IMAGE PROCESSING

Chapter 4: Filtering in the Frequency Domain

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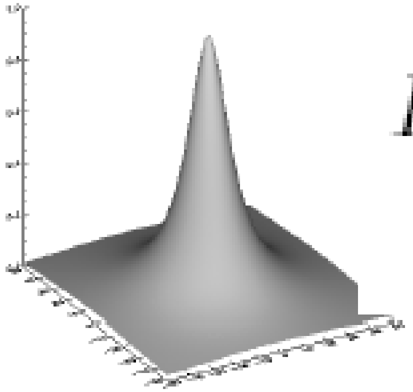
CONTENTS

Fourier Transform

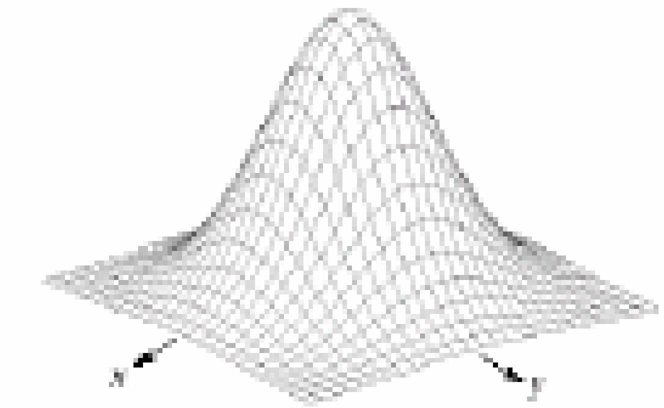
- Transform image into frequency domain, from the spatial domain
- Decompose into sine and cosine components
- Applications in: image analysis, image filtering, image reconstruction and image compression

- Application of Fourier transform
  - Removing noise:



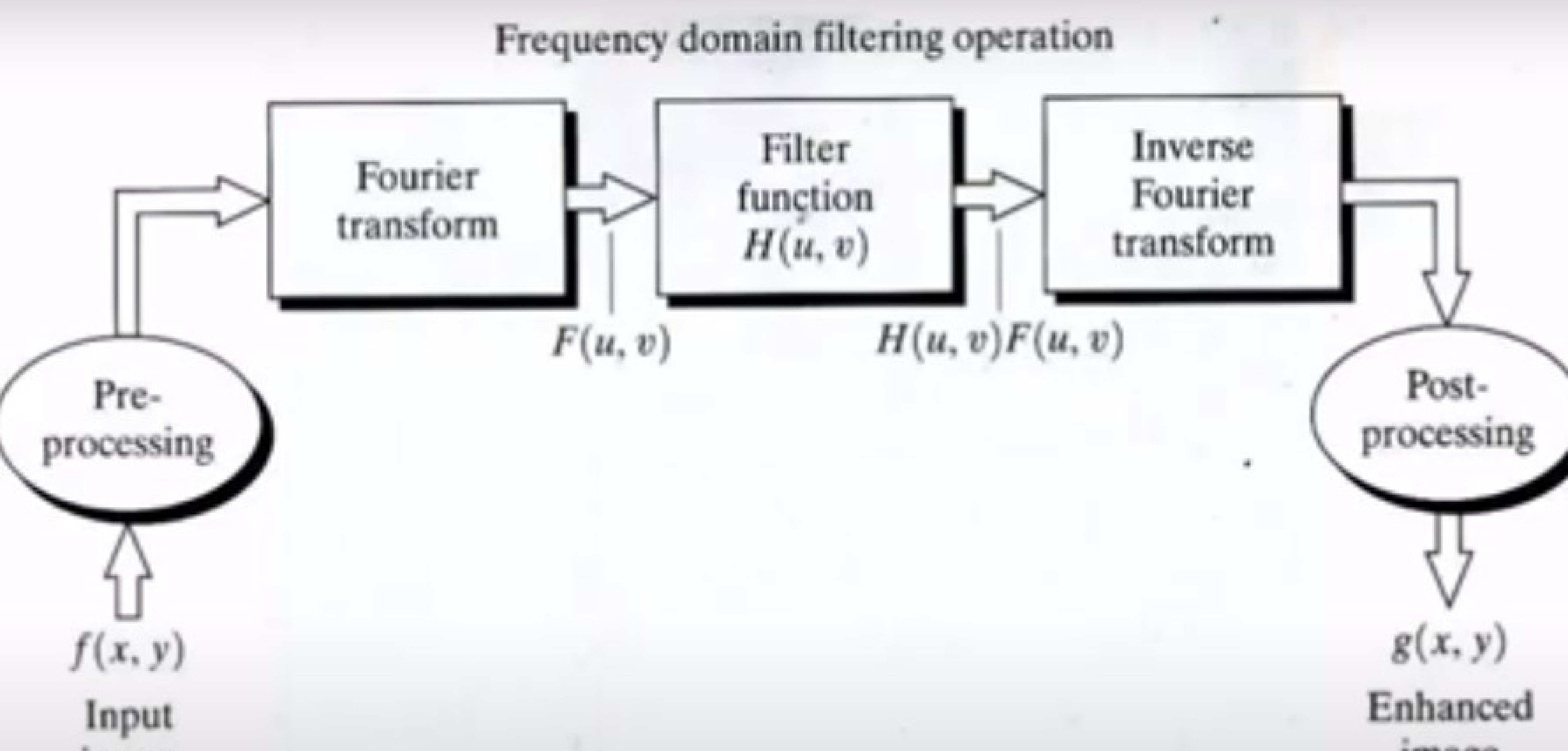
$$H(U,V) = \frac{1}{1 + \left(\frac{U^2+V^2}{D_0^2}\right)^P}$$

  - Extracting Edge:



−1	−1	−1
−1	8	−1
−1	−1	−1

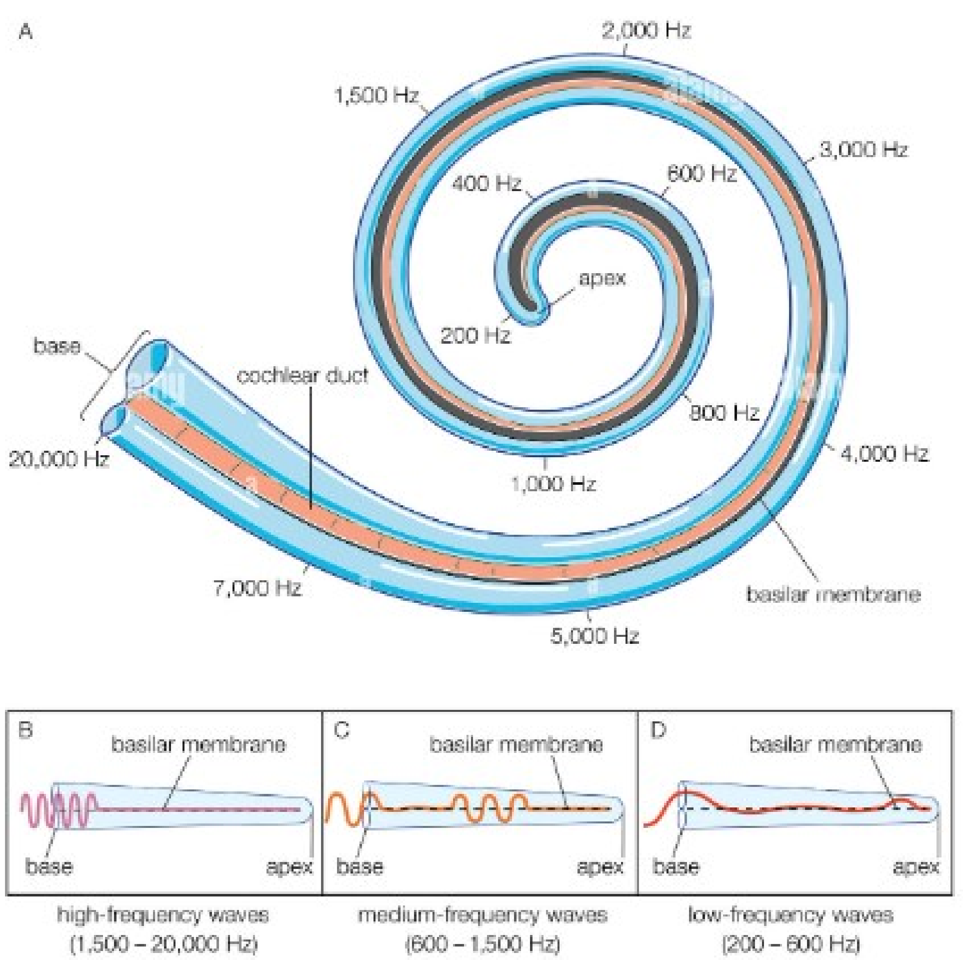
Frequency domain filtering operation



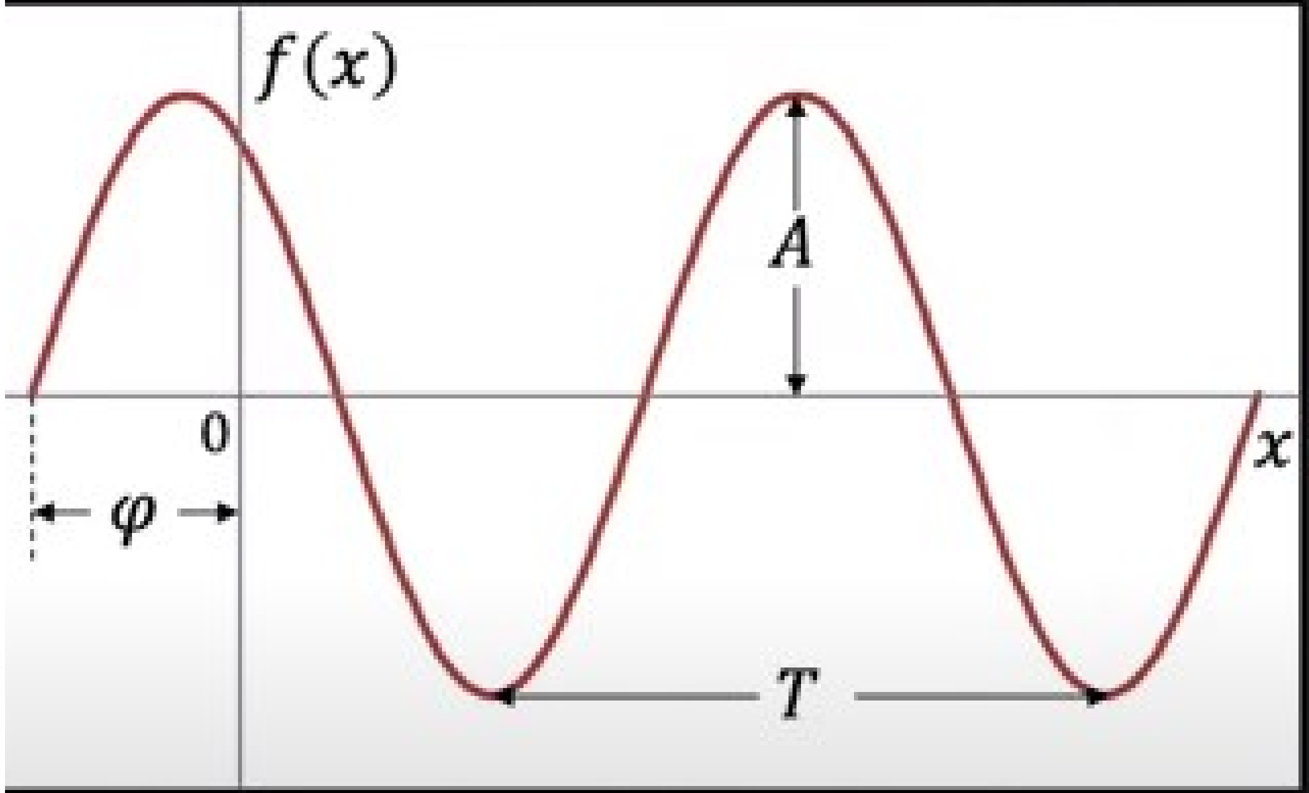
Properties of Fourier Transform

Property	Spatial Domain	Frequency Domain
Linearity	$\alpha f_1(x) + \beta f_2(x)$	$\alpha F_1(u) + \beta F_2(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x - a)$	$e^{-i2\pi ua} F(u)$

Sinusoid



$$f(x) = A\sin(2\pi u x + \varphi)$$



A: Amplitude

T: Period

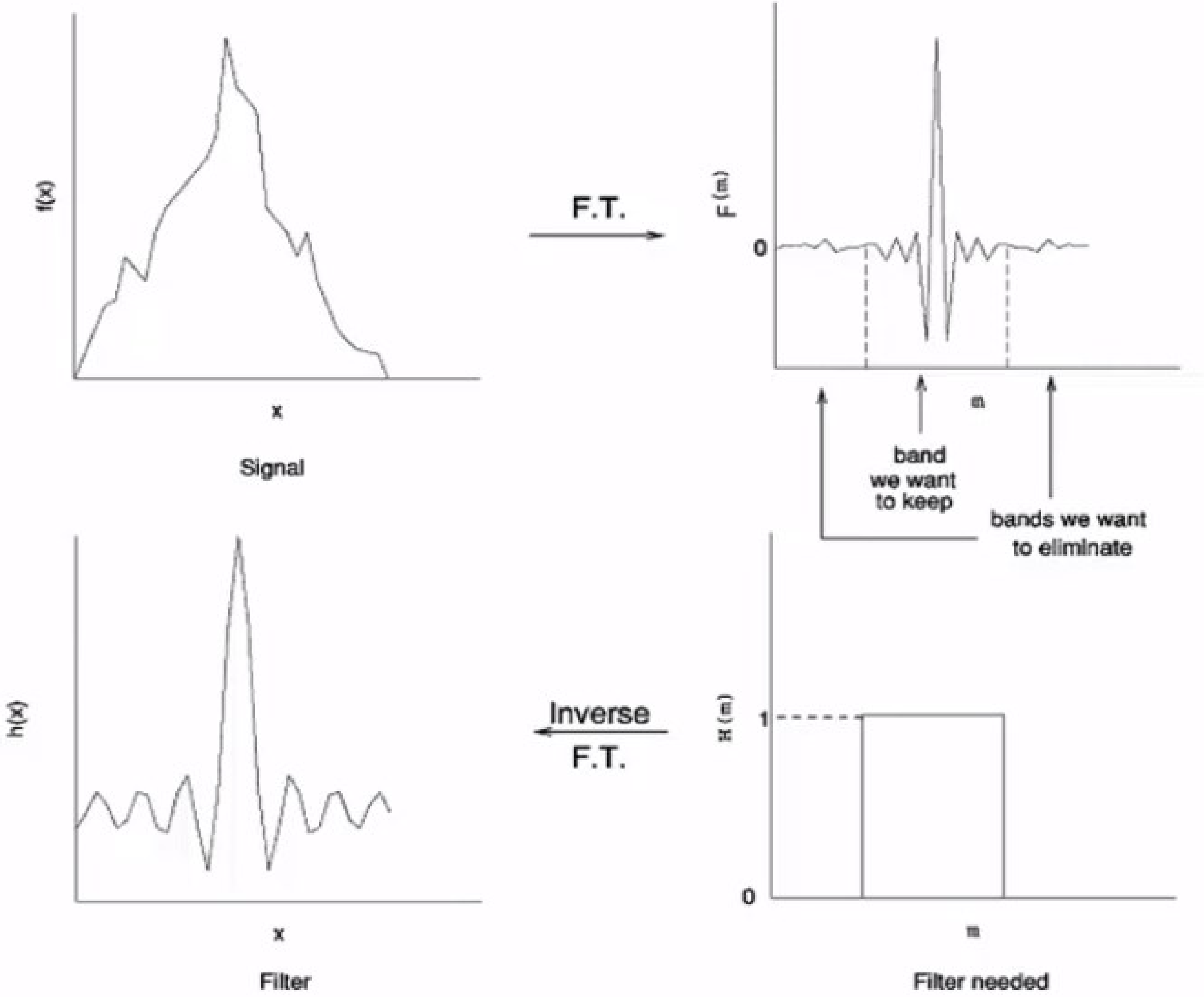
φ: Phase

u: Frequency $(1/T)$

1 DFT:

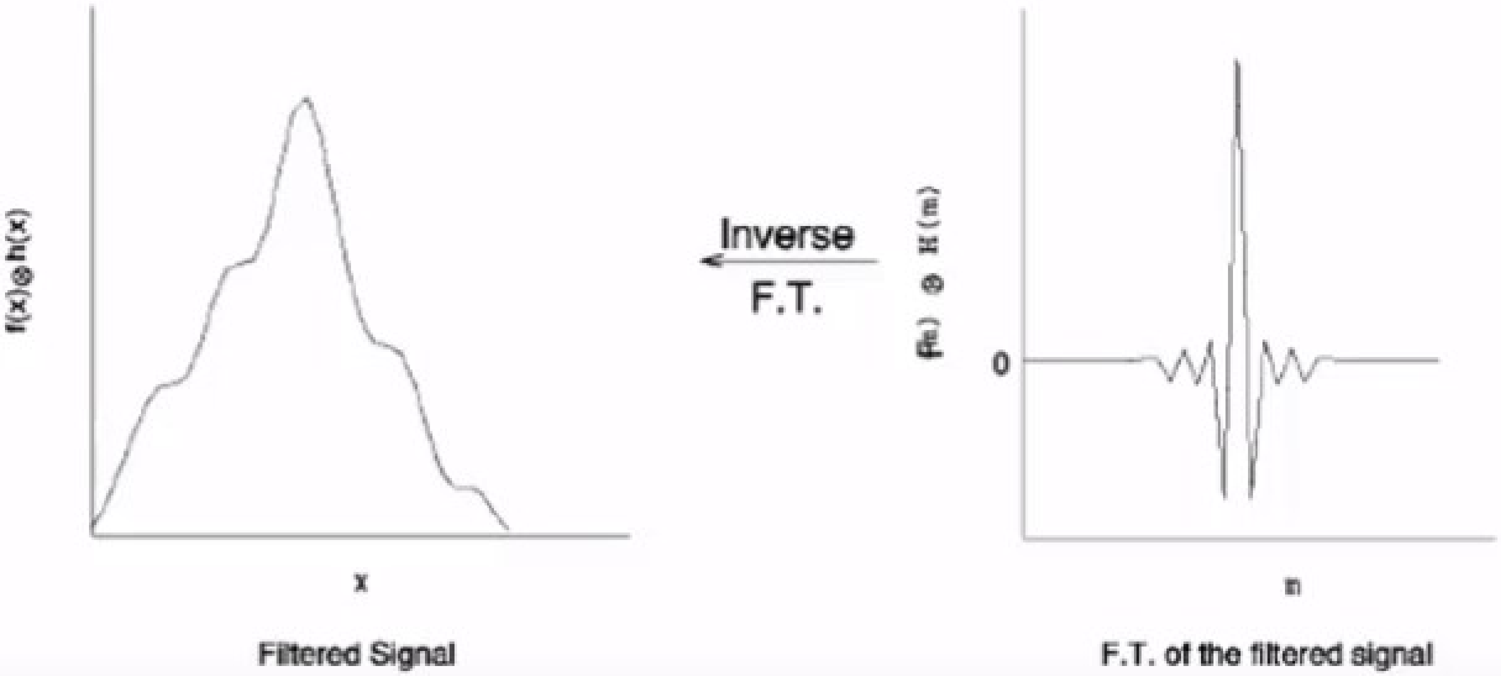
Fourier Transform

1D Fourier Transform



Top row: a signal and its Fourier transform. Middle row: the unit sample response function of a filter on the left, and the filter's system function on the right

1DFT



Bottom row: On the left the filtered signal that can be obtained by convolving the signal at the top with the filter in the middle. On the right the Fourier transform of the filtered signal obtained by multiplying the Fourier transform of the signal at the top, with the Fourier transform (system function) of the filter in the middle

One-Dimensional Discrete Fourier Transform

The Fourier transform of a discrete function of one variable, $f(x), x = 0, 1, 2, \ldots, M - 1$, is given by the equation

$$F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad \text{for } u = 0, 1, 2, \ldots, M - 1. \qquad (4.2\text{-}5)$$

This *discrete Fourier transform* (DFT) is the foundation for most of the work in this chapter. Similarly, given $F(u)$, we can obtain the original function back using the inverse DFT:

$$f(x) = \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad \text{for } x = 0, 1, 2, \ldots, M - 1. \qquad (4.2\text{-}6)$$

The $1/M$ multiplier in front of the Fourier transform sometimes is placed in front of the inverse instead. Other times (not as often) both equations are multiplied by $1/\sqrt{M}$. The location of the multiplier does not matter. If two multipliers are used, the only requirement is that their product be equal to $1/M$.

One-Dimensional Discrete Fourier Transform

Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta.$

Substituting this expression into Eq. (4.2-5), and using the fact that $\cos(-\theta) = \cos\theta$, gives us

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)[\cos 2\pi ux/M - j\sin 2\pi ux/M]$$

One-Dimensional Discrete Fourier Transform

In general, we see from Eqs. (4.2-5) or (4.2-8) that the components of the Fourier transform are complex quantities. As in the analysis of complex numbers, we find it convenient sometimes to express $F(u)$ in polar coordinates:

$$F(u) = |F(u)|e^{-j\phi(u)}$$

$$|F(u)| = \left[R^2(u) + I^2(u)\right]^{1/2}$$

is called the *magnitude* or *spectrum* of the Fourier transform, and

$$\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$$

is called the *phase angle* or *phase spectrum* of the transform.

$R(u)$ and $I(u)$ are the real and imaginary parts of $F(u)$

power spectrum, defined as the square of the Fourier spectrum:

$$\begin{aligned} P(u) &= |F(u)|^2 \\ &= R^2(u) + I^2(u). \end{aligned}$$

The term *spectral density* also is used to refer to the power spectrum.

Example : Consider the time signal shown in Fig. 2.1. The DFT of this signal is shown in Fig. 2.2.

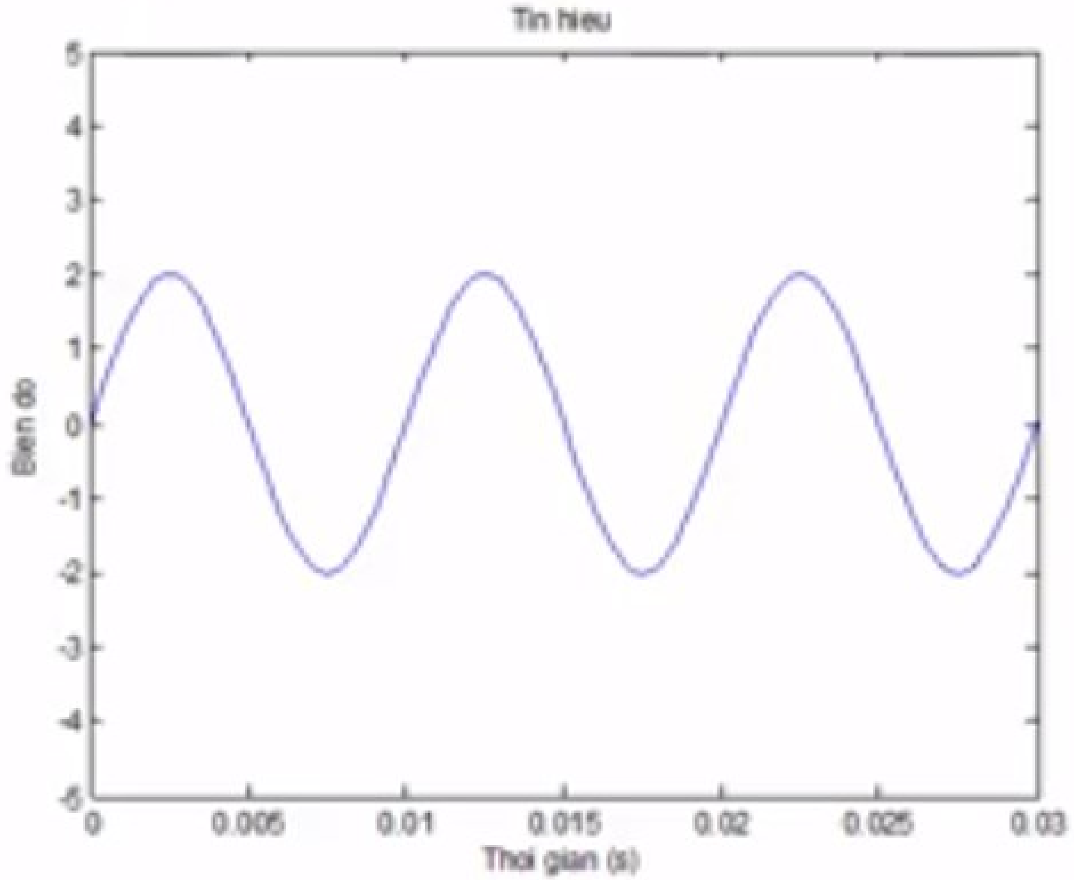


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Fig. 2.1: Signal x, defined in the time domain

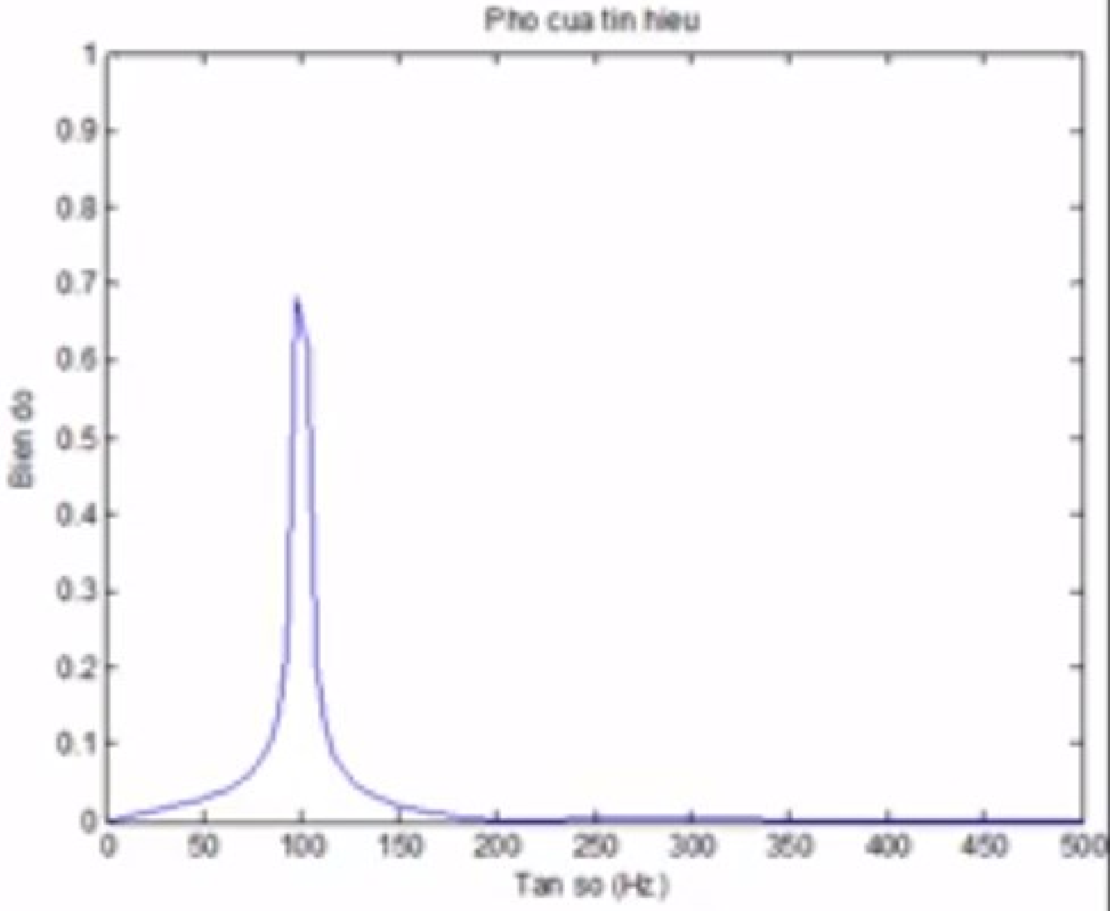
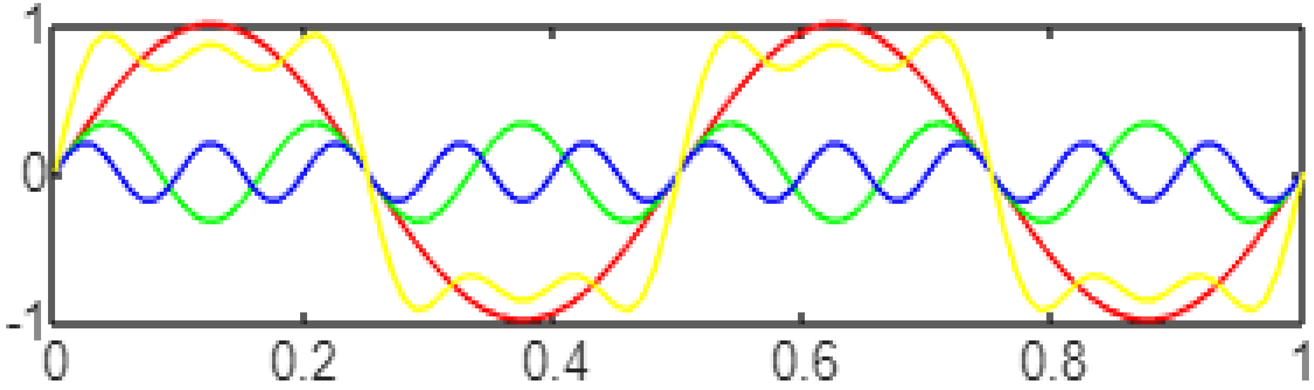


Fig. 2.2: Magnitude of the DFT of signal x

```
function test1DFT
x   = [0:.01:1]; fo = 2;
f1 = sin(2*pi*fo*x);
f2 = 1/3*sin(2*pi*3*fo*x)
f3 = 1/5*sin(2*pi*5*fo*x)
figure
subplot(3,1,1);

plot(x,f1,'r'); hold on
plot(x,f2,'g'); hold on
plot(x,f3,'b'); hold on
plot(x,f1+f2+f3,'y');
end
```



Two-Dimensional Discrete Fourier Transform

The 2D FT is a rather straightforward extension of the 1D transform. Mathematically, the 2D DFT is defined as:

$G($ ... $M-1\ N-1$ ... $-i\frac{2\pi(ux)}{}$ $-i\frac{2\pi(vy)}{N}$ (3.1)

Let $W_N = e^{-i\frac{2\pi}{N}} \Rightarrow W_N^2 = \left(e^{-i\frac{2\pi}{N}}\right)^2 = \left(e^{-i\frac{2\pi}{N/2}}\right) = W_{N/2}^2$

We have :

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)W_N^{ux} = \frac{1}{N}\left(\sum_{x=0}^{\frac{N}{2}-1} f(2x)W_N^{2ux} + \sum_{x=0}^{\frac{N}{2}-1} f(2x+1)W_N^{(2x+1)u}\right)$$

$$= \frac{1}{N}\left(\sum_{x=0}^{\frac{N}{2}-1} f(2x)\left(W_N^2\right)^{ux} + W_N^u\sum_{x=0}^{\frac{N}{2}-1} f(2x+1)\left(W_N^2\right)^{ux}\right)$$

$$= \frac{1}{N}\left(G(u) + W_N^u H(u)\right)$$

where $u=($ ... frequency axes, in which $G(u$ ...

Two-Dimensional Discrete Fourier Transform

The inverse transformation, the 2D IDFT is denoted as:

$$g(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}G(u,v)e^{j\frac{2\pi(ux)}{M}}e^{j\frac{2\pi(vy)}{N}} \qquad (3.2)$$

where $x=0,1,\ldots,M-1$ and $y=0,1,\ldots,N-1$ and $g(x,y)$ is an image after inverse transform.

Example of Fourier transform

$G(u,v)$ is the Fourier transformation of an image $g(x,y)$

$$G = U^T * g * V \tag{3.3}$$

where U, V are the matrices:

$$U(x,u) = e^{-\frac{j2\pi xu}{M}}; x,u = 0 : M-1 \tag{3.4}$$

$$V(y,v) = e^{-\frac{j2\pi yv}{N}}; y,v = 0 : N-1 \tag{3.5}$$

Example: Find the Fourier transform of the following 4x4 image

Solution:

- One has the row, M=4 ; the column, N=4
- Transform matrix $U(x,u)$ using the formula:

$$g(x,y)=\begin{bmatrix}0&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&0\end{bmatrix}$$

$$U(x,u)=e^{-\frac{j2\pi xu}{M}};x,u=0:M-1$$

$$U=\begin{bmatrix}e^{\frac{-j\times2\times\pi\times0\times0}{4}} & e^{\frac{-j\times2\times\pi\times0\times1}{4}} & e^{\frac{-j\times2\times\pi\times0\times2}{4}} & e^{\frac{-j\times2\times\pi\times0\times3}{4}}\\ e^{\frac{-j\times2\times\pi\times1\times0}{4}} & e^{\frac{-j\times2\times\pi\times1\times1}{4}} & e^{\frac{-j\times2\times\pi\times1\times2}{4}} & e^{\frac{-j\times2\times\pi\times1\times3}{4}}\\ e^{\frac{-j\times2\times\pi\times2\times0}{4}} & e^{\frac{-j\times2\times\pi\times2\times1}{4}} & e^{\frac{-j\times2\times\pi\times2\times2}{4}} & e^{\frac{-j\times2\times\pi\times2\times3}{4}}\\ e^{\frac{-j\times2\times\pi\times3\times0}{4}} & e^{\frac{-j\times2\times\pi\times3\times1}{4}} & e^{\frac{-j\times2\times\pi\times3\times2}{4}} & e^{\frac{-j\times2\times\pi\times3\times3}{4}}\end{bmatrix} \Rightarrow U=\begin{bmatrix}1&1&1&1\\1&-j&-1&j\\1&-1&1&-1\\1&j&-1&-j\end{bmatrix}$$

Using the formula of Euler, we have $e^{-j\varpi}=\cos\varpi-j\sin\varpi$

Similarly, one has the matrix, V

$$V = \begin{bmatrix} 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$G = U * g * V = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$G = \begin{bmatrix} 4 & -2-j2 & 0 & -2+j2 \\ -2-j2 & j2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ -2+2j & 2 & 0 & -j2 \end{bmatrix}$$

FFT: Fast Fourier Transform

$$F(\omega) = \frac{1}{N}\sum_{n=0}^{N-1} f_n(x).\, e^{-i2\pi\omega n/N}$$

$$f(x) = \sum_{n=0}^{N-1} F(\omega).\, e^{i2\pi\omega n/N}$$

FFT: Fast Fourier Transform

$$Let \quad \mathrm{W}_{\mathrm{N}} = e^{-i\frac{2\pi}{N}} \quad => \quad \mathrm{W}_{\mathrm{N}}^{2} = \left(e^{-i\frac{2\pi}{N}}\right)^{2} = \left(e^{-i\frac{2\pi}{N/2}}\right) = \mathrm{W}_{\mathrm{N/2}}^{2}$$

We have :

$$\mathrm{F(u)} = \frac{1}{\mathrm{N}}\sum_{x=0}^{N-1} f(x)W_N^{ux} = \frac{1}{\mathrm{N}}\left(\sum_{x=0}^{\frac{N}{2}-1} f(2x)W_N^{2ux} + \sum_{x=0}^{\frac{N}{2}-1} f(2x+1)W_N^{(2x+1)u}\right)$$

$$= \frac{1}{\mathrm{N}}\left(\sum_{x=0}^{\frac{N}{2}-1} f(2x)\left(W_N^{2}\right)^{ux} + W_N^{u}\sum_{x=0}^{\frac{N}{2}-1} f(2x+1)\left(W_N^{2}\right)^{ux}\right)$$

$$= \frac{1}{\mathrm{N}}\left(G(u) + W_N^{u}H(u)\right)$$

Matlab tool:

- fft2(I):
- fftshift(F):
- abs(F): Tính độ lớn của phổ tần số,
- F = F / max(F(:)):
- ifft2 biến đổi Fourier ngược.

DCT, DWT (haar)…