Homework for the Modular forms course
Qiuzhen college, Fall 2021

This is the homework for our Modular forms class, which consists of 9 questions for a total of 100 points. The quality of the writing will be taken into account for the grade. Please justify carefully your answers. You are free to use what we have done and proved in class. You are welcome to indicate your ideas and partial results even if you cannot fully resolve a question (credits may be given for this).

Background and notation: $\mathfrak{h}$ is the upper-half plane. If $z \in \mathbb{C}$, $\Re(z)$ is the real part of z. We let $q = e^{i\pi z}$ (note the difference with $e^{2i\pi z}$). We let $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathfrak{h}$ by Mobius transformations: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$. You are free to use the fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{h}$ seen in class. We let

$$E_2(z) = 1 - 24 \cdot \sum_{n=1}^{+\infty} \sigma_1(n) q^{2n}$$

where $\sigma_1(n) = \sum_{d|n} d$. Recall that we have seen in class that $E_2(-\frac{1}{z}) = z^2 E_2(z) - \frac{6i}{\pi} z$ for all $z \in \mathfrak{h}$.

1. (5 points) Let $\Gamma(2) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $a \equiv d \equiv 1$ (modulo 2) and $b \equiv c \equiv 0$ (modulo 2)$\}$. Show that $\Gamma(2)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$. What is (the isomorphism class of) the quotient group $\mathrm{SL}_2(\mathbb{Z})/\Gamma(2)$?

2. (5 points) Prove that $\Gamma(2)$ is generated by the matrices $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Remark: The group $\Gamma(2)/\pm 1$ is actually free with 2 generators, given by A and B.

3. (10 points) Let

$$\mathcal{F}_2 = \{z \in \mathfrak{h} \text{ such that } -1 < \Re(z) \leq 1, \mid z - \frac{1}{2} \mid \geq \frac{1}{2} \text{ and } \mid z + \frac{1}{2} \mid > \frac{1}{2}\} .$$

Show that $\mathcal{F}_2$ is a fundamental domain for the action of $\Gamma(2)$ on $\mathfrak{h}$.

4. (10 points) Let $\lambda : \mathfrak{h} \to \mathbb{C}$ given by

$$\lambda(z) = 16q \prod_{n=1}^{+\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}} \right)^8 .$$

Prove that λ is a well-defined, nowhere vanishing, holomorphic function on $\mathfrak{h}$. (You can use the fact that if $(f_n)_{n\geq 1}$ is a sequence of holomorphic functions on $\mathfrak{h}$ converging to f, and that the convergence is uniform on any given compact, then f holomorphic on $\mathfrak{h}$).

5. (10 points) Express the logarithmic derivative $\frac{\lambda'(z)}{\lambda(z)}$ of λ in terms of the function $E_2(z)$. Use this to prove that λ is $\Gamma(2)$-invariant, i.e. for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$, $f(\frac{az+b}{cz+d}) = f(z)$.

6. (15 points) Prove that $\lim_{z\to 0 \text{ s.t. } z\in\mathcal{F}_2} \lambda(z)$ exists and is a real number > 0; we denote this limit by $\lambda(0)$.
Remark: One can show that $\lambda(0) = 1$, although I am not aware of a short and simple proof (if you find one, you are welcome to write it down).

7. (5 points) Prove that $\lambda(z+1) = -\lambda(0) \cdot \frac{\lambda(z)}{\lambda(-\frac{1}{z})}$ for all $z \in \mathfrak{h}$.

8. (15 points) Prove that for any $z \in \mathfrak{h}$, we have $\lambda(-\frac{1}{z}) = \lambda(0) - \lambda(z)$.

9. (Hard, 25 points) Prove that $\lambda : \mathcal{F}_2 \to \mathbb{C} - \{0, \lambda(0)\}$ is a bijection. Hint: First consider the restriction of λ to the boundary of $\mathcal{F}_2$, and show that this restriction is a bijection onto $(-\infty, 0] \cup [\lambda(0), +\infty)$. Then, for any $c \in \mathbb{C} - ((-\infty, 0] \cup [\lambda(0), +\infty))$, show that there is a unique z in the interior of $\mathcal{F}_2$ such that $\lambda(z) = c$. You may apply the argument principle to a well-chosen contour integral.
Remark: This result can be used to prove that a non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C}$ takes as values all complex numbers except possibly at most one.