IBEHS 4A03

Assignment 1 – Modeling and ODEs

To be completed in GROUPS OF UP TO TWO (2)
Due Feb 09, 2025 @ 11:59pm via the A2L Drop Box
Grading: 6% of course grade (85 Points Available)

Purpose

This assignment is meant to give us some practice with ODEs and modeling and introduce a few practice opportunities with using the Laplace Transform.

Submission Instructions

Please submit this assignment *electronically* before the due date. Late submissions will **not** be accepted. Submit via the A2L dropbox. Be sure that you have the names and student numbers of all students on the front page of your submission. Submit your solutions as a **.pdf** file including all relevant figures, tables, and math. You may embed code in your solutions, but you **must submit your coded solutions**.

ONE group member is to submit a single .zip file that includes the solutions and code using the naming convention (for both the .zip and .pdf contained in the .zip):

`4A03_AXX_MACID1_MACID2`

Where `AXX` is the assignment number, and `MACID` is the McMaster ID (*e.g.* neasej, NOT your student number) of the submitting group member(s). Up to two McMaster IDs can be included on a single assignment.

Up to 10 points may be deducted from your submission for sloppy or otherwise unprofessional work. This is rare, but possible. The definition of unprofessional work may include:

- Low-resolution screenshots of figures and tables.
- Giving no context to an answer relating to the task (*i.e.,* "See code" or "113.289" with no units, context, or discussion whatsoever).
- Clear changes in author denoted by format changes, blatant writing style changes (including AI-generated discussion), or other factors that may deduct from the cohesiveness of the report.
- Failing to provide references for work that is obviously not yours (particularly bad cases will be considered as academic dishonesty).

Problem 1 [10]

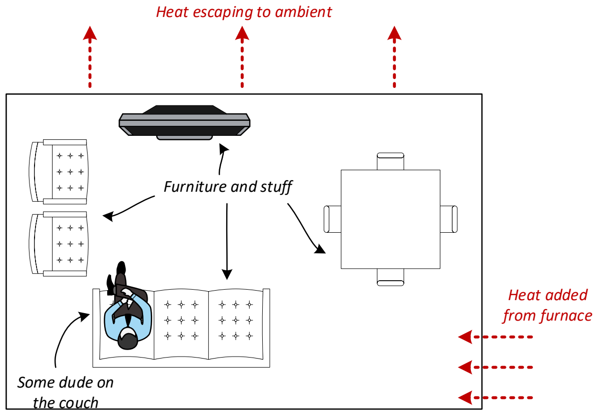
In our first class, we briefly ran through a few examples of control systems that attempt to regulate homeostasis in the human body. The purpose of this problem is to investigate a homeostasis control system of your choice in more detail and describe how it works. Write a brief (1 page maximum **including** figures) description of the system, being sure to identify the following:

- The **state/measured** variable(s) (what you can assume is the variable “under control”).
- The **sensor** and how it works.
- The **effector** and how it takes the measured signal from the sensor and produces…
- The **input** and how that affects the state.

Please make any other assumptions you require. The goal here is to demonstrate that you have a conceptual grip on a natural control system and are familiar with the terminology in 4A03. You will be graded on the accuracy and presentation of the information presented in your report.

Problem 2 **[25]**

In this problem, we want to explore the use of a "bang-bang" controller that is used to moderate the temperature in a typical room. Most non-smart thermostats use this control technique, where the furnace "trips on" to apply heat when a low temperature threshold is reached, and then "trips off" when the temperature reaches a sufficiently high number. We want to explore how different assumptions, ambient conditions, insulation levels, and other factors affect heating costs in a residence. I have drawn an **extremely helpful and attractive** sample figure for you at right.



To solve this problem, we'll need a few of **Assumptions**:

- The air in the room is well-mixed.
- There is no significant material (air) added to or removed from the room.
- Energy is *lost* to the outside environment and is proportional to the temperature difference between the room and the ambient outdoors.
- The outdoor temperature is constant.
- The effects of kinetic energy and people in the room are negligible.

With this in mind, we can derive the dynamic model that represents the **energy balance** in the room as follows:

$$\frac{dE(t)}{dt} = Q_f(t) - Q_l(t)$$

Where $E(t) = \rho c_p V T(t)$ is the energy contained in the room at any point in time t, ρ is the constant density of air in the room, V is the constant volume of air, c_p is the heat capacity of air, and $T(t)$ is the temperature in the room. Q_l is the energy loss to ambient and Q_f is the energy added to the room via the furnace:

$$Q_l(t) = UA(T(t) - T_a)$$

$$Q_f(t) = \begin{cases} 0 & \text{When } T(t) > 23°\text{C} \\ 1.5 \times 10^6 & \text{When } T(t) < 17°\text{C} \\ \textit{unchanged} & \text{For all } 17 \leq T(t) \leq 23°\text{C} \end{cases}$$

With UA being the overall heat transfer coefficient between the room and outside, and T_a is the ambient temperature. Note that when $T(t)$ is between the two thresholds, $Q_f(t)$ will be different depending on if it has been switched on. Putting it all together, we arrive at the following ordinary differential equation:

$$\frac{dT(t)}{dt} = \frac{Q_f(t) - UA(T(t) - T_a)}{\rho V c_p}$$

To simulate this scenario, we will need some **data**:

- The heat capacity of air is c_p = 0.17	[cal·g^{-1}·°C^{-1}]
- The density of air is ρ = 1190	[g·m^{-3}]
- The overall heat transfer coefficient is UA = 35×10^3	[cal·°C^{-1}·h^{-1}]
- The total volume of the room is V = 75	[m^3]
- When the furnace is running, Q_f is a constant 1.5×10^6	[cal·h^{-1}]
- The room is initially at $T(0)$ = 20	[°C]
- The furnace is initially in the OFF position	[-]
- The constant temperature outside is T_a = -5.0	[°C]

2.1. If there was no furnace (aka $Q_f(t) = 0$), show that the **steady-state** temperature in the room will be equal to the ambient temperature. [3]

2.2. Integrate this ODE system for a time domain of $[0, 0.001, \cdots, 2]$ hours in `MATLAB` using **Explicit Euler's Method** (see note below). Plot the results of $T(t)$ versus t on one set of axes, and $Q_f(t)$ versus t on another set of axes. Clearly label your figures. [10]

2.3. If your furnace produces thermal energy by combusting natural gas with 90% efficiency, how many standard cubic meters of natural gas were consumed during this simulation if the energy density of natural gas is 9×10^6 cal·m^{-3}? [3]

Consider now a simulation **for one 24-hour day**. At this point, it makes little sense to assume the outdoor ambient temperature T_a will remain constant for the duration of the simulation. Instead, we will assume that the ambient temperature is a function of time where the "low" occurs at midnight ($t = 0$) and is -5°C and the "high" occurs at noon ($t = 12$) and is +1°C. If we assume the temperature profile throughout the day is a smooth wave, we can model $T_a(t)$ as:

$$T_a(t) = -2 - 3\cos\left(\frac{\pi t}{12}\right)$$

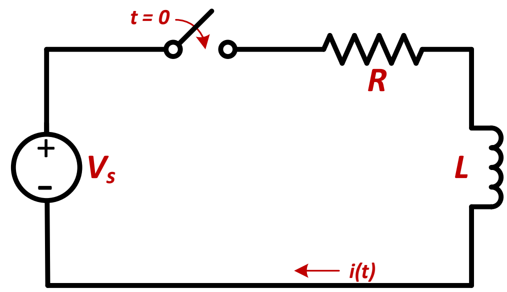
2.4. Integrate this ODE system given the new expression for $T_a(t)$ for a time domain of $[0, 0.001, \cdots, 24]$ hours in `MATLAB` using **Explicit Euler's Method** (see note below). Plot the results of $T(t)$ versus t on one set of axes, and $Q_f(t)$ versus t on another set of axes. Clearly label your figures. Briefly discuss any features of interest in the results, relating them back to the scenario. [9]

NOTE

Given a differential equation $\frac{dx}{dt} = f(x, t)$ with initial condition $x(t = 0) = x(0) = x_0$, one step of univariate Euler's method with fixed time step Δt can be completed as:

$$\begin{gathered} x_{k+1} = x_k + f(x_k, t_k)\Delta t \\ t_{k+1} = t_k + \Delta t \end{gathered}$$

Problem 3 **[15]**



In small electrical systems like those found in phone chargers, part of the process can be represented as a resistor-inductor loop with a fixed input voltage. For this example, consider the circuit drawn to right which consists of a DC voltage source $V_s(t)$ passing through a resistor R and inductor L, resulting in current $i(t)$. Initially, the switch is open and $i(t = 0) = 0$ A. At time $t = 0$, the switch is closed and a constant V_s is applied as an input to the system. According to Kirchhoff's Law, the ODE describing current through this system is:

$$V_s(t) = Ri(t) + L\frac{di(t)}{dt}$$

3.1. If the resistance R and inductance L are constant properties of the circuit's hardware, what is the expression that determines the **steady-state** value of $i(t)$? How is this useful when designing a circuit with a target current $i(t)$? [2]

3.2. For a circuit with $R = 50\ \Omega$, $L = 2$ H, and a constant voltage $V_s(t) = 5$ V, use `ode45` in `MATLAB` to numerically compute the state trajectory $i(t)$ from an initial condition of $i(t = 0) = 0$ A. Integrate the system for a time domain of $[0, 0.001, \cdots, 0.5]$ seconds. Plot the results of $i(t)$. Use the result to show your steady-state expression from part (1) holds. [5]

3.3. Based on the ODE in explicit form, which parameters (of R and L) do you expect to change the dynamic behaviour of $i(t)$ but **not change** the steady-state value of $i(t)$? Show your point by integrating the same circuit over the same time domain, but by changing the values of R and L (one at a time) and plotting the results. Discuss briefly. [4]

3.4. Integrate this system three times: once with a constant input $V_{s1}(t) = 5$ V (we'll call the result $i_1(t)$), once with a constant input $V_{s2}(t) = 10$ V ($i_2(t)$), and once with a constant $V_{s3}(t) = 15$ V ($i_3(t)$). Plot the results for $i_3(t)$ and the **sum** of $i_1(t) + i_2(t)$ on the same axes. Use a solid line for $i_3(t)$ and markers for $\{i_1(t) + i_2(t)\}$. Argue that this system is linear-time-invariant (LTI). [4]

Problem 4 [20]

In tutorial, we looked at a simplified Susceptible/Infected/Recovered (SIR) model that describes the propagation of a disease in a population. Consider now a modified SIR model that also includes vaccinations providing immunity to that disease. We'll call this the SIRV[1] model and it has four states:

- **Susceptible** $\triangleq \boldsymbol{S}$: The population subset that is at risk of contracting the disease.
- **Infected** $\triangleq \boldsymbol{I}$: The population subset that is currently infected.
- **Recovered** $\triangleq \boldsymbol{R}$: The population subset that has recovered from the disease.
- **Vaccinated** $\triangleq \boldsymbol{V}$: The population subset that is vaccinated against the disease.

The set of differential equations changes slightly from those provided in the tutorial. In this case, we have a fourth ODE and the introduction of a few more parameters:

$$\frac{dS(t)}{dt} = (1-\epsilon p)\mu N - \beta S(t)I(t) - \mu S(t) \tag{1}$$

$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t) - \mu I(t) \tag{2}$$

$$\frac{dR(t)}{dt} = \gamma I(t) - \mu R(t) \tag{3}$$

$$\frac{dV(t)}{dt} = \epsilon p \mu N - \mu V(t) \tag{4}$$

In the above equations:

- β is the contagious contact rate in [contagious contacts / person / year].
- γ is the recovery rate in [recoveries / year].
- μ is the death rate (due to natural causes) in [deaths / person / year]. Deaths in the community are "replaced" by births into either the vaccinated or susceptible groups.
- ϵ is the vaccine take (effectiveness or percentage the vaccine is effective) [unitless].
- p is the proportion of individuals born and vaccinated [unitless].
- N is the total population, which is assumed constant and equal to $S(t) + V(t) + I(t) + R(t)$.

We are going to use this model to study the impacts of a Measles outbreak in a fictitious country. For this scenario, we will use the following modeling parameters:

$$\beta = 0.00424\ \frac{\text{contacts}}{\text{person}\cdot\text{year}} \qquad \mu = 0.0155\ \frac{\text{deaths}}{\text{person}\cdot\text{year}} \qquad \gamma = 25\ \frac{\text{recoveries}}{\text{year}} \qquad N = 100{,}000 \text{ people}$$

4.1. Consider a Measles outbreak in this population consisting of 250 infected individuals. For this problem, we will assume that (initially) 85% of the population has herd immunity for Measles and start in the "recovered" category. The remaining population is susceptible, and no one is vaccinated. Moreover, assume that there is no vaccine available (hence $\epsilon = p = 0$).

[1] Scherer et al. British Medical Bulletin 2002; 62: 187

Integrate the system of ODEs (1)-(4) using `ode45` in `MATLAB`. Use a time span of 40 years. Plot the number of infections $I(t)$ and comment on the results. Note that you may want to avoid plotting $I(t)$, $S(t)$ and $R(t)$ on the same axes since they are of considerably different scales. [10]

4.2. Next, assume that **five years after the initial outbreak**, a vaccine is rolled out to the community. The vaccine is applied only to new births (already accounted for in the set of ODEs) and has an effective take of $\epsilon = 0.95$. Assume 75% of all new births receive the vaccination.

Integrate the system of ODEs (1)-(4) using `ode45` in `MATLAB` considering the vaccine rollout at time $t = 5$. Plot $I(t)$ and comment on these results. Comment briefly. [5]

4.3. Finally, consider the concept of **disease eradication**. The objective of a vaccine is to "starve" the disease until no outbreaks occur indefinitely. In our model, we will assume a disease is eradicated when no outbreaks occur within 50 years of the vaccine rollout. A measure of a disease's propagation is its so-called **reproductive number**:

$$R_0 = \frac{N\beta}{\mu + \gamma}$$

For a disease to be eradicated, enough people need to be effectively vaccinated (p_c) so the rate of vaccinated people exceeds the rate of transmission:

$$\epsilon p_c > 1 - \frac{1}{R_0}$$

Compute the critical vaccination rate p_c if ϵ remains constant at 0.95. Show that this eradicates the disease by integrating the ODE system in MATLAB for 55 years and plotting $I(t)$. [5]

Problem 5 [15]

Consider the second-order differential equation for some state x with $x(0) = 0$ and $\frac{dx(t)}{dt}\big|_{t=0} = -2$:

$$2\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} - 2x(t) = te^{-2t}$$

5.1. Determine the expression for the state in the Laplace domain as a rational polynomial. That is, determine $X(s) = \frac{N(s)}{D(s)}$ where $N(s)$ and $D(s)$ are polynomials. *Hint – the expression te^{-2t} can be found in the table of Laplace transforms*. [5]

5.2. Use partial fraction expansion and the table of Laplace transforms to determine an analytical expression for $x(t)$. [5]

5.3. Integrate the differential equation numerically for time $t = 0 \ldots 10$ using ode45. Plot the numerical and analytical results for $x(t)$ on the same axes to verify they give the same solution. [5]