COURSE:

DIGITAL SIGNAL PROCESSING

Instructor: Ninh Khanh Duy

CHAPTER 2:
DISCRETE-TIME
SIGNALS & SYSTEMS

Lecture 2: Discrete-time (DT) signals

Lecture 3: Discrete-time (DT) systems

Duration: 6 periods

Lecture 2

Discrete-time (DT) Signals

- **Duration: 3** periods
- **Outline:**

1. Representations of DT signals

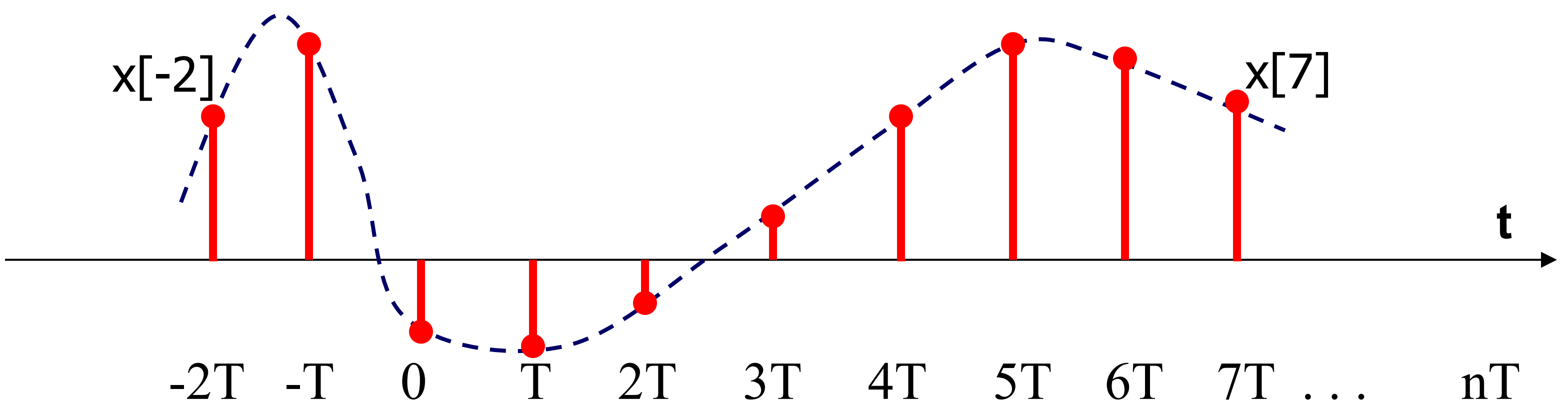
2. Some elementary DT signals

3. Simple manipulations of DT signals

Sampled signals (tin hieu lay mau)

CT: contiguous-time
DT: discrete-time

Converting a CT signal into a DT signal by sampling: given $x_a(t)$ to be a CT signal, $x_a(nT)$ is the value of $x_a(t)$ at **t = nT** → DT signal $x_a(nT)$ is defined only **for n an integer**



$$\boxed{x_a(t)\big|_{t=nT} = x_a(nT) \equiv x(n), \quad -\infty < n < \infty}$$

T: sampling period (s), t: contiguous time (s), n: discrete time = sample index

Representations of DT signals

1. Functional representation

$$x[n] = \begin{cases} 2^n, 1 \le n \le 3 \\ 0, \quad elsewhere \end{cases}$$

2. Tabular representation

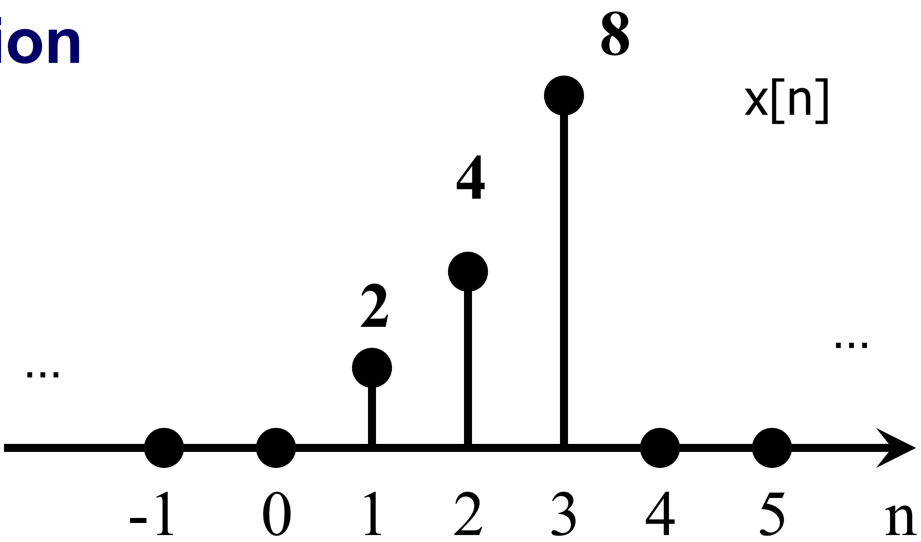
n	… -1	0	1	2	3	4 …
x[n]	… 0	0	2	4	8	0 …

Representations of DT signals

3. Sequence representation

$$x[n] = \{ \underset{\substack{\uparrow \\ n=0}}{0}, 2, 4, 8 \}$$

4. Graphical representation



Lecture 2

Discrete-time (DT) Signals

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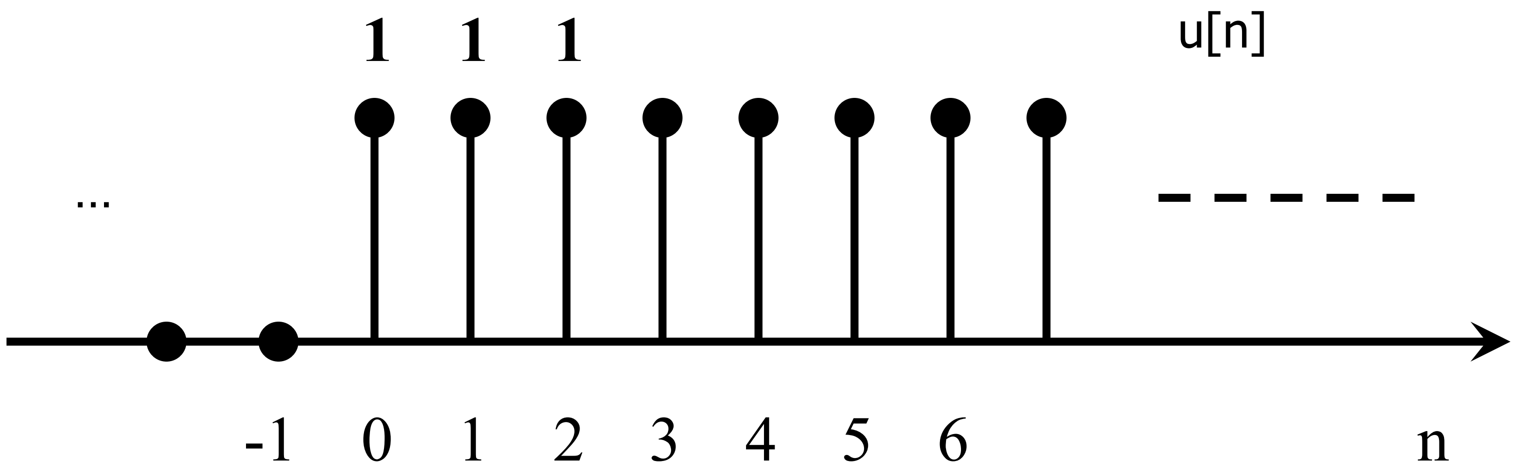
Some elementary DT signals

1. Unit step sequence
2. Unit impulse signal
3. Sinusoidal signal
4. Exponential signal

Unit step sequence

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

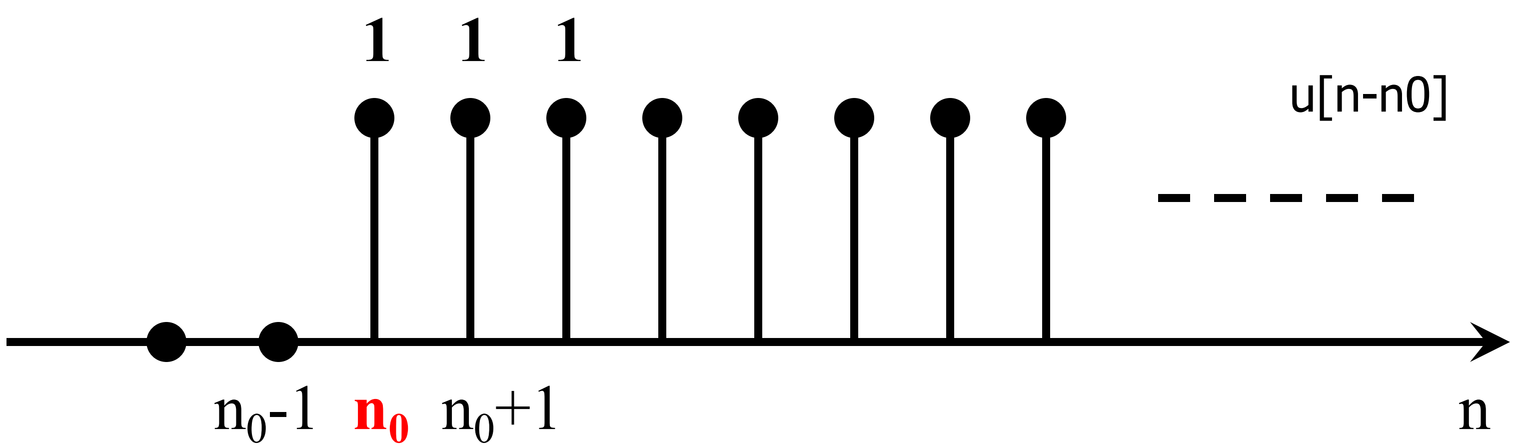


Q: x[n]=5.u[n]?

Time-shifted unit step

Replace n with n-n0 → $u[n-n_0]=\begin{cases}1, & n \geq n_0 \\ 0, & n < n_0\end{cases}$ n0: shift (an integer)

(is a n0-samples shifted version of the signal u[n])

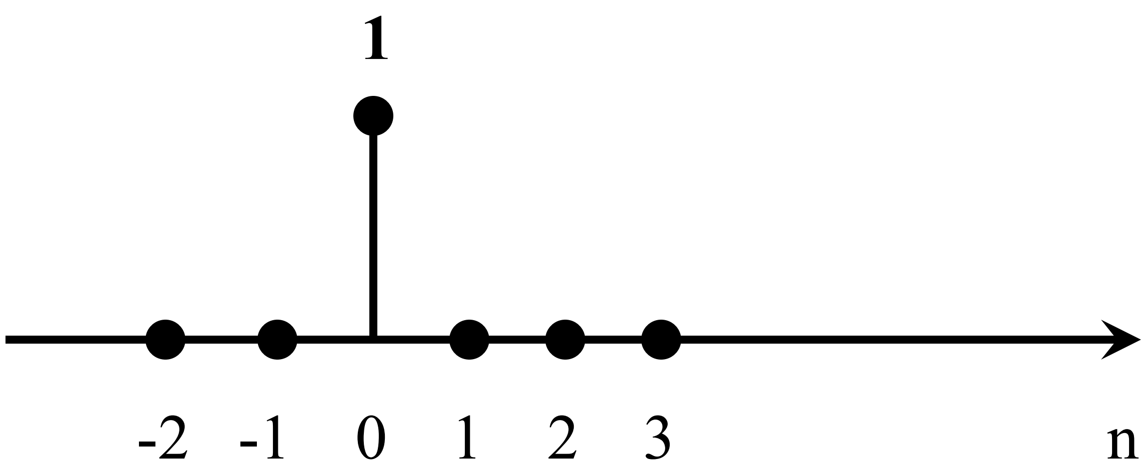


Q: x[n]=-5.u[n-2]?

Unit impulse sequence

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

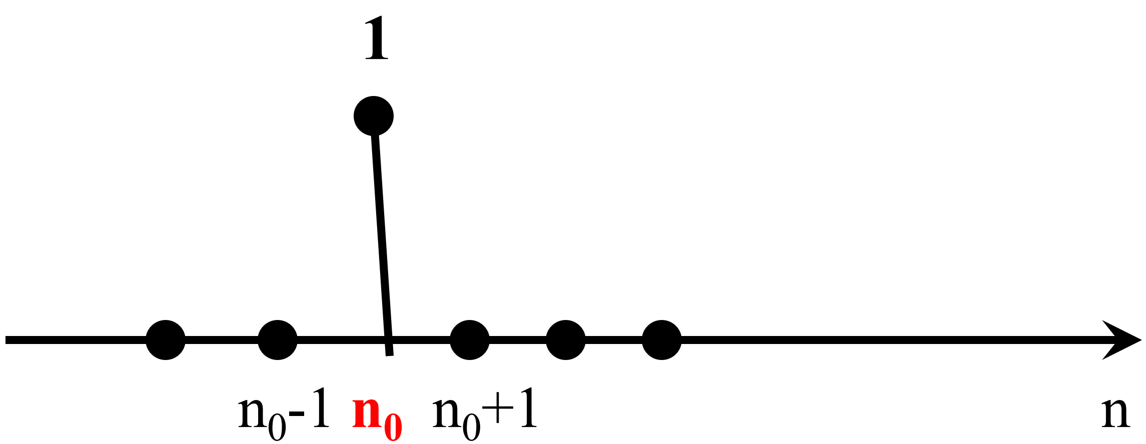
(/delta/)



Q: x[n]=-5 δ[n]?

Time-shifted unit impulse

$$\delta[n-n_0]=\begin{cases}1, & n=n_0\\ 0, & n\neq n_0\end{cases}$$



Q: x[n]=-5 δ[n+2]

Relation between unit step and unit impulse

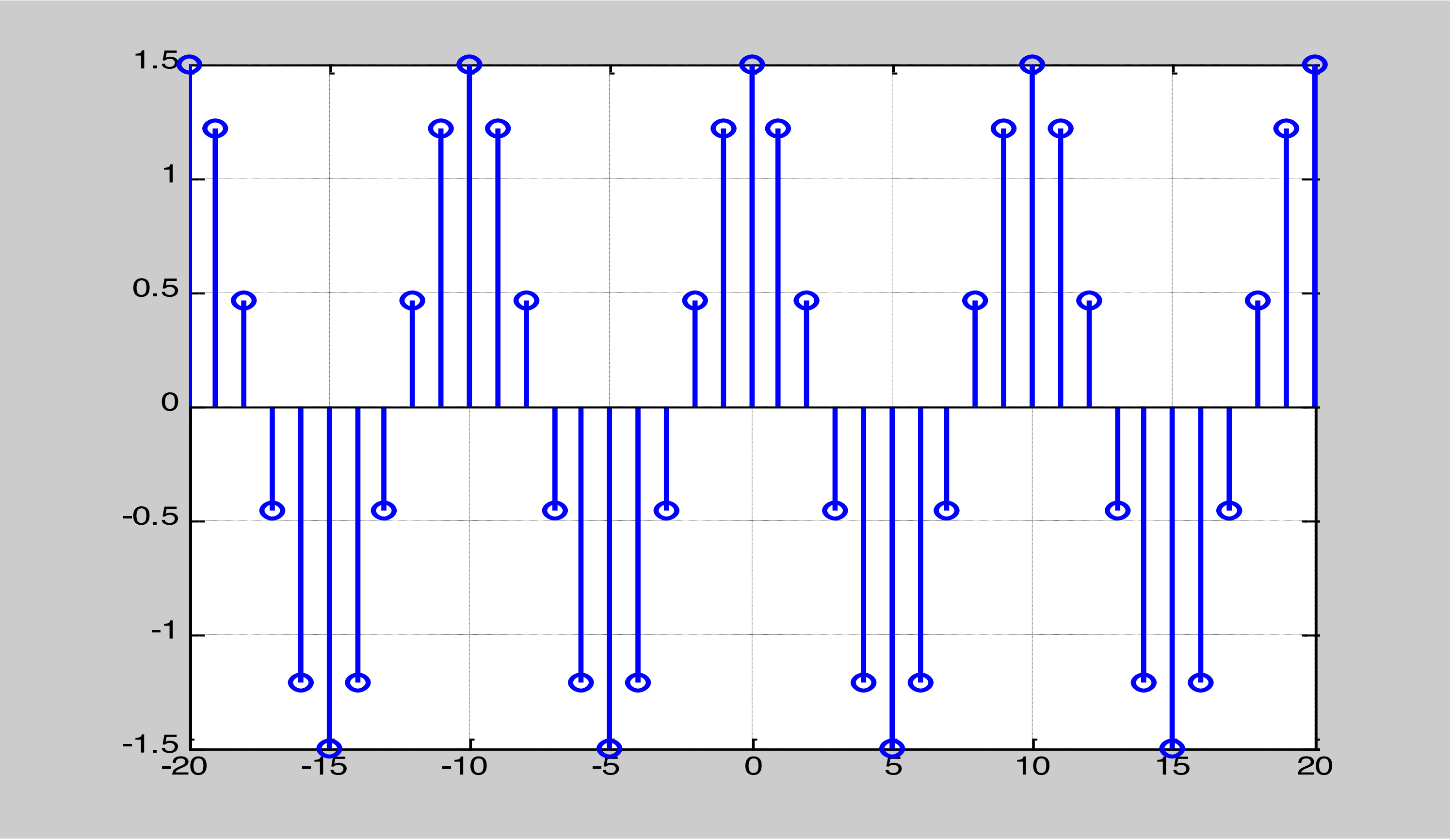
$$u[n]=\sum_{k=-\infty}^{n}\delta[k]$$ =d[n]+d[n-1]+d[n-2]+…+d[n-M]+… : running sum

$$\delta[n]=u[n]-u[n-1]$$: first difference

$$x[n]\delta[n-n_0]=x[n_0]\delta[n-n_0]$$: sample & hold

Sinusoidal signal

$$\begin{aligned} x(n) &= A\cos(\Omega n + \theta), \quad -\infty < n < +\infty \\ &= A\cos(2\pi F n + \theta), \quad -\infty < n < +\infty \end{aligned}$$



Exponential signal

$$x[n] = a^n$$

1. If *a* is real, then x[n] is a real exponential

$a > 1 \rightarrow$ growing exponential

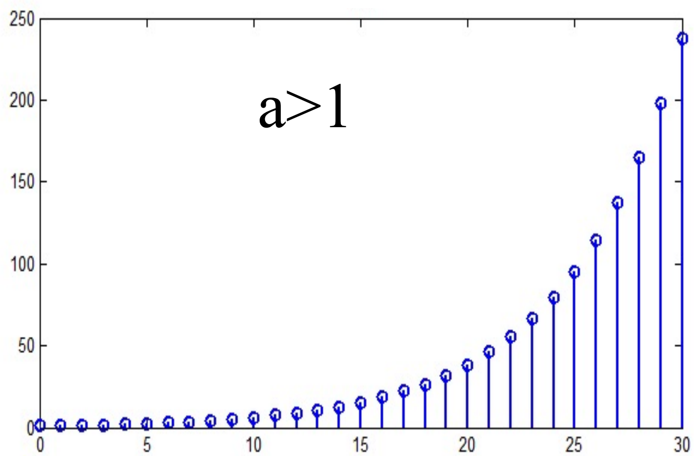
$0 < a < 1 \rightarrow$ shrinking exponential

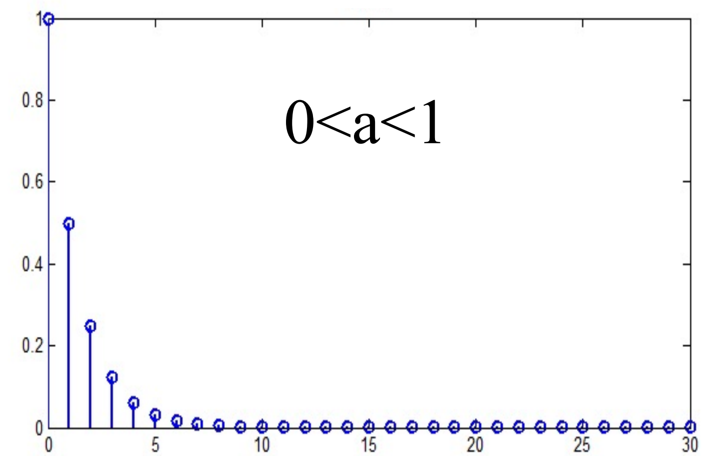
$-1 < a < 0 \rightarrow$ alternate and decay

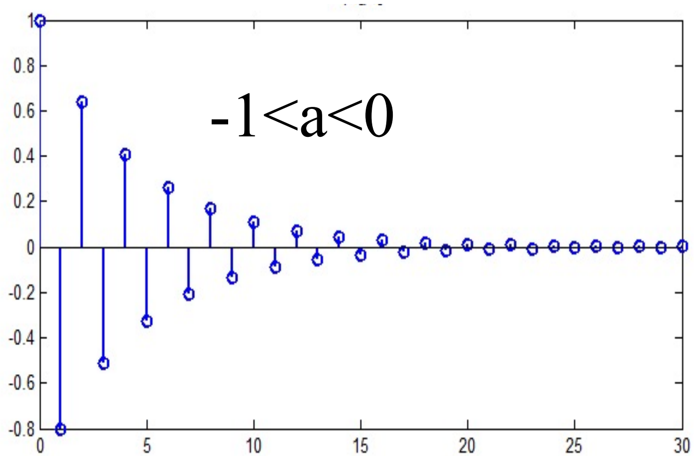
$a < -1 \rightarrow$ alternate and grows

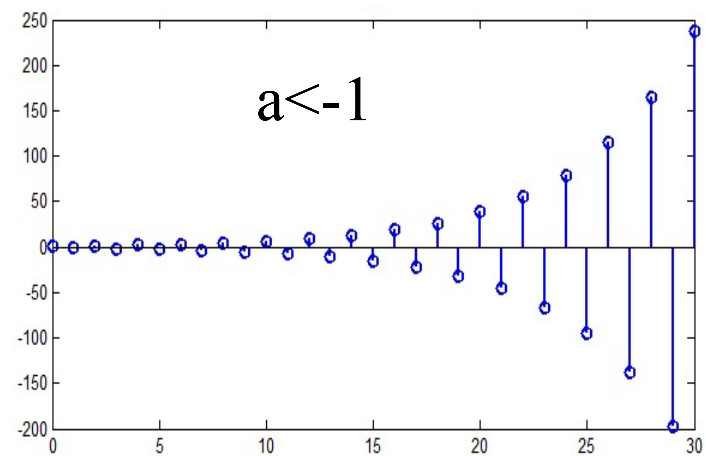
2. If *a* is complex, then x[n] is a complex exponential

Exponential signal - Real examples





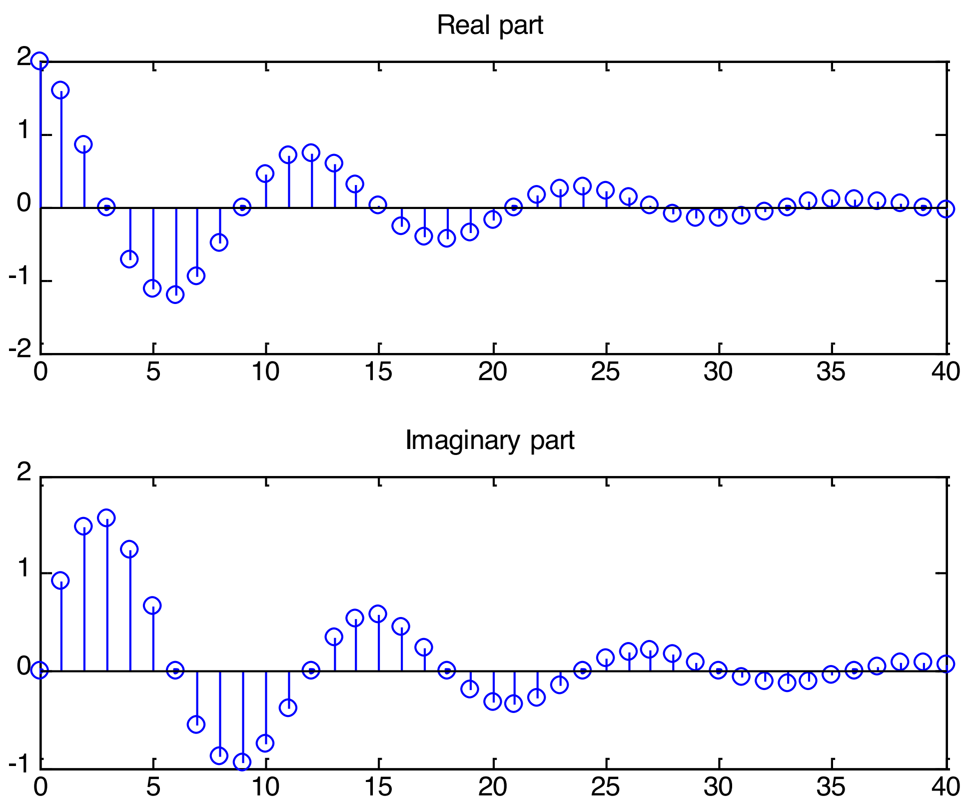




Exponential signal - Complex example

$$x[n] = 2e^{\left(-\frac{1}{12}+j\frac{\pi}{6}\right)n}$$

j: imaginary unit



Lecture 2

Discrete-time (DT) Signals

- **Duration: 3** periods
- **Outline:**

1. Representations of DT signals
2. Some elementary DT signals
3. **Simple manipulations of DT signals**

Simple manipulations of DT signals

- Transformation of time:

- Time shifting
- Time reversal

- Adding and subtracting signals

Time shifting a DT signal

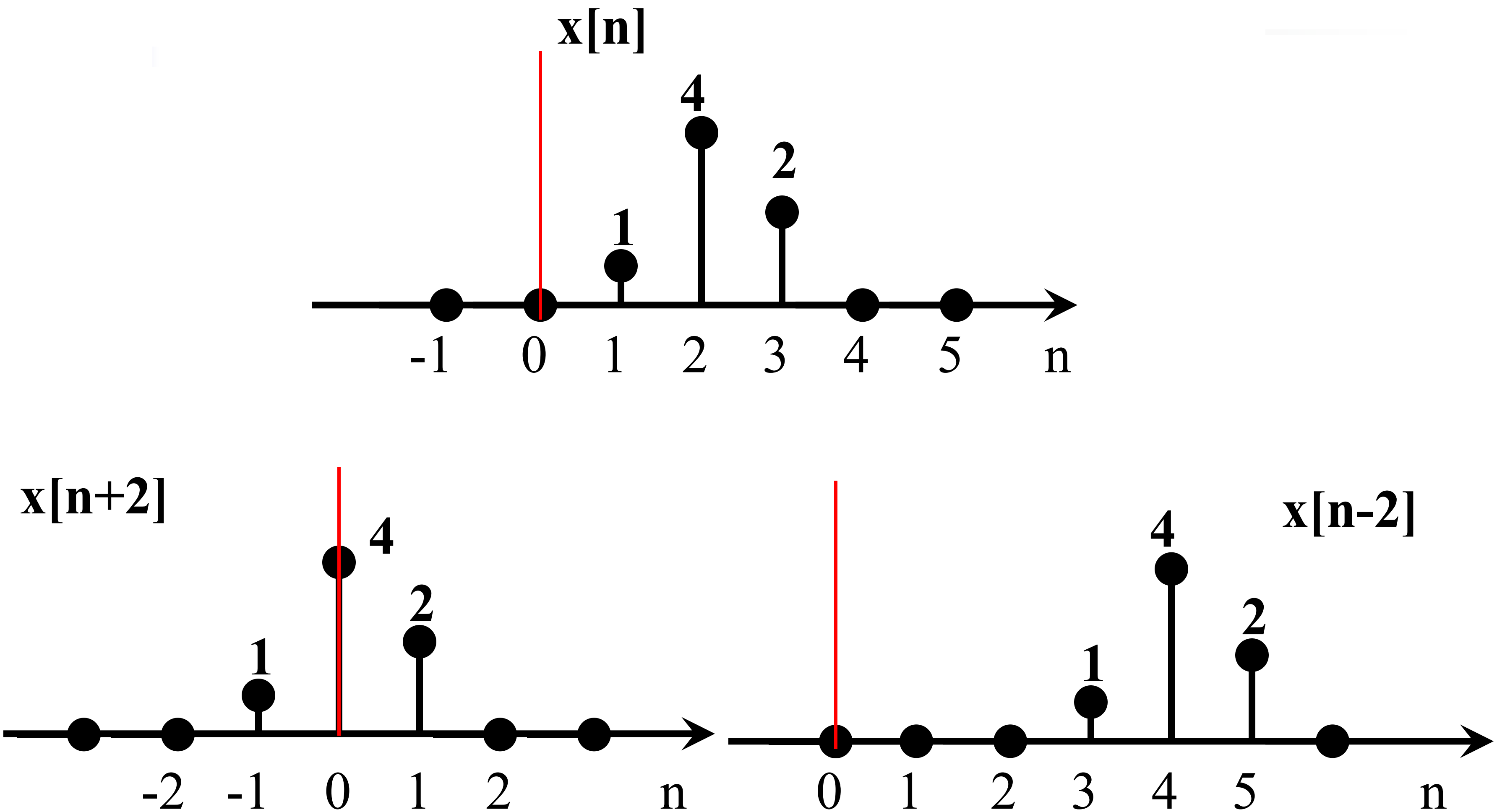
$x[n] \rightarrow x[n-k]$; k is an integer

- **<u>$k > 0$:</u>** right-shift $x[n]$ by $|k|$ samples

  (delay of signal)

- **<u>$k < 0$:</u>** left-shift $x[n]$ by $|k|$ samples
  (advance of signal)

Examples of time shifting



n=-1: x[-1+2]=x[1]=1

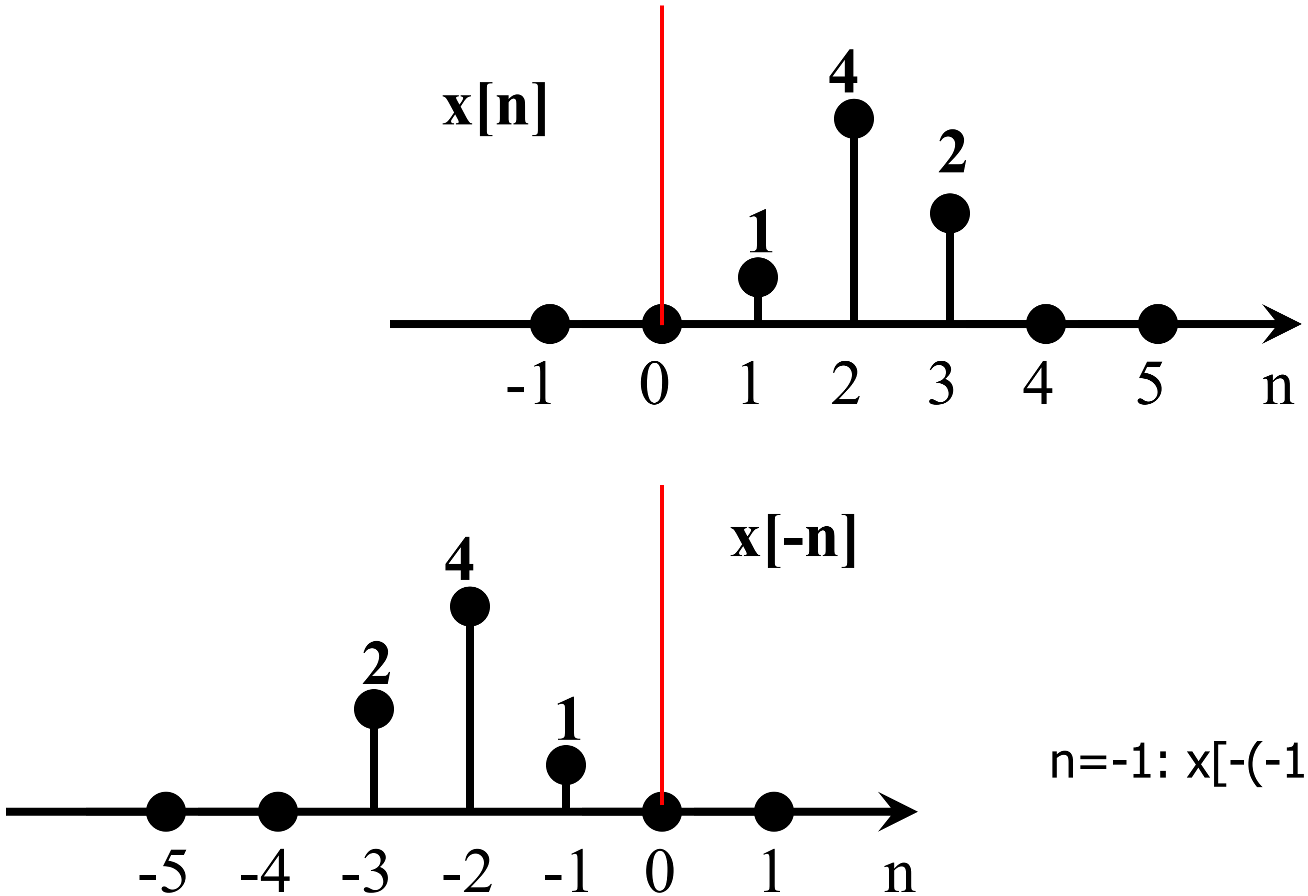
n=4: x[4-2]=x[2]=4

Time reversal a DT signal

$$x[n] \rightarrow x[-n]$$

Flip a signal x[n] about the vertical axis at n=0

Example of time reversal



n=-1: x[-(-1)]=x[1]=1

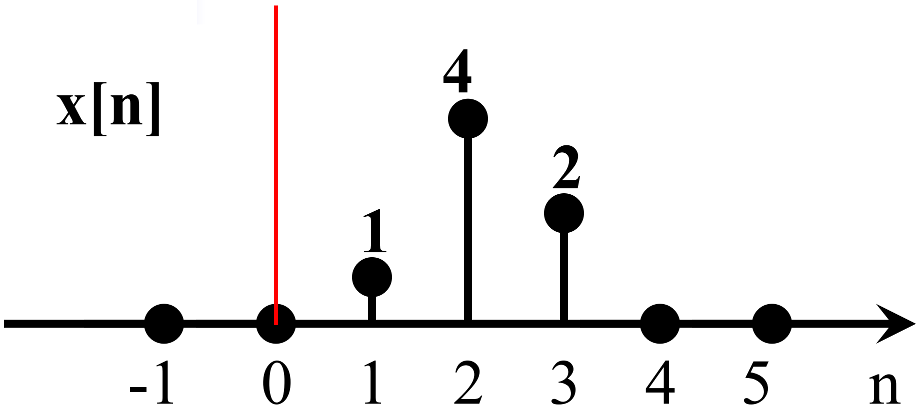
Combining time reversal and time shifting

$$\boxed{x[n] \rightarrow x[-n-k]}$$ **(k: an interger)**

Method 1: Flip first, then shift

Method 2: Shift first, then flip

Examples of combining time reversal and time shifting



x[-n+1]??

n<=-3: x[-n+1]=0

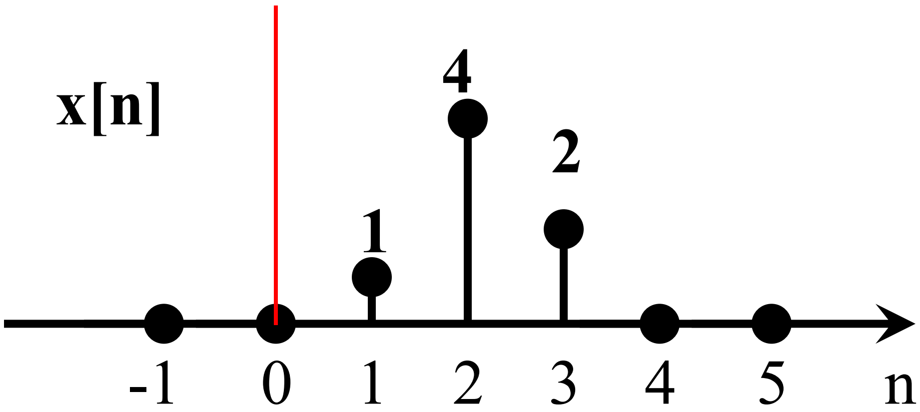
n=-2: x[-(-2)+1]=x[3]=2

n=-1: x[-(-1)+1]=x[2]=4

n=0: x[-0+1]=x[1]=1

n>=1: x[-n+1]=0

Examples of combining time reversal and time shifting



x[-n-1]??

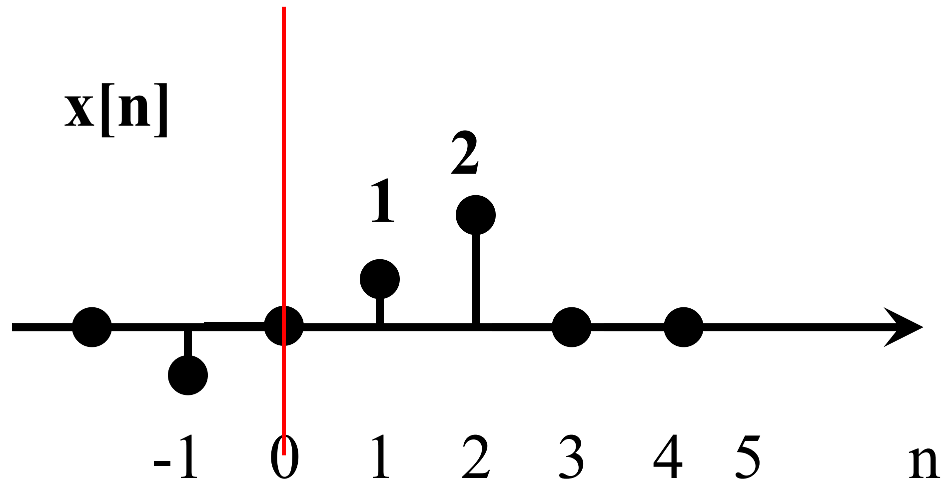
Adding and subtracting signals

- Do it "point by point"
- Can do using a table, or graphically, or by computer program
- Example: $x[n] = u[n] - u[n-4]$

n	<=-1	0	1	2	3	>=4
x[n]	0	1	1	1	1	0

Exercise

- Find $x[n]=(u[n+1]-u[n-5])(nu[2-n])$



(Hint: x1[n] = u[n+1]-u[n-5], x2[n]=n.u[2-n] → x[n]=x1[n].x2[n])

Homework (HW)

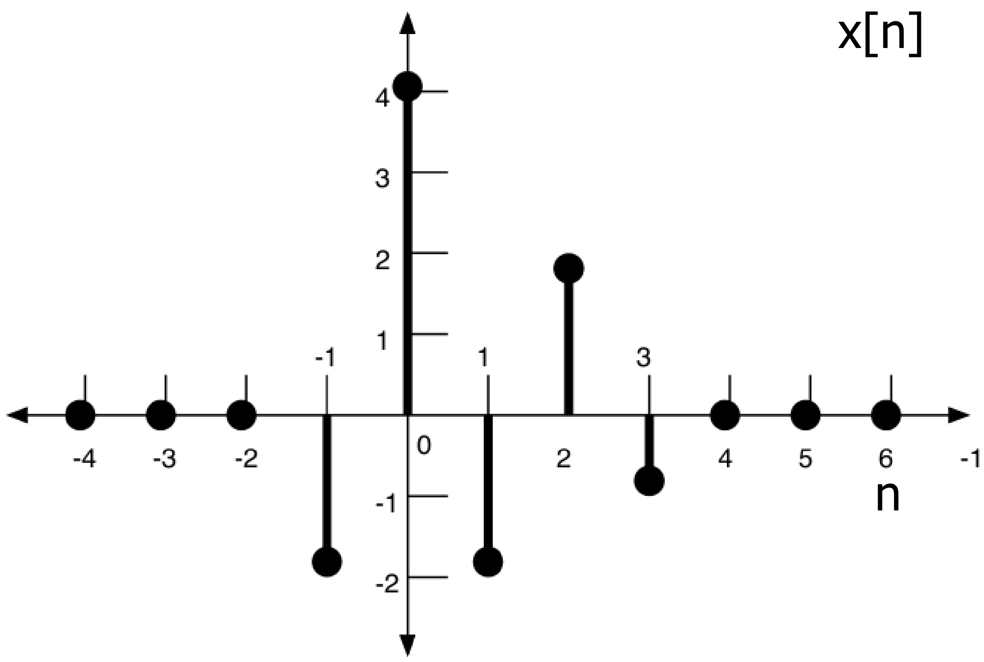
Prob.1 The following graph is of signal x[n].

Plot the following:

a) y[n]= 3x[-n-1]

b) y[n]=x[2n] - 1

c) y[n]= –x[n] + 2



HW

Prob.2 Sometimes signals can be decomposed into combinations of simple unit step sequences such as this one:

$$y[n] = 2u[n-2] - 2u[n-7] - 2u[-n] + 2u[4-n]$$

Sketch y[n] and the following signals:

a) 2-3y[n]

b) 3y[n-2]

c) 2-2y[-2+n]

HW

Prob.3 Write a program to do the followings:

- Read a recorded .wav file to a data vector x & a sample rate value Fs
- Plot the signal over time (s) and show its length in samples and in seconds
- Playback the signal with 3 different sampling rate: Fs, Fs/2, 2*Fs & give your remarks

(Hint: read the textbook
"Applied Digital Signal Processing -Theory and Practice_Manolakis-Ingle_2011"
at pages 15, 28 & 30.)

HW

Prob.4

Write a program to do the followings:

- Given an analog sine signal x(t) having its parameters (amplitude, frequency in Hz (denoted as F0), and phase)
- Sampling the signal with 2 dif. sampling frequencies: Fs1 = 3*F0 (→x1[n]), Fs2 = 1.5*F0 (→x2[n])
- Plot 2 resulting discrete signals x1[n] & x2[n] on the same time axis
- Playback the 2 discrete signals with its corresponding sampling rates : x1[n] with Fs1, x2[n] with Fs2
- Check if we can recover the original analog signal in 2 cases

(Hint: read the textbook
"Applied Digital Signal Processing -Theory and Practice_Manolakis-Ingle_2011"
at pages 15, 28 & 30.

- F0: an audible frequency in (1000Hz, 15000Hz)
- sampling x(t)=Acos(2*pi*F0*t+phi) to receive x[n]): t=n*Ts

Lecture 3
DT systems

- **Duration:** 3 periods
- **Outline:**

1. Input-output description of systems

2. DT system properties

3. Linear-time invariant (LTI) systems

Input-output description of DT systems

Think of a DT system as an operator on DT signals:

- It processes DT input signals, to produce DT output signals
- **Notation: y[n] = T{x[n]}:** y[n] is the response of the system T to the excitation x[n]
- Ex1: $$y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n-2]\}$$

→ a three-point moving average filter, which is often used to smooth a signal corrupted by additive noise

- Ex2: $$y[n] = \text{median}\{x[n-1], x[n-2], x[n], x[n+1], x[n+2]\}.$$

→ a five-point median filter, used to remove spikes from experimental data

System ←→ Filter

Lecture 3

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DT system properties

- Causality
- Stability
- Linearity
- Time-invariance

Causality

- The output of a causal system (at each time) **does not depend on** future inputs

(n_0: current time instant → n_0 -1: past, n_0 +1: future)

→ $y[n_0]$ (result) only depends on $x[n_0]$, $x[n_0 -1]$, $x[n_0 -2]$,…(cause)

for all n_0 interger

→ Imply if the system is physically implementable in online mode

Examples for causality

Determine which of the systems below are causal:

a) $y[n] = x[-n]$: Non-causal (eg. $y[-1]=x[1]$)

b) $y[n] = (n+1)x[n-1]$: causal since $y[n0]$ only depends on $x[n0-1]$

c) $y[n] = x[(n-1)^2]$: Non-causal (eg. $y[0]=x[1]$)

d) $y[n] = \cos(w_0 n+x[n])$: causal since $y[n]$ only depends on $x[n]$

e) $y[n] = 0.5y[n-1] + x[n-1]$:

causal since $y[n]$ only depends on $x[n-1]$

Stability

- If a system "blow up" it is **not stable**

  In particular, if a "well-behavior" signal (all values have finite amplitude) results in infinite magnitude output, the system is **unstable**

- BIBO stability: "**<u>b</u>ounded <u>i</u>nput – <u>b</u>ounded <u>o</u>utput**" – if you put finite signals in, you will get finite signals out

Examples for stability

Determine which of the systems below are BIBO stable:

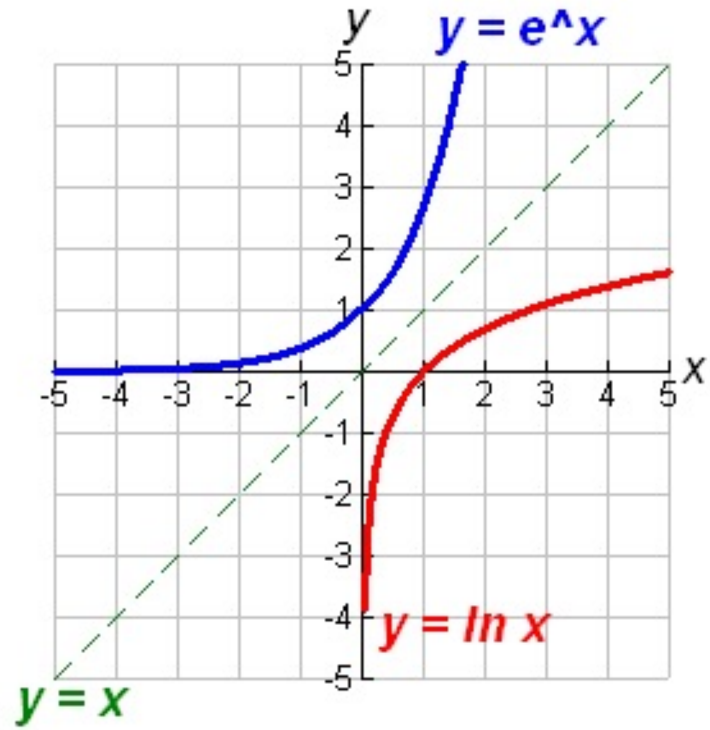
a) A unit delay system:

b) An accumulator:

c) y[n] = cos(x[n]):

d) y[n] = ln(x[n]):

e) y[n] =exp(x[n]):



Linearity

Scaling signals and adding them, then processing through the system

same as

Processing signals through system, then scaling and adding them

If $T(x_1[n]) = y_1[n]$ and $T(x_2[n]) = y_2[n]$

$\rightarrow T(ax_1[n] + bx_2[n]) = ay_1[n] + by_2[n]$

Only has theoretical meanings, but simplify modeling & analysis of real systems

Time-invariance

- If you time shift the input, get the **same** output, but with the **same** time shift
- The behavior of the system **doesn't change** with time

$$\text{If } T(x[n]) = y[n]$$

$$\text{then } T(x[n-n_0]) = y[n-n_0]$$

Only has theoretical meanings, but simplify modeling & analysis of real systems

Examples for linearity and time-invariance

Prob 0. Determine if the systems below are linear and/or time-invariant. Prove!

a) $y[n] = nx[n]$

Linear
Time-variant

Examples for linearity and time-invariance

Determine if the systems below are linear and/or time-invariant. Prove!

b) $y[n] = x^2[n]$

Non-linear
Time-invariant

Examples for linearity and time-invariance

Determine if the systems below are linear and/or time-invariant. Prove!

c)

$$y[n] = \sum_{r=0}^{M} b_r x[n-r]$$

Linear
Time-**i**nvariant **(LTI)**

Lecture 3
DT systems

- **Duration:** 3 periods
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1. Input-output description of systems

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3. LTI systems

Computing the response of DT LTI systems to arbitrary input

Method 1: based on the direct solution of the input-output equation for the system → not often used

Method 2:

- Decompose the input signal into a sum of elementary signals
- Find the response of system to each elementary signal
- Add those responses to obtain the total response of the system to the given input signal

→ often used

$$x[n] = \sum_k c_k x_k[n]$$

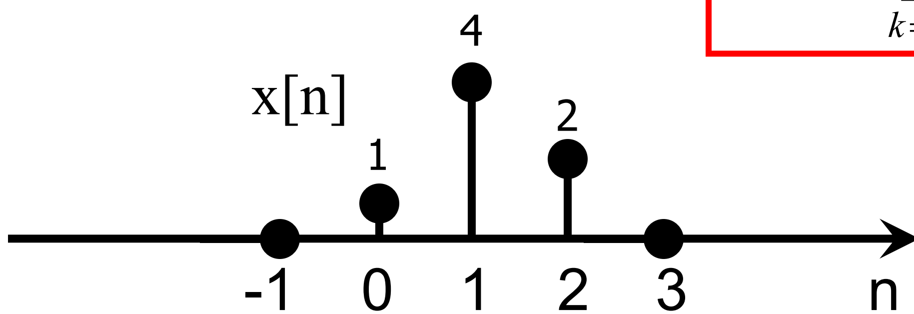
$$x_k[n] \quad \rightarrow \quad y_k[n]$$

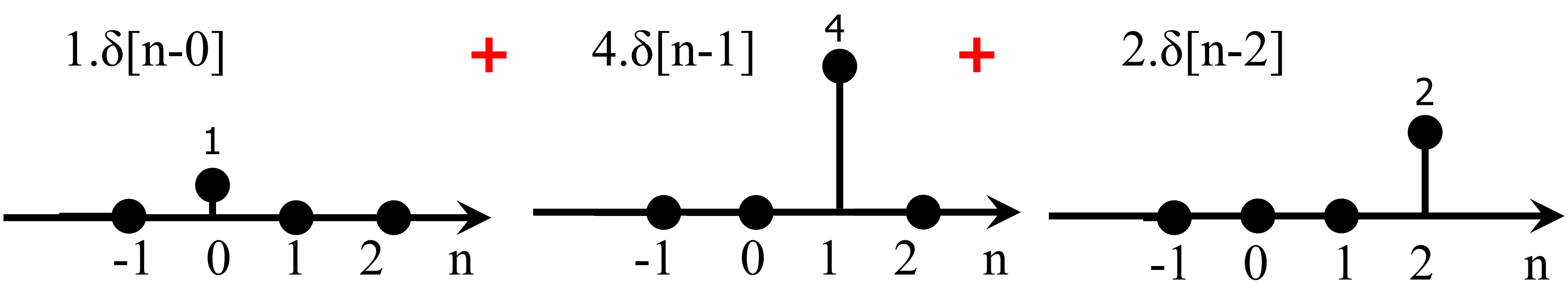
$$x[n] \quad \rightarrow \quad y[n] = \sum_k c_k y_k[n]$$

Impulse representation of DT signals

We can describe any DT signal x[n] as: $\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]}$

Example:





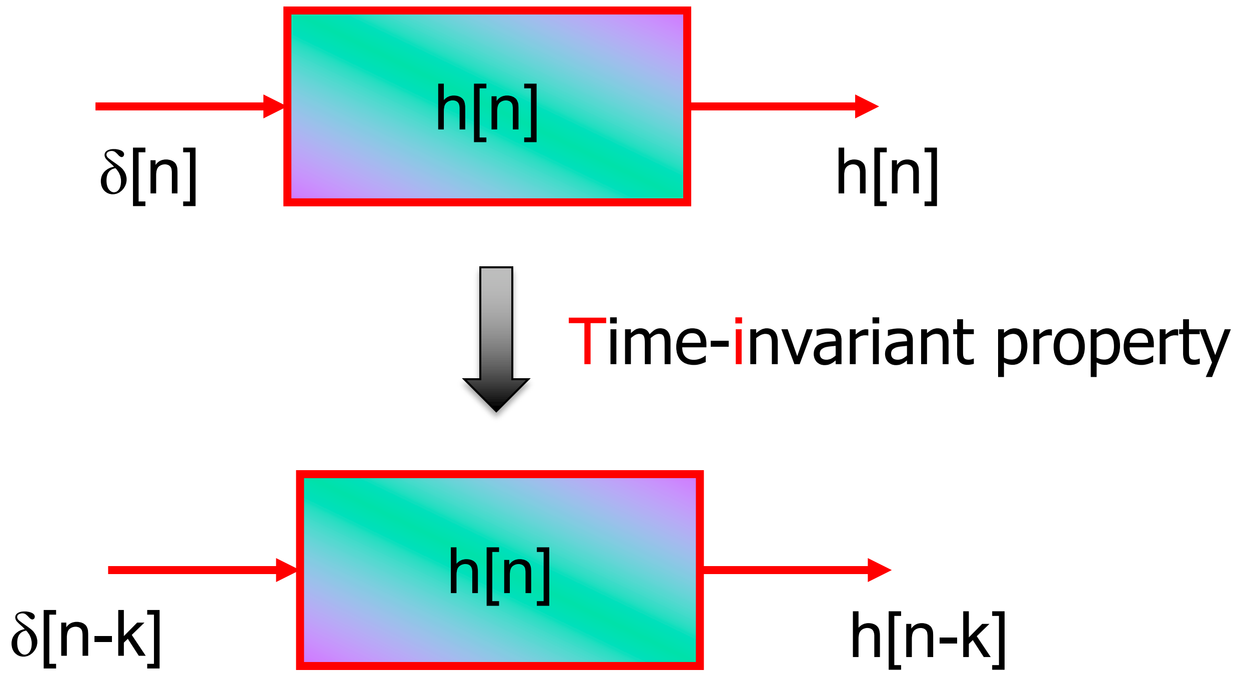
Impulse response of DT LTI systems

- Any DT LTI system is characterized by its impulse response h[n].
- Notation:

- Impulse response = response of the system to the (unit) impulse

$h[n] = y[n] \mid x[n] = \delta[n]$

Response of LTI system to delayed impulse



Response of LTI system to a DT signal

Time-invariant property

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Linear property

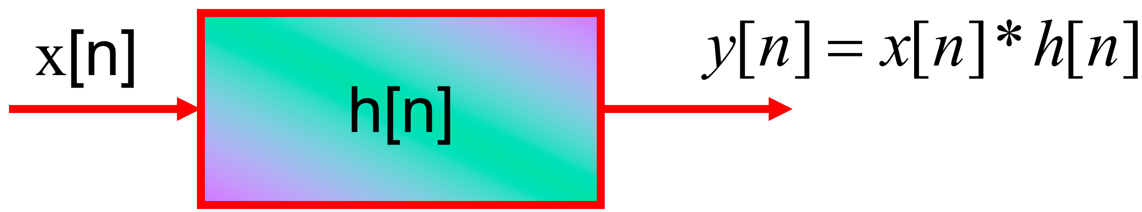
$$y[n] \triangleq x[n] * h[n]$$

Convolution sum

(an operation on 2 signals)

DT convolution formula

Convolution: an operation between the input signal to a system and its impulse response, resulting in the output signal



In DT systems: convolution of 2 signals involves summing the product of the 2 signals – where one of signals is flipped and shifted

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Computing the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Suppose to compute the output y[n] at time $n = n_0$.

1. **Flip** h[k] about k = 0, to obtain h[-k]

2. **Shift** h[-k] by n_0 to the right (left) if n_0 is positive (negative), to obtain $h[n_0-k]$

3. **Multiply** x[k] and $h[n_0-k]$ for all k, to obtain the product

$$x[k].h[n_0-k]$$

4. **Sum** up the product for all k, to obtain $y[n_0]$

Repeat from 2-4 fof all of n

The length of the convolution sum result

$$\boxed{y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}$$

Suppose:

Length of x[k] is $N_x \rightarrow N_1 \le k \le N_1 + N_x - 1$

Length of h[n-k] is $N_h \rightarrow N_2 \le n\text{-}k \le N_2 + N_h - 1$

$\rightarrow N_1 + N_2 \le n \le N_1 + N_2 + N_x + N_h - 2$

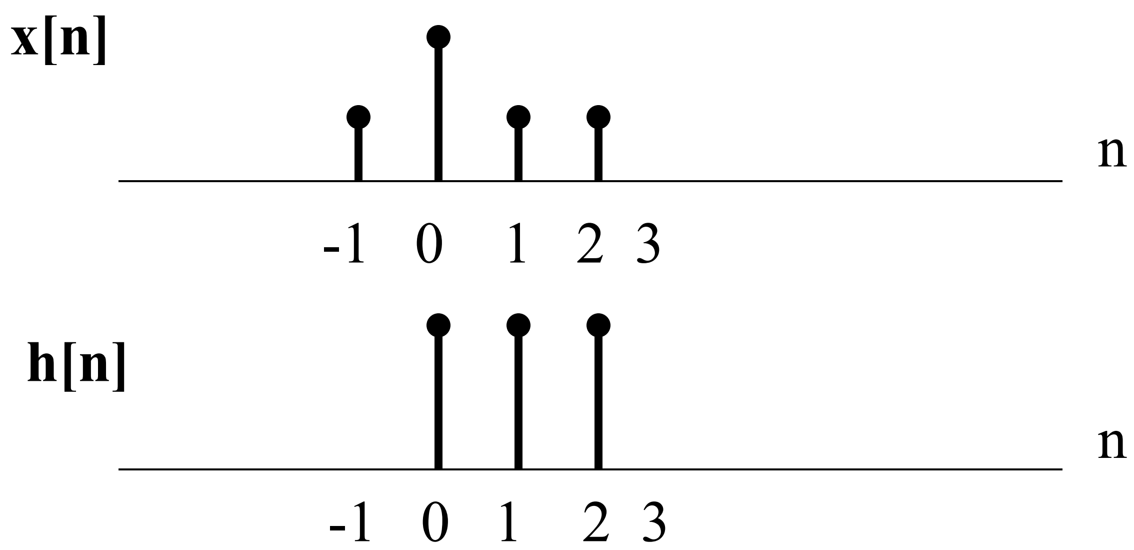
Length of y[n]:

$$\boxed{\mathbf{N_y = N_x + N_h - 1}}$$

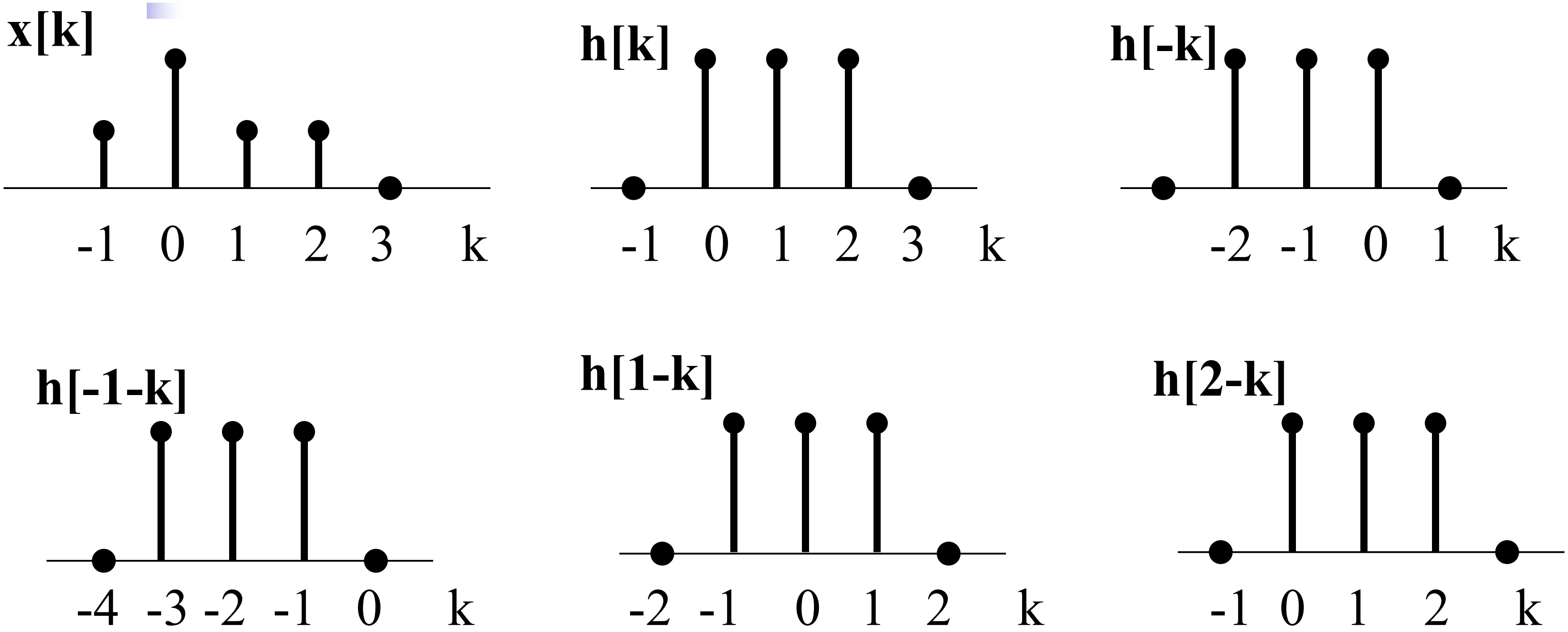
Examples of computing the convolution sum

Ex1. Find y[n] = x[n]*h[n] where

$$x[n] = u[n+1] - u[n-3] + \delta[n] \qquad h[n] = 2\left(u[n] - u[n-3]\right)$$



Ex1 (cont.)



$y[n<-1] = 0$; $y[-1] = 2$; $y[0] = 6$; $y[1] = 8$;

$y[2] = 8$; $y[3] = 4$; $y[4] = 2$; $y[n>4] = 0$

Examples of computing the convolution sum

Ex2. Find y[n] = x[n]*h[n] where $x[n] = a^n u[n]$ $h[n] = u[n]$

Examples of computing the convolution sum

Ex3. Find $y[n] = x[n]*h[n]$ where $x[n] = b^n u[n]$ and $h[n] = a^n u[n+2]$

$|a| < 1$, $|b| < 1$, $a \neq b$

DT LTI properties based on impulse response

1. Causal system: h[n] is zero for all time n<0

2. BIBO stable system:

$$\sum_{n=-\infty}^{+\infty} \left| h[n] \right| \;<\; \infty$$

- Finite impulse response (FIR) systems → always stable
- Infinite impulse response (IIR) systems → can be stable or not

Examples of DT LTI properties

1. Is $h[n] = 0.5^n u[n]$ BIBO stable? Causal?

2. Is $h[n] = 3^n u[n]$ BIBO stable? Causal?

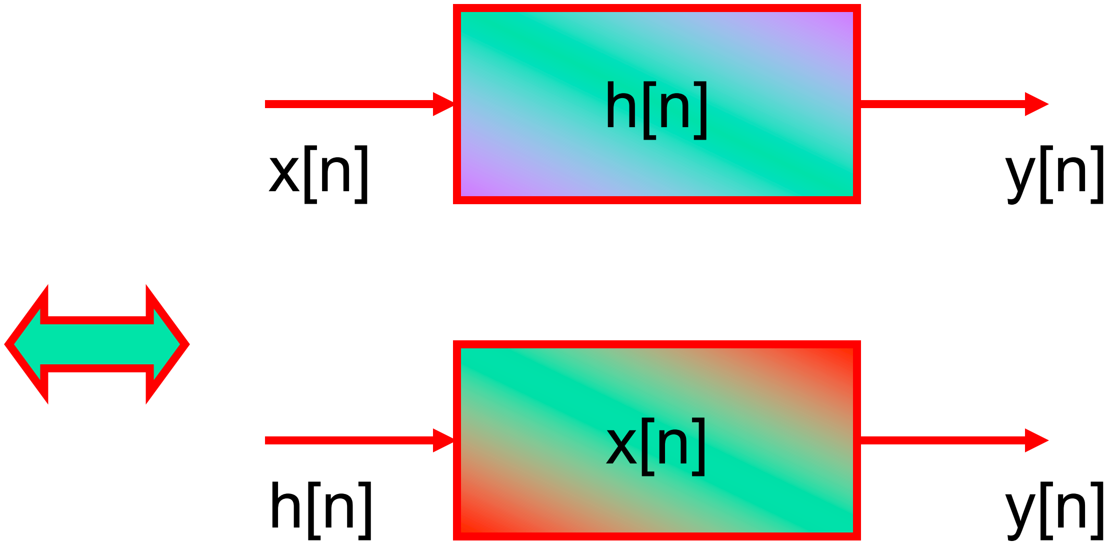
3. Is $h[n] = 3^n u[-n]$ BIBO stable? Causal?

Convolution sum properties

- Commutative law
- Associative law
- Distributive law

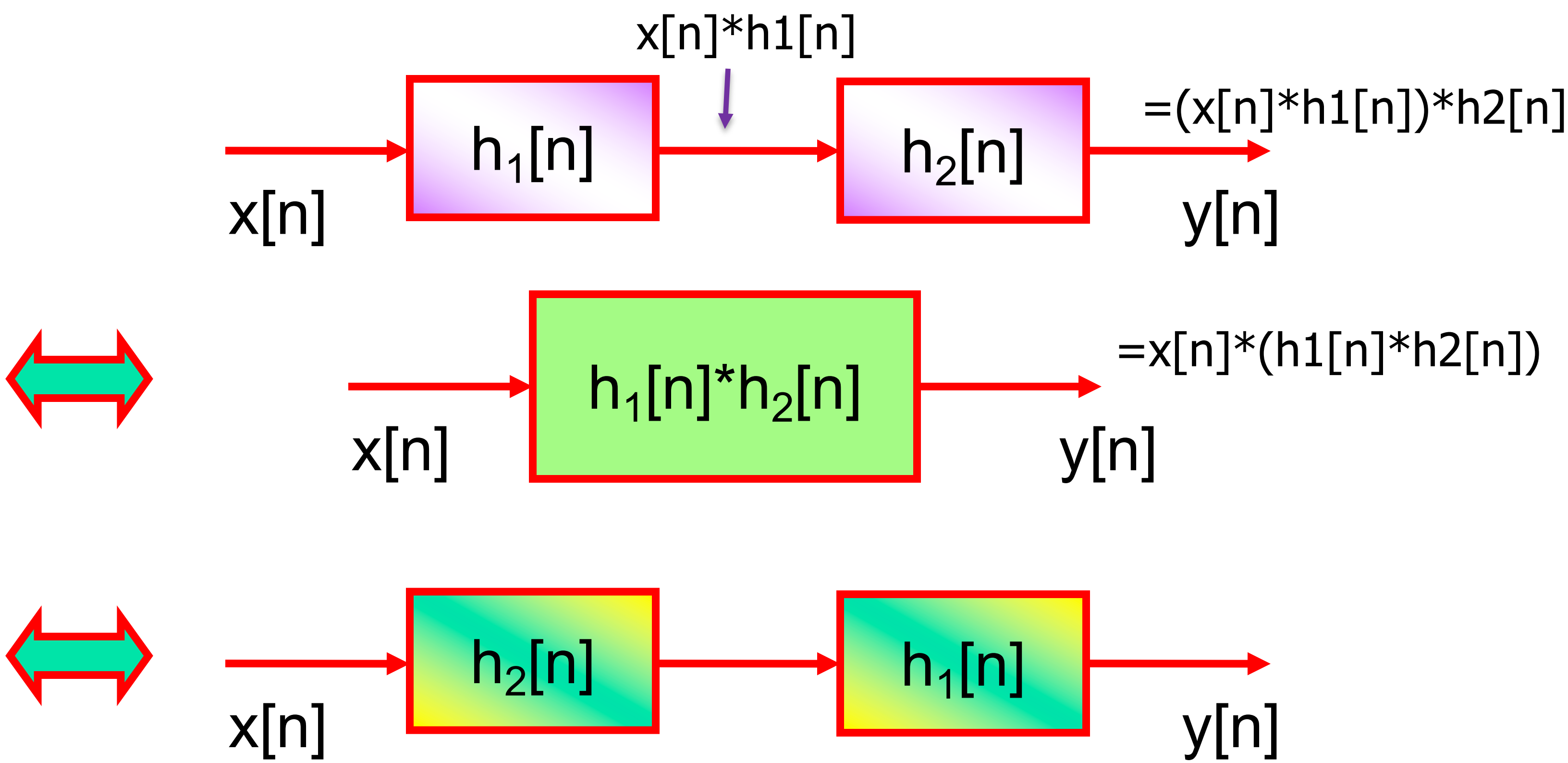
Commutative law

$$\boxed{x[n] * h[n] = h[n] * x[n]}$$



Associative law

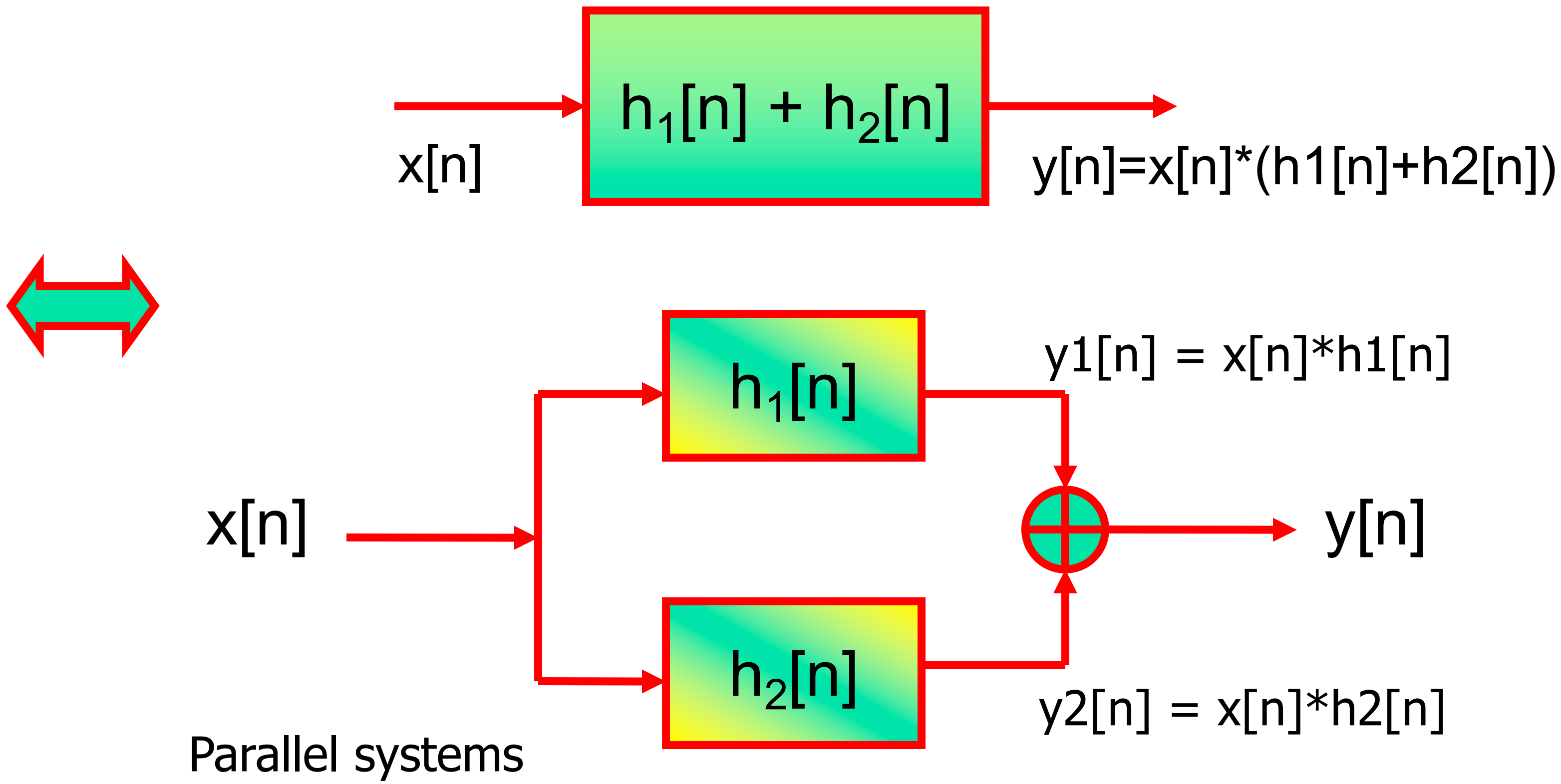
$$\boxed{(x[n]*h_1[n])*h_2[n]=x[n]*(h_1[n]*h_2[n])}$$



Directly connected systems

Distributive law

$$\boxed{x[n]*(h_1[n]+h_2[n]) = (x[n]*h_1[n]) + (x[n]*h_2[n])}$$



LTI systems characterized by linear constant coefficient difference equations

General form:

$$y[n]+a_1 y[n-1]+\ldots+a_N y[n-N]=b_0 x[n]+b_1 x[n-1]+\ldots+b_M x[n-M]$$

$$\boxed{\Leftrightarrow \sum_{k=0}^{N} a_k y[n-k]=\sum_{r=0}^{M} b_r x[n-r], \quad a_0=1}$$

N, M: non-negative integers

N: order of equation

a_k, b_r: constant real coefficients

HW

Prob.3 Conclude about the linearity, time-invariance, stability, causality of the following systems. Prove!

(a) $$y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n-2]\}$$

(b) $$y[n] = \text{median}\{x[n-1], x[n-2], x[n], x[n+1], x[n+2]\}.$$

(Hint: read pp. 32-35 textbook)

Trung vị (Median) là gì?

https://vietnambiz.vn/trung-vi-median-la-gi-vi-du-ve-trung-vi-2019110713491368.htm

HW

Prob.4 Determine the causality and the BIBO stability for the systems with the following impulse responses:

a) $h[n] = \sin(-n)u[n]$

b) $h[n] = e^{-n}u[-n]$

c) $h[n] = e^{n}u[n]$

d) $h[n] = \sin(n)u[-n]$

e) $h[n] = ne^{-n}u[n]$

f) $h[n] = e^{-n}\sin(n)u[n]$

HW

<u>Prob.5</u> Find $y[n] = x[n]*h[n]$ where:

a) $x[n] = a^n u[n]$ and $h[n] = u[n] – u[n-10]$

b) $x[n] = u[-n]$ and $h[n] = a^n u[n-2]$, $|a|<1$

c) $x[n] = 2\delta[n+2] + 2\delta[n+1] + 2\delta[n-1] + 2\delta[n-2] + 2\delta[n-3] + 2\delta[n-4]$

and $h[n] = \delta[n] – \delta[n-1] + \delta[n-2]$

d) $x[n] = u[-n+2]$ and $h[n] = a^n u[-n]$

e) $x[n] = 0.2^n u[n]$ and $h[n] = \delta[n] – 0.2\delta[n-1]$

HW

Prob.6 Consider the LTI system with the input and output related by: y[n] = 0.5x[n-1] + 0.7x[n]

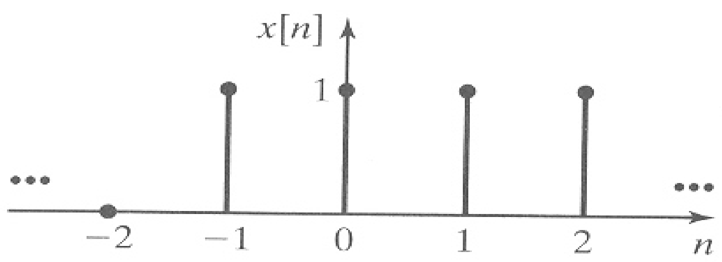
a) Find the impulse response h[n]

b) Is this system causal? Stable? Why?

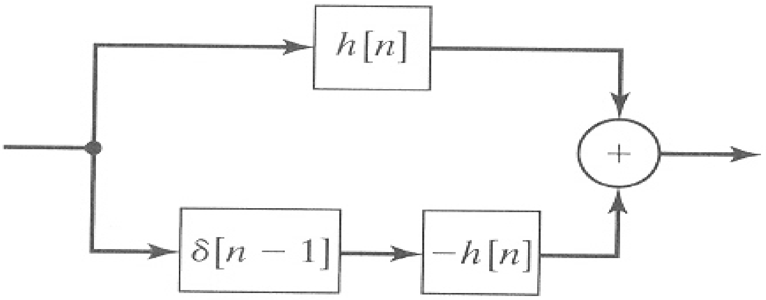
c) Determine the system response y[n] for the input shown in Fig. (a)

d) Consider the interconnections of the LTI systems given in Fig. (b). Find the impulse response of the total system

e) Solve for the response of the system of part (d) for the input of part (c)



(a)



(b)

HW

Denoising signal using moving averaging system:

Modify the moving_average_smoothing.m program so that it implements the system in Prob. (3a) (slide 66, Chapter 2) in 3 different ways:

- Averaging of time-shifted input signals
- Using conv() function
- Using filter() function