A brief answer to final exam

1. Notice that the global maximum can only be attained at points where gradient vanishes or at the boundary. Since f vanishes on the boundary, calculate the derivatives and we see that f attains global maximum at $x = y = \frac{\pi}{3}$, and equal to $\frac{3\sqrt{3}}{8}$.

2. Use Stokes theorem, the original integral is $\int_D \cos y dx \wedge dy \wedge dz = 4\pi(\sin 1 - \cos 1)$.

3. The original integral equals to

$$\int_{x^2+y^2\leq 4, y\leq x} \frac{dxdy}{\sqrt{1+x^2+y^2}} = \int_0^{\pi/4} d\theta \int_0^2 \frac{rdr}{\sqrt{1+r^2}} = \frac{\pi}{4}\left(\sqrt{5}-1\right) .$$

4. Direct calculation shows that $h(0,0,0,0) = (0,0)$, and

$$\frac{\partial h}{\partial u} = \begin{pmatrix} 2u_1 & 1 \\ e^{u_1} & 1 \end{pmatrix}$$

The determinant of it at the original point is -1 hence non-degenerate, and we can use the implicit function theorem to obtain f. And

$$Df(0,0) = -\left(\frac{\partial h}{\partial u}\right)^{-1}\left(\frac{\partial h}{\partial w}\right)|_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} .$$

5. Prove $\{F_n\}$ is uniformly bounded and equicontinuous, then use Arzela-Ascoli theorem, we obtain $\{F_n\}$ has a uniformly convergent subsequence or the closure of $\{F_n\}$ is sequentially compact hence has a convergent subsequence in $C([0,1])$.

6. We have

$$f^*(u,v,w) = \sup_{x,y,z\in\mathbb{R}} (xu + yv + zw - f(x,y,z)) = \frac{1}{4}\left(u^2 + \frac{v^2}{\alpha} + (w-\beta v)^2\right)$$

7. We can prove the estimation $xe^{-yx} \leq \frac{1}{ye}$, and notice that $\int_1^\infty \frac{1}{y^2 e}$ converges, the integral is uniformly convergent in $x \in [0, +\infty)$.

8. First, $d\eta = 0$ implies that $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ when $i \neq j$. Let $g : M_F \hookrightarrow \mathbb{R}^n \times \mathbb{R}^n$ be the

natural inclusion, we have two methods:

(1) Notice that

$$g^*\omega = \sum_{i=1}^{n} dx_i \wedge d(f_i) = \sum_{i=1}^{n} dx_i \wedge \left(\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j \right) = 0 \ .$$

(2) Notice that $g^*\omega(v_1, v_2) = \omega(dg(v_1), dg(v_2))$, and $dg = \begin{pmatrix} I_n \\ dF \end{pmatrix}$, after coordinate calculation we also get $g^\omega(v_1, v_2) = 0$.

Notice: Don't write $\omega = d\eta$, because ω is a differential form on M_F and $d\eta$ is a form on $\mathbb{R}^n$!

9. Notice that

$$\int \frac{x \ln x}{(1+x^2)^2} dx = \int \ln x d \left(-\frac{1}{1+x^2} \right) = -\frac{1}{2} \frac{\ln x}{1+x^2} + \frac{1}{2} \ln x - \frac{1}{4} \ln\left(1+x^2\right) + C \ .$$

10. $d\omega = 2dx \wedge dy$, $\omega \wedge d\omega = 2dx \wedge dy \wedge dz$.