IBEHS 4A03 Assignment #2
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1. Question 1

(1.1) The pole-zero maps for $Y_1(s)$ and $Y_2(s)$ help us see system stability.
- The left plot shows the pole-zero locations for $Y_1(s)$, which has poles in the left half-plane, indicating a stable system. The poles are: $s = 0, -2, -3$.
- The right plot for $Y_2(s)$ shows at minimum 1 pole in the right half-plane, meaning the system is unstable and will have an exponentially growing response. The poles are: $s = 0, 2, 1 + 2i, 1 - 2i$.

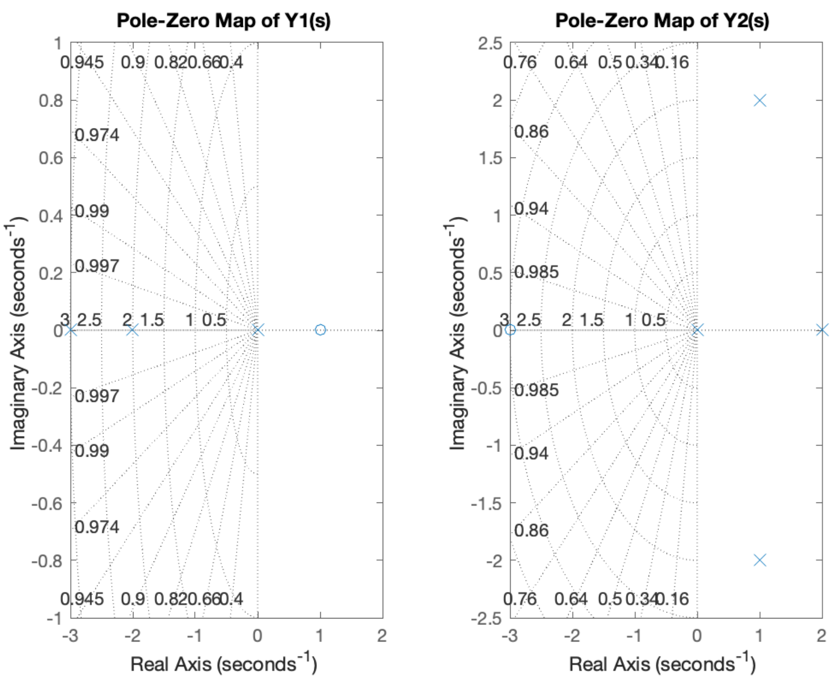

Figure 1: Pole-Zero Maps for $Y_1(s)$ and $Y_2(s)$

(1.2) The Final Value Theorem was used to determine the steady-state values (via hand calculation and using limit() in Matlab):

$$\lim_{t\to\infty} y_1(t) = \lim_{s\to 0} sY_1(s) = -\frac{1}{6}$$

$$\lim_{t\to\infty} y_2(t) = \lim_{s\to 0} sY_2(s) = -\frac{3}{10}$$

Q1

b)

$$\lim_{t\to\infty} y_1(t) = \lim_{s\to 0} sY_1(s) = \frac{\cancel{s}(s-1)e^{-s}}{\cancel{s}(s+2)(s+3)}$$

$$= \frac{-1}{6}$$

$$\lim_{t\to\infty} y_2(t) = \lim_{s\to 0} sY_2(s) = \frac{\cancel{s}(s+3)}{\cancel{s}(s-2)(s^2-2s+5)}$$

$$= \frac{3}{(-2)(5)}$$

$$= \frac{-3}{10}$$

Figure 2: Final Value Theorem for $Y_1(s)$ and $Y_2(s)$

The calculations confirm that $Y_1(s)$ settles to a finite negative value, whereas $Y_2(s)$, despite being unstable in impulse response, has a well-defined steady-state when analyzed using Final Value Theorem.

(1.3) The impulse response for both systems was computed to observe the effect of the pole locations.
- The response of $Y_1(s)$ looks to be bounded confirming stability.
- The response of $Y_2(s)$ grows exponentially, which aligns with its unstable pole.

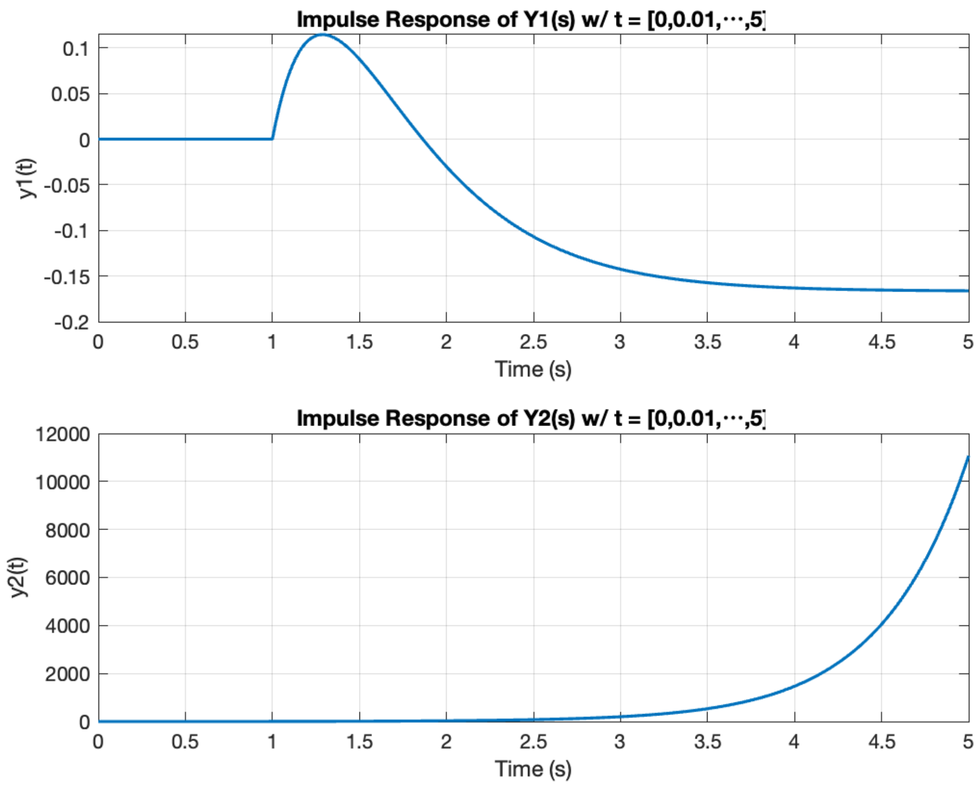

Figure 3: Impulse Response of $Y_1(s)$ and $Y_2(s)$ for $t \in [0, 5]$

Extending the time range further highlights the unstable growth of $Y_2(s)$. Not only that, if you zoom into $Y_1(s)$, you can see that the curve looks a little patchy since the time steps aren't small enough:

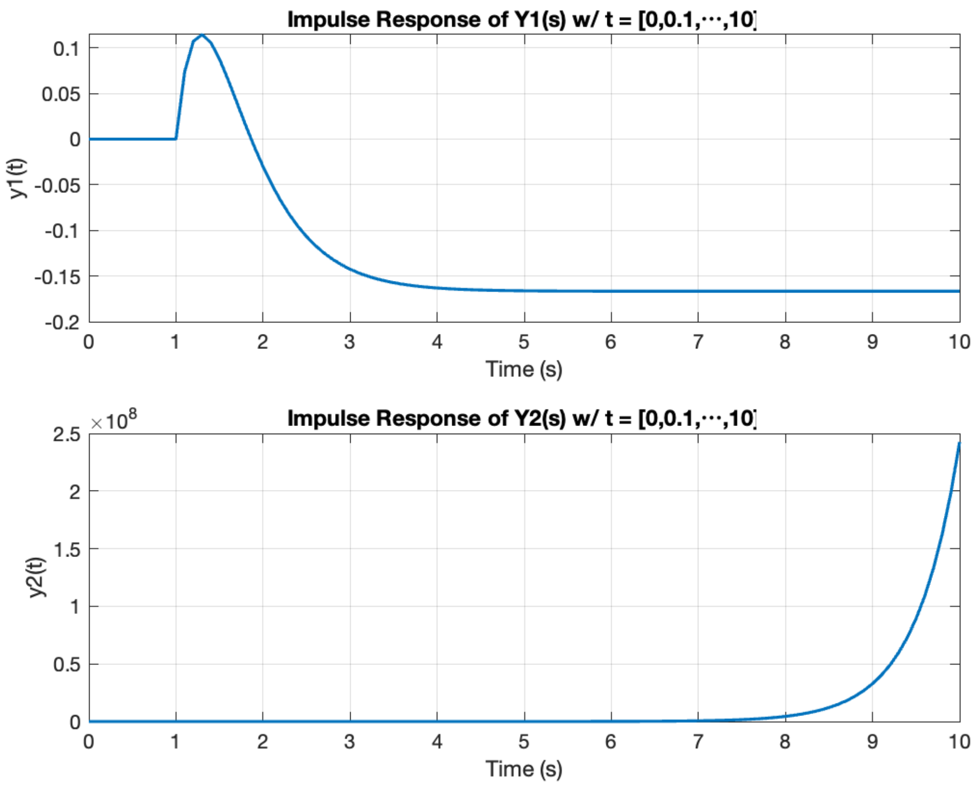

Figure 4: Impulse Response of $Y_1(s)$ and $Y_2(s)$ for $t \in [0, 10]$

2. Question 2

(2.1) Below we've derived the transfer function model that relates the change in temperature of the thermocoupled T to a change in the furnace temperature T_F.

2.1

$$\frac{dT}{dt} = \frac{\epsilon A \sigma}{m c_p}\left(T_F^4 - T^4\right) \quad \Big\} \; f(t)$$

$$\left.\frac{\partial f}{\partial T_F}\right|_{\bar{T}_F, \bar{T}} = \frac{4 \epsilon A \sigma}{m c_p}\bar{T}_F^3 \qquad \left.\frac{\partial f}{\partial T}\right|_{\bar{T}_F, \bar{T}} = \frac{-4 \epsilon A \sigma}{m c_p}\bar{T}^3$$

$$\frac{dT'}{dt} = \frac{4 \epsilon A \sigma}{m c_p}\bar{T}_F^3 \cdot T_F' - \frac{4 \epsilon A \sigma}{m c_p}\bar{T}^3 \cdot T'$$

$$= \frac{4 \epsilon A \sigma}{m c_p}\bar{T}^3 \left(T_F' - T'\right)$$

Laplace!

$$sT'(s) = \frac{4 \epsilon A \sigma}{m c_p}\bar{T}_F^3 T_F'(s) - \frac{4 \epsilon A \sigma}{m c_p}\bar{T}^3 T'(s)$$

$$T'(s)\left[s + \frac{4 \epsilon A \sigma}{m c_p}\bar{T}^3\right] = \frac{4 \epsilon A \sigma}{m c_p}\bar{T}_F^3 T_F'(s)$$

$$\frac{T'(s)}{T_F'(s)} = \frac{\frac{4 \epsilon A \sigma}{m c_p}\bar{T}_F^3}{s + \frac{4 \epsilon A \sigma}{m c_p}\bar{T}^3}$$

$$= \frac{\bar{T}_F^3}{\frac{m c_p}{4 \epsilon A \sigma} s + \bar{T}^3}$$

Figure 5: Hand calculation to find $T'(s)/T_F'(s)$

(2.2) The thermocouple's response to a 30°C drop in furnace temperature was simulated using the linearized transfer function. The step response of the thermocouple shows a gradual decrease in temperature, confirming that the thermocouple lags behind furnace temperature changes.

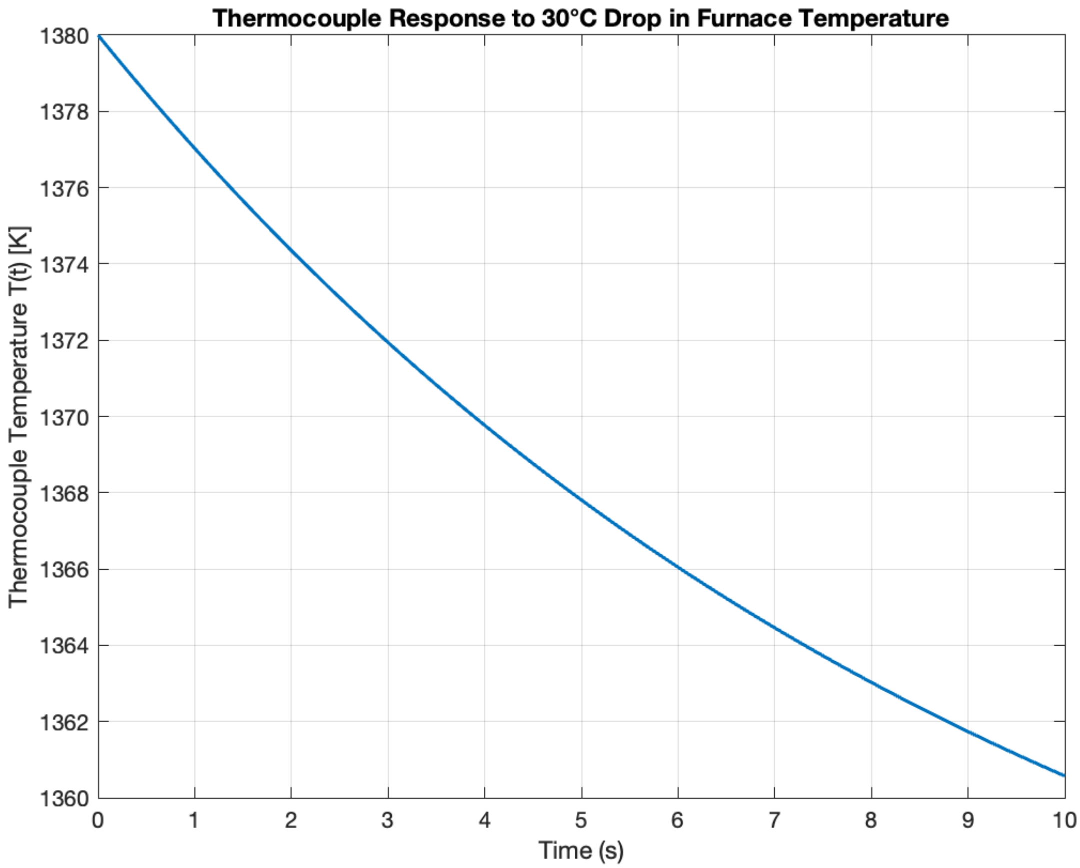

Figure 6: Thermocouple Response to 30°C Drop in Furnace Temperature

(2.3) The nonlinear ODE model for the thermocouple temperature was simulated using ode45(). The resulting response follows a slightly curved trajectory, showing a more gradual cooling effect compared to the linearized model. You can see here, that it is very similar to the linearized model, it is a little more rapid towards the start and then slows down as time goes on.

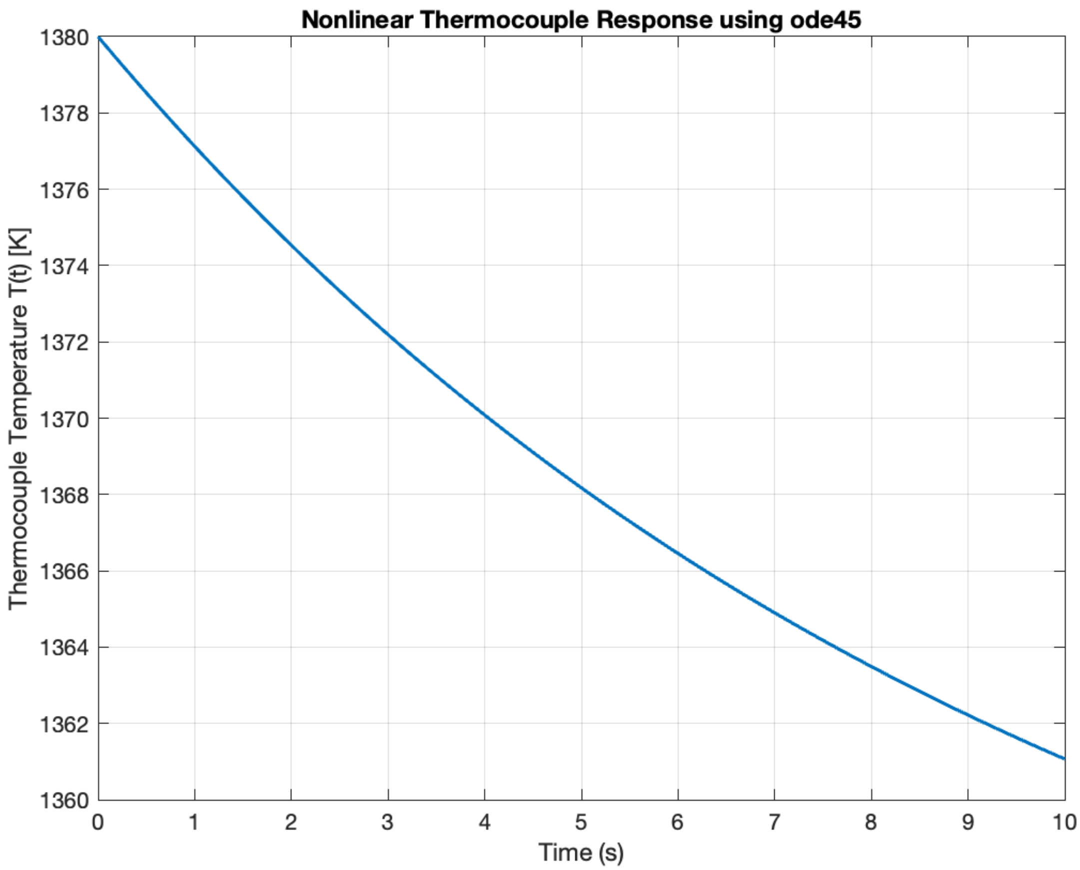

Figure 7: Nonlinear Thermocouple Response using ode45

(2.4) Radiative heat transfer follows the Stefan-Boltzmann Law, which is temperature-dependent and nonlinear. This means the rate of heat loss changes more significantly at higher temperatures compared to a simple linear approximation. Due to this, we expected to see that a comparison of the linear and nonlinear models would show that the linearized transfer function underestimates the cooling rate. What we actually saw was that with a -30°C change in temperature, the linearized model predicts a faster response by just a little compared to the non-linear model. With the +30°C change, the non-linear model was a little faster, but again very similar responses. This shows that the linearized model estimated the true temperature quite well.

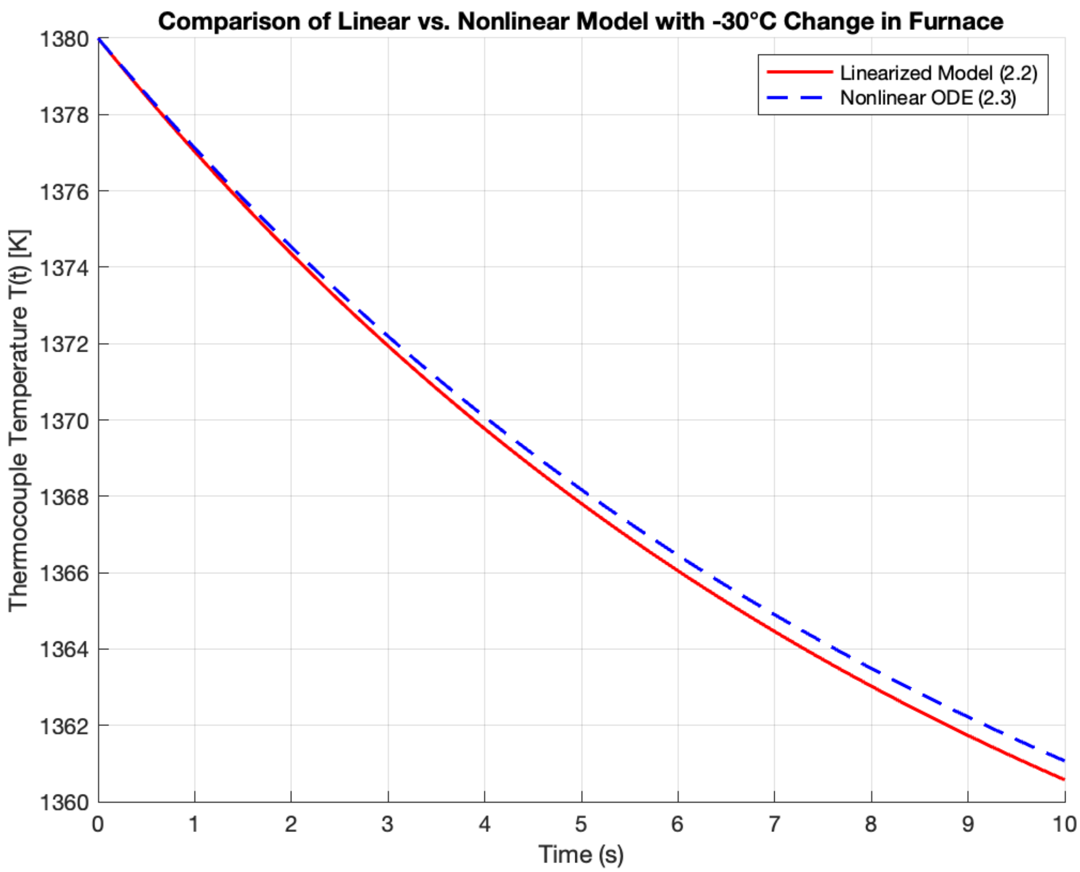

Figure 8: Comparison of Linear vs. Nonlinear Model with +30°C Change in Furnace

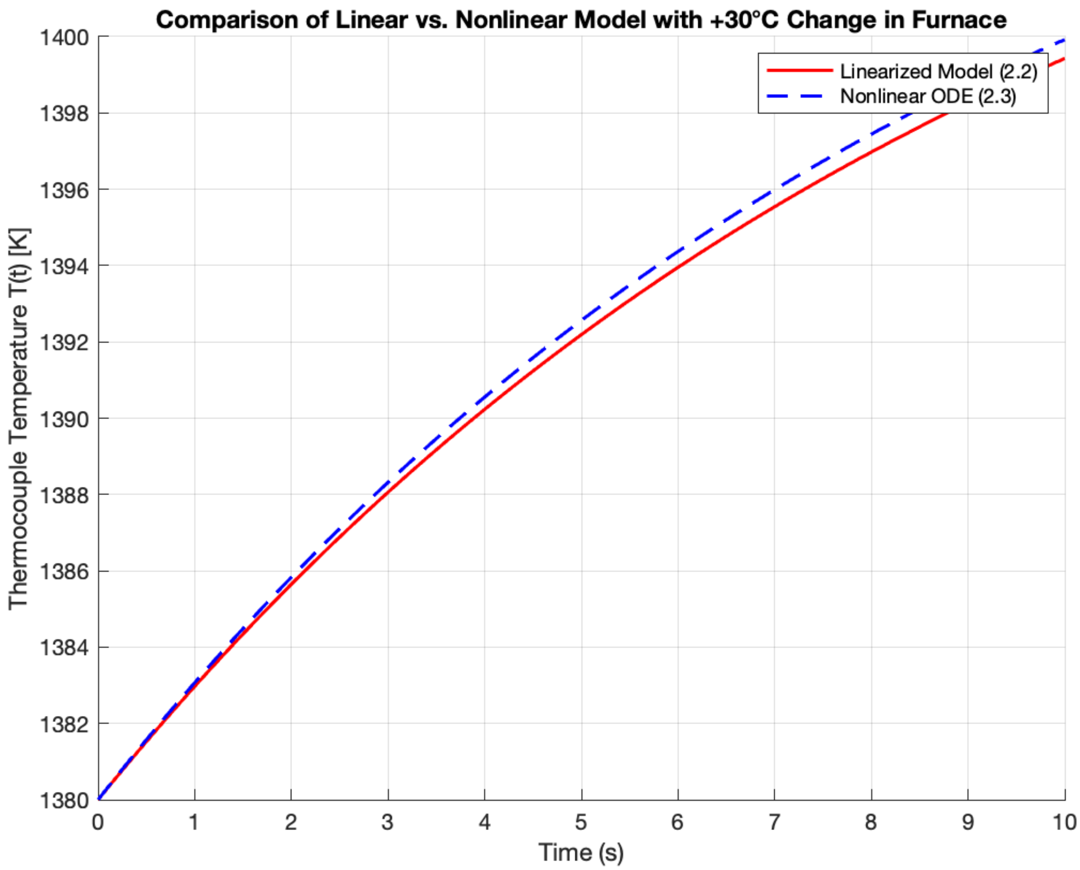

Figure 9: Comparison of Linear vs. Nonlinear Model with -30°C Change in Furnace

(2.5) In this case, we analyzed the effect of heat input $Q(t)$ on the thermocouple temperature. The furnace response to $Q(t)$ was modeled, and the thermocouple's response was calculated by multiplying the transfer functions:

$$\frac{T(s)}{Q(s)} = \frac{T(s)}{T_F(s)} \times \frac{T_F(s)}{Q(s)}$$

2.5 Furnice temp (T_F)

$$\frac{T_F(s)}{Q(s)} = \frac{1}{s+1} \qquad \frac{T(s)}{T_F(s)} = \frac{\bar{T}_F^{\,3}}{\frac{mc_p}{4\epsilon A\sigma}s + \bar{T}^3}$$

$$\frac{T_F(s)}{Q(s)} = \frac{1}{s+1} \qquad Q(s) = \frac{20}{s}$$

$$T_F(s) = 20\,\frac{1}{s(s+1)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$1 = A(s+1) + B(s)$$

$$s=0: \quad A = 1$$

$$s=-1: \quad B = -1$$

$$T_F(s) = 20\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

$$\mathcal{L}^{-1}\{T_F(s)\} = T_F(t) = 20(1 - e^{-t})$$

Measured Thermocoupled Response

$$\frac{T(s)}{Q(s)} = \frac{T(s)}{T_F(s)} \times \frac{T_F(s)}{Q(s)} = \left(\frac{\bar{T}_F^{\,3}}{\frac{mc_p}{4\epsilon A\sigma}s + \bar{T}^3}\right)\left(\frac{1}{s+1}\right)$$

Figure 10: Hand calculation to find $T_F(s)$ and $T(s)$ given the new $Q(s)$ input.

The final response shows a delayed and gradual increase in temperature, confirming that both the furnace and thermocouple introduce time lag in heat transfer.
The final response shows a delayed and gradual increase in temperature, confirming that both the furnace and thermocouple introduce time lag in heat transfer.
Notably, $T_F(t)$ responds much faster to the step change in heat input, reaching its

new steady-state value quickly. In contrast, $T(t)$ (thermocouple reading) lags significantly, indicating that the thermocouple has a much slower thermal response. Knowing about this delay is critical in control applications, as for example, in this case, it means that relying solely on the thermocouple reading for feedback could lead to incorrect adjustments due to outdated temperature measurements.

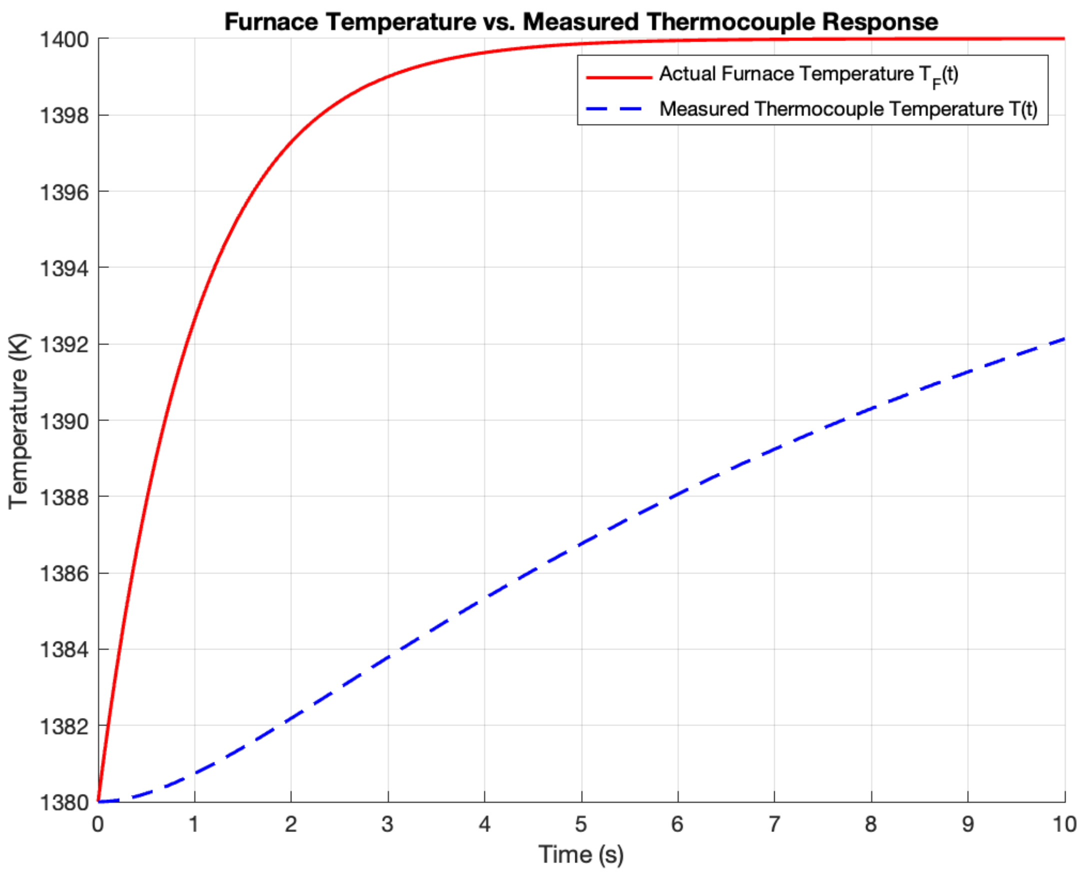

Figure 11: Furnace Temperature vs. Measured Thermocouple Response with Heat Input Q

(2.6) In this case, we analyzed the **effect of heat input $Q(t)$ on the thermocouple temperature**. The **furnace response to $Q(t)$ was modeled**, and the thermocouple's response was calculated using the cascaded transfer functions:

$$\frac{T(s)}{Q(s)} = \frac{T(s)}{T_F(s)} \times \frac{T_F(s)}{Q(s)}$$

The results show that the furnace temperature $T_F(t)$ increases rapidly in response to the applied heat input and reaches its steady-state value within a few seconds. In contrast, the thermocouple temperature $T(t)$ rises much more gradually indicating a significant time delay in its response.

This delay occurs because the thermocouple absorbs heat primarily through radiation, which is a slower process compared to the direct heating of the furnace. As a result, the thermocouple reading does not immediately reflect the actual furnace temperature, but instead lags behind, converging toward the final temperature over time (which it doesn't even reach in the 10 seconds).

In control applications, this measurement lag is critical to know because relying solely on $T(t)$ for feedback could lead to delayed or incorrect adjustments. To improve performance, a control system should account for this delay and maybe add some other elements or logic to work around it.

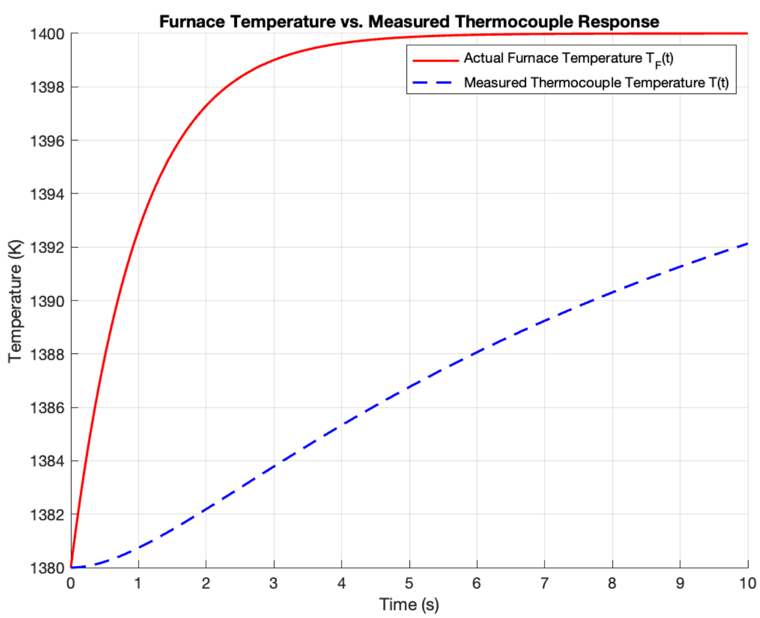

Figure 12: Furnace Temperature vs. Measured Thermocouple Response with Heat Input Q