Final Exam

Instructions: You can choose any **five** problems, from the list below, to solve.

Problem 1

Suppose the equations of motion of a dynamical system can be written as

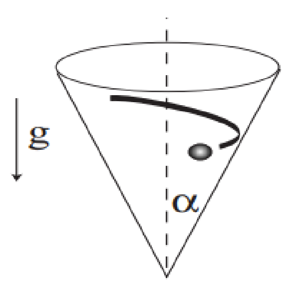
$$\frac{dL}{dt} = [L, M] := L \cdot M - M \cdot L$$

where L and M are square matrices whose entries are functions of the coordinates p and q.

1. Show that the quantities $\mathrm{Tr}(L^k)$ are conserved quantities for any $k \in \mathbb{N}$.
2. The matrices L and M are called a Lax pair. Are the matrices L and M unique?. Write two examples of transformations of M and L that will not change the equations of motion.
3. Show that the equations of the Kepler problem (particle in a gravitational potential, in $\mathbb{R}^3$) can be written in this form if we define
$$L := \frac{1}{2}\begin{pmatrix} -\langle p, x\rangle & \langle x, x\rangle \\ -\langle p, p\rangle & \langle x, p\rangle \end{pmatrix}$$
4. What is the meaning of $\mathrm{Tr}(L^2)$?

Problem 2

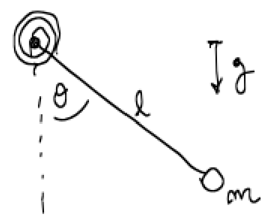
Consider a point-particle of mass m restricted to move on a cone of angle α as the Figure shows. Also assume that the only external force acting over the particle is gravity given by $F_{\text{grav}} = -mge_3$ (there is no friction)



1. Show that this problem is equivalent to a central force problem i.e. we can reduce it to the dynamics of a single variable in $\mathbb{R}_{\geq 0}$.
2. Find an explicit integral form for the orbit equation and compute it exactly in the limit $\alpha = \frac{\pi}{2}$.

Problem 3

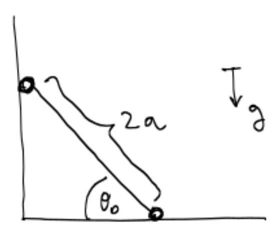
Consider a pendulum of length l and mass m whose pivot is attached to a torsional spring (see figure). We are familiar with the usual spring where the force follows Hooke's law, i.e. $F_{\text{Hooke}} = k\|x\|$. A torsional spring is a spring that applies a torque $\tau_{\mathcal{O}} = x \times F_{\text{Tor}}$ linear on the torsion angle, that is $\tau_{\mathcal{O}} = -\kappa\theta e_3$ where $\kappa \in \mathbb{R}_{>0}$ is a proportionality constant.



1. Assume all the forces and movement are resticted to a plane. Find the potential energy associated to the torsional spring (up to a constant).
2. Find the Lagrangian for the system of the pendulum attached to the torsional spring and solve its equation of motion for a small angle.

Problem 4

A massless stick of length $2a$ with two identical masses attached to its ends (i.e. a dumbbell) is leaning on the smooth vertical wall on one side, and lying on the smooth ground on the other side with the angle θ_0 (see figure). Release the stick from rest and so the stick slides down by gravity. Compute:



1. An expression (you do not have to solve the integral) for the time $t(\theta)$ that the dumbbell reach the angle θ before losing contact with the wall.
2. The angle θ that the dumbbell loses contact with the wall.

Problem 5

Consider the Kepler problem with potential $U = -\frac{m\alpha}{r}$ where $r = \sqrt{\sum_{i=1}^{3} x_i^2}$, and consider the Laplace-Runge-Lenz vector:

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - m^2\alpha\frac{\mathbf{x}}{r} \tag{1}$$

with $\mathbf{p}$ the momentum of the system, $\mathbf{L}$ the angular momentum.

1. Compute the Poisson bracket $\{p_i, L_j\}$ with $i = 1, 2, 3$.
2. Show that the Poisson bracket satisfy $\{fg, h\} = f\{g, h\} + \{f, h\}g$ for any set of functions f, g, h of the position and momenta.
3. Verify the Poisson bracket $\{A_i, H\}$ vanishes (for all i). **Hint**: It can be useful to use that the Poisson bracket is bilinear and the identity $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$.

Problem 6

There are certain materials that present what is known as a superconducting phase, below a certain temperature. When an external magnetic field $B : \mathbb{R}^4 \to \mathbb{R}^3$ is applied to these materials, is observed that, inside the material B vanishes i.e. $B(x,t) \sim 0$ for all time t and all $x \in R$ where $R \subset \mathbb{R}^3$ is the region occupied by the superconductor (this is known as the Meissner effect) and also is observed, that an electric current $J : \mathbb{R}^4 \to \mathbb{R}^3$ can exist inside R without any external electric field $E : \mathbb{R}^4 \to \mathbb{R}^3$ (this is why it is called a super-conductor). In 1935 F. and H. London, propose that, to explain the latter phenomeon we should have the following law:

$$\frac{\partial J}{\partial t} = \alpha E$$

where $\alpha \in \mathbb{R}_{>0}$ is a constant. The goal of this problem is to show that this also implies the Meissner effect.

1. Find the units of the constant α.
2. Using Maxwell equations show that B satisfy (in the region R)

$$\frac{\partial}{\partial t}\left(\nabla \times J + \alpha B\right) = 0.$$

3. Assume that $\nabla \times J + \alpha B = 0$ in R (as the simplest solution of the above equation). Then, using Maxwell equations show that, if $E = 0$ we have

$$\nabla^2 B = \frac{1}{\lambda^2}B$$

where the operator ∇^2 is defined as $(\nabla^2 B)_i e_i = \left(\sum_{j=1}^{3} \frac{\partial^2 B_i}{\partial x_j^2}\right) e_i$. Determine explicitly the value of λ. **Hint**: The identity $\nabla^2 A = \nabla(\mathrm{div} A) - \nabla \times (\nabla \times A)$ for vector valued functions A, can be useful.

4. Apply the result above for a superconductor located in an (infinite) region $R = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | -a \leq x_1 \leq a\}$ under an external magnetic field of the form $B = B_3(x_1)e_3$ where the function $B_3(x_1)$ satisfies

$$B_3(x_1) = \begin{cases} B_0 = \text{const.} & x_1 \geq a, x_1 \leq -a \\ B_3(x_1) & x_1 \in (-a, a) \end{cases}$$

and no external electric field ($E = 0$). Does your result represent the Meissner effect? what is the physical interpretation of λ?.