1. Question 1

(1.1) The open-loop step response of the system is shown in the Simulink model (Figure
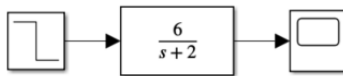1) below with the response shown in Figure 2.



Figure 1: Simulink model of the open-loop step response
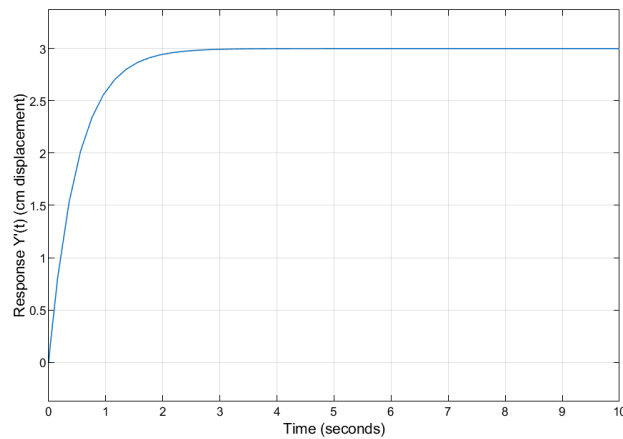


Figure 2: Open-loop step response

To identify the gain and time constant for this system, we can rearrange the
transfer function to isolate the gain K and time constant $\tau$:

The given transfer function is:

$$G_p(s) = \frac{6}{s+2}$$

To express this in the standard first-order form, which is:

$$G(s) = \frac{K}{\tau s + 1}$$

where $K$ is the gain and $\tau$ is the time constant.
We rearrange $G_p(s)$ as follows:

$$G_p(s) = \frac{6}{2(\frac{1}{2}s + 1)} = \frac{3}{\frac{1}{2}s + 1}$$

Thus, we identify the gain $K$ and the time constant $\tau$ as:

$$K = 3, \quad \tau = \frac{1}{2}$$

(1.2) The closed-loop feedback control with a proportional controller is shown below in Figure 3. The transfer function $G_c(s)$ can also be represented by a proportional gain $K_c$.
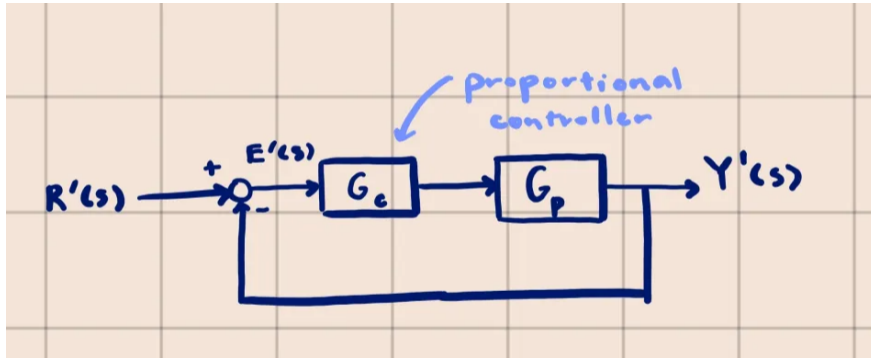


Figure 3: Simulink model of the closed-loop feedback control with a proportional controller

(1.3) The Simulink model is shown below in Figure 4. The response of the system is shown in Figure 5.



Figure 4: Simulink model of the closed-loop feedback control with a proportional controller
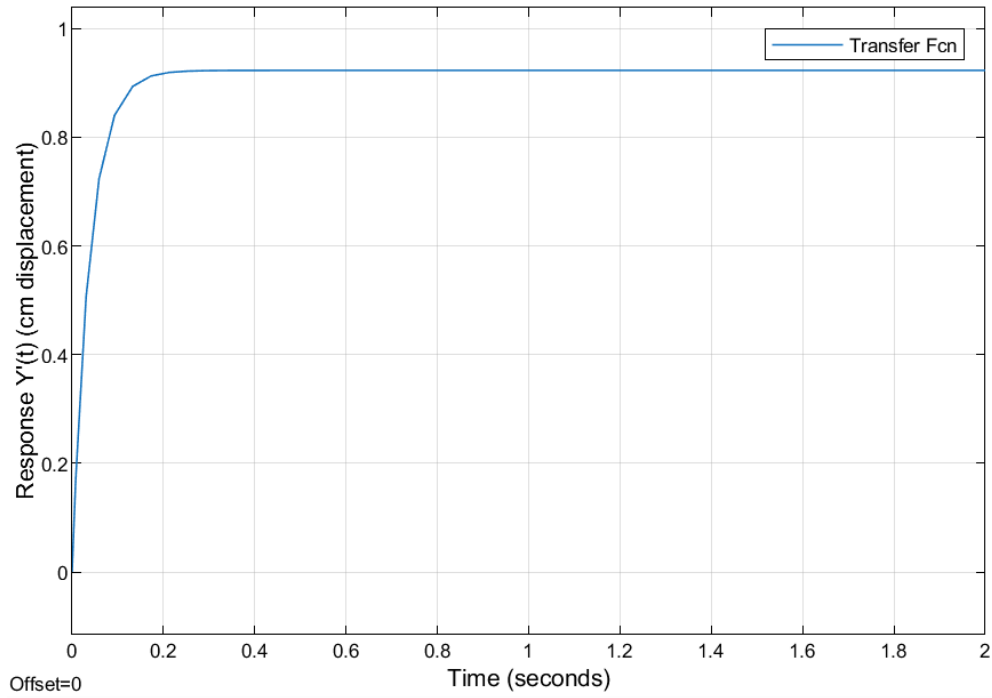


Figure 5: Closed-loop step response

(1.4) The closed-loop transfer function $T(s)$ is given by:

$$T(s) = \frac{Y'(s)}{R'(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

Where $G_c(s) = K_c = 4$ (proportional controller) and $G_p(s) = \frac{6}{s+2}$ (plant transfer

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function). Substituting the transfer functions:

$$T(s) = \frac{4 \cdot \frac{6}{s+2}}{1 + 4 \cdot \frac{6}{s+2}}$$

Simplifying:

$$T(s) = \frac{\frac{24}{s+2}}{1 + \frac{24}{s+2}} = \frac{24}{s+2+24} = \frac{24}{s+26}$$

Thus, the closed-loop transfer function is:

$$T(s) = \frac{24}{s+26}$$

We then apply the Final Value Theorem, which states that:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)$$

For a unit step input $R'(s) = \frac{1}{s}$, the Laplace transform of the output is:

$$Y'(s) = T(s)R'(s) = \frac{24}{s+26} \cdot \frac{1}{s}$$

Applying the FVT:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} s \cdot \frac{24}{(s+26)s} = \lim_{s \to 0} \frac{24}{s+26} = \frac{24}{26} = \frac{12}{13} \approx 0.923$$

The steady-state value of the output is approximately 0.923. Since the system is operating on displacement from a rock assembly with a step input, the set point is 1. Thus, the offset of this system is approximately 0.923. Referring to the step response in Figure 5, we can see the offset is approximately $1 - 0.923 = 0.077$.

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(1.5) The closed-loop system for controller gains $K_c = 0.25, 1, 2, 4, 10$ are shown in Figure 6. Observing the results, we can see that the offset of the system decreases as the proportional gain increases.
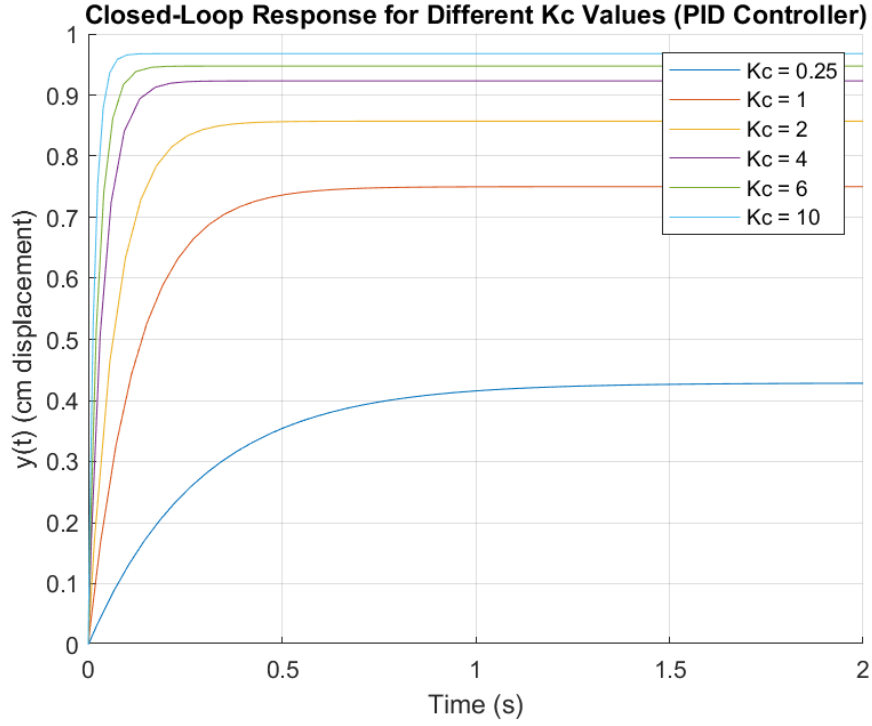


Figure 6: Closed-loop step response for different controller gains

(1.6) We want to find the controller gain $K_c$ such that the offset is less than 0.08. The offset is given by:

$$\text{Offset} = \lim_{t \to \infty} (r(t) - y(t))$$

Using the Final Value Theorem:

$$\text{Offset} = \lim_{s \to 0} \left( 1 - \frac{K_c \cdot G_p}{1 + K_c \cdot G_p} \right)$$

With $G_p = \frac{6}{s+2}$, we evaluate at $s = 0$:

$$\text{Offset} = 1 - \frac{K_c \cdot \frac{6}{0+2}}{1 + K_c \cdot \frac{6}{0+2}} = 1 - \frac{3K_c}{1 + 3K_c}$$

We require that the offset be less than 0.08:

$$0.08 > 1 - \frac{3K_c}{1 + 3K_c}$$

Rearranging the inequality:

$$\frac{3K_c}{1 + 3K_c} > 1 - 0.08 = 0.92$$

$$3K_c > 0.92(1 + 3K_c)$$

$$3K_c > 0.92 + 2.76K_c$$

$$(3 - 2.76)K_c > 0.92$$

$$0.24K_c > 0.92$$

$$K_c > \frac{0.92}{0.24}$$

$$K_c > 3.8333...$$

Therefore, the controller gain $K_c$ must be greater than approximately 3.83 to achieve an offset less than 0.08.

(1.7) We change from the proportional controller to the proportional-integral controller by adjusting the transfer function $G_c(s)$ to include an integral term. The transfer function for the PI controller is given by:

$$G_c(s) = K_c \left( 1 + \frac{1}{\tau_I s} \right)$$

Where $K_c = 3$ and $\tau_I = 0.05$. The response of the system is shown in Figure 7.
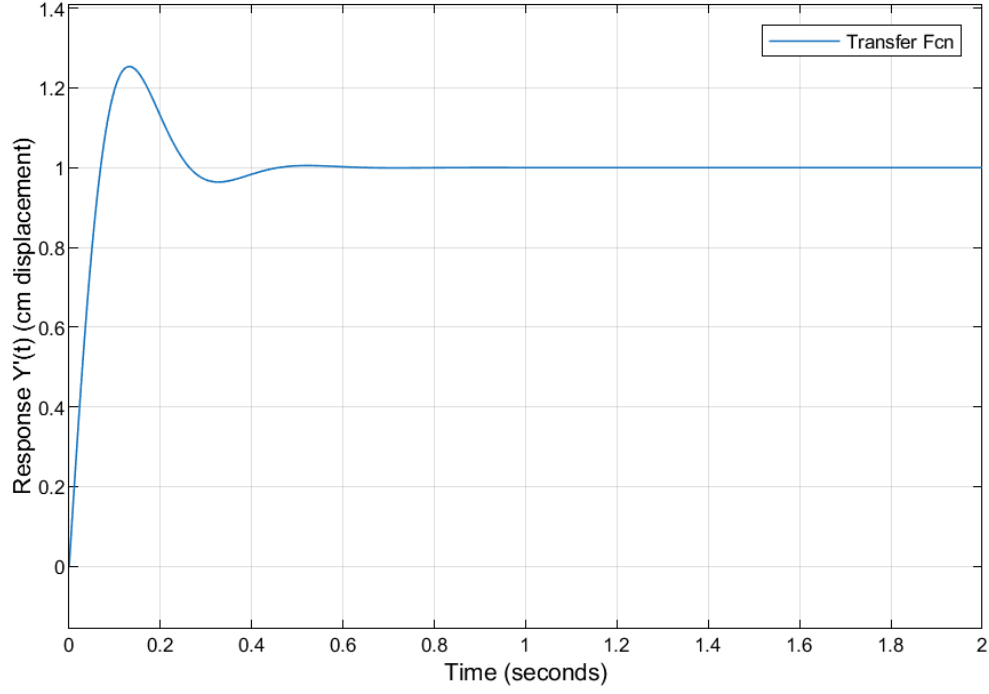
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Figure 7: Closed-loop step response with a PI controller

(1.8) Given the below two transfer functions in our system:

$$G_c = K_c \left( 1 + \frac{1}{\tau_I s} \right)$$

$$G_p = \frac{6}{s+2} = \frac{K_p}{\tau_p s + 1}$$

where,

$$K_p = 3, \quad \tau_p = \frac{1}{2}, \quad K_c = 3, \quad \tau_I = 0.05$$

Together, the closed-loop transfer fucntion for the system becomes:

$$\frac{Y'(s)}{R'(s)} = \frac{\tau_I s + 1}{\frac{\tau_I \tau_p}{K_c K_p} s^2 + \tau_I \frac{(1+K_c K_p)}{K_c K_p} s + 1}$$

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We can use the equations as presented in the lecture to calculate the time constant $\tau$ and the damping factor $\zeta$.

$$\tau = \sqrt{\frac{\tau_I \tau_p}{K_c K_p}} = \sqrt{\frac{(0.05)(\frac{1}{2})}{(3)(3)}} = \sqrt{0.00278}$$

$$\tau \approx 0.0527$$

$$\zeta = \frac{1}{2}(1 + K_c K_p)\sqrt{\frac{\tau_I}{K_c K_p \tau_p}} = \frac{1}{2}(1 + 9)\sqrt{\frac{0.05}{(3)(3)(\frac{1}{2})}}$$

$$\zeta \approx 0.526$$

We also derive the closed-loop gain $K$.

$$K = \tau_I s + 1 = 0.05s + 1$$

The calculated time constant is approximately $\tau \approx 0.0527$, which suggests the system should settle relatively quickly. The expected settling time, approximately $4\tau = 0.21s$, can be seen in Figure 7, where it settles around that point.

The damping factor was found to be $\zeta \approx 0.526$, suggesting a slightly underdamped response. This means the system should exhibit small oscillations but should not overshoot    excessively,    as    observed    in    the    above    figure.

The closed-loop gain is given by $K = 0.05s + 1$, which means the gain varies with frequency rather than being a simple constant. Evaluating it at steady-state $(s = 0)$, we get $K(0) = 1$, which confirms that the system should track a step input without steady-state error. Over time, our system approaches 1 as expected.

2. Question 2

   (2.1) placeholder