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1. (Soft margin SVM)

(Primal) $\begin{cases} \min\limits_{w,b,\xi} & \frac{1}{2}\|w\|_2^2 + C_1\sum_{i=1}^n \xi_i + C_2\sum_{i=1}^n \xi_i^2 \\ \text{s.t.} & y_i(x_i^T w + b) \geq 1-\xi_i, \ \forall i \\ & \xi_i \geq 0, \ \forall i \end{cases}$ (Assume $C_1, C_2 > 0$)

The Lagrangian:

$$\mathcal{L}(w,\xi,b,\alpha) = \frac{1}{2}\|w\|_2^2 + C_1\sum_{i=1}^n \xi_i + C_2\sum_{i=1}^n \xi_i^2 + \sum_{i=1}^n \alpha_i\left[1-\xi_i - y_i(x_i^T w + b)\right]$$

$$= \underbrace{\frac{1}{2}\|w - \sum_{i=1}^n \alpha_i y_i x_i\|_2^2}_{A_1} - \frac{1}{2}\sum_{i,j=1}^n \alpha_i\alpha_j y_i y_j x_i^T x_j + \sum_{i=1}^n \alpha_i - \underbrace{\sum_{i=1}^n \alpha_i y_i b}_{A_2}$$

$$+ \underbrace{C_2\sum_{i=1}^n\left(\xi_i + \frac{C_1-\alpha_i}{2C_2}\right)^2 - \sum_{i=1}^n \frac{(C_1-\alpha_i)^2}{4C_2}}_{A_3}$$

$\min\limits_{w,\xi,b} \mathcal{L}(w,\xi,b,\alpha)$ with $\xi_i \geq 0 \ \forall i$ $\qquad \sum (C_1-\alpha_i)\xi_i$

$\min\limits_{w} A_1 \Rightarrow w = \sum_{i=1}^n \alpha_i y_i x_i \qquad A_1 = 0$

$\min\limits_{b} A_2 \Rightarrow \sum_{i=1}^n \alpha_i y_i = 0 \qquad A_2 = 0$

$\min\limits_{\xi_i} A_3 \Rightarrow \begin{cases} \xi_i = -\frac{C_1-\alpha_i}{2C_2} & \alpha_i > C_1 \\ \xi_i = 0 & 0 \leq \alpha_i \leq C_1 \end{cases}$ $\qquad A_3 = -\sum_{i=1}^n \frac{(C_1-\alpha_i)^2}{8C_2}\left[1+\text{sign}(\alpha_i - C_1)\right]$

(Dual) $$\begin{cases} \max\limits_{\alpha} \ \sum\limits_{i=1}^{n} \alpha_i - \frac{1}{2}\sum\limits_{i,j=1}^{n} \alpha_i\alpha_j y_i y_j x_i^T x_j - \sum\limits_{i=1}^{n} \frac{(C_1-\alpha_i)^2}{8C_2}\left[1+\mathrm{sign}(\alpha_i - C_1)\right] \\ \text{s.t.} \ \sum\limits_{i=1}^{n} \alpha_i y_i = 0 \\ \quad 0 \le \alpha_i \ , \forall i \end{cases}$$

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2. X is finite Set

(a). if $F: 2^X \mapsto \mathbb{R}_+$ is submodular,

that means $F(S)+F(T) \geqslant F(S\cup T) + F(S\cap T)$
$\forall S, T \subseteq X$

then $F(S)+F(T) \geqslant F(S\cup T) + F(S\cap T) \geqslant F(S\cup T)$

so. F is subadditive.

(b).

① F is submodular ② F_S is subadditive

The statement ①⇔② isn't true!

Counter example: If $A\subseteq B$. $A\cup S \neq S$, but $F(A\cup S) < F(S)$

- If F_S is subadditive, $F_S(A) + F_S(B) \geqslant F_S(A\cup B) = F_S(B)$

$\Rightarrow F(A\cup S) - F(S) \geqslant 0$ conflict!

With the addition condition $\boxed{F \text{ is monotone}}$ OR $\boxed{F_S \text{ is non-negative}}$ one can finish the proof.

Pf: ①⇒② F is submodular, then let $T\subset M$, $e \in X/M$

$$F_S(T\cup\{e\}) - F_S(T) = F(S\cup T\cup\{e\}) - F(S\cup T)$$
$$\geq F(S\cup M\cup\{e\}) - F(S\cup M)$$
$$= F_S(M\cup\{e\}) - F_S(M)$$

So, F_S is submodular, with the addition condition

F_S is subadditive

② ⇒ ① Let $S \subseteq T \subseteq X$, $e \in X \setminus T$. $T = S \cup T'$

$$F(T\cup\{e\}) - F(T) = F(S\cup T'\cup\{e\}) - F(S\cup T')$$
$$= F_S(T'\cup\{e\}) - F_S(T')$$
$$\leq F_S(T') + F_S(\{e\}) - F_S(T')$$
$$= F_S(\{e\}) = F(S\cup\{e\}) - F(S)$$

So, $F(T\cup\{e\}) - F(T) \leq F(S\cup\{e\}) - F(S)$

$\Rightarrow F$ is submodular

Q.E.D.

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3. $f(S) =$ # of edge $e = (u,v) \in E$ $u \in S$, $v \in V/S$.

Def: $E(X,Y) = \{(x,y) \in E \mid x \in X, y \in Y\}$

then $f(S) = |E(S, V/S)|$, $|\cdot|$ means the number of elements

☆ we have $E(X\cup\{a\}, Y) = E(X,Y) \cup E(\{a\},Y)$

$E(X,Y) = E(X, Y/\{b\}) \cup E(X,\{b\})$

if $a \notin X$, $|E(X\cup\{a\},Y)| = |E(X,Y)| + |E(\{a\},Y)|$

if $b\in Y$, $|E(X,Y)| = |E(X,Y/\{b\})| + |E(X,\{b\})|$

What's more, if $A\cap X=\emptyset$, $|E(X\cup A,Y)| = |E(X,Y)| + |E(A,Y)|$

if $B\subseteq Y$ $\quad |E(X,Y)| = |E(X,Y/B)| + |E(X,B)|$

Let $A\subset B\subseteq V$, $x\in V/B$, $B=A\cup S$, $\hat{B}=B\cup\{x\}$, $\hat{A}=A\cup\{x\}$, $\hat{S}=S\cup\{x\}$.

$$
\begin{aligned}
f(\hat{B})-f(B) &= |E(\hat{B},V/\hat{B})| - |E(B,V/B)| \\
&= |E(\hat{A},V/\hat{B})| + |E(S,V/\hat{B})| - |E(A,V/B)| - |E(S,V/B)| \\
&= |E(\hat{A},V/\hat{A})| - |E(\hat{A},S)| - |E(A,V/A)| + |E(A,S)| \\
&\quad + |E(S,V/\hat{B})| - |E(S,V/B)| \\
&= f(\hat{A}) - f(A) - 2|E(\{x\},S)| \\
&\le f(\hat{A}) - f(A)
\end{aligned}
$$

$\Rightarrow f(B\cup\{x\}) - f(B) \le f(A\cup\{x\}) - f(A)$

f is submodular

4.

s	a	s'	r	$p(s',r\mid s,a)$
high	search	high	r_{search}	α
high	search	low	r_{search}	$1-\alpha$
high	wait	high	r_{wait}	1.
high	wait	low	-	0

low	search	high	-3	$1-\beta$
low	search	low	r_{search}	β
low	wait	low	r_{wait}	1
low	wait	high	-	0
low	recharge	high	0	1
low	recharge	low	-	0

5. The Bellman equation for q_*

$$q_*(s,a) = \sum_{s',r'} p(s',r|s,a)\left[r + \gamma \max_a q_*(s,a) \right].$$

For cycling robot, state $= \{h, l\}$, action $= \{s, w, re\}$.

$$\begin{cases} q_*(l,s) = \beta\left(r_{search} + \gamma \max_{a'} q_*(l,a')\right) + (1-\beta)\left(-3 + \gamma \max_{a'} q_*(h,a')\right) \\ q_*(l,w) = \left(r_{wait} + \gamma \max_{a'} q_*(l,a')\right) \\ q_*(l,re) = \gamma \max_{a'} q_*(h,a') \\ q_*(h,s) = \alpha\left(r_{search} + \gamma \max_{a'} q_*(h,a')\right) + (1-\alpha)\left(r_{search} + \gamma \max_{a'} q_*(l,a')\right) \\ q_*(h,w) = r_{wait} + \gamma \max_{a'} q_*(h,a') \end{cases}$$