分析2期中考试

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几点声明:

1、此试卷为考生回忆版,不保证问题顺序的准确性与问题陈述的严谨性。

2、考试时间为2022年4月18日9:50至12:15。

3、满分为108分。

1.Let $(X, \mathcal{B}, \mu)$ be a measure space and $f_n : X \to \mathbb{C}$ be a sequence of measurable functions.

(1)(10 points)Show that $S = \{x \in X; \lim\limits_{n\to\infty} f_n(x)\, exists\}$ is measurable.

(2)(10 points)Suppose $g : X \to [0, \infty]$ is a integral function such that$|f_n(x)| \leqslant g(x), \forall x \in X, n \in \mathbb{N}$.Suppose $f : X \to \mathbb{C}$ is another measurable function.Show that f_n pointwise converges to f almost everywhere if and only if f_n almost uniformly converges to f.

2.(1)(8 points)Use the Dominated Convergence Theorem to find:

$$\lim_{n\to\infty} \int_{2022}^{\infty} \frac{x^4 cos(n^2\pi x^2) + x^{4.8}}{27 + nx^6}\, dx.$$

(2)(10 points)Use the fact $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ to prove:

$$\int_0^{\infty} \frac{x}{e^{2x} - 1}\, dx = \frac{\pi^2}{24}$$

3.Let $\mathcal{A}$ be the set of all subsets of $\mathbb{R}$ which are the finite union of the left-closed right-open intervals like $[a, b)$.$\mu_0 : \mathcal{A} \to [0, \infty]$ satisfying $\mu_0(E) = \infty, \forall \phi \neq E \in \mathcal{A}$ and $\mu_0(\phi) = 0$.

(1)(6 points)Show that μ_0 is a pre-measure.

(2)(6 points)Show that $\langle \mathcal{A} \rangle$ is actully the Borel algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$.

(3)(6 points)Let $\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ be the Kolmogorov measure of μ_0.Show that $\mu(E) = \infty, \forall \phi \neq E \in \mathcal{B}(\mathbb{R})$.

(4)(6 points)Construct a measure $\mu' : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ such that μ' is a extension of μ_0 but $\mu' \neq \mu$.

4.(1)(10 points)Suppose $f : \mathbb{R} \to [0,\infty]$ is a measurable function such that $\int_{\mathbb{R}} f\,dm < \infty$.Let $f_k(x) = f(x+k), \forall k \in \mathbb{N}$ and $h = \liminf_{k\to\infty} f_k$.
Show that

$$m(\{x \in \mathbb{R}; h(x) \neq 0\}) = 0.$$

(2)(10 points)Suppose a Lebesgue measurable set $E \subseteq [0,1]$ satisfying:there exists a constant $C > 0$,such that $\forall$ interval $I \subseteq [0,1]$,

$$m(E \cap I) \geqslant C\,m(I).$$

Determine whether $m(E) = 1$ always holds.

5.(1)(5 points)For each subset E of $\mathbb{R}$,define

$$D(E) := \{x - y; x, y \in E\}.$$

Show that if E is Lebesgue measurable with $m(E) > 0$,then $D(E)$ contains a neighborhood of 0.
(2)(5 points)We know that $G = \{n + m\pi; n, m \in \mathbb{Z}\}$,the group generated by 1 and π,is dense in $\mathbb{R}$.Suppose

$$V = \{x_C \in [0,1]; C \in \mathbb{R}/G\}$$

where x_C is a representation element of C.
Show that V is a non-measurable set.
(3)(5 points)We know that $G_0 = \{2n + m\pi; n, m \in \mathbb{Z}\}$ is a dense subgroup of G,and $G \backslash G_0$ is dense in $\mathbb{R}$.Suppose

$$\mathcal{S} = V + G_0.$$

Show that for each measurable subset F of $\mathcal{S}$,$D(F) \cap (G \backslash G_0) = \phi$.
(4)(5 points)Show that $\mathcal{S}$ is a non-measurable set.
(5)(6 points)Show that every measurable subsets of $\mathcal{S}$ are null sets.