資工導論 Data Representation in Computer Systems

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What are data?

Inside computer

- 文字 text ABC天地玄黃 01100001011000…
- 數字 number 123 3.14159 00000001000000…
- 聲音 sound  01001100010101…
- 圖像 image  10001001010100...
- 影片 video 00110000001001…

Binary system

- Computers use a binary system, 0 and 1, to represent and store all kinds of data.
- Why binary?
  - We need to find physical objects/phenomenon to store, transmit, and process data. Binary is the most straightforward representation.

有無 上下 黑白 真偽 勝負

北南 負正 錯對 陽陰 關開 1 0

Some jargons

- **Bit:** a binary digit (0 or 1)
- **Byte**: 8 bits
  - Basic storage unit in computer system
- **Hexadecimal notation:**
  - Represents each 4 bits by a single symbol
  - Example: A3 denotes
    1010 0011

Bit pattern	Hexadecimal
0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7
1000	8
1001	9
1010	A
1011	B
1100	C
1101	D
1110	E
1111	F

More jargons

- **Kilobyte (KB):** 2^{10} bytes = 1024 bytes ≈ 10^3 bytes
  - Example: 3 KB ≈ 3×10^3 bytes
- **Megabyte (MB):** 2^{20} bytes ≈ 10^6 bytes
  - Example: 3 MB ≈ 3×10^6 bytes
- **Gigabyte (GB):** 2^{30} bytes ≈ 10^9 bytes
  - Example: 3 GB ≈ 3×10^9 bytes
- **Terabyte (TB):** 2^{40} bytes ≈ 10^{12} bytes
  - Example: 3 TB ≈ 3×10^{12} bytes
- **Petabyte (PB):** 2^{50} bytes ≈ 10^{15} bytes
  - Example: 3 PB ≈ 3×10^{15} bytes

Text data

- Each character is assigned to a unique bit pattern.
- ASCII code
  - **A**merican **S**tandard **C**ode for **I**nformation **I**nterchange
  - Uses **7-bits** to represent most symbols used in English text

```
 !"#$%&'()*+,-./
0123456789:;<=>?
@ABCDEFGHIJKLMNO
PQRSTUVWXYZ[\]^_
`abcdefghijklmno
pqrstuvwxyz{|}~
```

01001000	01100101	01101100	01101100	01101111	00101110
H	e	l	l	o	.

  - Quiz: how many different bit patterns can be represented by 7 bits?

Big5 code

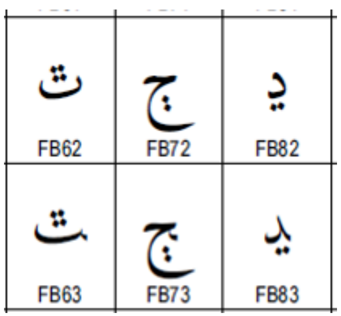
- For Chinese character encoding
- Uses 16 bits to represent a Chinese character
- Example

我	身	騎	白	馬
A7DA	A8AD	C34D	A5D5	B0A8

1010 0111 1101 1010

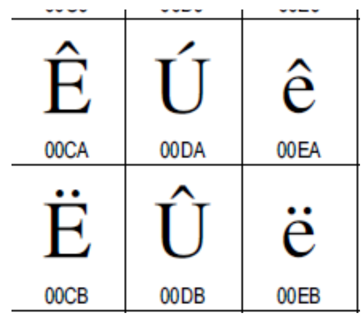
Unicode

- Uses 16-bits to represent the major symbols used in languages worldwide



Arabic char

CJK char



Latin char

Indic char

- Quiz: how many different bit patterns can be represented by 16 bits?

Encoding

- You need correct encoding to read meaningful texts.
- For example, on browser, you can choose the encoding.
- Most webpages have specified their encoding.

For HTML4:

```
<meta http-equiv="Content-Type" content="text/html;charset=ISO-8859-1">
```

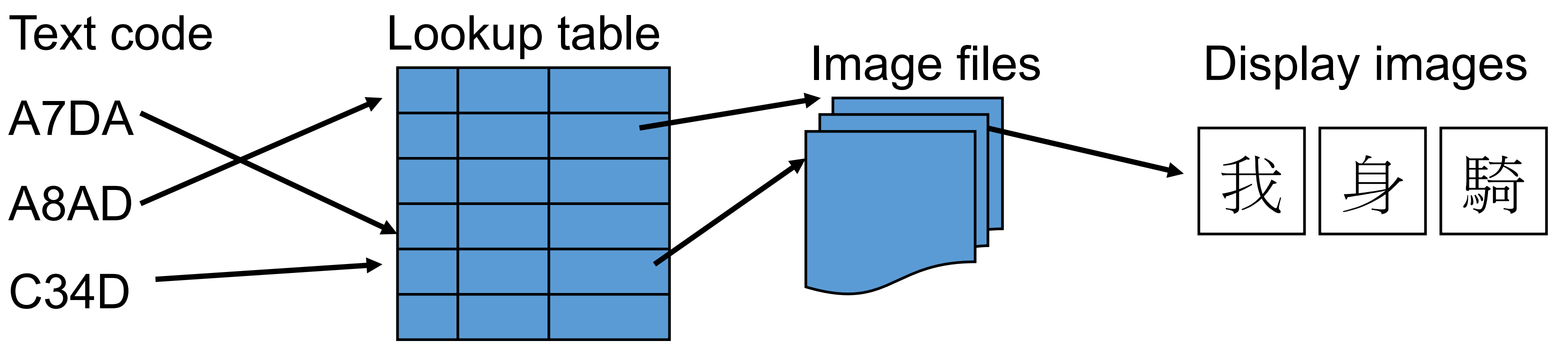
For HTML5:

```
<meta charset="UTF-8">
```

æ ¹æ“šè□¯é‚¦æ³•é™¢æ–
‡ä»¶é¡¯ç¤ºï¼Œå□°ç□£é‡□é‡□ç´šä
¼□æ¥å®¶ã€□ä¸–
ç•ŒçŸ¥å□□æ‰‹æ©Ÿå¤§å»
ã€Œå®□é□”é»ã€□(HTC)å…¬å□¸
è‘£äº‹é•·ç□‹é›ªç´…ï¼Œé□-
å^°æ•™æœƒäººå£«æ…ˆå–

Display characters

- Computer doesn’t show the codes directly to us. It displays what we can read.



- Those images for displaying characters are called fonts.
  - We will talk about images later.

Numbers

- Like texts, we can use 4 bits to represent decimal digits 0,1,2,3,4,5,6,7,8,9
  - This is called "Binary-coded decimal" (BCD)
  - EX: 123 is coded 0001 0010 0011
- Problems
  - 4 bits binary can have 16 different patterns, but BCD only uses 10. It wastes 6 bit-patterns.
  - Difficult to do calculation (+-*/)

	BCD
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001

Example of adding BCDs

- Using lookup table
- EX: 5+7
- In BCD:

```
      0101
     +0111
----------------
 0001  0010
```

a	b	carry	sum
:	:	:	:
0101	0110	0001	0001
0101	0111	0001	0010
0101	1000	0001	0011
0101	1001	0001	0100
:	:	:	:

Binary numeral system

- Uses bits to represent a number in base-2 system

Base 10 system

3	7	5

$3\times10^2+7\times10^1+5\times10^0$

Base 2 system

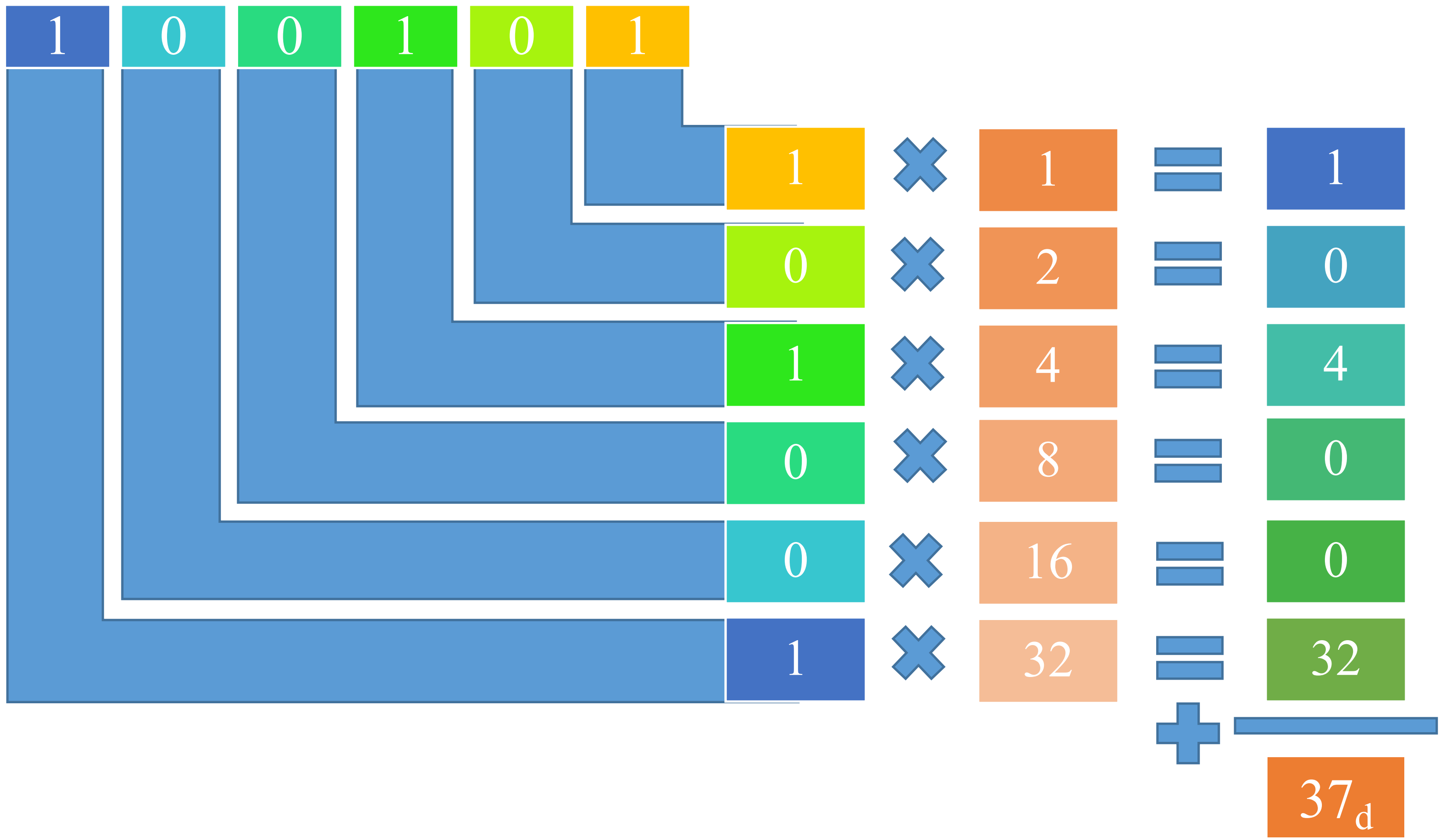
1	0	0	1

$1\times2^3+0\times2^2+0\times2^1+1\times2^0$

- We put a subscript b to a number for binary, and a subscript d for decimal.
  - 10_d is number ten, and 10_b is number two.

Binary to decimal

- What is the decimal number of 100101_b?



Decimal to binary

- What is the binary number representation of 13_d?
- First, how many bits do we need for 13.
  - Since $2^3=8<13<16=2^4$, we need at least 4 bits.

$$13 = b_3 \; b_2 \; b_1 \; b_0$$

$$= b_3 \times 8 + b_2 \times 4 + b_1 \times 2 + b_0 \times 1$$

- Remember b_3, b_2, b_1, b_0 are either 0 or 1.
- Can we decide what b_0 is? 0 or 1.
  - Because $b_3 \times 8 + b_2 \times 4 + b_1 \times 2$ is an even number, and 13 is an odd number, b_0 must be 1.

Decimal to binary

- What is b_1?

$$13 = b_3 \times 8 + b_2 \times 4 + b_1 \times 2 + b_0 \times 1$$

$$- \; b_0 = - \; b_0$$

$$12 = b_3 \times 8 + b_2 \times 4 + b_1 \times 2$$

$$\div \; 2 = \div \; 2$$

$$6 = b_3 \times 4 + b_2 \times 2 + b_1 \times 1$$

- It becomes to decide the binary representation of 6.
    - Since 6 is an even number, b_1 must be 0.

Decimal to binary

- What is b_2?
- We can follow the same procedure.

$$\boxed{6} = \boxed{b_3} \times 4 + \boxed{b_2} \times 2 + \boxed{b_1} \times 1$$

$$- \boxed{b_1} = \quad - \boxed{b_1}$$

$$\boxed{6} = \boxed{b_3} \times 4 + \boxed{b_2} \times 2$$

$$\div\ 2 = \div\ 2$$

$$\boxed{3} = \boxed{b_3} \times 2 + \boxed{b_2} \times 1$$

- Since 3 is an odd number, b_2 must be 1.

Decimal to binary

- What is b_3?

$$3 = b_3 \times 2 + b_2 \times 1$$
$$- b_2 = - b_2$$
$$2 = b_3 \times 2$$
$$\div 2 = \div 2$$
$$1 = b_3 \times 1$$

- It shows b_3 is 1.
- So $13_d = b_3 b_2 b_1 b_0 = 1101_b$.

Algorithm for decimal to binary

- An algorithm is a procedure to describe the steps for computers to solve some problems.
- Let x be the decimal number in question and y is the corresponding binary number representation.
- What we did can be described as the following steps.

Step 1	If x equals to 0 or 1, $y = x \circ y$.
Step 2	If x is even, b is 0. Otherwise b is 1. Let $y = b \circ y$.
Step 3	Let $x = (x - b)/2$ and go to step 1.

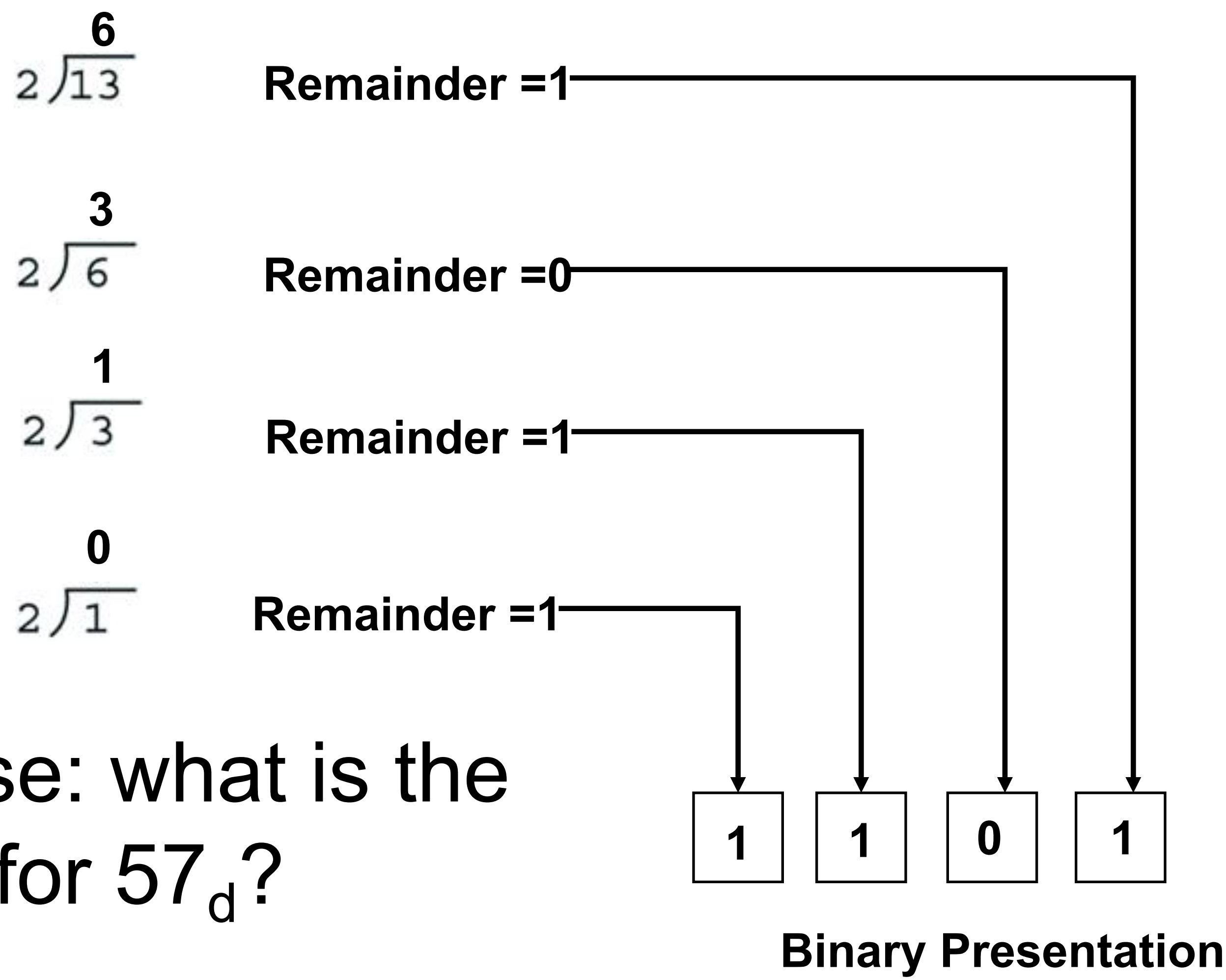
  - The initial value of y is an empty bit pattern.
  - The $\circ$ operator means concatenation.

Algorithm for decimal to binary

- Problem: how a computer knows a number is even or odd?
- Use "integer division"
  - x **div** $2 = q \times 2 + r$, where q is quotient and r is remainder.
- The previous algorithm can be modified as follows

Step 1	If x equals to 0 or 1, $y = x \circ y$.
Step 2	Let x **div** $2 = q \times 2 + r$ and $y = r \circ y$.
Step 3	Let $x = q$ and go to step 1.

Example



Exercise: what is the binary for 57_d?

Binary number calculations

- Binary number representations are easy for calculations
- For example, the one bit addition

$$
\begin{array}{r} 0 \\ +\,0 \\ \hline 0 \end{array}
\qquad
\begin{array}{r} 1 \\ +\,0 \\ \hline 1 \end{array}
\qquad
\begin{array}{r} 0 \\ +\,1 \\ \hline 1 \end{array}
\qquad
\begin{array}{r} 1 \\ +\,1 \\ \hline 10 \end{array}
$$

- So, what is 5_d+9_d in the binary number form?

$$
\begin{array}{rl} 0\ 1\ 0\ 1 & \quad 5 \\ +\ 1\ 0\ 0\ 1 & \quad 9 \\ \hline 1\ 1\ 1\ 0 & \quad 14 \end{array}
$$

Another example

- Remember the rules of one bit addition

$$\begin{array}{r} 0 \\ +0 \\ \hline 0 \end{array} \qquad \begin{array}{r} 1 \\ +0 \\ \hline 1 \end{array} \qquad \begin{array}{r} 0 \\ +1 \\ \hline 1 \end{array} \qquad \begin{array}{r} 1 \\ +1 \\ \hline 10 \end{array}$$

- What is $00111010_b + 00011011_b$?

$$\begin{array}{rr} & \color{red}{111\ 1\ \ \ } \\ & 00111010 \\ + & 00011011 \\ \hline & 01010101 \end{array}$$

Negative numbers

- How to represent -1, -2, … on a computer?
- Solution 1: use an extra bit to represent the negatives sign.
  - It is called *the sign bit*, in front of numbers.
  - Usually, 0 is for positives; 1 is for negatives.
  - Example: 1 0001 is -1 and 0 0100 is +4
- Problem 1: It will have 2 zeros, +0 and –0.
- Problem 2: how can we to do the addition (-1) + (4) efficiently using the signed notation?

Example of -1+4 with sign notation

- One bit subtraction

$$
\begin{array}{r}
0 \\
-\ 0 \\
\hline
0
\end{array}
\qquad
\begin{array}{r}
1 \\
-\ 0 \\
\hline
1
\end{array}
\qquad
\begin{array}{r}
1 \\
-1 \\
\hline
0
\end{array}
\qquad
\begin{array}{rr}
 & 0 \\
 & -\ 1 \\
\hline
-1 & 1
\end{array}
= -1*2+1*1=-1
$$

$$
\begin{array}{rcccc}
 & & \color{red}{-1} & \color{red}{-1} & \\
 & 0 & 1 & 0 & 0 \\
- & 0 & 0 & 0 & 1 \\
\hline
 & 0 & 0 & 1 & 1
\end{array}
$$

It can be very complicated !!

Solution 2

- The negative sign "–" just means the "opposite" or the "inverse".
  - Ex, the opposite of East is West.
- For addition, the inverse of a number d, denoted I(d), has the property: I(d)+d=0.
- We can use this property to define negative numbers.
  - For example, let x be the representation of -1. It should satisfy the property $x + 1 = 0$.

What is -1?

- If we use four bits to represent a number, zero is 0000, and one is 0001. What is -1?
- Find b_3, b_2, b_1, b_0 such that

$$
\begin{array}{rcccc}
 & 1 & 1 & 1 & 1 \\
 & b_3 & b_2 & b_1 & b_0 \\
+ & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 0
\end{array}
$$

- b_0 must be 1, and it will create a carry 1.
- b1 must be 1, and it will create a carry 1.
- b2 must be 1, and it will create a carry 1.
- b3 must be 1, and it will create a carry 1.

This 1 will be "truncated " since it is a 4 bits numbering system.

Finite digit systems

- One common example of finite digit system is clocks.
- What is the next second after 23:59:59?

Two's complement

- Such number representation for negative numbers is called two's complement.
- The right table lists 4 bits 2's complement representation.
- Only one zero.
- Easy for addition.
- The leading bit of negative numbers is 1. (signed bit)

Bit pattern	Values
0111	7
0110	6
0101	5
0100	4
0011	3
0010	2
0001	1
0000	0
1111	-1
1110	-2
1101	-3
1100	-4
1011	-5
1010	-6
1001	-7
1000	-8

Calculation with 2's complement

- Calculation can be made easily for two's complement representation.

Problem in base ten	Problem in two's complement	Answer in base ten
3 + 2	0011 + 0010 0101	5
– 3 – 2	1101 + 1110 1011	-5
7 – 5	0111 + 1011 0010	2

Computing 2’s complement

- A simple algorithm to compute the 2’s complement of a number
  1. Change each bit 0 to 1 and bit 1 to 0
  2. Add 1.

$$6_d = 0110_b \to 1001_b$$

$$\begin{array}{r} 1001_b \\ +\ 0001_b \\ \hline 1010_b = -6_d \end{array}$$

truncated

$$\begin{array}{rl} 0110_b & (\ 6) \\ +\ 1010_b & (-6) \\ \hline 1\ 0000_b & (\ 0) \end{array}$$

Exercises

- What are the decimal numbers for the following 2's complement representations?

  (a) 00000001 (b) 01010101 (c) 11111001
  (d) 10101010 (e) 10000000 (f) 00110011

- Find the negative value represented in 2's complement for each number

Overflow

- What is 5+4?

$$5_d+4_d=0101_b + 0100_b=1001_b$$

- This is called overflow
  - Adding two positive numbers results a negative number; or adding two negative numbers results a positive number.
  - A 4 bits 2's complement system can only represent –8~7

Bit pattern	Values
0111	7
0110	6
0101	5
0100	4
0011	3
0010	2
0001	1
0000	0
1111	-1
1110	-2
1101	-3
1100	-4
1011	-5
1010	-6
1001	$\boxed{-7}$
1000	-8

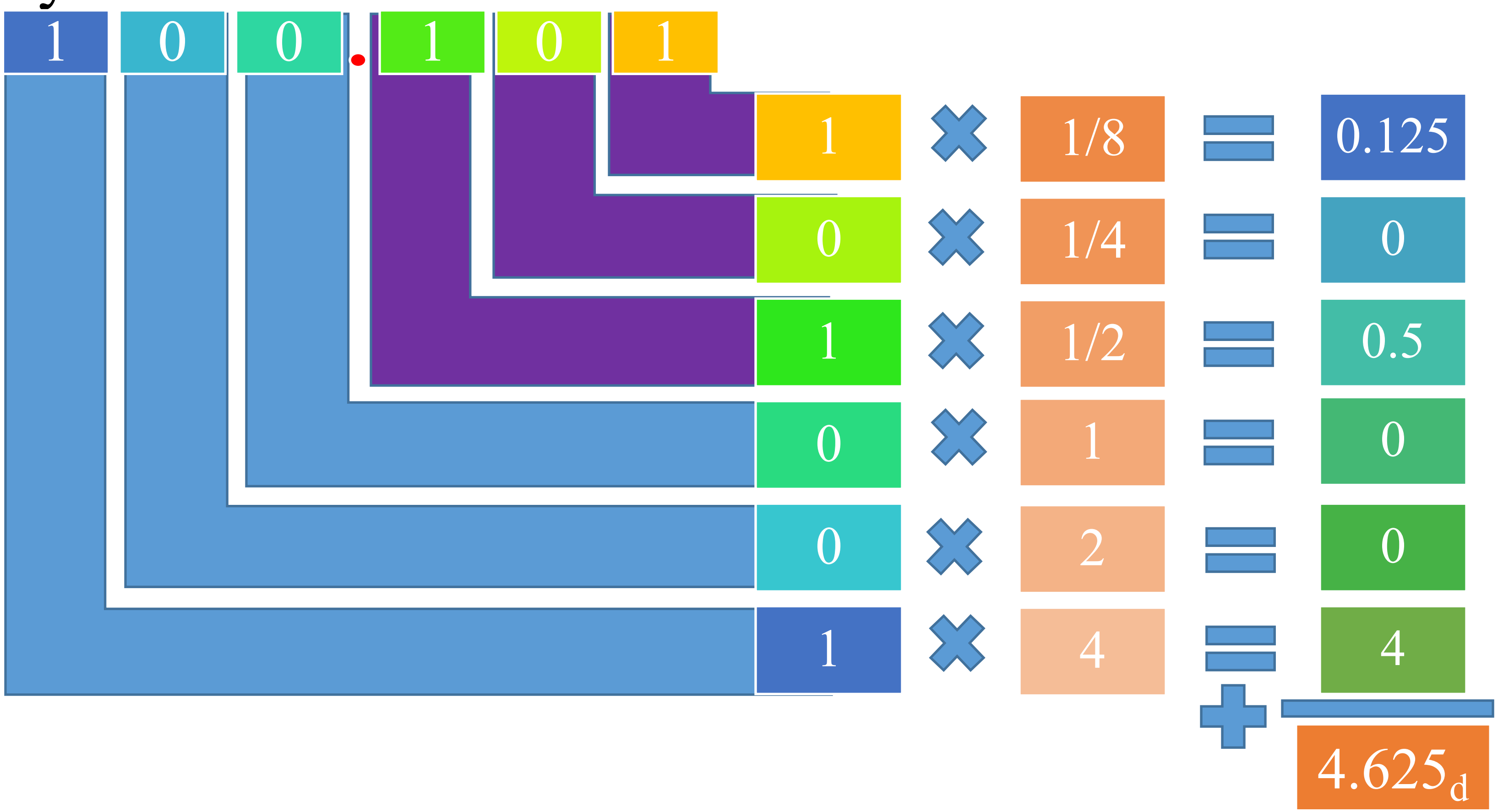
Data types

- How do we know 1001_b is 9 or -7?
- In computer system, every datum needs a data type, which describes the format of the data.
- For example, in C, there are many data types for integers.

Data types	Size	Range
char	1 byte	-128 to 127
unsigned char	1 byte	0 to 255
int	4 bytes	-2147483648 to 2147483647
unsigned int	4 bytes	0 to 4294967295

Numbers with the decimal point

- Real numbers can be represented by the binary system too.



Fixed point representation

- This representation is called fixed point, because the decimal point is at the fixed position.

1 0 0 . 1 0 1

- Fixed point representation is good for calculations, but the representable range is very limited.
  - Think about 1000000 or 0.0001. They will be all 0 for 6 bit fixed point representation.

Floating point

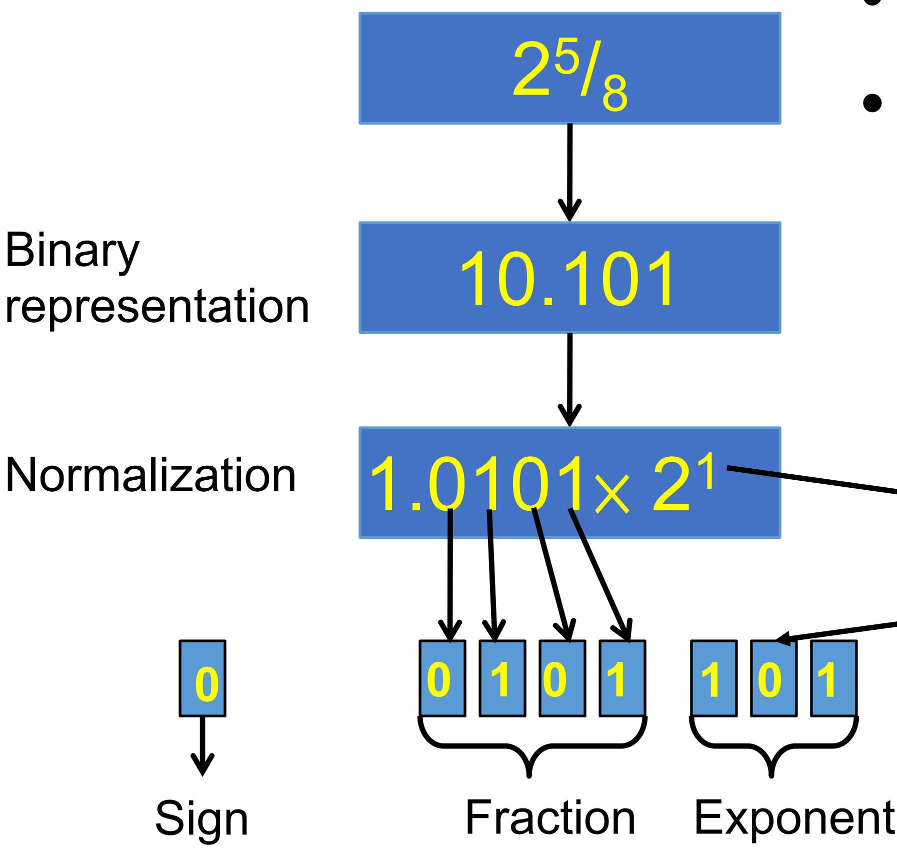
- To represent a wide range of numbers, we allow the decimal point to “float”.
- It uses the scientific notation of numbers.

$$40.1_d = 4.01_d \times 10^1 = 401_d \times 10^{-1} = 0.401_d \times 10^2$$

$$101.101_b = +1.01101_b \times 2^{2_d} = \underset{\text{sign}}{+}1.\underset{\text{fraction}}{01101_b} \times 2^{\overset{\text{exponent}}{10_b}}.$$

- This is called the floating point representation of real numbers.

Ex: floating point of $2^5/_8$



- Sign bit: 0 for + and 1 for –.
- Exponent uses excess representation.

Excess notation	Bit pattern
3	111
2	110
1	101
0	100
-1	011
-2	010
-3	001
-4	000

Why excess notation?

- Why excess notation?
- Why moving the exponent in front of the fraction?
- Both are for the same reason: easy to compare the numbers.
  - For two normalized numbers x and y, if x's exponent is larger than y's exponent, x is larger than y.
  - We can use the circuit of comparing integers to compare floating point numbers

Excess notation	Bit pattern
3	111
2	110
1	101
0	100
-1	011
-2	010
-3	001
-4	000

Truncation error

- Consider the decimal number 0.1_d.
  - $0.1 = {}^{1}/_{16} + {}^{1}/_{32} + {}^{1}/_{256} + {}^{1}/_{512} + \ldots$
  - So the binary representation of 0.1 is 0.000110011…
  - It is a infinite sequence in the binary notation.
- After normalization, it is $+1.10011\ldots \times 2^{-4}$. The 8 bit floating point format can only represent $+1.1001 \times 2^{-4} = 25/256 = 0.09765625$
- The difference $|0.1 - 0.09765625| = 0.00234375$ is called truncation error.

IEEE Standard 754

William Morton Kahan

- IEEE (Institute of Electrical and Electronics Engineers) has made a floating point number standard, IEEE-754, so that different computers can have a common ground.
- Two commonly used formats of floating numbers

	Sign	Exponent	Fraction	Total
Single	1 bit	8 bits	23 bits	32 bits
Double	1 bit	11 bits	52 bits	64 bits

Exercises

- What are the fractions for the following floating number representations?
  - Suppose 1 bit for sign, 3 bits for exponent (using excess notation), 4 bits for mantissa

(a) 01001010 (b) 01101101 (c) 11011100 (d) 10101011

- If direct truncation is used, what are the ranges of their possible values?