Mathematical Analysis, Final Exam

January 3, 2022

1. Find the global maximum of the function $f(x, y) = \sin x \sin y \sin(x + y)$ on the domain $\{x \geq 0,\ y \geq 0,\ x + y \leq \pi\}$.

2. Compute the integral
$$\int_S z dy \wedge dz + \sin y dz \wedge dx + dx \wedge dy,$$
where S is the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$.

3. Compute the integral
$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1 + x^2 + y^2}} dx dy.$$

4. Let $h : \mathbb{R}^{2+2} \to \mathbb{R}^2$ be given by
$$h(u_1, u_2, w_1, w_2) = (u_1^2 + u_2 + w_1^2, e^{u_1} - 1 + u_2 + w_2).$$
Show that $h(0, 0, 0, 0) = (0, 0)$ and that one can apply the implicit function theorem in order to obtain some small neighborhood U of $(0, 0)$ in $\mathbb{R}^2$ and a C^1 function $f : U \to \mathbb{R}^2$, such that
$$h(f(w), w) = 0,\ \forall\ w = (w_1, w_2) \in U.$$
Find $Df(0, 0)$.

5. Let $f_n :\ [0, 1] \to \mathbb{R}$ be a sequence of uniformly bounded Riemann integrable functions and let $F_n :\ [0, 1] \to \mathbb{R}$ be a sequence defined by $F_n(x) = \int_0^x f_n(t)\, dt$. Prove that the sequence $\{F_n\}$ has a convergent subsequence in $C([0, 1])$.

6. Compute the Legendre transformation of the following function, where α, β are positive constants,
$$f(x, y, z) = x^2 + \alpha(y + \beta z)^2 + z^2.$$

7. Prove that f(x) convergence uniformly
$$f(x) = \int_1^\infty \frac{x}{y} e^{-yx} dy,\ x \geq 0$$

8. Let $\mathbb{R}^n \times \mathbb{R}^n$ be given coordinates $(x_1, \dots, x_n; y_1, \dots, y_n)$. Conisder the 2-form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ on $\mathbb{R}^n \times \mathbb{R}^n$. Let $\eta = \sum_{i=1}^n f_i(x) dx_i$ be a closed 1-form, where $f_i : \mathbb{R}^n \to \mathbb{R}$ is smooth and vanishes outside a bounded region, $i = 1, \dots, n$. We next think $F = (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ as a vector-valued function. Thus the graph $\{(x, F(x)) \mid x = (x_1, \dots, x_n) \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}^n$ is a manifold of dimension n, denoted by M_F. Prove that for any $z \in M_F$ and any $v_1, v_2 \in T_z M_F$, we have $\omega(v_1, v_2) = 0$.

9. Evaluate the indefinite integral
$$\int \frac{x \ln x}{(1 + x^2)^2} dx.$$

10. Given differential form $\omega = dz + x dy - y dx$ on $\mathbb{R}^3$, compute $d\omega$ and $\omega \wedge d\omega$.