Physics-1 (Yagi) Midterm Exam

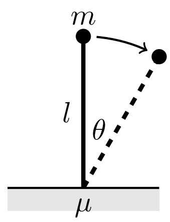
1. Using Newton's laws of motion,

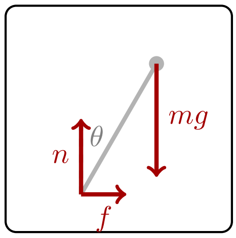
   (a) prove the conservation of mechanical energy for a conservative force;

   (b) prove the conservation of momentum for an isolated system.

   See lecture notes.

2. A particle of mass m is attached to a massless stick of length l. The stick stands at rest vertically on a table. The coefficient of static friction between the stick and the table is μ. You give a gentle push to the particle so that the stick starts to fall over. (By "a gentle push" we mean that the starting speed of m can be regarded as zero.)





(a) Let's consider the stick and the particle as one system. Identify all *external* forces acting on this system when the stick has tilted by angle θ. Draw the forces in a picture similar to the one in the box above.

(b) Until θ reaches some angle θ_s, the particle moves along a circular path. Find its radial acceleration a_r and tangential acceleration a_t as a function of θ.

(c) Find the angle θ_s at which the stick starts to slip.

(a) External forces are gravity mg on the particle, a normal force n from the floor, and friction f from the floor. See the above picture.

(b) Using energy conservation, we find

$$a_r = \frac{v^2}{l} = 2g(1 - \cos\theta) \quad \text{(pointing inward)},$$

From $\tau = I\alpha$, $\tau = mgl\sin\theta$, $I = ml^2$ and $a_t = l\alpha$, we find

$$a_t = g\sin\theta .$$

Note that we only used the external forces for $\tau = I\alpha$, so we don't need to worry about the force on the particle from the stick.

(c) We need two conditions:

$$n \geq 0 , \qquad f \geq \mu n .$$

Equations of motion for the center of mass are

$$f = ma_x = m(a_t\cos\theta - a_r\sin\theta) ,$$
$$mg - n = m(a_t\sin\theta + a_r\cos\theta) ,$$

From the second equation and $n \geq 0$,

$$\cos\theta(2 - 3\cos\theta) \geq 0 .$$

Since $\cos\theta > 0$ in this problem,

$$\cos\theta \geq \frac{2}{3} .$$

Fro $f \geq \mu n$ and the equations of motion,

$$a_t\cos\theta - a_r\sin\theta \geq \mu(g - a_t\sin\theta - a_r\cos\theta) .$$

Assuming $\cos\theta > 2/3$, we get

$$\tan\theta \le \mu$$

Noting that $\cos\theta > 2/3$ is equivalent to $\tan\theta < \sqrt{5}/2$, we obtain

$$\theta_s = \begin{cases} \tan^{-1}\mu & (\mu \ge \sqrt{5}/2)\,; \\ \tan^{-1}\sqrt{5}/2 & (\mu > \sqrt{5}/2)\,. \end{cases}$$

3. HAT-P-13 is a star in the constellation Ursa Major. It is known to have two planets, HAT-P-13b and HAT-P-13c. The nearest planet HAT-P-13b orbits around a circle. Let r_b be its radius and v_b be the orbital speed of HAT-P-13b. The orbit of HAT-P-13c has a high eccentricity (about 0.66). If v_c and v_c' are HAT-P-13c's lowest and highest speeds along its elliptic orbit, what is the smallest distance between HAT-P-13c and the star HAT-P-13?

   Let r_c and r_c' be the distance of HAT-P-13b when it has speed v_c and v_c', respectively. By energy conservation, the speed is lowest when the distance is largest, and highest when the distance is smallest. We want to find r_c'.

   From HAT-P-13b's circular motion,

$$\frac{v_b^2}{r_b} = \frac{GM}{r_b^2}\,,$$

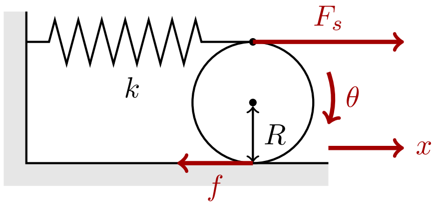
   where M is the mass of the star HAT-P-13. From energy conservation and angular conservation for HAT-P-13c, we have

$$\begin{aligned} &\frac{1}{2}v_c^2 - \frac{GM}{r_c^2} = \frac{1}{2}(v_c')^2 - \frac{GM}{(r_c')^2} \\ &r_c v_c = r_c' v_c'\,. \end{aligned}$$

   (Note that the velocity is perpendicular to the position vector at the points of smallest and largest distance.) From these we find

$$r_c' = \frac{2v_b^2 r_b}{v_c'(v_c + v_c')}\,.$$

4. A uniform solid cylinder of mass M and radius R is placed on a horizontal floor. One end of a spring with spring constant k is attached to the top of the cylinder, and the other end is attached to a wall at height $2R$. You push the cylinder slightly with your finger and let it go. Then, the cylinder starts to oscillate without slipping. What is the period of this small oscillatory motion?



Let's take the x-direction and θ-direction as in the above picture. The horizontal forces acting on the cylinder are the spring force F_s and the friction f from the floor.

When the cylinder rolls by angle θ, the center of mass of the cylinder moves by $x_{\mathrm{CM}} = R\theta$, and the top part moves by $x_{\mathrm{top}} = 2R\theta$. The spring force is then $F_s = -2kR\theta$. Equations of motion are

$$F_x = ma_x \implies -2kR\theta - f = MR\ddot{\theta}$$

and

$$\tau = I\alpha \implies -2kR\theta + f = \frac{1}{2}MR^2\ddot{\theta}\,.$$

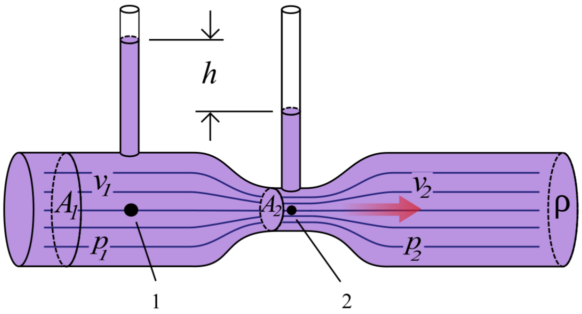
From these we obtain

$$\ddot{\theta} = -\frac{8}{3}\frac{k}{M}\theta\,.$$

The angular frequency is therefore $\omega = \sqrt{8k/3M}$ and the period is

$$T = 2\pi\sqrt{\frac{3M}{8k}}\,.$$

5. A fluid of density ρ flows through a Venturi tube, as shown in the picture. Vertical tubes are attached above point 1 and point 2, which allows you to measure the pressure difference $p_1 - p_2$ between point 1 and point 2. The cross-sectional areas are A_1 at point 1 and A_2 at point 2. You know the flow speed is v_1 at point 1. Assuming that this is an ideal fluid flow, what is the height difference h?



By the continuity equations

$$A_1 v_1 = A_2 v_2 \,.$$

By Bernoulli's equation

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2 \,.$$

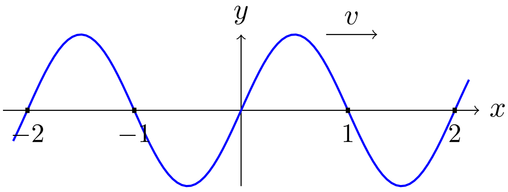
We have $p_1 - p_2 = \rho g h$, so

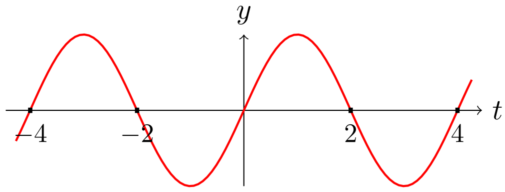
$$h = \frac{(A_1^2 - A_2^2) v_1^2}{2 g A_2^2} \,.$$

6. A sinusoidal wave is propagating along a rope. The rope extends in the x-direction, and each point of the rope is vibrating in the y-direction with amplitude A. The wave is travelling in the ~~positive~~ negative x-direction at speed v. The first graph shows the wave form at time $t = 0$. The second graph describes the oscillation of the point $x = 0$ of the rope as a function of time.

   (a) Write down the wave function of this wave.

   (b) Draw a graph representing the wave form at $t = 1$.





(a) The wave function takes the form $y(x, t) = A \sin(k(x + vt) + \phi)$. From the first graph, we see $k = \pi$ and $\phi = 0$. From the second graph we see $v = 1/2$. So we have

$$y(x, t) = A \sin\left(\pi x + \frac{\pi}{2} t\right) .$$

(b) $y(x, 1) = A \cos(\pi x)$: