Feuille d'exercice 6 - Systèes de déduction

Exercice 1 – Séquents valides

1. $(p \Rightarrow q \Rightarrow r), (p \Rightarrow q) \vdash \neg p, r$
   formule associée : $(p \Rightarrow q \Rightarrow r) \wedge (p \Rightarrow q) \Rightarrow \neg p \vee r$ Valide

$$\dfrac{\dfrac{\dfrac{}{(p \Rightarrow q), p \vdash p, r}\,\mathit{hyp}}{(p \Rightarrow q) \vdash \neg p, r, p}\,\neg d \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash r, q, p}\,\mathit{hyp}}{\vdash \neg p, r, q, p}\,\neg p \qquad \dfrac{}{q \vdash, \neg p, r, q}\,\mathit{hyp}}{(p \Rightarrow q) \vdash, \neg p, r, q}\,\Rightarrow g \qquad \dfrac{}{r, (p \Rightarrow q) \vdash \neg p, r, q}\,\mathit{hyp}}{(q \Rightarrow r), (p \Rightarrow q) \vdash \neg p, r}\,\Rightarrow q}{(p \Rightarrow q \Rightarrow r), (p \Rightarrow q) \vdash \neg p, r}\,\Rightarrow g$$

   Valide

2. $\vdash (p \wedge q) \vee (\neg p \wedge \neg q)$
   formule associée : $(p \wedge q) \vee (\neg p \wedge \neg q)$ Non Valide

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\mathit{hyp} \qquad p \vdash q}{p \vdash (p \wedge q)}\,\wedge d}{\vdash (p \wedge q), \neg p}\,\neg d \qquad \dfrac{\vdots}{\vdash (p \wedge q), \neg q}\,\neg d}{\vdash (p \wedge q), (\neg p \wedge \neg q)}\,\wedge d$$

   Non valide

3. $p, \bot \vdash$
   formule associée : $p \wedge \bot \Rightarrow \bot$ Valide

$$\dfrac{}{p, \bot \vdash}\,\bot g$$

   Valide

4. $\top \vdash$
   formule associée $\top \Rightarrow \bot$ Non valide

$$\dfrac{\vdash}{\top \vdash}\,\top g$$

   Non Valide

5. $\top \vdash$
   formule associée $\top \Rightarrow \bot$ Non valide

$$\vdash$$

   Non Valide

Exercice 2 – Preuve dans le système G

1. $\vdash ((p \wedge q) \Rightarrow r) \Rightarrow (\neg p \vee q) \Rightarrow (p \Rightarrow r)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{((p \wedge q) \Rightarrow r), p \vdash p, r}\,\mathit{hyp}}{((p \wedge q) \Rightarrow r), \neg p, p \vdash r}\,\neg g \qquad \dfrac{\dfrac{}{r, q, p \vdash r}\,\mathit{hyp} \qquad \dfrac{\dfrac{}{q, p \vdash r, p}\,\mathit{hyp} \qquad \dfrac{}{q, p \vdash r, q}\,\mathit{hyp}}{q, p \vdash r, (p \wedge q)}\,\wedge d}{((p \wedge q) \Rightarrow r), q, p \vdash r}\,\Rightarrow g}{((p \wedge q) \Rightarrow r), (\neg p \vee q), p \vdash r}\,\vee g}{((p \wedge q) \Rightarrow r), (\neg p \vee q) \vdash (p \Rightarrow r)}\,\Rightarrow d}{((p \wedge q) \Rightarrow r) \vdash (\neg p \vee q) \Rightarrow (p \Rightarrow r)}\,\Rightarrow d}{\vdash ((p \wedge q) \Rightarrow r) \Rightarrow (\neg p \vee q) \Rightarrow (p \Rightarrow r)}\,\Rightarrow d$$

2. $\vdash ((p \wedge q) \Rightarrow r) \wedge (\neg p \vee q) \Rightarrow p$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(\neg p \vee q) \vdash p, (p \wedge q)} \qquad \dfrac{\dfrac{r \vdash p}{r, \neg p \vdash p}\,\neg g \qquad r, q \vdash p}{r, (\neg p \vee q) \vdash p}\,\vee g}{((p \wedge q) \Rightarrow r), (\neg p \vee q) \vdash p}\,\Rightarrow d}{((p \wedge q) \Rightarrow r) \wedge (\neg p \vee q) \vdash p}\,\wedge g}{\vdash ((p \wedge q) \Rightarrow r) \wedge (\neg p \vee q) \Rightarrow p}\,\Rightarrow d$$

Exercice 3 – Séquents et quantificateurs

1. $\exists x, \forall y, \mathcal{R}(x,y) \vdash \forall z, \exists t, \mathcal{R}(t,z)$ Valide

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \mathcal{R}(x,z) \vdash \Delta, \mathcal{R}(x,z)}\ hyp}{\Gamma \vdash \Delta, \mathcal{R}(x,z)}\ \forall g}{\Gamma \vdash \Delta := (\exists t, \mathcal{R}(t,z))}\ \exists d}{\Gamma := \forall y, \mathcal{R}(x,y) \vdash \forall z, \exists t, \mathcal{R}(t,z)}\ \forall d}{\exists x, \forall y, \mathcal{R}(x,y) \vdash \forall z, \exists t, \mathcal{R}(t,z)}\ \exists g$$

2. $\forall y, \exists x, \mathcal{R}(x,y) \vdash \exists t, \forall z, \mathcal{R}(t,z)$ Non Valide

$$\dfrac{\dfrac{\dfrac{\dfrac{\color{red}{(\exists x, \mathcal{R}(x,z)), \mathcal{R}(x',z) \vdash \mathcal{R}(t,z), \Delta}}{\exists x, \mathcal{R}(x,z) \vdash \mathcal{R}(t,z), \Delta}\ \exists d}{\forall y, \forall x, \mathcal{R}(x,y) \vdash \mathcal{R}(t,z), \Delta}\ \forall g}{\forall y, \exists x, \mathcal{R}(x,y) \vdash \forall z, \mathcal{R}(t,z), \Delta}\ \forall d}{\forall y, \exists x, \mathcal{R}(x,y) \vdash \Delta := \exists t, \forall z, \mathcal{R}(t,z)}\ \exists d$$

Exercice 4 – x, y, z des varaibles propositionnelles

1. — $(x \Leftrightarrow y)$

$$\begin{aligned}(x \Leftrightarrow y) &\equiv (x \Rightarrow y) \wedge (y \Rightarrow x)\\ &\equiv (\neg x \vee y) \wedge (\neg y \vee x)\\ &\equiv (\neg x \wedge (\neg y \vee x)) \vee (y \wedge (\neg y \vee x))\\ &\equiv (\neg x \wedge \neg y) \vee (\neg x \wedge x) \vee (y \wedge x) \vee (y \wedge \neg y)\\ &\equiv (\neg x \wedge \neg y) \vee (y \wedge x)\end{aligned}$$

   — $\neg(x \Leftrightarrow y)$

$$\begin{aligned}\neg(x \Leftrightarrow y) &\equiv \neg((x \Rightarrow y) \wedge (y \Rightarrow x))\\ &\equiv \neg(x \Rightarrow y) \vee \neg(y \Rightarrow x)\\ &\equiv (x \wedge \neg y) \vee (y \wedge \neg x)\end{aligned}$$

2. $x \Leftrightarrow (y \Leftrightarrow z)$

x	y	z	$y \Leftrightarrow z$	$x \Leftrightarrow (y \Leftrightarrow z)$
0	0	0	1	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	1	1
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

3. (a) — $\dfrac{\Gamma, p, \vdash \Delta, q \quad \Gamma, q, \vdash \Delta, p}{\Gamma \vdash \Delta, p \Leftrightarrow q} \Leftrightarrow d$

Γ	Δ	p	q	$\Gamma, p \vdash \Delta, q$	$\Gamma, q \vdash \Delta, p$	$\Gamma \vdash \Delta, p \Leftrightarrow q$
F	_	_	_	V	V	V
V	V	_	_	V	V	V
V	F	F	F	V	V	V
V	F	F	V	V	F	F
V	F	V	F	F	V	F
V	F	V	V	V	V	V

   — $\dfrac{\Gamma, p, q, \vdash \Delta \quad \Gamma, \vdash \Delta, p, q}{\Gamma, p \Leftrightarrow q \vdash \Delta} \Leftrightarrow g$

Γ	Δ	p	q	$\Gamma, p, q, \vdash \Delta$	$\Gamma, \vdash \Delta, p, q$	$\Gamma, p \Leftrightarrow q \vdash \Delta$
F	–	–	–	V	V	V
V	V	–	–	V	V	V
V	F	F	F	V	F	F
V	F	F	V	V	V	V
V	F	V	F	V	V	V
V	F	V	V	F	V	F

(b) Véritable avec la table de vérité au dessus

(c) $\vdash x \Leftrightarrow (y \Leftrightarrow (x \Leftrightarrow y))$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x, y \vdash y}\,hyp \quad \dfrac{}{x \vdash y, x}\,hyp}{x, (x \Leftrightarrow y) \vdash y}\Leftrightarrow g \quad \dfrac{\dfrac{}{x, y \vdash x}\,hyp \quad \dfrac{}{x, y \vdash y}\,hyp}{x, y \vdash (x \Leftrightarrow y)}\Leftrightarrow d}{x \vdash y \Leftrightarrow (x \Leftrightarrow y)}\Leftrightarrow d \quad \dfrac{\dfrac{\dfrac{}{y, x \vdash x}\,hyp \quad \dfrac{}{y \vdash x, y}\,hyp}{y, (x \Leftrightarrow y) \vdash x}\Leftrightarrow g \quad \dfrac{\dfrac{}{y \vdash x, y}\,hyp \quad \dfrac{}{x \vdash x, y}\,hyp}{\vdash x, y, (x \Leftrightarrow y)}\Leftrightarrow g}{y \Leftrightarrow (x \Leftrightarrow y) \vdash x}\Leftrightarrow g}{\vdash x \Leftrightarrow (y \Leftrightarrow (x \Leftrightarrow y))}\Leftrightarrow d$$

(d) La formule est valide (démonstration au dessus).

Exercice 5 – 1. Construction des clauses associés à E_1 et E_2

— $E_1 = \{(\neg p \vee q), (p \vee \neg q), (p \vee q), (\neg q \vee \neg p)\}$

— $E_2 = \{(\neg p \vee q), \neg q, q, p\}$

2. Réfutation pour chacun de ces ensembles :

— E_1 :

$$\dfrac{\dfrac{p \vee \neg q \quad p \vee q}{p}\,q \quad \dfrac{\neg p \vee q \quad \neg p \vee \neg q}{\neg p}\,q}{\bot}\,p$$

— E_2 :

$$\dfrac{\neg q \quad q}{\bot}\,q$$

Exercice 6 – 1. `valide`(A) :

— `valide`$(A) =$ `deductionG`(A)

— `valide`$(A) =$ `refutation(clausale(`$\neg A$`))`

— `valide`$(A) =$ `non(sat(clausale(`$\neg A$`)))`

2. — `satsifiable`(A) :

  — `satsifiable`$(A) =$ `non(deductionG(`$\neg A$`))`

  — `satsifiable`$(A) =$ `non(refutation(clausale(`A`)))`

  — `satsifiable`$(A) =$ `sat(clausale(`A`))`

— `insatisfiable`(A) :

  — `insatisfiable`$(A) =$ `non(deductionG(`A`))`

  — `insatisfiable`$(A) =$ `refutation(clausale(`A`))`

  — `insatisfiable`$(A) =$ `non(sat(clausale(`A`)))`

Exercice 7 –

$$\dfrac{C \vee \neg p \vee \neg q \quad C' \vee p \vee q}{C \vee C'}$$

La règle n'es pas correcte :

$$\dfrac{\dfrac{C \vee \neg p \vee \neg q \quad C' \vee p \vee q}{C \vee C' \vee \neg p \vee p}\,q}{\top}$$

Ou sinon

$$\begin{array}{ll} I \models C \vee \neg p \vee q & q(V) \\ I \models C' \vee p \vee \neg q & p(V) \\ I \not\models C \vee C' & C(F), C'(F) \end{array}$$

Exercice 8 – Unification

1. $f(x,x,y) \stackrel{?}{=} f(f(y,y,z), f(y,a,z), a)$

$$\begin{array}{rlr} \{\} & \left\{x \stackrel{?}{=} f(y,y,z); x \stackrel{?}{=} f(y,a,z); y \stackrel{?}{=} a\right\} & 7 \\ \{y \leftarrow a\} & \left\{x \stackrel{?}{=} f(a,a,z)\right\} & 4 \\ \{y \leftarrow a; x \leftarrow f(a,a,z)\} & \{\} & 4 \end{array}$$

2. $f(x,x,y) \stackrel{?}{=} f(f(y,y,z), f(y,x,z), a)$

$$\begin{array}{rlr} \{\} & \left\{x \stackrel{?}{=} f(y,y,z); x \stackrel{?}{=} f(y,x,z); y \stackrel{?}{=} a\right\} & 7 \\ \{y \leftarrow a\} & \left\{x \stackrel{?}{=} f(a,a,z) x \stackrel{?}{=} f(a,x,z)\right\} & 4 \\ \{y \leftarrow a; x \leftarrow f(a,a,z)\} & \left\{f(a,a,z) \stackrel{?}{=} f(a, f(a,a,z), z)\right\} & 4 \\ \{y \leftarrow a; x \leftarrow f(a,a,z)\} & \left\{a \stackrel{?}{=} a, a \stackrel{?}{=} f(a,a,z), z \stackrel{?}{=} z\right\} & 7 \\ \{y \leftarrow a; x \leftarrow f(a,a,z)\} & \left\{a \stackrel{?}{=} f(a,a,z)\right\} & 2 \times 2 \\ \{y \leftarrow a; x \leftarrow f(a,a,z)\} & \left\{a \stackrel{?}{=} f(a,a,z)\right\} & \text{échec } 6 \end{array}$$

Exercice 9 – Soit les formules :

$$H_1 = (\forall x, P(x) \vee Q(x)) \quad H_2 = (\exists x, P(x) \Rightarrow Q(x)) \quad C = \exists x, Q(x)$$

1. — $H_1 \rightsquigarrow \{P(x) \vee Q(x)\}$
   — $H_2 \rightsquigarrow \{\neg P(a) \vee Q(a)\}$
   — $\neg C \rightsquigarrow \{\neg Q(x)\}$
   — $\{H_1, H_2, \neg C\} \rightsquigarrow \{P(x) \vee Q(x), \neg P(a) \vee Q(a), \neg Q(x)\}$
2. $\{P(a) \vee Q(a), \neg P(a) \vee Q(a), \neg Q(a)\}$

$$\dfrac{\dfrac{P(a) \vee Q(a) \quad \neg P(a) \vee Q(a)}{Q(a)}\, P(a) \qquad \neg Q(a)}{\perp}\, Q(a)$$

3.

$$\dfrac{\dfrac{\dfrac{P(x) \vee Q(x)}{P(a) \vee Q(a)}\,\text{[x]a} \quad \neg P(a) \vee Q(a)}{Q(a)}\, P(a) \qquad \dfrac{\neg Q(x)}{\neg Q(a)}\,\text{[x]a}}{\perp}\, Q(a)$$

Exercice 10 – *Résolution (exame session12020)*

1. $A, B, C, \neg D \models \perp$
   — $A \stackrel{def}{=} \forall x, (\forall y, P(x,y) \Rightarrow V(y)) \Rightarrow H(x)$

$$\begin{aligned} A &\equiv \forall x, (\neg(\forall y, P(x,y) \Rightarrow V(y))) \vee H(x) \\ &\equiv \forall x, (\exists y, P(x,y) \wedge \neg V(y)) \vee H(x) \\ &\rightsquigarrow \forall x, (P(x, f(x)) \wedge \neg V(f(x))) \vee H(x) \\ &\rightsquigarrow (P(x, f(x)) \wedge \neg V(f(x))) \vee H(x) \\ &\equiv (P(x, f(x)) \vee H(x)) \wedge (\neg V(f(x)) \vee H(x)) \end{aligned}$$

$\{(P(x, f(x)) \vee H(x)), (\neg V(f(x)) \vee H(x))\}$

— $B \overset{def}{=} \forall x, B(x) \Rightarrow V(x)$

$$\begin{aligned} B &\equiv \forall x, \neg B(x) \vee V(x) \\ &\rightsquigarrow \neg B(x) \vee V(x) \end{aligned}$$

$\{\neg B(x) \vee V(x)\}$

— $C \overset{def}{=} \forall x, (\exists y, B(y) \wedge P(y,x)) \Rightarrow B(x)$

$$\begin{aligned} C &\equiv \forall x, (\neg(\exists y, B(y) \wedge P(y,x))) \vee B(x) \\ &\equiv \forall x, (\forall y, \neg B(y) \vee \neg P(y,x)) \vee B(x) \\ &\equiv \forall x, \forall y, \neg B(y) \vee \neg P(y,x) \vee B(x) \\ &\rightsquigarrow \neg B(y) \vee \neg P(y,x) \vee B(x) \end{aligned}$$

$\{\neg B(y) \vee \neg P(y,x) \vee B(x)\}$

— $D \overset{def}{=} \forall x, B(x) \Rightarrow H(x)$

$$\begin{aligned} \neg D &\equiv \neg(\forall x, \neg B(x) \vee H(x)) \\ &\equiv \exists x, B(x) \wedge \neg H(x) \\ &\rightsquigarrow B(a) \wedge \neg H(a) \end{aligned}$$

$\{B(a), \neg H(a)\}$

Pour montrer $A, B, C \models D$ on va utiliser la forme clausale de l'ensemble de formules :
$\{A, B, C, \neg D\} =$
$\{(P(x, f(x)) \vee H(x)), (\neg V(f(x)) \vee H(x)), (\neg B(x) \vee V(x)), (\neg B(y) \vee \neg P(y,x) \vee B(x)), B(a), \neg H(a)\}$

$$\dfrac{\neg H(a) \qquad \dfrac{B(a) \qquad \dfrac{\dfrac{P(x, f(x \vee H(x))}{P(a, f(a)) \vee H(a)} \qquad \dfrac{\dfrac{\dfrac{\neg B(y) \vee \neg P(y,x) \vee B(x)}{\neg B(y) \vee \neg P(y, f(a)) \vee B(f(a))} \qquad \dfrac{\dfrac{V(x) \vee \neg B(x)}{V(f(a)) \vee \neg B(f(a))} \qquad \dfrac{\neg H(a) \qquad \dfrac{\neg V(f(x)) \vee H(x)}{\neg V(f(a)) \vee H(a)}}{\neg V(f(a))}}{\neg B(f(a))}}{\neg B(y) \vee \neg P(y, f(a))}}{\neg B(a) \vee \neg P(a, f(a))}}{\neg B(a) \vee H(a)}}{H(a)}}{\bot}$$

2. On a le dommaine $\mathcal{D}\,\{Puff, Draco, Saphira\}$
   — $B_I = \{Puff, Draco\}$
   — $H_I = \{Puff, Saphira\}$
   — $V_I = \{Draco\}$

Exercice 11 – Relation Binaire
— Totale : $\forall x, \exists y, R(x,y)$
— Symétrique : $\forall x, y, R(x,y) \Rightarrow R(y,x)$
— Transitive : $\forall x, y, z, R(x,y) \wedge R(y,z) \Rightarrow R(x,z)$
— Reflexive : $\forall x, R(x,x)$
On ve montrer $Total, Symtrique, Transitive \models Reflexive$
Pour cela on ve montrer $Total, Symtrique, Transitive, \neg Reflexive \models \bot$
On a donc la clause : $\{R(x, f(x)), \neg R(y,x) \vee R(y,x), \neg R(x,y) \vee \neg R(y,z) \vee R(x,z), R(a,a)\}$

$$\dfrac{R(a,a) \qquad \dfrac{\dfrac{}{R(x,z)}}{R(a,a)}}{\bot}$$