TOPOLOGY MIDTERM

Exam time: 2:00 PM-4:30 PM. 10 problems, 200 points in total. Even if you can't solve part (1), you can still assume its conclusion and work on part (2).

Problem 1 (10+10 points): Consider the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

(1) Verify that d is a metric on $\mathbb{R}^2$. You need to check all the axioms.
(2) Show that the metric topology defined by d is the standard topology on $\mathbb{R}^2$.

Problem 2 (10+10 points): Consider the projection map $p : X \times Y \to X$ defined by $p(x, y) = x$. We use the product topology on $X \times Y$.
(1) Is p always an open map? If yes, prove it. If no, give a counter example.
(2) Is p always a closed map? If yes, prove it. If no, give a counter example.

Problem 3 (10+10 points): Recall that the finite complement topology $\mathcal{T}_{fc}$ on $\mathbb{R}$ is defined as:

$$U \text{ is open} \iff \mathbb{R} \backslash U \text{ is a finite set.}$$

Let X be the product space $(\mathbb{R}, \mathcal{T}_{fc}) \times ([0, 1], \mathcal{T}_{st})$. Here $\mathcal{T}_{st}$ denotes the standard topology on $[0, 1]$.
(1) Write down a basis for X. (You don't need to prove that it is actually the basis for the product topology.)
(2) Let $f : X \to \mathbb{R}$ be a continuous map. Show that f has a maximum and a minimum.

Problem 4 (10+10 points): Let X be a compact, Hausdorff space.
(1) Prove that X is normal. (Recall that normal means that X satisfies the T_4-axiom.)
(2) Show that given any disjoint, closed subsets A, B, there exists a neighborhood U of A and a neighborhood V of B such that $\overline{U} \cap \overline{V} = \varnothing$.

Problem 5 (10+10 points): Let A be a nonempty, bounded subset of $\mathbb{R}^n$.
(1) Consider the subspace

$$B = \{x \in \mathbb{R}^n \mid \exists y \in A \text{ such that } |y - x| \leqslant 1\}$$

Suppose A is compact. Show that B is also compact.
(2) Consider the subspace

$$B' = \{x \in \mathbb{R}^n \mid \exists y \in A \text{ such that } |y - x| < 1\}$$

Show that B' is never compact (regardless whether A is compact or not).

Problem 6 (10+10 points): Let X be a compact space and let $f : X \to [0, 1]$ be

a surjective, continuous map.
(1) Show that f is a quotient map.
(2) Suppose $f^{-1}(t)$ is connected for any $t \in [0,1]$. Show that X is connected.

Problem 7 (10+10 points): Consider the topological group $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with the group structure given by the multiplication of complex numbers.
(1) Does there exists a free action of S^1 on S^3? If yes, construct one. If no, give a proof.
(2) Does there exist a free action of S^1 on S^2? If yes, construct one. If no, give a proof.

Problem 8 (20 points): Consider the following three homeomorphisms on $\mathbb{R}^2$:

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (x+1, y)$;
- $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(x,y) = (x, y+1)$;
- $h : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(x,y) = (-y, x)$.

Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ generated by the following relations:

$$f(x,y) \sim (x,y), \quad g(x,y) \sim (x,y), \quad h(x,y) \sim (x,y) \quad \text{for any } (x,y) \in \mathbb{R}^2.$$

Find the homeomorphism type of $\mathbb{R}^2/\sim$.

Problem 9 (10+10 points): Let X be a normal topological space and let V be a closed subspace of X.
(1) Consider a continuous map $f : V \to \mathbb{R}^n$. Show that f can be extended to a continuous map $\tilde{f} : X \to \mathbb{R}^n$ (i.e. $\tilde{f}|_V = f$)
(2) Consider a continuous map $g : V \to S^n$. Show that there exists a neighborhood U of V such that g can be extended to a continuous map $\tilde{g} : U \to S^n$.

Problem 10 (20 points). Recall that $U(2)$ is the topological group

$$\{2 \times 2 \text{ complex matrix } A \mid AA^* = I\}.$$

Here $A^* = \overline{A}^T$ and I denotes the identity matrix. To define the topology on $U(2)$, we treat it as a subspace of $\mathbb{C}^4$ and use the subspace topology of the standard topology. Show that $U(2)$ is homeomorphic to $S^1 \times S^3$ as a topological space.