IBEHS 4A03 Assignment #1
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1. Question 1

   (1.1) yapyapyap

2. Question 2

(2.1) The ordinary differential equation (ODE) governing the temperature $T(t)$ is:

$$\frac{dT(t)}{dt} = \frac{Q_f(t) - UA(T(t) - T_a)}{\rho V c_p}$$

For this problem, the furnace is off, so $Q_f(t) = 0$. Substituting this into the equation:

$$\frac{dT(t)}{dt} = \frac{-UA(T(t) - T_a)}{\rho V c_p}$$

At steady state, the temperature $T(t)$ no longer changes with time, so:

$$\frac{dT(t)}{dt} = 0$$

Substitute this condition into the ODE:

$$0 = \frac{-UA(T(t) - T_a)}{\rho V c_p}$$

Solve:

$$T(t) - T_a = 0$$

Thus:

$$T(t) = T_a$$

(2.2) Using the derived equation for $T(t)$, the behaviour of the furnace as defined below, and the given definition of one step of the univariate Euler's method, we can plot Figure 1.

$$Q_f(t) = \begin{cases} 0 & \text{When } T(t) > 23°\text{C}, \\ 1.5 \times 10^6 & \text{When } T(t) < 17°\text{C}, \\ \text{unchanged} & \text{For all } 17 \leq T(t) \leq 23°\text{C}. \end{cases}$$

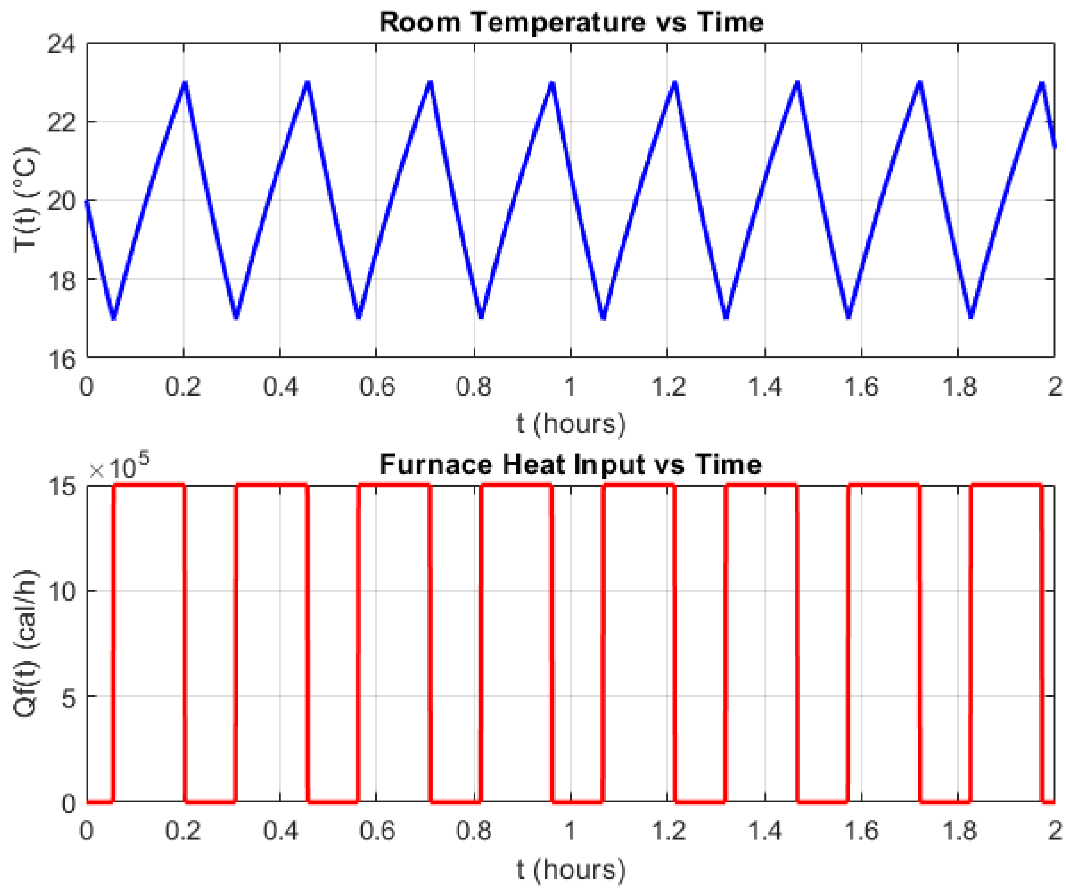

Figure 1: Temperature and Furnace Input Over Time

(2.3) To calculate the number of standard cubic meters of natural gas consumed, we follow the below equation:

Define $t_{on} \triangleq$ time the furnace was on in hours. The Matlab code says it was 1.182 hours

Define $Q_{f,on} \triangleq$ as the furnace heat input.

Define $\rho_e \triangleq$ as the energy density.

Define $e \triangleq$ as the efficiency.

$$\text{Volume of gas consumed} = \frac{t_{on} Q_{f,\text{on}}}{\rho_e e} = \frac{(1.182)(1.5 \times 10^6)}{(9 \times 10^6)(0.9)} = 0.2188888$$
$$\approx 0.219 \, \text{standard cubic meters}$$

We also confirm with unit analysis that we get cubic meters.

$$\text{Volume of gas consumed} = \frac{t_{\text{on}} Q_{f,\text{on}}}{\rho_e e} = \frac{\text{hour} \cdot \frac{\text{cal}}{\text{hour}}}{\frac{\text{cal}}{\text{m}^3}} = \text{m}^3$$

(2.4) With a new oscillating defintion of Ta, we get Figure 2 below demonstrating the temperature and furnace input over time.

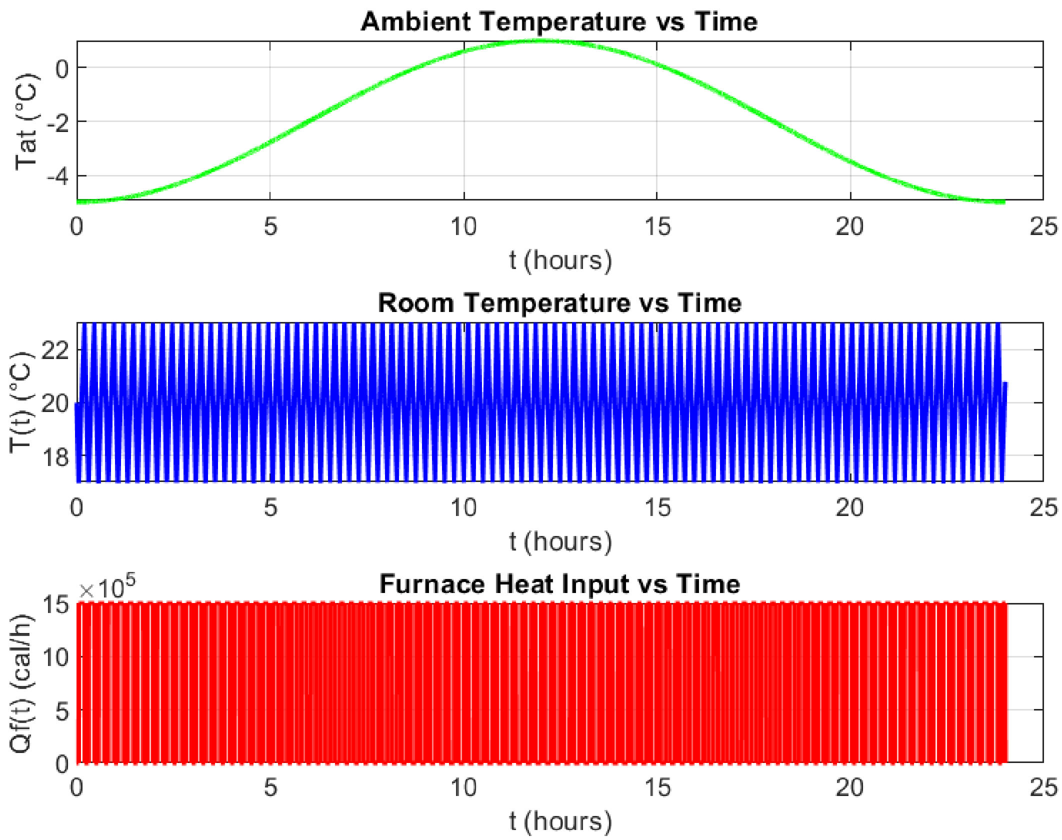

Figure 2: Temperature and Furnace Input Over Time

The simulation results align with the expected behavior of the system. The room temperature $T(t)$ is maintained within the desired range of 17°C to 23°C as the furnace responds to changes in the ambient temperature $T_a(t)$. When $T_a(t)$ is lower, the furnace operates more frequently to offset increased heat loss, while at higher $T_a(t)$, the furnace operates less often due to reduced heat loss. This behavior reflects the thermal dynamics of the system and the influence of the bang-bang control logic.

3. Question 3

(3.1) We are given the following ODE describing the current going thorugh the system.

$$V_s(t) = Ri(t) + L\frac{di(t)}{dt}$$

During steady state, the current $i(t)$ is constant, so the derivative of $i(t)$ with respect to time is zero. Substituting this into the ODE:

$$V_s(t) = Ri(t) + L(0)$$

Rearranging, we get the following equation for $i(t)$:

$$i(t) = \frac{V_s(t)}{R}$$

This is useful when designing a cirtcuit with a target current i(t) because you can control the voltage source to achieve the desired current while the resistance will always be a constant property of the circuit's hardware.

(3.2) The results of the Matlab code was plotted below in Figure 3. You can see that it plateaus to a value of 0.1. We know $V_s(t)$ was defined as a constant $5V$ and R as 50Ω. We know the steady state is $\frac{V_s(t)}{R}$ which in this case would give us $0.1A$, so we can see that the plot shows our steady state expression holds.

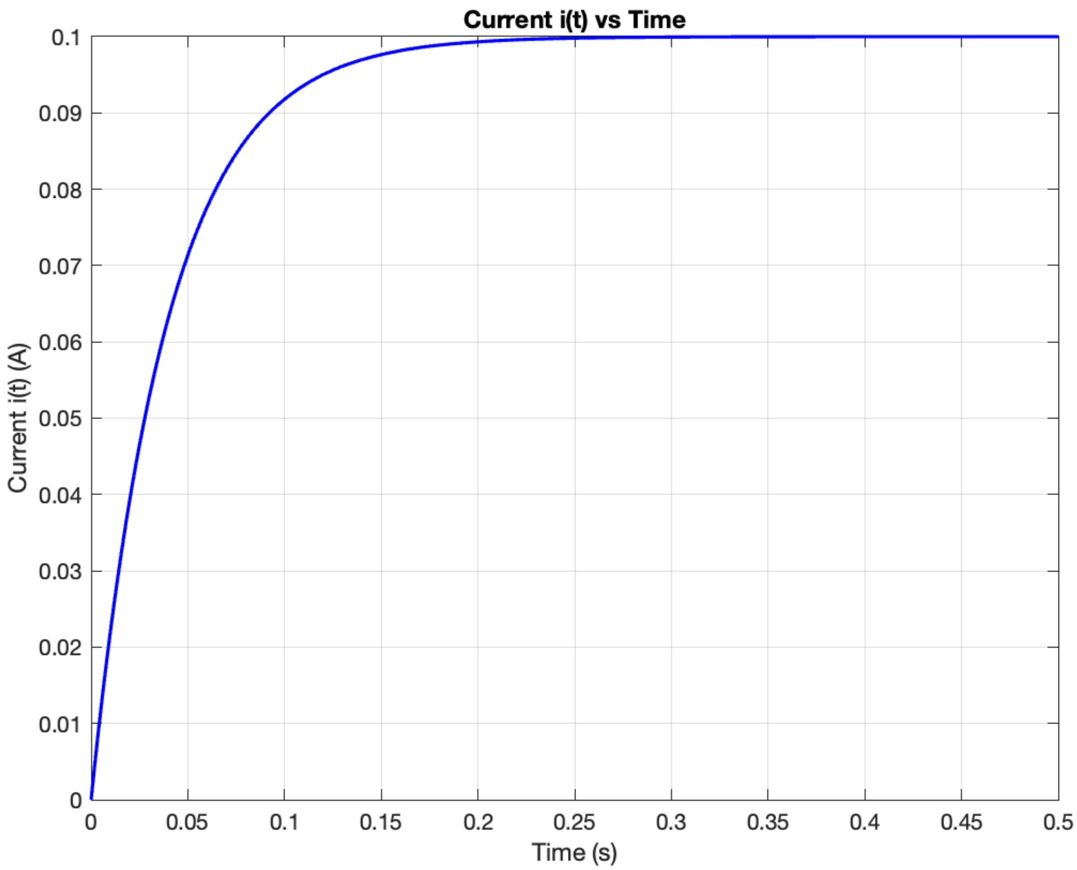

Figure 3: Current Over Time

(3.3)

$$V_s(t) = Ri(t) + L\frac{di(t)}{dt}$$

Based on the ODE above, inductance L should have no effect on the steady state current $i(t)$ because the derivative of $i(t)$ with respect to time is zero, and inductance is a coeffecient on the derivative. This means that the inductance L term in the ODE will not affect the current $i(t)$ when the system reaches steady state, and the current $i(t)$ will be determined solely by the voltage source $V_s(t)$ and the resistance R in the circuit. This can be seen below in Figure 4, where changing R causes the steady state current $i(t)$ to change, while changing L has no effect on the steady state current $i(t)$.

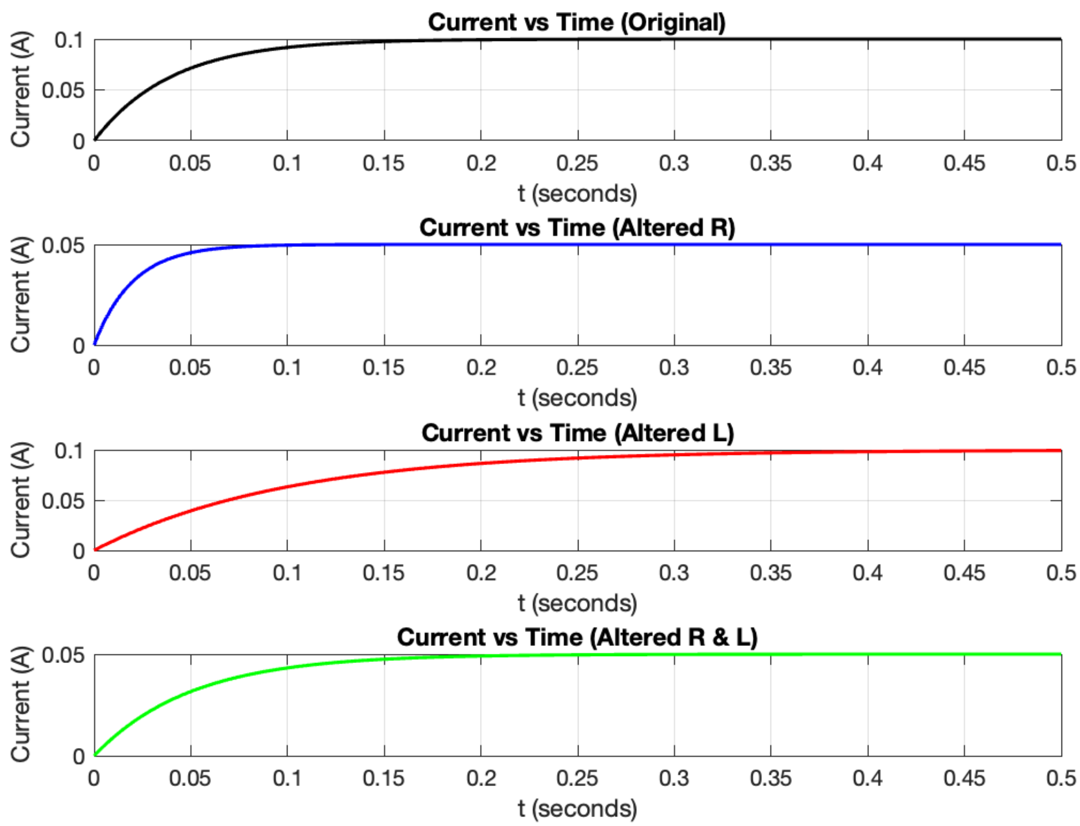

Figure 4: Current Over Time

(3.4) Figure 5 and 6 below demonstrates the LTI nature of the system. An LTI system is defined as a system that is both Linear and Time Invariant. A linear system is one that follows the condition that if $x(t) = \alpha x_1(t) + \beta x_2(t)$, then $y(t) = \alpha y_1(t) + \beta y_2(t)$. That is, the output of a system is a linear combination of the inputs. We can observe in Figure 5 that the combination of the inputs $V_s1 = 5V$ and $V_s2 = 10V$ results in the outputs $i_1(t)$ and $i_2(t)$, which is equal to $i_3(t).i_3(t)$ is the resultant output from the combination input of $V_s3 = V_s1 + V_s2 = 15V$. This is exhibits the linear property. We can also observe the system is time-invariant by looking at the equation — the equation has no coeffecients that change over time. We can also see in Figure 6 that when taking the input and adding time delay, it is equivalent to taking the original output and time shifting that. This proves the equation is time-invariant.

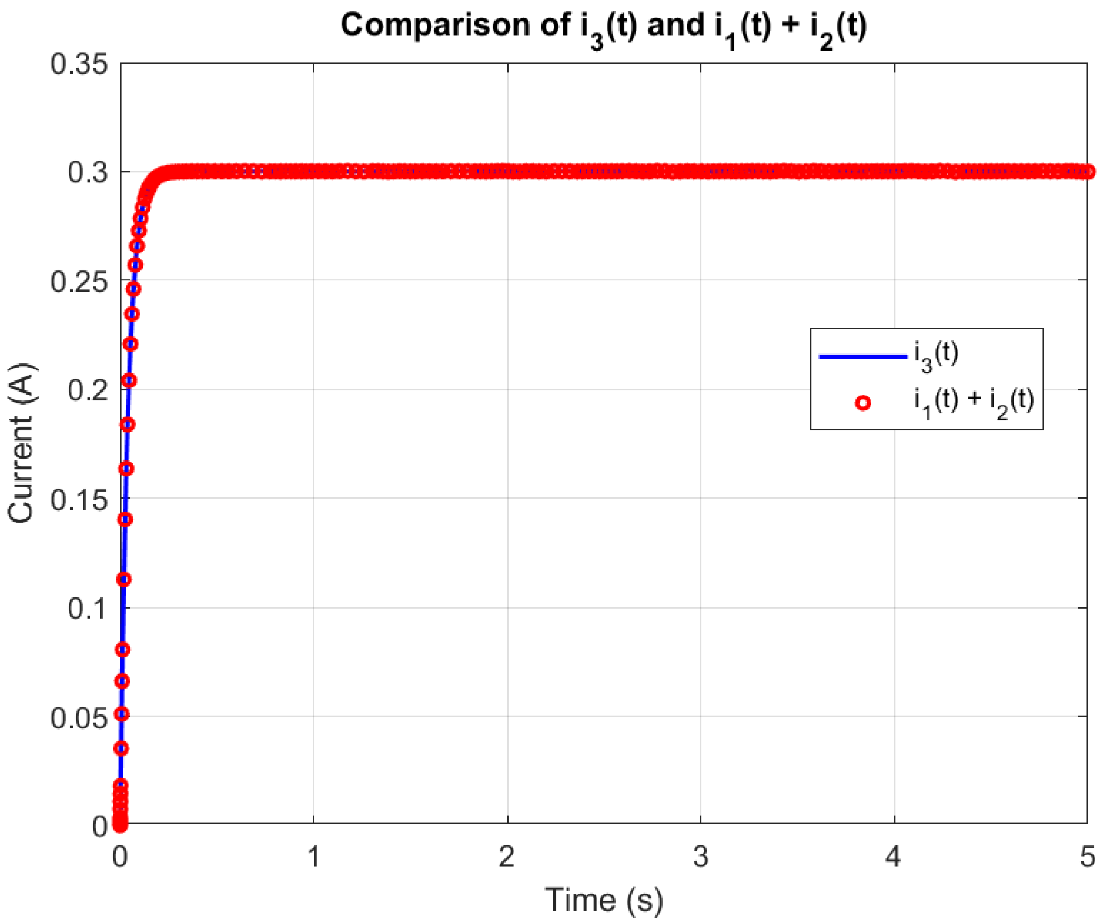

Figure 5: Comparing Linear Combination of Inputs to an Equivalent Output Over Time

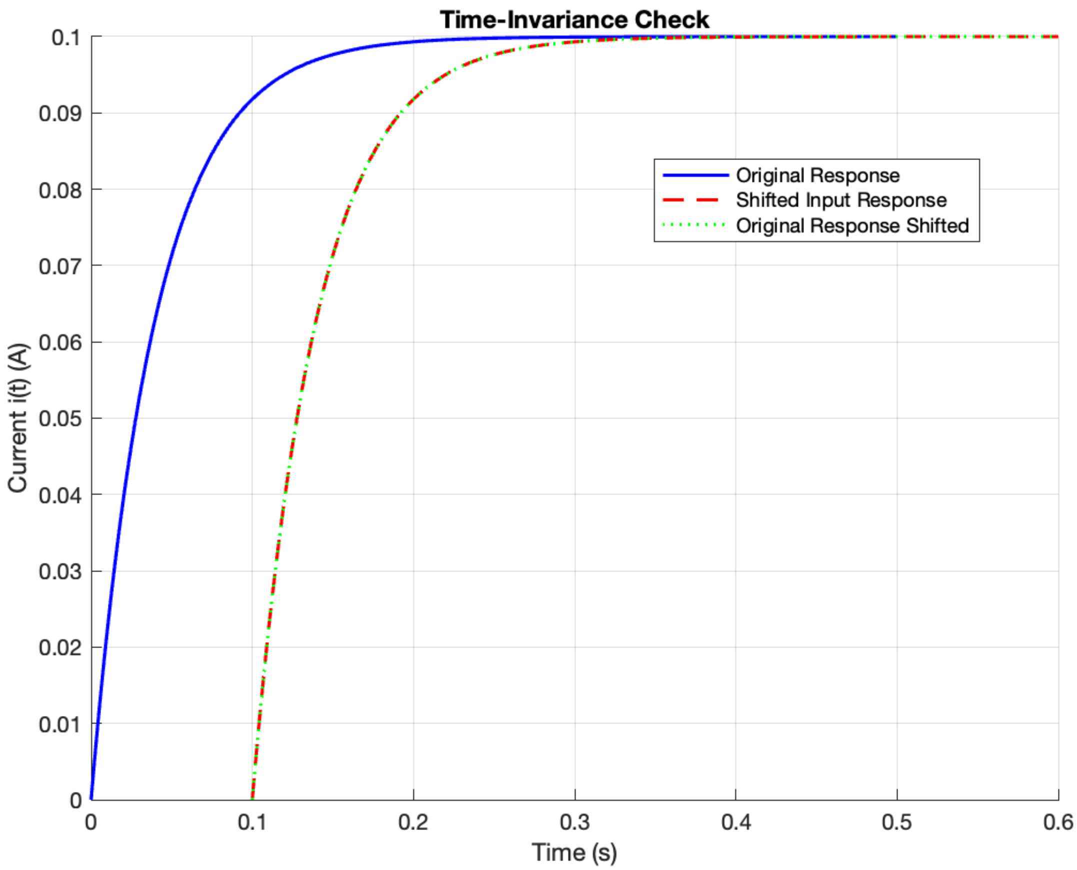

Figure 6: Comparing a Time-Shifted Input with the Time-Shifted Output of the Original Signal

4. Question 4
   (4.1) Part A
   SOLUTION
   (4.2) Part B
   SOLUTION

5. Question 5

(5.1) Part A

We are given the second-order differential equation:

$$2\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} - 2x(t) = te^{-2t}$$

Applying the Laplace transform to the differential equation:

$$2\left[s^2X(s) - sx(0) - \dot{x}(0)\right] + 3\left[sX(s) - x(0)\right] - 2X(s) = \mathcal{L}\left(te^{-2t}\right)$$

Given initial conditions $x(0) = 0$ and $\dot{x}(0) = -2$, this simplifies to:

$$2\left[s^2X(s) + 2\right] + 3sX(s) - 2X(s) = \frac{1}{(s+2)^2}$$

$$2s^2X(s) + 4 + 3sX(s) - 2X(s) = \frac{1}{(s+2)^2}$$

$$X(s)\left(2s^2 + 3s - 2\right) + 4 = \frac{1}{(s+2)^2}$$

$$X(s)\left[(s+2)(2s-1)\right] + 4 = \frac{1}{(s+2)^2}$$

$$X(s) = \frac{1 - 4(s+2)^2}{(s+2)^3(2s-1)}$$

(5.2) **Step 1: Partial Fraction Expansion**

We decompose $X(s)$ as:

$$X(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{2s-1}$$

Multiply through by the common denominator $(s+2)^3(2s-1)$:

$$1 - 4(s+2)^2 = (s+2)^2(2s-1)A + (s+2)(2s-1)B + (2s-1)C + (s+2)^3D$$

Step 2: Determine the Coefficients

To find A, B, C, and D, we use strategic substitutions for s and equate coefficients:
1. Substitute $s = -2$:

$$1 - 4(s+2)^2 = 0 \Rightarrow C = -\frac{1}{5}$$

2. Substitute $s = \frac{1}{2}$:

$$D = -\frac{192}{125}$$

3. Substitute $s = 0$:

$$A = \frac{96}{125}$$

4. Substitute $s = 1$:

$$B = -\frac{2}{25}$$

To find A, B, C, and D, we substitute strategic values of s and equate coefficients of powers of s.

1. Substitute $s = -2$ to find C

$$1 - 4(s+2)^2 = (s+2)^2(2s-1)A + (s+2)(2s-1)B + (2s-1)C + (s+2)^3 D$$

When $s = -2$:

$$s + 2 = 0 \quad \Rightarrow \quad (s+2)^2 = 0 \quad \text{and} \quad (s+2)^3 = 0$$

This simplifies the equation to:

$$(2s-1)C = 1 - 4(0)$$

At $s = -2$, $2(-2) - 1 = -5$, so:

$$-5C = 1 \quad \Rightarrow \quad C = -\frac{1}{5}$$

2. Substitute $s = \frac{1}{2}$ to find D

When $s = \frac{1}{2}$:

$$2s - 1 = 0 \quad \Rightarrow \quad (2s-1) = 0$$

$$(s+2)^3 D = 1 - 4\left(\frac{5}{2}\right)^2$$

$$s + 2 = \frac{1}{2} + 2 = \frac{5}{2}$$

$$1 - 4\left(\frac{5}{2}\right)^2 = 1 - 4 \cdot \frac{25}{4} = 1 - 25 = -24$$

$$\left(\frac{5}{2}\right)^3 D = -24 \quad \Rightarrow \quad \frac{125}{8} D = -24$$

$$D = -\frac{24 \cdot 8}{125} = -\frac{192}{125}$$

3. Substitute $s = 0$ to find A

When $s = 0$:

$$1 - 4(s+2)^2 = (s+2)^2(2s-1)A + (s+2)(2s-1)B + (2s-1)C + (s+2)^3 D$$

$$s + 2 = 2 \quad \text{and} \quad 2s - 1 = -1$$

$$1 - 4(2)^2 = (2)^2(-1)A + (2)(-1)B + (-1)C + (2)^3 D$$

$$1 - 16 = 4(-1)A - 2B - C + 8D$$

$$-15 = -4A - 2B - C + 8D$$

Substitute $C = -\frac{1}{5}$ and $D = -\frac{192}{125}$:

$$-15 = -4A - 2B - \left(-\frac{1}{5}\right) + 8\left(-\frac{192}{125}\right)$$

$$-15 = -4A - 2B - \frac{307}{125}$$

$$-1568 = -500A - 250B$$

$$\frac{1568}{250} = 2A + B$$

Simplify:

$$\frac{784}{125} = 2A + B$$

4. Substitute $s = 1$ to find the second equation for A and B

When $s = 1$:

$$1 - 4(s+2)^2 = (s+2)^2(2s-1)A + (s+2)(2s-1)B + (2s-1)C + (s+2)^3D$$

Substitute into the equation:

$$1 - 4(3)^2 = (3)^2(1)A + (3)(1)B + (1)C + (3)^3D$$
$$-35 = 9A + 3B + C + 27D$$

Substitute $C = -\frac{1}{5}$ and $D = -\frac{192}{125}$:

$$-35 = 9A + 3B - \frac{1}{5} + 27\left(-\frac{192}{125}\right)$$
$$-35 = 9A + 3B - \frac{5210}{125}$$
$$-\frac{167}{75} = 3A + B$$

5. Solve the system of equations

We have:

$$2A + B = \frac{784}{125}$$
$$3A + B = -\frac{167}{75}$$

Subtract the first equation from the second:

$$(3A + B) - (2A + B) = -\frac{167}{75} - \frac{784}{125}$$
$$A = -\frac{2}{25}$$

Substitute $A = -\frac{2}{25}$ into $2A + B = \frac{784}{125}$:

$$2\left(-\frac{2}{25}\right) + B = \frac{784}{125}$$
$$B = \frac{784}{125} + \frac{4}{25} = -\frac{2}{25}$$

These are the final values for the partial fraction:

$$A = \frac{96}{125}, B = -\frac{2}{25}, C = -\frac{1}{5}, D = -\frac{192}{125}$$

Step 3: Inverse Laplace Transform

We now take the inverse Laplace transform of each term:

$$X(s) = \frac{96}{125} \cdot \frac{1}{s+2} - \frac{2}{25} \cdot \frac{1}{(s+2)^2} - \frac{1}{5} \cdot \frac{1}{(s+2)^3} - \frac{192}{125} \cdot \frac{1}{2s-1}$$

Using the Laplace table, performed the inverse Laplace and got the time domain solution:

$$x(t) = \frac{96}{125}e^{-2t} - \frac{2}{25}te^{-2t} - \frac{1}{10}t^2e^{-2t} - \frac{192}{250}e^{t/2}$$

(5.3) As you can see in Figure 7 below, plotting both the numerical solution (using ode45) and the analytical solution (the time domain solution we solved in 5.2), they are equivalent over the time shown.

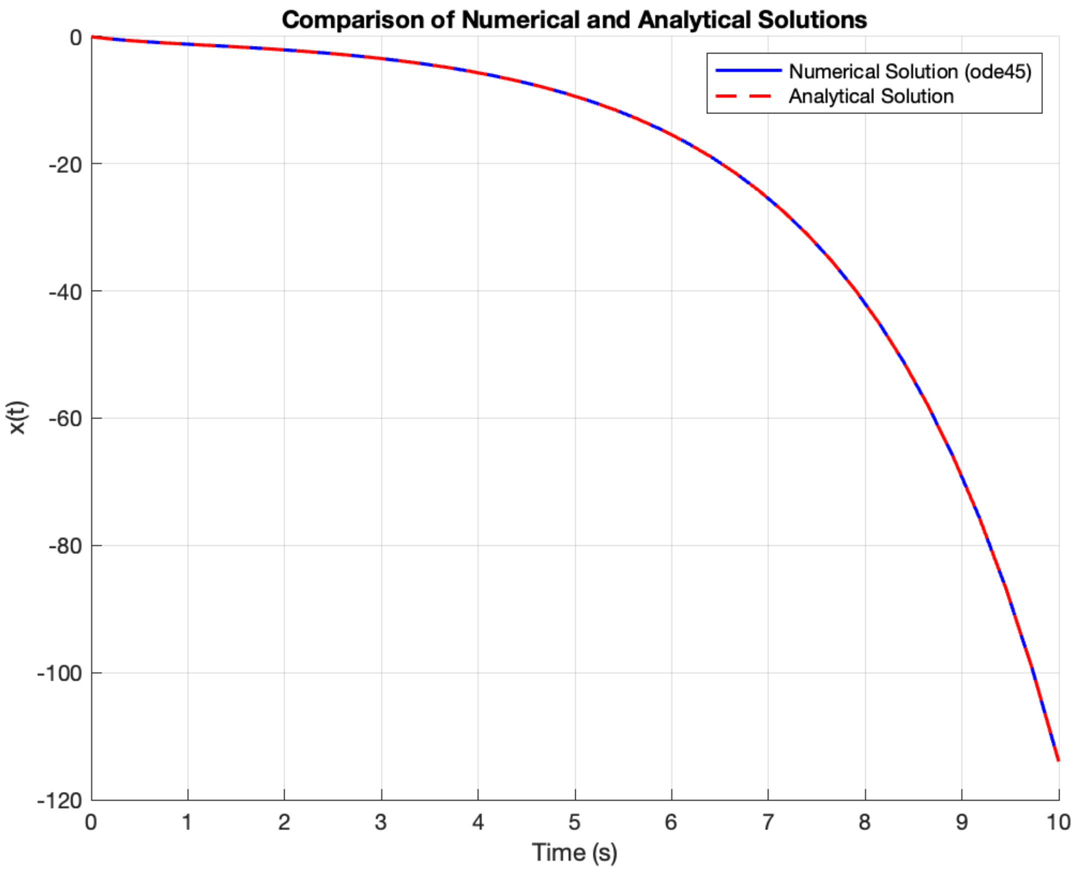

Figure 7: Plotting the Numerical vs Analytical Solution

References

[1] MATLAB Documentation: `ode45`. Available at: https://www.mathworks.com/help/matlab/ref/ode45.html.