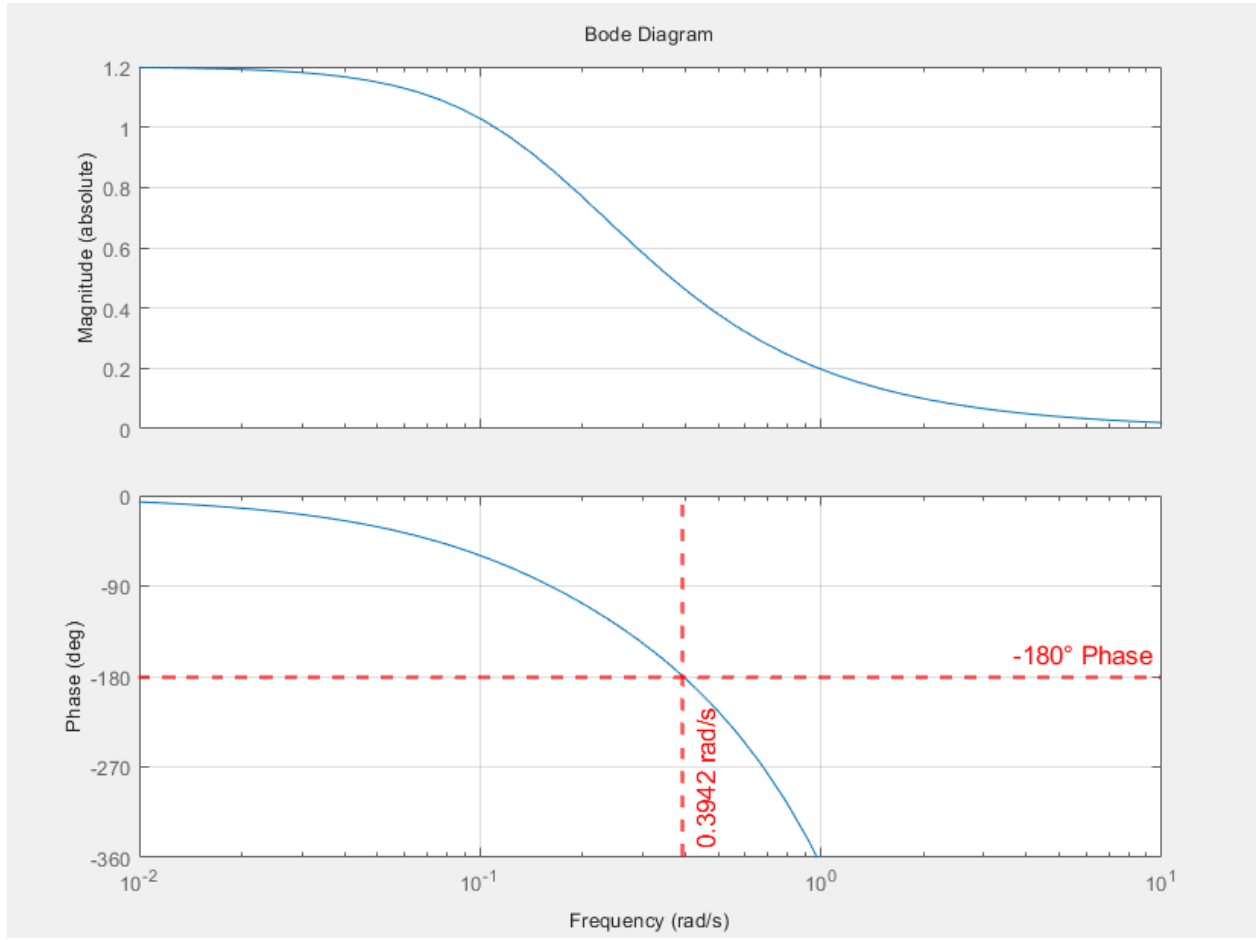


Figure 15: Bode plot of the closed-loop system with transport delay.

From the Bode plot and the associated Matlab code, we derive the Critical Frequency and the amplitude at said frequency as 0.3942 rad/s and 0.4673. With these numbers, we calculate the ultimate gain and the ultimate period.

$$K_u = \frac{1}{\text{Critical Amplitude}} = \frac{1}{0.3942} \approx 2.537$$

$$T_u = \frac{2\pi}{\omega_c} = \frac{2\pi}{0.4673} \approx 13.445$$

We then calculate the PID parameters K_c , τ_I and K_D using the Ziegler-Nichols tuning rules.

$$K_c = \frac{K_u}{1.7} = \frac{2.537}{1.7} \approx 1.492$$

$$\tau_I = \frac{T_u}{2} = \frac{13.445}{2} \approx 6.7225$$

$$K_D = \frac{T_u}{8} = \frac{13.445}{8} \approx 1.6806$$