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5.11 原问题可改写为：

$$(P)\begin{cases} \min\limits_{x,y} & \sum_{i=1}^{N} \|y_i\|_2 + \frac{1}{2}\|x-x_0\|_2^2 \\ \text{s.t} & y_i = A_i x + b_i \ , \ i=1,\dots,N \end{cases}$$

则：

$$\mathcal{L}(x,\{y\},\{z\}) = \sum_{i=1}^{N}\|y_i\|_2 + \frac{1}{2}\|x-x_0\|_2^2 + \sum_{i=1}^{N} z_i^T(y_i - A_i x - b_i) \quad , \{y\} = (y_1,\dots,y_N)$$

$$\{z\} = (z_1,\dots,z_N)$$

$$\Rightarrow \quad \inf_{y_i}(\|y_i\|_2 + z_i^T y_i) = \begin{cases} 0 & , \|z_i\|_2 \le 1 \\ -\infty & , \text{o.w.} \end{cases}$$

$$\operatorname*{argmin}_x \left(\frac{1}{2}\|x-x_0\|_2^2 - \sum_{i=1}^{N} z_i^T A_i x\right) = x_0 + \sum_{i=1}^{N} z_i^T A_i$$

$$\Rightarrow \quad g(\{z\}) = \begin{cases} -\sum_{i=1}^{N} z_i^T (A x_0 + b_i) - \frac{1}{2}\left\|\sum_{i=1}^{N} A_i^T z_i\right\|_2^2 \\ -\infty & , \text{o.w} \end{cases}$$

则对偶问题：

$$\begin{cases} \max & -\sum_{i=1}^{N}(A_i x_0 + b_i)^T z_i - \frac{1}{2}\left\|\sum_{i=1}^{N} A_i^T z_i\right\|_2^2 \\ \text{s.t.} & \|z_i\|_2 \le 1 \ , \ i=1,\dots,N \end{cases}$$

5.17 robust LP $\begin{cases} \min & c^T x \\ \text{s.t.} & \sup_{a\in P_i} a^T x \le b_i, \ i=1\sim m \end{cases}$ where $P_i = \{a \mid C_i a \le d_i\}$

$P_i \ne \phi$

robust LP 可化为 $\begin{cases} \min\limits_{x} & c^T x \\ \text{s.t.} & f_i(x) \le b_i, \ i=1\sim m \end{cases}$ 其中 $f_i(x)$ 为 (LP) $\begin{cases} \max\limits_{a} & x^T a \\ \text{s.t.} & C_i a \le d_i. \end{cases}$

的最优值.

该(LP)可化为其对偶形式: $\begin{cases} \min & d_i^T z \\ \text{s.t.} & C_i^T z = x \\ & z \ge 0 \end{cases}$ 由于 LP的最优值 $= f_i(x)$.

$\Rightarrow$. $f_i(x) \le b_i$ iif $\begin{cases} d_i^T z \le b_i \\ C_i^T z = x \\ z \ge 0 \end{cases}$

因此 robust LP 等价于 $\begin{cases} \min\limits_{x,z} & c^T x \\ \text{s.t.} & d_i^T z_i \le b_i \\ & C_i^T z_i = x \quad i=1\sim m \\ & z_i \ge 0 \end{cases}$

5.19. $f(x)=$ the sum of r largest element of x

(a). 为简化记号，假设 $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n$

则 LP $\begin{cases} \max & x^T y \\ \text{s.t.} & 0 \leq y \leq \vec{1} \\ & \vec{1}^T y = r \end{cases}$

的最优值 $= \sum_{i=1}^{r} x_i$，且 $y_1 = y_2 = \cdots = y_r = 1$，$y_{r+1} = \cdots y_n = 0$

即为 r 个最大的元素

因此 $f(x)$ 与该 LP 的最优值等价

(b). 将原 LP 改写为 $\begin{cases} \min & -x^T y \\ \text{s.t.} & 0 \leq y \leq 1 \\ & 1^T y = r \end{cases}$

则有 $\mathcal{L}(y, \lambda, u, t) = -x^T y - \lambda^T y + u^T (y - \vec{1}) + t(1^T y - r)$

$= y^T(-x - \lambda + u + t\vec{1}) - u^T \vec{1} - tr \quad (\lambda \geq 0, u \geq 0$

$g(\lambda, u, t) = \begin{cases} -1^T u - tr & x = u + t\vec{1} - \lambda \\ -\infty & \text{o.w.} \end{cases}$

因此对偶问题：$\begin{cases} \max & -1^T u - tr \\ \text{s.t.} & u + t\vec{1} - \lambda = x \\ & u \geq 0, \lambda \geq 0 \end{cases}$

消去 λ 并且转为 minimize 问题，则有

$$\begin{cases} \min & 1^T u + tr \\ \text{s.t.} & u + t\vec{1} \geq x \\ & u \geq 0 \end{cases}$$

(c) 已知 diversification 约束：

$$\sum_{i=1}^{\lfloor 0.1n \rfloor} x_{[i]} \leq 0.8$$

根据前述讨论，可转化为 LP 约束

$$\begin{cases} \lfloor n/10 \rfloor t + 1^T u \leq 0.8 \\ u + t\vec{1} \geq x \\ u \geq 0 \end{cases}$$

综合得：QP

$$\begin{cases} \min & x^T \Sigma x \\ \text{s.t} & \bar{p}^T x \geq r_{min} \\ & 1^T x = 1, \quad x \geq 0 \\ & \lfloor n/10 \rfloor t + 1^T u \leq 0.8 \\ & u + t\vec{1} \geq x \\ & u \geq 0 \end{cases}$$

with variable x, u, t

13.9 (a) 原问题 $\begin{cases} \min & -5x_1 - x_2 \\ \text{s.t.} & x_1 + x_2 \le 5 \\ & 2x_1 + \frac{1}{2}x_2 \le 8 \\ & x \ge 0 \end{cases}$ 增加 x_3, x_4 可改写为:

标准形式 $\begin{cases} \min & -5x_1 - x_2 \\ \text{s.t.} & x_1 + x_2 + x_3 = 5 \\ & 2x_1 + \frac{1}{2}x_2 + x_4 = 8 \\ & \hat{x} \ge 0 \end{cases}$ 其中 $\hat{x} = (x_1, x_2, x_3, x_4)$

$\Leftrightarrow \begin{cases} \min & c^T\hat{x} \\ \text{s.t} & A\hat{x} = b \\ & \hat{x} \ge 0 \end{cases}$ 其中 $c = (-5, -1, 0, 0)^T$, $b = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$
$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & \frac{1}{2} & 0 & 1 \end{bmatrix}$

(b) 根据 procedure 13.1, 选取 $\mathcal{B} = \{3, 4\}$. , $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

① 则 $x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$, $\lambda = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $S_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

$c^T\hat{x} = 0$

选取 $q = 1$, 则 $d = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $x_q^+ = 4$, $p = 4$

$\Rightarrow x_B^+ = x_B - x_q^+ d = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $x_N^+ = (4, 0)$

$B^+ = \{3,1\},\ N = \{2,4\}$

② $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $C_B = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$, $C_N = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $N = \begin{bmatrix} 0 & 1 \\ 1 & \frac{1}{2} \end{bmatrix}$

则 $X_B = B^{-1}d = \begin{bmatrix} X_3 \\ X_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $\lambda = B^{-T}C_B = \begin{bmatrix} 0 \\ -\frac{5}{2} \end{bmatrix}$

$S_N = \begin{bmatrix} S_4 \\ S_2 \end{bmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{1}{4} \end{pmatrix} > 0$ (stop!)

$C^T\hat{X} = -20$

14.8 由 Algorithm 14.2 得: $\alpha_k \in [0,1]$ 为满足下列条件的最大 α

$(x^k(\alpha), \lambda^k(\alpha), s^k(\alpha)) \in \mathcal{N}_{-\infty}(\gamma)$

其中 $(x^k(\alpha), \lambda^k(\alpha), s^k(\alpha)) = (x^k, \lambda^k, s^k) + \alpha(\Delta x^k, \Delta\lambda^k, \Delta s^k)$

$\mathcal{N}_{-\infty}(\gamma) = \{(x,\lambda,s) \in \mathcal{F}^o \mid x_i s_i \geq \gamma\mu\}$, $\mathcal{F}^o$ 为严格可行集合, $\mu = \frac{x^Ts}{n}$

已知 $(x^k, \lambda^k, s^k) \in \mathcal{N}_{-\infty}(\gamma)$, 则 $(x^k, \lambda^k, s^k) \in \mathcal{F}^o$, $x_i^k s_i^k \geq \gamma\mu_k$

即: $\begin{cases} Ax^k = b \\ A^Ty^k + s^k = c \\ (x_i^k s_i^k) > 0 \end{cases}$ 而 $(\Delta x^k, \Delta\lambda^k, \Delta s^k)$ 由 $\begin{cases} A\Delta x = b - Ax^k \\ A^T\Delta y + \Delta s = c - A^Ty^k - s^k \\ x_i^k \Delta s_i + \Delta x_i s_i^k = \sigma_k\mu_k - x_i^k s_i^k \end{cases}$ 给出

$\Rightarrow \begin{cases} A\Delta x^k = 0 \\ A^T\Delta y^k + \Delta s^k = 0 \end{cases}$ 故 $\forall \alpha \in [0,1]$，有：$\begin{cases} Ax^k(\alpha) = b \\ A^T y^k(\alpha) + s^k(\alpha) = c \end{cases}$ 恒成立

考察：$x_i^k(\alpha) s_i^k(\alpha) \geq \gamma\mu_k(\alpha)$ （略去二阶项）得：

$$\mu_k(\alpha) = \mu_k + \alpha(\sigma_k - 1)\mu_k$$

代入 $\Rightarrow \quad x_i^k s_i^k + \alpha(\sigma_k\mu_k - x_i^k s_i^k) \geq \gamma\mu_k + \alpha(\sigma_k - 1)\gamma\mu_k$

$\Leftrightarrow \quad \alpha\left((\gamma-1)\sigma_k\mu_k + x_i^k s_i^k - \gamma\mu_k\right) \leq x_i^k s_i^k - \gamma\mu_k$

由于 $(\gamma-1)\sigma_k\mu_k \leq 0$，$\alpha \in [0,1]$，故对 $\forall \alpha \in [0,1]$

有 $x_i^k(\alpha) s_i^k(\alpha) \geq \gamma\mu_k(\alpha)$ 恒成立

因此由 $\begin{cases} \max \ \alpha \\ \text{s.t.} \ x^k(\alpha) > 0 \\ \quad\ \ s^k(\alpha) > 0 \\ \quad\ \ \alpha \in [0,1] \end{cases}$ 给出最大的 α

由 $x^k(\alpha) > 0$，$s^k(\alpha) > 0$ 得：

$\alpha < \min\limits_{i,\Delta x_i^k < 0} \dfrac{-x_i^k}{\Delta x_i^k}$ 且 $\alpha < \min\limits_{i,\Delta s_i^k < 0} \dfrac{-s_i^k}{\Delta s_i^k}$，记 $\alpha_1 = \min\limits_{i,\Delta x_i^k < 0} \dfrac{-x_i^k}{\Delta x_i^k}$

$\alpha_2 = \min\limits_{i,\Delta s_i^k < 0} \dfrac{-s_i^k}{\Delta s_i^k}$

综上：

$\Rightarrow \quad \alpha^k = \max(0, \min(\alpha_1, \alpha_2, 1))$