Final Exam

Algebra 1

1. (4 points) Let A be an $(n \times n)$-matrix with entries in a field k, let $\lambda_1, \ldots, \lambda_n$ be all the eigenvalues of A (counted with multiplicities) in an algebraic closure $\bar{k}$ of k, and let $f(t) \in k[t]$ be a polynomial. Prove that the characteristic polynomial of $f(A)$ is

$$(t - f(\lambda_1)) \cdots (t - f(\lambda_n)).$$

2. (4 points) Consider a block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are respectively $m \times m$, $m \times n$, $n \times m$, $n \times n$ matrices. We say the block matrix is block-upper-triangular (resp. block-lower-triangular) if $C = 0$ (resp. $B = 0$). We say a block-triangular matrix is elementary if A and D are identity matrices: $A = I_m$, $D = I_n$, and either $B = 0$ or $C = 0$
   (1) Find an elementary block-triangular matrix P so that $\begin{pmatrix} tI_m & B \\ C & I_n \end{pmatrix} \cdot P$ is block-upper-triangular.
   (2) Find an elementary block-triangular matrix Q so that $\begin{pmatrix} tI_m & B \\ C & I_n \end{pmatrix} \cdot Q$ is block-lower-triangular.
   (3) Find the relationship between the characteristic polynomial of BC and that of CB.

3. (10 points) Let $T : V \to V$ be a linear operator on a finite dimensional vector space V over an algebraically closed field k. For any eigenvalue λ of T, define the *eigenspace* V_λ and the *generalized eigenspace* V'_λ for the eigenvalue λ by

$$\begin{aligned} V_\lambda &= \{v \in V : Tv = \lambda v\}, \\ V'_\lambda &= \{v \in V : (\lambda I - T)^m(v) = 0 \text{ for some } m \geq 1\}. \end{aligned}$$

   (1) Prove V_λ and V'_λ are subspaces of V, $V_\lambda \subset V'_\lambda$, and

$$V = \bigoplus_\lambda V'_\lambda,$$

   where λ goes over the set of eigenvalues of T.
   (2) Let $T' : V \to V$ be an operator. Define $[T, T'] = TT' - T'T$. Prove that if $[T, T'] = 0$, then V_λ and V'_λ are invariant subspaces of T'.
   (3) Suppose $[T, T'] = qT'$ for some $q \in k$. Prove that

$$T'(V_\lambda) \subset V_{\lambda+q}, \quad T'(V'_\lambda) \subset V'_{\lambda+q}.$$

   Suppose furthermore q is nonzero. Prove that T' must be nilpotent, that is, $T'^N = 0$ for some positive integer N.

4. (10 points) Let $a_0, \ldots, a_{d-1} \in \mathbb{C}$. We try to find a formula for sequences $x = (x_0, x_1, \ldots)$ of complex numbers defined by the recursive formula

$$x_{n+d} = -a_0 x_n - \cdots - a_{d-1} x_{n+d-1}$$

for all $n \geq 0$. Let V be the space of all such sequences. It is a vector space. For any sequence $x = (x_0, x_1, \ldots)$, let

$$T(x) = (x_1, x_2, \ldots).$$

T defines a linear operator on V

(1) Calculate the characteristic polynomial, the invariant factors, and the rational canonical form of T.

(2) Suppose the polynomial

$$\Delta(t) = t^d + a_{d-1}t^{d-1} + \cdots + a_0 = (t - \lambda_1) \cdots (t - \lambda_d)$$

has no multiple root. Find an explicit basis of V consisting of eigenvectors of T. Using this basis, find a formula for x_n expressed in terms of the roots of $\lambda_1, \ldots, \lambda_d$, and the initial values $x_0, \ldots, x_{d-1}$ for any $x \in V$.

5. (18 points) We continue to work on the Problem 4 and keep the notations there. Let $C^\infty(\mathbb{R})$ be the space all smooth complex valued function on $\mathbb{R}$. For any $f(x) \in C^\infty(\mathbb{R})$ let $f^{(i)}$ be the i-th higher order derivative of f, and let $\mathcal{V}$ be the space of solutions of the ordinary differential equation

$$f^{(d)} + a_{d-1} f^{(d-1)} + \cdots + a_1 f' + a_0 f = 0.$$

A fundamental theorem in the theory of ordinary differential equations says that for any given complex numbers $x_0, x_1, \ldots, x_{d-1}$, there exists one and only one solution of the above differential equation such that

$$f(0) = x_0, \quad f'(0) = x_1, \quad f^{(2)}(0) = x_2, \quad \ldots, \quad f^{(d-1)}(0) = x_{d-1}.$$

(1) Prove that we have an isomorphism $\phi : \mathcal{V} \to V$ so that for any $f(x) \in \mathcal{V}$, we have $T(\phi(f(x))) = \phi(f'(x))$.

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\phi", "\cong"'] \arrow[d, "\frac{d}{dx}"'] & V \arrow[d, "T"] \\ \mathcal{V} \arrow[r, "\phi", "\cong"'] & V \end{tikzcd}$$

(2) What are the eigenvalues of the operator

$$\frac{d}{dx} : \mathcal{V} \to \mathcal{V}, \quad f(x) \mapsto f'(x).$$

For each eigenvalue λ, find an explicit eigenvector for $\frac{d}{dx}$.

(3) Suppose the polynomial

$$\Delta(t) = t^d + a_{d-1}t^{d-1} + \cdots + a_0 = (t - \lambda_1) \cdots (t - \lambda_d)$$

has no multiple root. Find an explicit basis of $\mathcal{V}$ consisting of eigenvectors of $\frac{d}{dx}$.

(4) Consider the general case where $\Delta(t)$ may have multiple roots. Write

$$\Delta(t) = (t-\lambda_1)^{m_1}\cdots(t-\lambda_l)^{m_l}.$$

where $\lambda_1,\ldots,\lambda_l$ are distinct. Find an explicit basis of $\mathcal{V}$ with respect to which the nonzero entries of the matrix of $\frac{d}{dx}:\mathcal{V}\to\mathcal{V}$ lie on the diagonal or adjacent to the diagonal. Write down this matrix. Calculate the Jordan canonical form of $\frac{d}{dx}:\mathcal{V}\to\mathcal{V}$

(5) In the case where $\Delta(t)$ may have multiple roots, find an explicit basis of V, and find a formula for x_n expressed in terms of the roots of $\lambda_1,\ldots,\lambda_l$ of $\Delta(t)$, their multiplicities $m_1,\ldots,m_l$, and the initial values $x_0,\ldots,x_{d-1}$ for any $x\in V$

6. (2 points) Let R be an integral domain and let K be the fraction field of R. We say R is *normal* if for any $x\in K$ satisfying a polynomial equation

$$x^n + a_1x^{n-1}+\cdots+a_n = 0$$

such that $a_1,\ldots,a_n\in R$, we have $x\in R$. Prove that if R is UFD, then R is normal.

7. (9 points) Let R be an integral domain, K the fraction field of R, $\mathfrak{m}$ a maximal ideal of R, and

$$f(t) = t^n + a_1t^{n-1}+\cdots+a_n$$

a monic polynomial such that $a_1,\ldots,a_n\in R$. Suppose

$$a_1,\ldots,a_n\in\mathfrak{m},\quad a_n\notin\mathfrak{m}^2.$$

(1) Prove that $f(t)$ is an irreducible element in $R[t]$.
(2) Suppose furthermore that R is a UFD. Prove that $(f(t))$ is a prime ideal of $R[t]$.

8. (10 points) Let D be an integer with no square factors. Consider

$$\mathbb{Q}[\sqrt{D}] = \{a+b\sqrt{D} : a,b\in\mathbb{Q}\}.$$

(1) Prove that $\mathbb{Q}[\sqrt{D}]$ is a subfield of $\mathbb{C}$.
(2) Prove that

$$\sigma:\mathbb{Q}[\sqrt{D}]\to\mathbb{Q}[\sqrt{D}],\quad a+b\sqrt{D}\mapsto a-b\sqrt{D}$$

is an isomorphism.
(3) For any $z\in\mathbb{Q}[\sqrt{D}]$, define the trace and the norm of z by

$$\mathrm{Tr}(z) = z+\sigma(z),\quad N(z) = z\sigma(z).$$

Prove that $\mathrm{Tr}(z), N(z)\in\mathbb{Q}$. We say z is an *algebraic integer* if it satisfies a polynomial equation

$$z^n + a_1z^{n-1}+\cdots+a_n = 0$$

such that $a_1,\ldots,a_n$ are integers. Prove that z is an algebraic integer if and only if $\mathrm{Tr}(z), N(z)\in\mathbb{Z}$.

9. (18 points) Let k be a field of characteristic $\neq 2$, and let $p(x)\in k[x]$ be a polynomial with no square factors. Denote by $k(x)$ the fraction field of $k[x]$. Consider

$$k(x)[\sqrt{p(x)}] = k(x)[y]/(y^2-p(x)).$$

(1) Prove that $k(x)[y]/(y^2 - p(x))$ is a field, and any element in $k(x)[y]/(y^2 - p(x))$ can be written as $a(x) + yb(x)$ such that $a(x), b(x) \in k(x)$.

(2) Prove that we have an isomorphism

$$\sigma : k(x)[y]/(y^2 - p(x)) \to k(x)[y]/(y^2 - p(x))$$

such that $\sigma(a(x)) = a(x)$ for any $a(x) \in k(x)$ and $\sigma(y) = -y$.

(3) For any $f \in k(x)[y]/(y^2 - p(x))$, define the trace and the norm of f by

$$\mathrm{Tr}(f) = f + \sigma(f), \quad N(f) = f\sigma(f).$$

Prove that $\mathrm{Tr}(f), N(f) \in k(x)$. We say f is *integral* if it satisfies an equation

$$f^n + a_1(x)f^{n-1} + \cdots + a_n(x) = 0$$

for some polynomials $a_1(x), \ldots, a_n(x) \in k[x]$. Prove that f is integral if and only if $\mathrm{Tr}(f), N(f) \in k[x]$.

(4) Prove that $k[x,y]/(y^2 - p(x))$ is an integral domain, its fraction field can be identified with the field $k(x)[y]/(y^2 - p(x))$, and the integral domain $k[x,y]/(y^2 - p(x))$ is normal. (*This shows that the algebraic curve $y^2 = p(x)$ is normal.*)

10. (15 points) Consider the ring

$$A = \mathbb{R}[x,y]/(x^2 + y^2 - 1) = \mathbb{R}[x]\left[\sqrt{1 - x^2}\right].$$

It is a normal integral domain by Problem 9 (4). For any polynomial $f \in \mathbb{R}[x,y]$, denote also by f the image of f in A.

(1) Prove that y is not a prime element in A, and there exist prime ideals P $\boxed{\mathfrak{p} \text{ and } \mathfrak{q} \text{ in } A}$ such that such that $(y) = \mathfrak{p}\mathfrak{q}$. Describe explicit families of generators for $\mathfrak{p}$ and $\mathfrak{q}$.

(2) Prove that $x + 1$ and $x - 1$ are not prime elements in A. Calculate $\mathfrak{p}^2$ and $\mathfrak{q}^2$, and show that they are equal to $(x - 1)$ and $(x + 1)$.

(3) $x + 1, x - 1, y$ are irreducible elements in A. To save time, choose one element from this list and prove it is irreducible.

(4) Prove that the two factorizations

$$-(x+1) \cdot (x-1), \quad y \cdot y$$

for y^2 in A are non-associate factorizations into products of irreducible elements. So A is not a UFD. (*A is a Dedekind domain. Each nonzero ideal in the Dedekind domain A has a unique factorization into a product of prime ideals. But elements in A can have different factorization into product of irreducible elements. The prime factorization for ideals corresponding to $(x+1) \cdot (x-1) = (y)^2$ is*

$$\mathfrak{p}^2 \cdot \mathfrak{q}^2 = (\mathfrak{p}\mathfrak{q}) \cdot (\mathfrak{p}\mathfrak{q}).$$

So there is no conflict between the non-uniqueness of factorization of elements, and the uniqueness of factorization of ideals.)