Assignment 2 – Modeling and ODEs

To be completed in GROUPS OF UP TO TWO (2)
Due: Mar 02, 2025 @ 11:59pm via the A2L Drop Box
Grading: 6% of course grade (70 Points Available)

Purpose

In this assignment, we will use the Laplace Transform to help us analyze some exemplary and realistic dynamic systems. You will demonstrate the importance of transfer function poles and dynamics as they relate to stability. You will use linearization tools and comment on their accuracy using simulations.

Submission Instructions

Please submit this assignment *electronically* before the due date. Late submissions will **not** be accepted. Submit via the A2L dropbox. Be sure that you have the names and student numbers of all students on the front page of your submission. Submit your solutions as a **.pdf** file including all relevant figures, tables, and math. You may embed code in your solutions, but you **must submit your coded solutions**.

ONE group member is to submit a single .zip file that includes the solutions and code using the naming convention (for both the .zip and .pdf contained in the .zip):

`4A03_AXX_MACID1_MACID2`

Where `AXX` is the assignment number, and `MACID#` is the McMaster ID (*e.g.* neasej, NOT your student number) of the submitting group member(s). Up to three McMaster IDs can be included on a single assignment.

Up to 10 points may be deducted from your submission for sloppy or otherwise unprofessional work. This is rare, but possible. The definition of unprofessional work may include:

- Low-resolution screenshots of figures and tables.
- Giving no context to an answer relating to the task (*i.e.,* "See code" or "113.289" with no units, context, or discussion whatsoever).
- Clear changes in author denoted by format changes, blatant writing style changes (including AI-generated discussion), or other factors that may deduct from the cohesiveness of the report.
- Failing to provide references for work that is obviously not yours (particularly bad cases will be considered as academic dishonesty).

Problem 1 – Transfer Functions [10]

Consider the Laplace-domain functions below, which represent an output $Y(s)$ responding to an impulse response $U(s)$. For each expression $Y(s)$, please perform the following analyses.

- Plot the poles of $Y(s)$. Is the response oscillatory? Does $y(t)$ converge or diverge? [**4**]
- Use the final value theorem to compute $\lim_{t\to\infty} y(t)$, if it exists. If it does not exist, mention why. [**2**]
- Using the built-in `MATLAB` transfer function system and the analysis function `impulse()`, Plot $y(t)$ for both systems for time domains $t_1 = [0, 0.01, \cdots, 5]$ and $t_2 = [0, 0.1, \cdots, 10]$ for $y_1(t)$ and $y_2(t)$, respectively. Discuss the results briefly, especially if anything seems out of place. The use of these tools is covered in Tutorial 2. [**4**]

$$Y_1(s) = \frac{(s-1)e^{-s}}{s(s+2)(s+3)} \qquad Y_2(s) = \frac{(s+3)}{s(s-2)(s^2-2s+5)}$$

Problem 2 – First-Order Analysis and Linearization [20]

A thermocouple is a common measurement instrument for high-temperature applications. Consider the thermocouple from midterm 1 with area $A = 0.1\ \mathrm{cm}^2$, mass $m = 0.1\ \mathrm{g}$ and heat capacity $c_p = 0.4\ \mathrm{J \cdot g^{-1} \cdot K^{-1}}$ placed in a furnace of temperature T_F. At high temperatures, radiative heat transfer dominates and thus the energy balance that describes the thermocouple temperature T is:

$$mc_p \frac{dT}{dt} = \epsilon A \sigma (T_F^4 - T^4)$$

In the above expression, $\epsilon = 0.7$ is the emissivity (or "view angle") of the thermocouple and $\sigma = 5.669 \times 10^{-8}\ \mathrm{W \cdot m^{-2} \cdot K^{-4}}$ is the Stefan-Boltzmann constant. It is our objective to understand how quickly the thermocouple will respond to a change in furnace temperature.

2.1. Derive a transfer function model that relates the *change* in temperature of the thermocouple T to a *change* in the furnace temperature T_F. Linearize when necessary, assuming a linearization around some steady state $\bar{T} = \bar{T}_F$. [**4**]

2.2. If the furnace and thermocouple are typically at $\bar{T}_F = \bar{T} = 1380\ K$, plot the temperature of the thermocouple $T(t)$ for $t = [0, 0.01, 0.02, \cdots 10]$ seconds after the temperature in the furnace suddenly decreases by 30°C according to your transfer function model. You may use a built-in `MATLAB` function like `step()` or determine the time-domain expression for $T(t)$ at your discretion. What is the temperature of the thermocouple 10 seconds after the change? [**3**]

2.3. If the furnace and thermocouple are typically at $\bar{T}_F = \bar{T} = 1380\ K$, determine the temperature of the thermocouple $T(t)$ for $t = [0, 0.01, 0.02, \cdots 10]$ seconds after the temperature in the furnace suddenly decreases by 30°C by integrating the original nonlinear ODE. Integrate the ODE using `ode45()`. What is the temperature of the thermocouple 10 seconds after the change? [**5**]

2.4. Plot the simulated responses for $T(t)$ using the linear transfer function and nonlinear ODE on the same set of axes. Comment on any differences. If the typical temperature fluctuations in the furnace are within ± 30°C, is the linearized model appropriate for controller design? Explain. [**3**]

2.5. This system describes the response of a sensor to a change in state. It is important to distinguish that our goal when implementing a controller is to control the state $T_F(t)$, NOT the sensor temperature $T(t)$. Unfortunately, we rely on the measurement of $T(t)$ as a surrogate to the real temperature $T_F(t)$, which we can never know exactly.[1]

Consider a scenario where the furnace temperature is influenced by an external energy source (gas, electricity, or another application) $Q(s)$ such that $\frac{T_F(s)}{Q(s)} = \frac{1}{s+1}$. For $Q(s) = \frac{20}{s}$, plot $T_F(t)$ and $T(t)$ for $t = [0, 0.01, 0.02, \cdots 10]$ seconds on the same axes. Explain briefly how relying on the thermocouple measurement might affect a controller's ability to choose $Q(t)$ to control the furnace temperature. [**5**]

[1] Information we use in life and controllers is only as good as the measurement device we have. However, the concept is often overlooked, even by senior students in my home department. We typically go by the adage cribbed from George Orwell: "All sensors lie. Some sensors are useful."

Problem 3 – Second-Order Analysis [15]

Frequently as engineers and medical professionals, we are presented with a "black box" system that we know follows certain dynamics, but we need to use our own insight and understanding to model the system mathematically. We will often perform diagnostic tests to help us understand the system dynamics. For this problem, consider a catheter used to monitor a patient's distal blood pressure. The catheter senses a change in pressure $P(t)$, which it then converts to a voltage signal $V(t)$ before sending it to a piece of monitoring equipment (interestingly, it is then converted back to pressure units on-screen, but the information is transmitted entirely as an electrical signal).

Catheters are known to follow second-order dynamics that depend on the properties of the catheter itself as well as the fluid it is measuring. The voltage **output** of the catheter relates to the pressure **input** via the transfer function:

$$\frac{V(s)}{P(s)} = \frac{AK_{(P\to V)}}{\rho A L s^2 + bs + c}$$

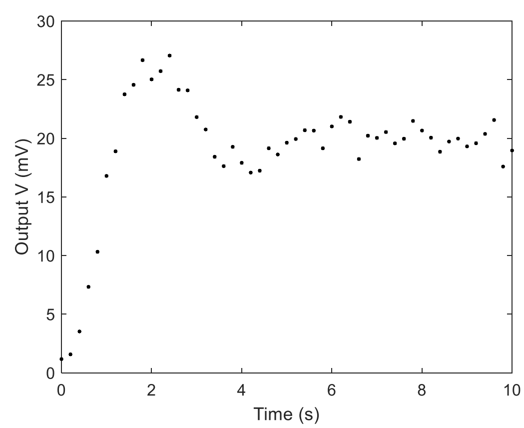
The output of the catheter depends on three properties, all influenced by the materials of construction:

- The lineal friction of the fluid moving through the catheter, c.
- The viscous friction of the fluid moving through the catheter, b.
- The gain that relates a change in output voltage to a change in pressure, $K_{P\to V}$.

For this catheter, we can assume the following physical dimensions and properties of the fluid. You may assume all units are appropriate (no unit conversions required).

- A is the catheter's cross-sectional area. $A = 0.20$
- L is the length of the catheter. $L = 6$
- ρ is the density of the fluid. $\rho = 0.62$

You perform a "step test" by increasing the pressure in the catheter by 2 units. The resulting voltage profile is given in the plot to the right. Using this information, you would like to determine the characteristics of this catheter, namely the modeling coefficients c, b and $K_{(P\to V)}$.



3.1. Re-write the transfer function above in standard form. [1]

Included in this assignment is a data set called `Empirical_Data.csv`. This is the data collected every 0.2 seconds for this test and is reproduced in the figure above. We seek to find the values of K, τ and ζ that **minimize the error** between the data and the transfer function (TF) model for $V(s)$.

3.2. Make a convincing argument that the process gain K for this system is $K = 10$. [1]

Assuming that $K = 10$, we desire τ and ζ that result in a dynamic response as close to the measured values of $y(t)$ as possible. We will do this by **minimizing the sum-of-squared-errors (SSE) between the data and the model**. This means we want to minimize the following objective:

$$\min_{\zeta,\tau} SSE = \sum_{i=1}^{N} (\hat{y}(t_i) - y_i)^2$$

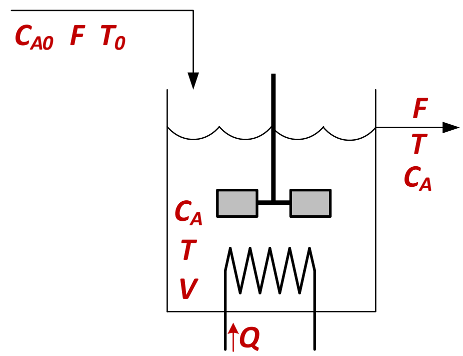
Where y_i are the data provided in `Empirical_Data.csv` and $\hat{y}(t_i)$ are the model predictions of $y(t)$ according to our second-order transfer function.

3.3. Using [some loops in] `MATLAB`, you are to find the values of τ and ζ that minimize SSE as it is defined above, thus fitting an optimal transfer function model to the data. To do this, create a transfer function with `tf()` and simulate it using `step()` to obtain $\hat{y}_i$ for your TF model. You know from experience that τ and ζ will both be between 0.1 and 1, and you want to find them to the nearest 0.01. Report the optimal values of K, τ, and ζ. Plot the model response as a continuous line and the data as markers on the same figure to demonstrate the quality of fit. [10]

3.4. Based on your results, determine the physical parameters b, c, and $K_{(P\rightarrow V)}$. [3]

Problem 4 – Linearization of Multi-Output Systems [25]

Consider the non-isothermal stirred tank bioreactor to the right. In this system, a reaction of the form $A \rightarrow B$ takes place that consumes component A. The reaction is **exothermic**, which means it releases energy in the form of heat as the reaction proceeds. The heat released by the reaction is ΔH [energy per mole reacted] and $\Delta H < 0$ for exothermic reactions. We may make all other assumptions that we have had for well mixed systems to date, such as constant ρ and constant c_p. Furthermore, the flow rate entering and leaving the system (F) is constant, and thus V is also constant.



The reaction follows first-order Arrhenius kinetics in the form $r_A = -k_0 e^{-\frac{E}{RT}} C_A$, where E, R, and k_0 are constants depending on the reaction (let the chemists worry about that, we do not care) and the reaction rate notably **depends on temperature** T.[2]

Luckily, I have completed the nonlinear material and energy balances on this process for you. The differential equation system for this reactor is:

$$\frac{dC_A(t)}{dt} = \frac{F}{V}\left(C_{A0}(t) - C_A(t)\right) - k_0 C_A(t) e^{\left(-\frac{E}{RT(t)}\right)} \quad (1)$$

$$\frac{dT(t)}{dt} = \frac{F}{V}\left(T_0 - T(t)\right) - \frac{k_0 C_A(t) \Delta H}{\rho c_p} e^{\left(-\frac{E}{RT(t)}\right)} + \frac{Q(t)}{\rho c_p V} \quad (2)$$

In this system, there are **two states** ($C_A(t)$ and $T(t)$) and **two inputs** ($C_{A0}(t)$ and $Q(t)$).

4.1. Derive linearized differential equations for $\frac{dC_A'(t)}{dt}$ and $\frac{dT'(t)}{dt}$ (both in deviation form) linearizing around the centering point $\bar{C}_A$, $\bar{T}$, $\bar{C}_{A0}$, and $\bar{Q}$. Do not substitute any numbers yet. [**5**]

For this reactor, we have the following data. All units presented are compatible (as in, no unit conversions are required for this question).

V = 0.5 [m^3]	R = 8.314 [$kJ \cdot kmol^{-1} \cdot K^{-1}$]	T_0 = 352.6634 [K]
ΔH = -4.78×10^4 [$kJ \cdot kmol^{-1}$]	k_0 = 65×10^9 [min]	E = 8.314×10^4 [$kJ \cdot kmol^{-1}$]
c_p = 0.329 [$kJ \cdot kg^{-1} \cdot K$]	ρ = 1000 [$kg \cdot m^3$]	F = 0.2 [$m^3 \cdot min^{-1}$]

4.2. If the inputs C_{A0} and Q are held long-term at values $\bar{C}_{A0} = 0.6767$ [$kmol \cdot m^{-3}$] and $\bar{Q} = 0$ [$kJ \cdot min^{-1}$] respectively, show the steady-state values for C_A and T are $\bar{C}_A = 0.0199$ [$kmol \cdot m^{-3}$] and $\bar{T} = 448.09$ [K]. You will need to solve a nonlinear system of two equations and two unknowns. [**5**]

[2] I'm aware it has been a while since first-year chemistry – fear not, this is all a formality that does not impact your ability to solve the problem.

4.3. Use `ode45()` in `MATLAB` to simulate your linearized system of ODEs **and** the original nonlinear system of ODEs from the initial condition $C_A(0) = \bar{C}_A$ and $T(0) = \bar{T}$ with constant $C_{A0} = \bar{C}_{A0}$ and $\boldsymbol{Q = -1000}$ (as in, the reactor is being cooled at 1000 kJ·min^{-1}). Plot the results for each simulation on the same axes (one set of axes each for $C_A(t)$ and $T(t)$) and comment on the accuracy of the linearized model. [**10**]

4.4. Using the linearized ODEs and by substituting all parameter values, determine the transfer functions for this system:

$$\frac{C_A'(s)}{C_{A0}'(s)} \qquad \frac{C_A'(s)}{Q'(s)} \qquad \frac{T'(s)}{C_{A0}'(s)} \qquad \frac{T'(s)}{Q'(s)}$$

Draw a block diagram where $C_{A0}(t)$ and $Q(t)$ are the inputs. We will revisit it later when we design some controllers. [**5**]