Midterm on Mathematical Analysis

November 2, 2021

1. A perfect set is defined to be a set all whose points is an accumulation point. Prove that a nonempty perfect subset of $\mathbb{R}$ is uncountable.

2. Let $\{f_n\}$ be a sequence of functions such that each $f_n : (0,\infty) \to \mathbb{R}$ is uniformly continuous. Suppose f_n converges uniformly to $f : (0,\infty) \to \mathbb{R}$. Prove that f is uniformly continuous.

3. Prove that the Legendre polynomial $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}((x^2-1)^n)$ has n distinct real roots.

4. Suppose $u(x,y) : D \to \mathbb{R}$ is C^2 and $F(s,t) : \mathbb{R}^2 \to \mathbb{R}$ is C^1, where $D \subset \mathbb{R}^2$ is the unit disc $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$. Suppose we have

$$F\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0, \ \left(\frac{\partial F}{\partial s}\right)^2 + \left(\frac{\partial F}{\partial t}\right)^2 \neq 0$$

everywhere on D and $\mathbb{R}^2$ respectively. Prove that

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2$$

holds on D.

5. Find out all the local maximum points of the function $f(x,y) = x^3 - 3x^2 - (x-y)^2$ and determine if each is a global maximum or not.

6. Determine if the series

$$\sum_{n=1}^{\infty}\left(e - (1+\frac{1}{n})^n\right)$$

is convergent or not.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function such that

$$\alpha_0 := \sup|f(x)|, \ \alpha_2 := \sup|f''(x)|$$

are both finite. Prove that $\alpha_1 := \sup|f'(x)|$ is also finite and we have the bound

$$\alpha_1 \leq 2\sqrt{\alpha_0\alpha_2}.$$

8. Let (X, d) be a metric space defined as follows. Let X be the space of all sequences consisting of zeros or ones, i.e.

$$X = \{(x_1, x_2, x_3, \ldots) \mid x_i \in \{0, 1\},\ i = 1, 2, 3, \ldots\}.$$

A metric d on X is defined as

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n},$$

for $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ in X. Prove:

(1) the metric space (X, d) is compact.

(2) Define the shift map $\sigma: X \to X$ as

$$\sigma(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Then prove that there exists a point $z \in X$ such that the orbit $\{\sigma^n(z),\ n \in \mathbb{N}\}$ is dense in X.

9. Recall in the construction of the Canter middle-third set that, in each step, we divide each interval from the last step equally into three pieces and discard the middle open interval. Let us now modify this construction slightly. In each step and for each interval we get from the last step, we divide it into three pieces, *not necessarily equally*, and discard the middle open interval. After infinitely many steps, we result in a set called generalized Cantor set. Suppose we start the construction from the $[0, 1]$ interval and define the measure of the generalized Cantor set as 1 minus the total length of the removed open intervals. Please construct a generalized Cantor set of measure 1/2.

10. Let $f, g: [a, b] \to \mathbb{R}$ be C^1 functions satisfying $f(a) = f(b) = 0$ and

$$\det \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \neq 0$$

for all $x \in [a, b]$. Prove that g has a zero in (a, b).