1. Question 1

  (1.1) The open-loop step response of the system is shown in the Simulink model (Figure
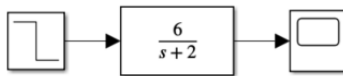   1) below with the response shown in Figure 2.

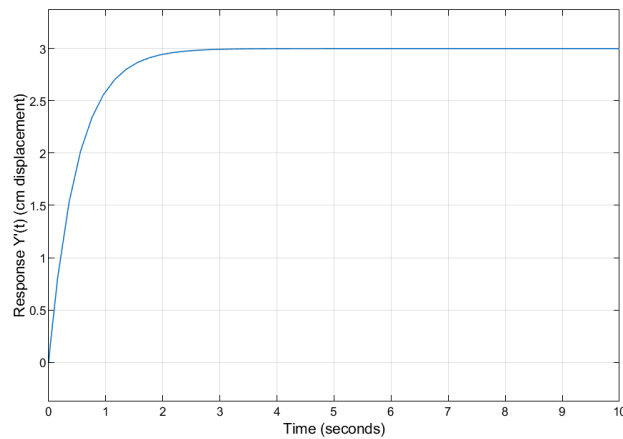Figure 1: Simulink model of the open-loop step response

Figure 2: Open-loop step response

To identify the gain and time constant for this system, we can rearrange the
transfer function to isolate the gain K and time constant τ:

The given transfer function is:

$$G_p(s) = \frac{6}{s+2}$$

To express this in the standard first-order form, which is:

$$G(s) = \frac{K}{\tau s + 1}$$

where K is the gain and τ is the time constant.
We rearrange $G_p(s)$ as follows:

$$G_p(s) = \frac{6}{2(\frac{1}{2}s + 1)} = \frac{3}{\frac{1}{2}s + 1}$$

Thus, we identify the gain K and the time constant τ as:

$$K = 3, \quad \tau = \frac{1}{2}$$

(1.2) The closed-loop feedback control with a proportional controller is shown below in Figure 3. The transfer function $G_c(s)$ can also be represented by a proportional gain K_c.
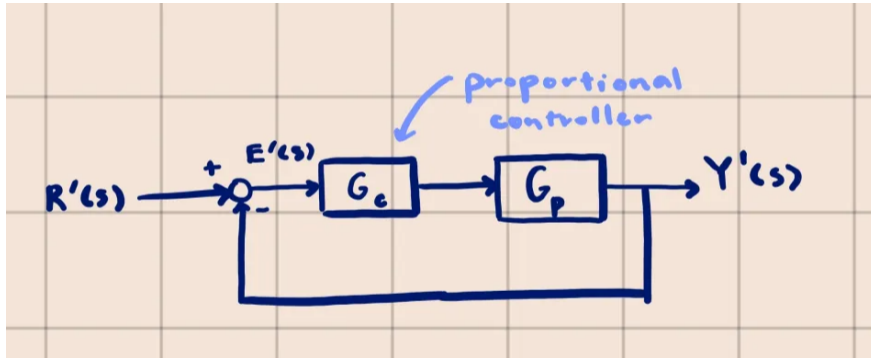

Figure 3: Simulink model of the closed-loop feedback control with a proportional controller

(1.3) The Simulink model is shown below in Figure 4. The response of the system is shown in Figure 5.

Figure 4: Simulink model of the closed-loop feedback control with a proportional controller
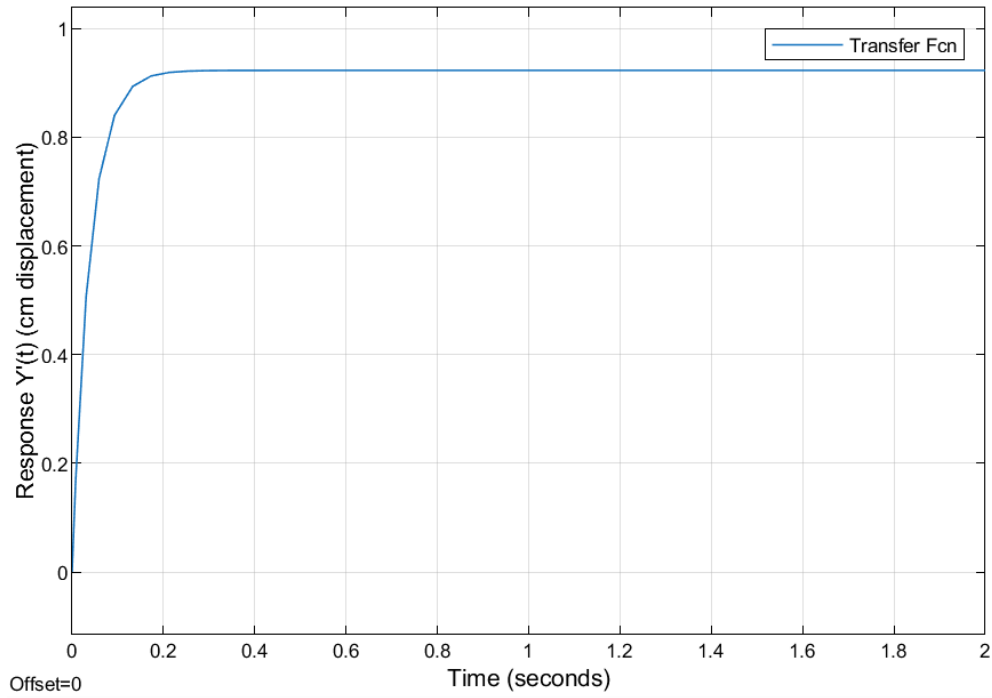

Figure 5: Closed-loop step response

(1.4) The closed-loop transfer function $T(s)$ is given by:

$$T(s) = \frac{Y'(s)}{R'(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

Where $G_c(s) = K_c = 4$ (proportional controller) and $G_p(s) = \frac{6}{s+2}$ (plant transfer

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function). Substituting the transfer functions:

$$T(s) = \frac{4 \cdot \frac{6}{s+2}}{1 + 4 \cdot \frac{6}{s+2}}$$

Simplifying:

$$T(s) = \frac{\frac{24}{s+2}}{1 + \frac{24}{s+2}} = \frac{24}{s + 2 + 24} = \frac{24}{s + 26}$$

Thus, the closed-loop transfer function is:

$$T(s) = \frac{24}{s + 26}$$

We then apply the Final Value Theorem, which states that:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)$$

For a unit step input $R'(s) = \frac{1}{s}$, the Laplace transform of the output is:

$$Y'(s) = T(s)R'(s) = \frac{24}{s + 26} \cdot \frac{1}{s}$$

Applying the FVT:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} s \cdot \frac{24}{(s + 26)s} = \lim_{s \to 0} \frac{24}{s + 26} = \frac{24}{26} = \frac{12}{13} \approx 0.923$$

The steady-state value of the output is approximately 0.923. Since the system is operating on displacement from a rock assembly with a step input, the set point is 1. Thus, the offset of this system is approximately 0.923. Referring to the step response in Figure 5, we can see the offset is approximately $1 - 0.923 = 0.077$.

(1.5) The closed-loop system for controller gains $K_c = 0.25, 1, 2, 4, 10$ are shown in Figure 6. Observing the results, we can see that the offset of the system decreases as the proportional gain increases.
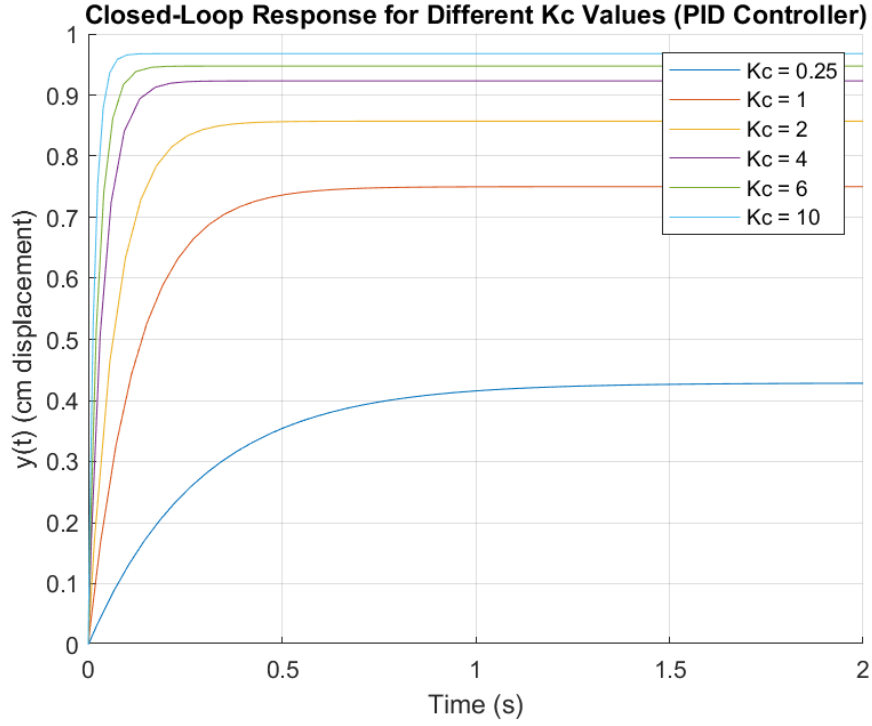

Figure 6: Closed-loop step response for different controller gains

(1.6) We want to find the controller gain K_c such that the offset is less than 0.08. The offset is given by:

$$\text{Offset} = \lim_{t \to \infty} (r(t) - y(t))$$

Using the Final Value Theorem:

$$\text{Offset} = \lim_{s \to 0} \left(1 - \frac{K_c \cdot G_p}{1 + K_c \cdot G_p} \right)$$

With $G_p = \frac{6}{s+2}$, we evaluate at $s = 0$:

$$\text{Offset} = 1 - \frac{K_c \cdot \frac{6}{0+2}}{1 + K_c \cdot \frac{6}{0+2}} = 1 - \frac{3K_c}{1 + 3K_c}$$

We require that the offset be less than 0.08:

$$0.08 > 1 - \frac{3K_c}{1 + 3K_c}$$

Rearranging the inequality:

$$\frac{3K_c}{1 + 3K_c} > 1 - 0.08 = 0.92$$

$$3K_c > 0.92(1 + 3K_c)$$

$$K_c > 3.8333...$$

Therefore, the controller gain K_c must be greater than approximately 3.83 to achieve an offset less than 0.08.

(1.7) We change from the proportional controller to the proportional-integral controller by adjusting the transfer function $G_c(s)$ to include an integral term. The transfer function for the PI controller is given by:

$$G_c(s) = K_c \left(1 + \frac{1}{\tau_I s} \right)$$

Where $K_c = 3$ and $\tau_I = 0.05$. The response of the system is shown in Figure 7.
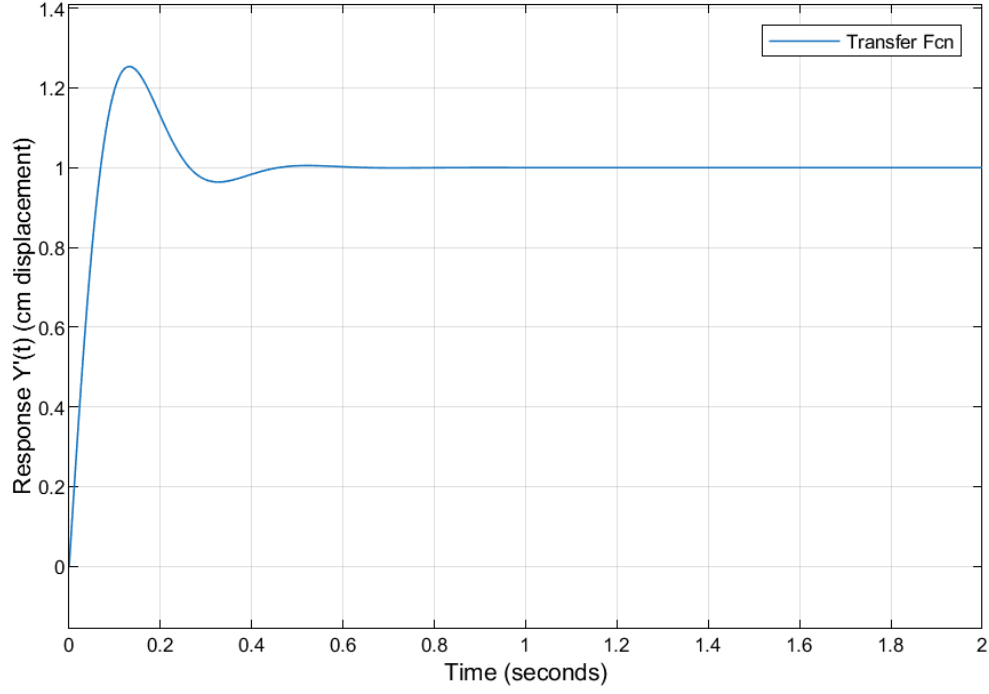
Figure 7: Closed-loop step response with a PI controller

(1.8) Given the below two transfer functions in our system:

$$G_c = K_c \left(1 + \frac{1}{\tau_I s} \right)$$

$$G_p = \frac{6}{s+2} = \frac{K_p}{\tau_p s + 1}$$

where,

$$K_p = 3, \quad \tau_p = \frac{1}{2}, \quad K_c = 3, \quad \tau_I = 0.05$$

Together, the closed-loop transfer function for the system becomes:

$$\frac{Y'(s)}{R'(s)} = \frac{\tau_I s + 1}{\frac{\tau_I \tau_p}{K_c K_p} s^2 + \tau_I \frac{(1+K_c K_p)}{K_c K_p} s + 1}$$

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We can use the equations as presented in the lecture to calculate the time constant τ and the damping factor ζ.

$$\tau = \sqrt{\frac{\tau_I \tau_p}{K_c K_p}} = \sqrt{\frac{(0.05)(\frac{1}{2})}{(3)(3)}} = \sqrt{0.00278}$$

$$\tau \approx 0.0527$$

$$\zeta = \frac{1}{2}(1 + K_c K_p)\sqrt{\frac{\tau_I}{K_c K_p \tau_p}} = \frac{1}{2}(1 + 9)\sqrt{\frac{0.05}{(3)(3)(\frac{1}{2})}}$$

$$\zeta \approx 0.526$$

We also derive the closed-loop gain K.

$$K = \tau_I s + 1 = 0.05s + 1$$

The calculated time constant is approximately $\tau \approx 0.0527$, which suggests the system should settle relatively quickly. The expected settling time, approximately $4\tau = 0.21s$, can be seen in Figure 7, where it settles around that point.

The damping factor was found to be $\zeta \approx 0.526$, suggesting a slightly underdamped response. This means the system should exhibit small oscillations but should not overshoot excessively, as observed in the above figure.

The closed-loop gain is given by $K = 0.05s + 1$, which means the gain varies with frequency rather than being a simple constant. Evaluating it at steady-state $(s = 0)$, we get $K(0) = 1$, which confirms that the system should track a step input without steady-state error. Over time, our system approaches 1 as expected.

(1.9) The closed-loop system for integral times $\tau_I = 0.01, 0.05, 0.1, 0.5, 1, 2$ are shown in Figure 8. As the integral time decreases, the system has an increased speed of response but also a higher relative overshoot.
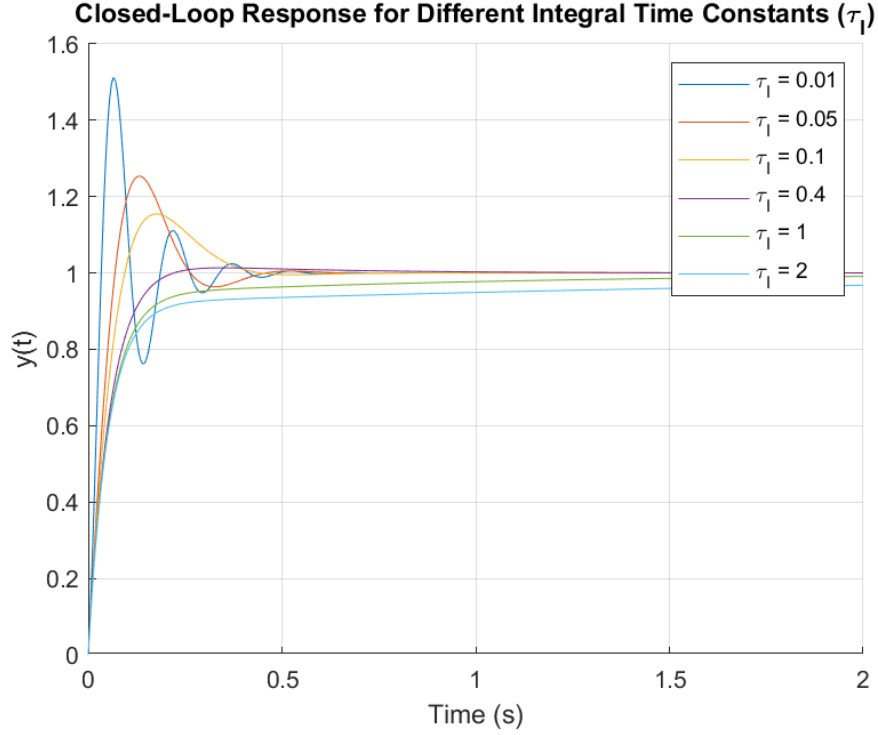
Figure 8: Closed-loop step response for different integral times

From comparing the different integral time constants, we can see that τ_I dictates in a PI controller the trade-off between response speed and stability. Small values (e.g., $\tau_I = 0.01$) yield fast responses but can cause excessive overshoot and oscillations. Larger values (e.g., $\tau_I = 2$) improve stability with less overshoot but result in slower responses. Tuning τ_I aims to find an optimal value, like $\tau_I = 0.4$, that balances speed and stability according to the specific process requirements. This is due to the mathematical relationship between τ_I and the damping factor and time constant, which affect the system's response characteristics.

(1.10) The surface plots are shown below in Figure 9 and Figure 10. The surface plots show the relationship between the integral time constant τ_I and the damping factor ζ with the settling time and the relative overshoot.
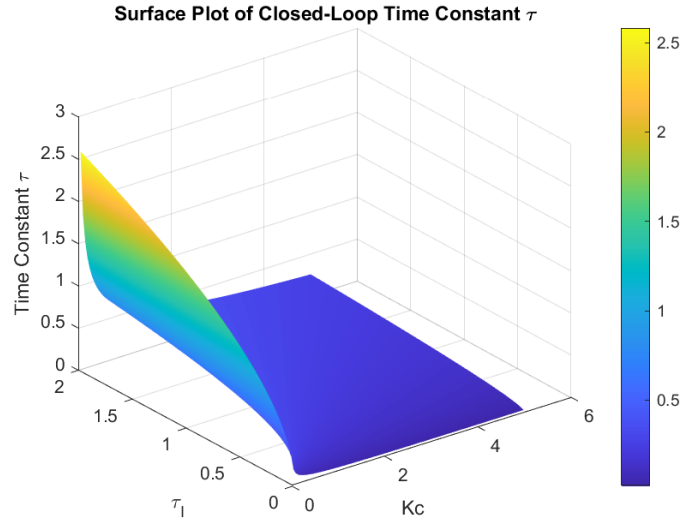
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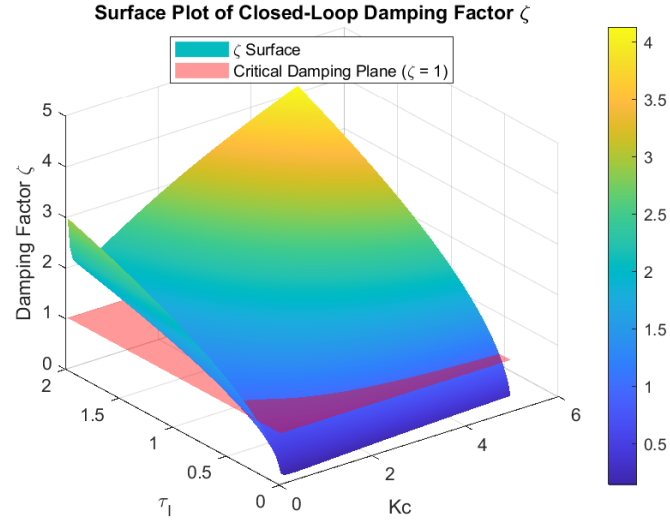Figure 9: Surface plot of settling time and integral time constant

Figure 10: Surface plot of relative overshoot and integral time constant

The surface plots for the closed-loop time constant (τ) and damping factor (ζ) reveal important insights into the system's dynamics and the trade-offs between

speed and stability. The time constant plot shows that τ decreases as the proportional gain (K_c) increases, indicating a faster system response. However, very high K_c values can lead to instability, as seen in the damping factor plot. Similarly, smaller integral time constants (τ_I) reduce τ, but overly small τ_I can destabilize the system. The goal is to minimize τ for a fast response while ensuring the system remains stable.

The damping factor plot highlights that a value of ζ close to 1 is ideal, as it represents critical damping, where the system responds quickly without oscillations. For small K_c, the system is underdamped $(\zeta < 1)$, leading to oscillations, while very large K_c results in overdamping $(\zeta > 1)$, causing a sluggish response. Larger τ_I values increase ζ, while smaller τ_I reduce it, potentially leading to instability.

To optimize the system, we aim to balance speed and stability by selecting K_c and τ_I values that minimize τ while keeping ζ close to 1. From the plots, moderate K_c values (e.g., 2–4) and small-to-moderate τ_I values (e.g., 0.5–1) are likely to achieve this balance, providing a fast and stable response.

(1.11) The curve that represents the intersection between the surface $\zeta = f(K_c, \tau_I)$ and plane $\zeta = 1$. The curve is shown in Figure 11.
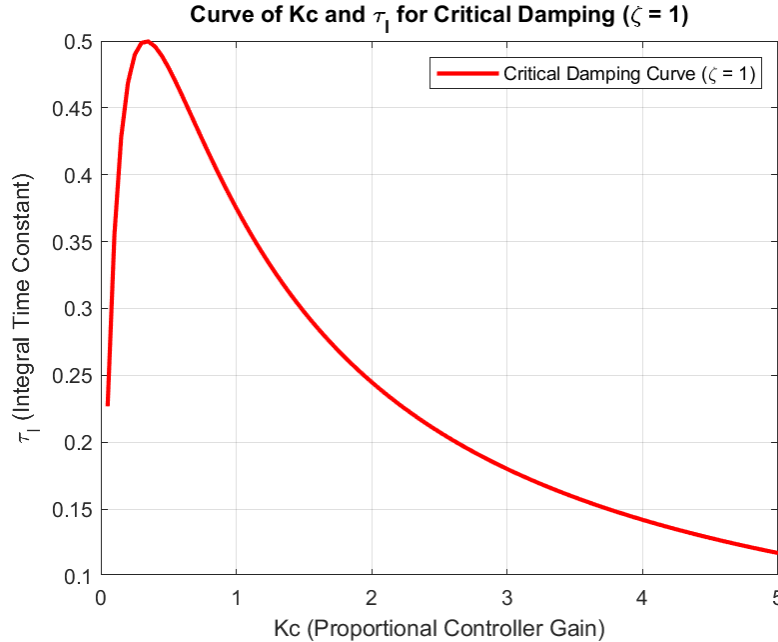

Figure 11: Intersection curve between the surface and plane

(1.12) To optimize the system, we want to choose parameters that also have the lowest closed-loop time constant, as well as having a damping factor of 1. The closed-loop time constant is thus overlayed over the Figure 11 in Figure 12.
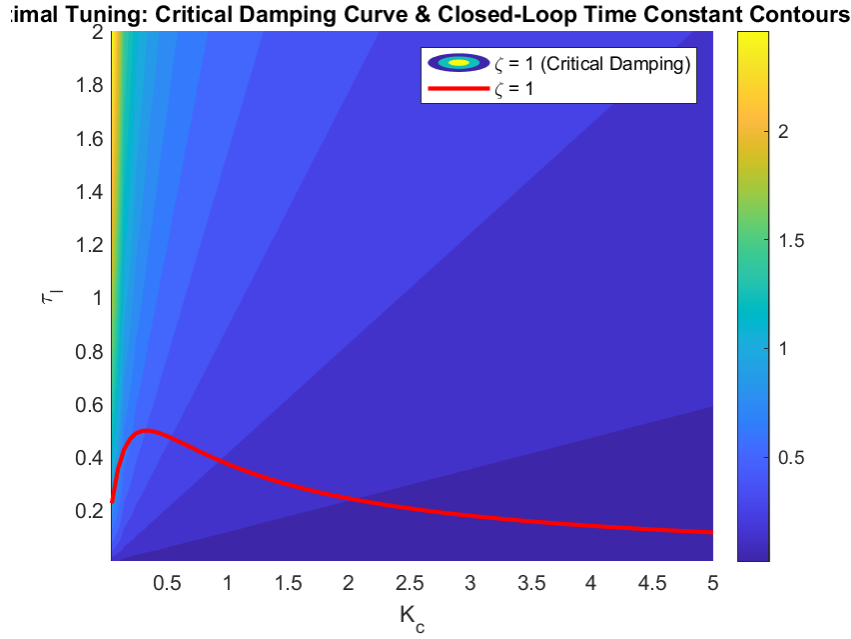

Figure 12: Intersection curve between the surface and plane with the closed-loop time constant

Given the constrainst that $K_c \leq 5$, the optimal values that follows the rational above occurs in Figure 12 at $K_c = 5$ and $\tau_I = 0.117187$. This point represents the optimal balance between speed and stability, ensuring a fast response without oscillations.

(1.13) Graphing the above optimal tuning parameters, we get the following results for yt. It exhibits the system characteristics that we were optimizing for in thh previous questions – we have the quickest response that could occur while critically damped, as shown in Figure ??.
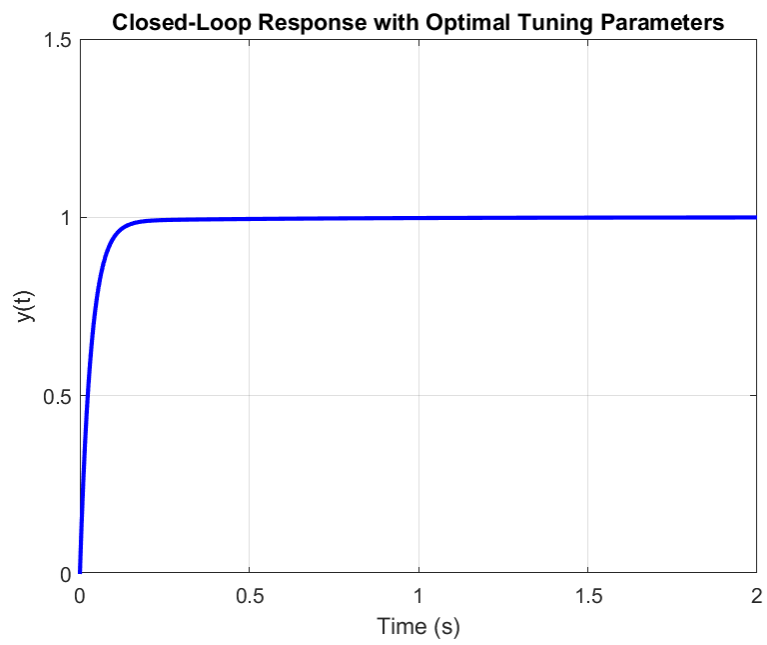
Figure 13: System response with optimal tuning parameters

2. Question 2

   (2.1) placeholder